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AGGREGATE EXPECTED CONSUMER SURPLUS AS A WELFARE INDEX WITH AN APPLICATION TO PRICE STABILIZATION

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I. INTRODUCTION

Since the 1940s, economists have used aggregate expected consumer surplus in order to examine the question of price stabilization [2,3,6,7,8,9]. Unfortunately, they have devoted little consideration to the assumptions underlying its use. The implicit assumption seems to have been that the condition sufficient for consumer surplus to be a welfare measure in a world of certainty (constant marginal utility of income with respect to price in the relevant market on the part of all individuals) is also sufficient for expected consumer surplus to be a welfare measure in a world or risk. This paper shows that this assumption is untrue in general.

Fortunately for researchers in applied fields, especially for those in agricultural economics, the assumption is true for the case where all the stochastic variation in prices originates from the supply side of the market. However, if variation in prices also originates from the demand side, then additional assumptions are required. What is particularly unfortunate is that in some cases of demand induced stochastic price variation, these assumptions are inconsistent with the ability of stochastic price variation to originate from the demand side in the first place.

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II. ASSUMPTIONS AND DEFINITIONS[1]

Consider an economy with m goods, one firm, and n consumers. Adopting a partial equilibrium view of market one, assume that prices of other goods, $\{P_j\}_{j=2}^m$, are exogenous random variables. As well, assume the existence of a random variable, V, which affects supply of good one, and n random variables, $\{W_i\}_{i=1}^n$, W_i affecting the i^{th} consumer's preferences over goods. Finally, assume that the income of the i^{th} consumer, M_i, is also a random variable. Large case letters will always be used to denote random variables, while the smaller case of the same letter will denote a particular realization of the random variable.

The following notation can now be introduced:

x_{ij},	Amount of good j consumed by consumer i.
M_i,	Income of consumer i.
$u_i(x_{i1},\ldots,x_{in},w_i)$	Utility function of i^{th} consumer which is compatible with the expected utility hypothesis.
$d_{ij}(p_1,\ldots,p_m,m_i,w_i)$,	Demand function for j^{th} good by i^{th} consumer.

$g_i(p_1,\ldots,p_m,m_i,w_i)$,	Indirect utility function of i^{th} consumer.
d_j,	equals $\sum_{i=1}^{n} d_{ij}$.
$\delta_i(p_1,\ldots,p_m,m_i,w_i)$,	marginal utility of income function. for i^{th} consumer.
$s_j(p_1,\ldots,p_m,v)$	supply function of j^{th} good.
M,	the vector $(M_1,\ldots,M_n)$
W,	the vector $(W_1,\ldots,W_n)$
P,	the vector $(P_2,\ldots,P_n)$
m,	the vector $(m_1,\ldots,m_n)$
w,	the vector $(w_1,\ldots,w_n)$
p,	the vector $(p_2,\ldots,p_n)$

Note in particular that P and p are vectors of prices excluding price of the first good.

Equality of supply and demand in market one determines the price in this market. Since the demand and supply for good one are affected by the above 2n + m random variables, the price of good one is a function of these random variables and is thus itself a random variable. That is, price in market one is determined by

$$d_1(p_1,p,m,w) = s_1(p_1,p,v).$$

Given the usual assumptions,[2] the implicit function theorem implies the existence of a function ϕ defined implicitly by the above.

$$p_1 = \phi\ (p,m,w,v)$$

We can compare this case to the case where government can enter the market by buying and selling so as to stabilize p_1 at some constant, p^*. This price is chosen so that government will buy and sell equal amounts over the long run and thus hold on average "zero" buffer stocks. The first case, that of no government interference, involves p_1 being a random variable determined by ϕ. The second case, that of government stabilization, involves p_1 being constant at p^*. More generally, if we allowed government the policy option of only partially stabilizing prices, the second case would involve p_1 being a random variable determined by ψ (P,W,V,M) where ψ is some function.

III. NECESSARY AND SUFFICIENT CONDITIONS FOR AGGREGATE EXPECTED CONSUMER SURPLUS TO BE A PARETO WELFARE MEASURE

Consider the general situation where the set of all possible states of the world is Ω and there are n individuals with preferences over Ω. The i^{th} individual's preferences are represented by the real valued function γ_i defined on Ω. That is, for x,y and Ω, we have x preferred by individual i to y if and only if $\gamma_i(x) > \gamma_i(y)$. Notions of social welfare almost inevitably involve interpersonal comparisons; there are some losers and winners. Certainly, however, a minimum requirement for any real valued function over Ω purported to represent social welfare is that it be consistent with the Pareto criterion. That is, the index should rank one state as being better (worse) than another if all individuals evaluate[3] it as being no worse (better) and at least one individual evaluates it as being better (worse).

Definition:

Let $A \subseteq \Omega$. Then a real valued function on Ω is called Pareto on A if it is consistent with the Pareto criterion over A.

In our case Ω can be viewed as all probability distribution functions over $R+^n x R^n x R+^n$. The distribution functions correspond to random vectors (P_1,P,W,M). The individuals' preferences are of course represented by the functionals which assign the expected

value of g_i under F to the distribution F. Let G be any distribution function over the last 3n-1 coordinates of $R+^n x R^n x R+^n$. Then let Ω_G be the set of all elements of Ω having marginal distribution G over the last 3n-1 coordinates. We need a social welfare index to compare elements of Ω within the same Ω_G. P,W and M are fixed random variables. We compare the results of having P_1 be $\phi(P,W,M,V)$ to P_1 being $\psi(P,W,M,V)$. That is, for fixed exogenous behavior of other prices, we compare the alternatives of having P_1 be the random variable generated by market forces or of having P_1 be some other random variable generated by government action. Therefore, any welfare index we use to make our decision should be Pareto over Ω_G for every G[4].

Cast in these terms, we want to know if the expected value of aggregate consumer surplus is Pareto over Ω_G for every G. The question is best answered by first considering the individual. Let E be the expectated value operation, EC_i be the expected value of consumer surplus for individual i viewed as a function from Ω to R, and EC be the expected value of aggregate consumer surplus. We will call a distribution function constant if it assigns a probability of one to a single point. A distribution function will be called constant over a subset of the variables it is defined over if the relevant marginal distribution function is constant. Consider the whole class of distributions over all but the first coordinate of $R+^n x R^n x R+^n$. These are interpreted as distributions for the random vector (P,W,M). For any S, a subset of $\{p_2,\ldots,p_n,w_1,\ldots w_n,m_1,\ldots m_n\}$, we can select out the distribution functions which are constant for this subset. Let Δ_S be these distributions. We can now state the theorem concerning individuals.

Theorem One:

Fix any $i \in \{1,\ldots,n\}$.

Fix any $S \subseteq \{p_2,\ldots,p_n,w_i,m_i\}$.

Then I and II are equivalent.

I. EC_i represents consumer i's preferences over Ω_G for every G in Δ_S.

II. (i) $\frac{D\delta_i}{Dp_1} = 0$

(ii) $\frac{D\delta_i}{Dp_j} = 0$ if p_j is not in S.

(iii) $\frac{D\delta_i}{Dw_i} = 0$ if w_i is not in S.

(iv) $\frac{D\delta_i}{Dm_i} = 0$ if m_i is not in S.

Proof:

Only II $\Rightarrow$ I will be proved here. The reverse is more difficult and is left to an appendix.

Let F_1 and F_2 be any two elements of Ω_G for some G in Δ_S. We want to show that using expected consumer surplus to compare F_1 and F_2 gives the same result as using expected utility. The latter, by definition, represents the i^{th} consumer's preferences.

$$EC_i(F_1) \gtreqless EC_i(F_2)$$

$$\iff \int\left(\int_{p_1}^{\infty} d_{i1}(z,p,w_i,m_i)\,dz\right)dF_1 \gtreqless \int\left(\int_{p_1}^{\infty} d_{i1}(z,p,w_i m_i)\,dz\right)dF_2$$

$$\iff \int\left(\frac{g_i(p_1,p,w_i,m_i) - g_i(\infty,p,w_i,m_i)}{\delta_i(p_1,p,w_i,m_i)}\right)dF_1$$

$$\gtreqless \int\left(\frac{g_i(p_1,p,w_i,m_i) - g_i(\infty,p,w_i,m_i)}{\delta_i(p_1,p,w_1,m_i)}\right)dF_2$$

This last step is by Roy's duality theorem. See Quirk [4], page 55, for a derivation. Now for each argument of δ_i, either δ_i is constant with respect to it because the derivative of δ_i with respect to it is zero or both F_1 and F_2 have constant marginals for the variable. Therefore by evaluating δ_i at the points of probability one and calling this number δ_i^*, we have

$$EC_i(F_1) \gtreqless EC_i(F_2)$$

$$\iff \frac{1}{\delta_i^*}\int\left(g_i(p_1,p,w_i,m_i) - g_i(\infty,p,w_i,m_i)\right)dF_1$$

$$\gtreqless \quad \frac{1}{\delta_i^*} \int \left(g_i(p_1,p,w_i,m_i) - g_i(\infty,p,w_i,m_i) \right) dF_2$$

$$\Longleftrightarrow \quad \frac{1}{\delta_i^*} \int g_i(p_1,p,w_i,m_i)\, dF_1 \gtreqless \frac{1}{\delta_i^*} \int g_i(p_1,p,w_i,m_i)\, dF_2$$

$$\Longleftrightarrow \quad \int g_i(p_1,p,w_i,m_i)\, dF_1 \gtreqless \int g_i(p_1,p,w_i,m_i)\, dF_2$$

The second to the last step is possible because the marginals of F_1 and F_2 are the same over (p,w_i,m_i). The last step is possible because standard consumer theory predicts δ_i^* is always positive.

□

The translation to sufficient conditions for aggregate expected consumer surplus to be Pareto over Ω_G for every G is now immediate. The question of necessary conditions is discussed in the appendix. The sufficient conditions are, in a practical sense, "close" to being necessary. This should be kept in mind during the discussion in section IV.

<u>Theorem Two:</u>

Fix any $S \subseteq \{p_2,\ldots,p_n,w_1,\ldots,w_n,m_1,\ldots m_n\}$.

Suppose the following hold for every i.

(i) $\frac{D\delta_i}{Dp_1} = 0$

(ii) $\frac{D\delta_i}{Dp_j} = 0$ if p_j is not in S

(iii) $\frac{D\delta_i}{Dw_i} = 0$ if w_i is not in S

(iv) $\frac{D\delta_i}{Dm_i} = 0$ if m_i is not in S

Then EC is a Pareto welfare index over Ω_G for every G in Δ_S.

<u>Proof:</u>

Let F_1 and F_2 be elements of Ω_G for some G. Then

$$EC(F_1) \gtreqless EC(F_2)$$

$$\Longleftrightarrow \quad \int \left(\int_{p_1}^{\infty} \sum_{i=1}^{n} d_{i1}(z,p,w_i,m_i)\, dz \right) dF_1$$

$$\gtreqless \quad \int \left(\int_{p_1}^{\infty} \sum_{i=1}^{n} d_{i1}(z,p,w_i,m_i)\, dz \right) dF_2$$

$$\iff \sum_{i=1}^{n} \int \left(\int_{P_1}^{\infty} d_{i1}(z,p,w_i,m_i)\; dz \right) dF_1$$

$$\gtreqless \sum_{i=1}^{n} \int \left(\int_{P_1}^{\infty} d_{i1}(z,p,w_i,m_i)\; dz \right) dF_2$$

$$\iff \sum_{i=1}^{n} \frac{1}{\delta_i^*} \int g_i(p_1,p,w_i,m_i)\; dF_1$$

$$\gtreqless \sum_{i=1}^{n} \frac{1}{\delta_i^*} \int g_i(p_1,p,w_i,m_i)\; dF_2$$

This last step uses the results in theorem one. Therefore, within a given Ω_G, EC operates like a positive weighted sum of the individual utility indexes. A function of this type is of course Pareto.

□

IV INTERPRETATION

Theorem One states that a necessary and sufficient condition for expected consumer surplus to represent a consumer's preferences over changes in the random variable P_1 is that the marginal utility of income be constant not only with respect to P_1 but also with respect to any other factor which is a random variable. Theorem Two points out that if this condition is satisfied for every consumer, then aggregate expected consumer surplus is merely a positive weighted sum of individual expected consumer surplus. Therefore, aggregate expected surplus is Pareto.

The simplest case is that where V is the only random element; P,W, and M are all constant. In terms of the supply and demand curves for market one, the supply curve is shifting but demand is stationary. In this case, the sufficient condition for EC to be a Pareto measure is that every consumer's marginal utility of income be constant with respect to P_1. This is, of course, the condition for the case of certainty as well. Therefore, if all stochastic variation originates from the supply curve, the condition sufficient for aggregate consumer surplus to be Pareto in a riskless world is also sufficient for EC to be Pareto in a world of risk.

Two particular cases of demand induced stochastic variation can be shown to never satisfy the sufficient conditions for EC to be Pareto. First is the case where fluctuations in income cause demand to shift.[5] In this case, the sufficient conditions for EC to be Pareto imply that income changes could not produce demand changes. That is, observation of income induced demand shifts in and of itself constitutes evidence that the sufficient conditions cannot hold. In principle, therefore, aggregate expected consumer surplus cannot be used as a welfare measure for cases of income induced stochastic price variation. This argument is proved in theorem three.

Theorem Three:

If $\frac{D\delta_i}{Dm_i} = 0$ and $\frac{D\delta_i}{Dp_1} = 0$

then $\frac{Dd_{i1}}{Dm_i} = 0$

Proof:

We prove the contrapositive. From Roy's duality theorem (dropping the subscript i for consumer i).

$$d_1 = \frac{-\frac{Dg}{Dp_1}}{\frac{Dg}{Dm}} = \frac{-\frac{Dg}{Dp_i}}{\delta}$$

Differentiate both sides with respect to m, which yields

$$\frac{D^2g}{Dp_1Dm} = -\frac{D\delta}{Dm}\,x_1 - \delta\,\frac{Dd_1}{Dm}$$

The result now follows immediately.

□

The same type of problem arises for a special case where W is random. If W_i affects the consumer's marginal utility with respect to the j^{th} good, then the consumer's demand for x_j will vary randomly as a function of W_i. As a consequence, if we assume that $\frac{D^2g_i}{Dw_iDp_j}$ is unequal to zero for some i, we should also allow p_j to be a nonconstant random variable.[6] In this situation, if W affects each marginal utility in the same direction.

($\frac{D^2g_i}{Dw_iDp_j}$ is non-negative for every j or is non-positive for every j), then the sufficient conditions for EC to be a Pareto measure cannot occur. The following proof of this fact will also make clear that in the general case, the assumption that $\frac{D\delta_i}{Dw_i} = 0$ amounts to a restriction of the vector

$$\left(\frac{D^2g_i}{Dw_iDp_1}, \ldots, \frac{D^2g_i}{Dw_iP_n}\right)$$ to a particular hyperplane in R^n. Only very special cases of W affecting utility are thus consistent with the sufficient conditions for EC to be a correct measure.

Theorem Four:

If $u_i = f_i(x_{i1},\ldots,x_{in}, w_i)$ and

$$\frac{D^2u_i}{Dw_iDx_{ij}} < 0 \text{ for } j = 1,\ldots,\ell$$

$$= 0 \text{ for } j = \ell+1,\ldots,n$$

or $\frac{D^2 u_i}{Dw_i Dx_{ij}} > 0$ for $j=1,\ldots,\ell$

$= 0$ for $j=\ell+1,\ldots,n$

then $\frac{D\delta_i}{Dw_i} = 0 \Rightarrow \frac{D\delta_i}{Dp_j} \neq 0$ for some $j=1,\ldots,\ell$.

Proof:

By totally differentiating the first order conditions we obtain (dropping the subscript i for consumer i)

$$\begin{bmatrix} dx_1 \\ \cdot \\ \cdot \\ \cdot \\ dx_n \\ d\lambda \end{bmatrix} = \left(\frac{A_{ji}}{|A|}\right) \begin{bmatrix} -\lambda dp_1 - u_{1w}dw \\ \cdot \\ \cdot \\ \cdot \\ -\lambda dp_n - u_{nw}dw \\ -\sum_{i=1}^{n} x_i dp_i + dm \end{bmatrix}$$

where $A = \begin{bmatrix} u_{11} & \cdots & u_{1n} & p_1 \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ u_{n1} & \cdots & u_{nn} & p_n \\ p_1 & \cdots & p_n & 0 \end{bmatrix}$

and A_{ij} is the ij^{th} cofactor of A, $-\lambda$ is the marginal utility of income, and $u_{iw} = \frac{D^2 u}{DwDp_i}$

Now suppose $\frac{D\lambda}{Dp_j} = 0 \;\forall\, j = 1,\ldots\ell$.

Then $0 = \frac{D\lambda}{Dp_j}$

$$= -\lambda \frac{A_{j,n+1}}{|A|} - x_j \frac{A_{n+1,n+1}}{|A|}.$$

Therefore $\frac{A_{j,n+1}}{|A|}$ has the same sign for every $j = 1,\ldots\ell$.

Now $\frac{D\lambda}{Dw_i} = -\sum_{j=1}^{\ell} \frac{A_{j,n+1}}{|A|} u_{jw}$.

Since the u_{jw} also all have the same sign, $\frac{D\lambda}{Dw}$ consists of the sum of a group of similarly signed non-zero elements. Therefore $\frac{D\lambda}{Dw} \neq 0$.

□

V. AN EXAMPLE

The following is an example of a one consumer world where expected consumer surplus ranks the alternatives of stabilization vs. nonstabilization differently than does expected utility. The source of random variation is the price of good x and government is considering stabalizing the price of good y. Let the consumer's utility function be

$$u(x,y) = 2y^{1/2} + x$$

With corresponding indirect utility function

$$g(p_x, p_y, m) = \frac{p_x^2 + mp_y}{p_x p_y} .$$

Let P_x be the random variable

$$P_x = \begin{cases} 1, \text{ with probability } .7 \\ 4, \text{ with probability } .3 \end{cases}$$

and let the supply curve for y be

$$s = p_y^2 .$$

The price of good y determined by supply and demand equilibriating is then

$$P_y = \sqrt{P_x} .$$

The buffer stock price, p^*, of good y is that price which makes the expected value of government purchases, b, zero.

$$E(b) = 0 \Rightarrow E\left(\frac{P_x^2}{p_y^2} - p_y^2\right) = 0$$

$$\Rightarrow \frac{E(P_x^2)}{p_y^2} - p_y^2 = 0$$

$$\Rightarrow p^* = \sqrt[4]{E(P_x^2)}$$

We now need to calculate four numbers: the expected utility from each policy and the expected consumer surplus from each policy.

Expected Utility From Stabilization:

$$Eg(P_x,p^*) = E\left(\frac{P_x^2 + mp^*}{P_x p^*}\right)$$

$$= \frac{E(P_x)}{\sqrt[4]{E(P_x^2)}} + m\,E(\frac{1}{P_x})$$

Expected Utility from No Stabilization:

$$Eg(P_x,P_y) = E\ \frac{P_x^2 + mP_y}{P_x P_y}$$

$$= E\ (\sqrt{P_x}) + m\ E(\frac{1}{P_x})$$

Expected Consumer Surplus from Stabilization

$$EC(P_x,p^*) = E\left(\int_{p^*}^{\infty} \frac{P_x^2}{P_y^2}\, dp_y\right)$$

$$= E\left(\frac{P_x^2}{p^*}\right) = \frac{E(P_x^2)}{\sqrt[4]{E(P_x^2)}}$$

$$= E(P_x^2)^{3/4}$$

Expected Consumer Surplus from No Stabilization:

$$EC(P_x,P_y) = E\int_{P_y}^{\infty} \frac{P_x^2}{P_y}\, dp_y$$

$$= E\int_{\sqrt{P_x}}^{\infty} \frac{P_x^2}{P_y}\, dp_y$$

$$= E\ \frac{P_x^2}{\sqrt{P_x}} = E(P_x^{3/2})$$

Therefore the gains from stabilization according to the expected utility index are

$$\frac{E(P_x)}{\sqrt[4]{E(P_x^2)}} - E(\sqrt{P_x})$$

Substituting in yields the answer -.0593. The expected utility criterion thus says that government should not stabilize prices. The gains from stabilization according to the expected consumer surplus index are

$$[E\ (P_x^2)\ ]^{3/4} - E(P_x^{\ 3/2}).$$

Substituting in yields +.4914. The expected consumer surplus index thus says that government should stabilize prices, which contradicts the conclusion drawn from the expected utility index.

VI. CONCLUSION

The use of aggregate expected consumer surplus to analyze the welfare implications of stochastic price variation requires no assumptions other than those that are required for the use of this method under conditions of certainty provided that the source of price variation lies solely in the supply curve of the problem. In the case of demand side variation, however, consumer surplus yields an ordering consistent with the Pareto criterion only when further assumptions are made. If variation in other prices causes demand side variation, then it is sufficient to additionally assume that the marginal utilities of income of all consumers be constant with respect to these prices. If variations in natural events such as rainfall or sunshine cause demand variation by directly affecting utility, the additional sufficient assumption amounts to be very restrictive condition on the nature of the effect of the event on preferences. Finally, in the case where random variation of income induces stochastic demand variation, observation of this phenomena in and of itself constitutes evidence that the sufficient conditions for EC to be a correct measure cannot occur.

In general, therefore, it seems that aggregate expected consumer surplus is most useful when stochastic varation results primarily from the supply side and does not significantly affect prices of other goods either indirectly through general equilibrium effects or directly through affecting supply curves of related markets.

APPENDIX

The main purpose of this appendix is to derive a necessary and sufficient condition for a function over R^n to represent a consumer's preferences over density functions over R^n when comparisons between density functions are restricted to those where the density functions have the same marginal distributions with respect to the last t variables. As a corollary, we will have a necessary and sufficient condition for expected consumer surplus to represent a consumer's preferences between price stabilization policies. Furthermore, we will be able to discuss the correctness of aggregate expected consumer surplus in similar situations.

First , some notation and definitions must be introduced. Let $Z = R^{s+t}$, $X = R^s$, and $Y = R^t$. Call Ω the set of all density functions on Z. Allow Ω to include discrete random variables (or mixtures of discrete and continuous random variables) by interpreting integration as summation when necessary. Call Ωg the set of all elements of Ω with marginal distribution g over Y. For any real valued measurable function u on Z and $f \varepsilon \Omega$, define Eu(f) by

$$Eu(f) = \int u(x,y)\, f(x,y)\, dxdy.$$

Consider any relation $\lesssim$ on Ω. We will say u represents $\lesssim$ on Ω if

$$Eu(f_1) \gtreqless Eu(f_2) \text{ iff } f_1 \gtrsim\!\!\!\!\!< f_2$$

for every f_1, f_2 ε Ω. Representation over Ωg is defined in an analogous fashion.

The content of the expected utility theorem[7] is that we can make a number of assumptions about $\lesssim$ which allow us to conclude that there is a real valued function u over Z which represents $\lesssim$. Furthermore, for any other real-valued function, u*, on Z, u* represents $\lesssim$ if and only if u* = au+b, where a and b are constants and a > 0. We are interested in finding necessary and sufficient conditions for u* to represent $\lesssim$ on Ωg for every g. This is a weaker requirement than representing $\lesssim$ on Ω. In the former u* only has to be correct when comparing density functions with the same marginal distributions. In the latter, u* has to be correct when comparing any two density functions. Therefore we expect a somewhat weaker necessary and sufficient conditions than u* = au+b where a > 0 to emerge. Two definitions are needed to state the theorem. For y ε Y, we call the fibre of y the set X × y. For two elements y_0 and y_1 of Y, we say that there is a preference reversal between the fibres of y_0 and y_1 if there are two density functions f' and f" over x such that

$$(f',y_0) > (f",y_0)$$

and $(f",y_1) > (f',y_1)$ where one of the > signs may be $\gtrsim$. The ordered pair (f,y) denotes the density function with marginal density f over x and discrete marginal density of y with probability 1 over Y. A preference reversal between two fibres simply

means that the consumer's preferences between two densities over x depend on what value of Y he receives.

Theorem 5:

Let $\lesssim$ be a relation over Ω and let u be a real valued function representing $\lesssim$. Then for any measurable real valued function u* defined on Z, the following three statements are equivalent.

I u* represents $\lesssim$ over Ω_g for every g.

II $u^*(x,y) = a(y) \cdot u(x,y) + b(y)$

where

(i) a(y) > 0 for every y.

(ii) $a(y_0) = a(y_1)$ if there is a preference reversal between the fibres of y_0 and y_1.

III $u^*(x,y) = a(y) \cdot u(x,y) + b(y)$

where

(i) a(y) > 0 for every y.

(ii) a(y) is constant over y if there is one instance of a preference reversal between two fibres.

Proof:

II ⟹ III:

Suppose there is at least one instance of a preference reversal between the fibres of two points y_0 and y_1. Then $a(y_0) = a(y_1)$. Consider any point $y \in Y$. There must be a preference reversal between y and one of y_0 and y_1. Therefore $a(y) = a(y_0) = a(y_1)$.

III ⟹ I:

$$Eu^*(f_1) \gtrless Eu^*(f_2)$$

$$\Longleftrightarrow \int u^*(x,y)\ f_1(x,y)\ dxdy \gtreqless \int u^*(x,y)\ f_2(x,y)\ dxdy$$

$$\Longleftrightarrow \int a(y)\ u(x,y)\ f_1(x,y)\ dxdy \gtreqless \int a(y)\ u(x,y)\ f_2(x,y)\ dxdy$$

because b is not a function of x and the marginal distributions of f_1 and f_2 with respect to y are the same.

$$\Longleftrightarrow \int a(y)\ c_1(y)\ g(y)\ dy \gtreqless \int a(y)\ c_2(y)\ g(y)\ dy (*)$$

where $c_i(y)$ is the conditional expectation of $u(x,y)$ using $f_i(x,y)$. Now if $c_1(y) \geq c_2(y)$ for every y we know that since $a(y) > 0$ that in fact * is equivalent to

$$\int c_1(y)\ g(y)\ dy \gtreqless \int c_2(y)\ g(y)\ dy. \quad (**)$$

Similarly if $c_1(y) \leq c_2(y)$ for every y, we know * is equivalent to ** However, ** is equivalent to

$$Eu(f_1) \gtrless Eu(f_2)$$

and we are done. This leaves the case where there are $y', y'' \in Y$ such that

$$c_1(y') > c_2(y')$$

$$\text{and} \quad c_2(y'') > c_1(y'')$$

where one of the inequalities need not be strict. However, this is precisely the condition for there to be a preference reversal between the fibres of y' and y". Therefore a(y) is constant on Y. It is now clear that * is once again equivalent to **.

I ⇒ II:

Part I:

First we will show that $u^*(x,y) = a(y)\, u(x,y) + b(y)$ where $a(y) > 0$ for every y. This is because, by assumption, for any fixed y, u* and u both represent the same order of density functions over x. Therefore, for a fixed y

$$u^*(x,y) = a\, u(x,y) + b$$

where $a > 0$, by the regular expected utility theorem. When y varies, then a and b depend on it in general.

Part II:

We now show that if there is a preference reversal between the fibres of y_1 and y_2 that $a(y_1) = a(y_2)$.

Suppose, for contradiction, that there are points y_1, y_2 in Y such that there is a preference reversal between their fibres but $a(y_1) \neq a(y_2)$. Since there is a preference reversal we know there are density functions f' and f" on x such that

$$(f',y_1) > (f'',y_1)$$

$$\text{and} \quad (f'',y_2) > (f',y_2)$$

where one of the inequalities need not be strict.

Now the procedure will be to construct two density functions over Z by taking a convex combination of y_1 and y_2 and pairing it alternately with f' and f". It will be shown that u ranks the two densities as equal yet u* ranks one as preferred to the other. Then we are done for u and u* cannot represent the same order over Ωg where g is the

convex combination of y_0 and y_1, which is a contradiction. To do this, let $\delta y_1 + (1-\delta)y_2$ denote the density function over Y

$$\delta y_1 + (1-\delta)\, y_2 = \begin{cases} y_1, & \text{with probability } \delta \\ y_2, & \text{with probability } (1-\delta) \end{cases}$$

Our two density functions over Z are then $(f', \delta y + (1-\delta)\, y_2)$ and $(f'', \delta y_1 + (1-\delta)\, y_2)$ where the marginals are independent. Now we choose δ so u is indifferent between them. That is, we solve

$$\delta\, Eu(f',y_1) + (1-\delta)\, Eu(f',y_2)$$
$$- \delta \quad Eu(f'',y_1) - (1-\delta) \quad (f'',y_2) = 0$$

This yields

$$\delta = \frac{Eu(f'',y_2) - Eu(f',y_2)}{(Eu(f'',y_2) - Eu(f',y_2)) + (Eu(f',y_1) - Eu(f'',y_1))}$$

But by the preference reversal assumption, it is clear that $\delta \in (0,1)$ As well, since $a(y_1) \neq a(y_2)$, it is easy to see that

$$\delta \quad Eu^*(f',y_1) + (1-\delta) \quad Eu^*(f',y_2)$$
$$- \quad \delta \quad Eu^*(f'',y_1) - (1-\delta) \quad Eu^*(f'',y_2) \neq 0$$

Therefore we have constructed the desired density functions and are done.

□

Suppose that there are a number of preference relations $\lesssim i$ corresponding to different consumers. It is clear that a sufficient condition for the aggregate index $E(\sum_i u_i^*(x,y))$ to be Pareto [8] on Ωg for every g is that u_i^* represent $\lesssim i$ on Ωg for every g and i. In this case $E(\sum_i u_i^*(x,y)$ is equivalent to $E(\sum_i u_i(x,y))$, which is obviously Pareto since it is a positive weighted sum of expected utilities. It is not strictly necessary that u_i^* represent $\lesssim i$ on Ωg for every i and g in order for $E(\sum_i u_i^*(x,y))$ to be correct on every Ωg, however. For example, suppose that Eu_i^* and Eu_i agree on all pairs of density functions in which one Pareto dominates the other and both are in the same Ωg. Then it is fairly easy to prove that $E(\sum_i u_i^*(x,y))$ is Pareto on Ωg for every g even though u_i^* does not necessarily represent $\lesssim i$ on Ωg for any i or g. The extreme case of this is where no distribution dominates another one. Then every index on the space is Pareto.

However, there are definitely collections of preferences in which it is necessary for u_i^* to represent $\lesssim i$ on Ωg for every i and g in order for $E(\sum_i u_i^*(x,y))$ to be Pareto (the case where all consumers are identical). Therefore if we want to specify a method for constructing an aggregate welfare index of the form $E(\sum_i u_i^*(x,y))$

which will yield a Pareto welfare index on Ωg when applied to all possible collections of preferences, it is necessary that each u_i^* represent $\lesssim i$ on each Ωg. As a practical matter then, the preference representation condition is also necessary. This will remain so until someone demonstrates that there is a broad class of economies (containing all the ones we are likely to run across) in which it is not necessary.

We can now apply this theory to our case of interest. The subscript i for consumer i will be dropped until we discuss aggregate consumer surplus. We let Z represent prices with X representing p_1 and Y representing p_2 through p_n. The utility function representing $\lesssim$ is the consumer's indirect utility function,

$$g(p_1 \ldots p_n).$$

For this interpretation I will assume income remains constant and events such as rainfall do not directly affect utility, for notational simplicity, although they can easily be included as part of Y.

Note that the existence of a preference reversal on two fibres now can be interpreted as an instance of non-neutrality of risk with respect to changes in p_2 through p_n. If only the expected value of p_1 mattered to the consumer we would expect the random variable with the lower expected value to be chosen on any fibre. No preference reversals would then occur. However, if other aspects of the distribution counted, then there would be a possibility for preference reversals. For example, suppose p_1 and p_1' have the same expected value but p_1' has a higher variance. When p_2 through p_n are

are extremely low the consumer might then possess enough extra income to prefer a gamble and select p_1' over p_1. However at higher prices of p_2 through p_n he might prefer the alternative with less risk. Therefore a preference reversal consists of a case where the consumer's attitudes towards the higher moments of the distribution of p_1 changed enough with changes in p_2 through p_n to affect his choice. This is a very natural definition of being risk non-neutral with respect to changes in p_2 through p_n. For the rest of this discussion I will assume that the consumer is not totally risk neutral with respect to p_2 through p_n. This allows me to assume that a(y) in Theorem Five is constant.

Now, letting CS be the consumer surplus function on R^n, we can easily prove the following corollary from Theorem Five. The proof is unaltered if we allow more than one price to change and use line integrals.

Corollary 5-a:

Suppose the consumer is not totally risk neutral with respect to p_2 through p_n. Then CS is a representation of $\precsim$ over Ωg for every g if and only if the marginal utility of income is constant with respect to all prices.

Proof:

CS is a representation of $\precsim$ over Ωg for every g

$$\iff \frac{DCS}{Dp_1}(p_1,\ldots,p_n) = a \cdot \frac{Dg}{Dp_1}(p_1,\ldots,p_n)$$

where a is some positive constant.

$$\iff \frac{D}{Dp_1}\int_{p_1}^{\infty} x_1(p,p_2,\ldots p_n)\,dp = -a \cdot \lambda(p_1,\ldots,p_n)\cdot x_1(p_1,\ldots,p_n)$$

where a is some positive constant and λ is the marginal utility of income.

$$\iff -x_1(p_1,p_2,\ldots,p_n) = -a \cdot \lambda(p_1,\ldots,p_n)\cdot x_1(p_1,\ldots,p_n)$$

where a is some positive constant.

$$\iff a\lambda(p_1,\ldots,p_n) = 1 \quad \text{for some positive constant } a .$$

$$\iff \lambda(p_1,\ldots,p_n) \text{ is a constant } (\lambda \text{ is always positive}).$$

□

This result is interesting in that it adds another separate reason for requiring that λ be constant. Furthermore it places much stronger requirements on λ. Just considering CS as a utility index, we know that in the multivariable consumer surplus case (where more than one price is allowed to change) the integral used to calculate consumer surplus is path independent if and only if λ is constant with respect to the prices which are changing.[9] However, when only one price is allowed to change there is no such problem. In fact, if the demand curve slopes downward, consumer surplus is obviously a utility index for the consumer over fibres of $p_2\ldots p_n$. Both CS and g slope downward, so there is obviously a monotone transformation relating them regardless of whether or not λ is fixed. However, corollary 5-a states that in the case where prices are all random variables, λ must be constant even in the one variable case as well as in the multivariate case. Furthermore, λ must be constant with respect to every price which experiences stochastic variation, not merely with respect to those which are changed.

We therefore know that a constant marginal utility of income with respect to prices that are random variables is sufficient for aggregate expected consumer surplus to be a correct welfare measure. Furthermore, it is also necessary to assume that the marginal utility of income is constant with respect to prices which are random variables if we want to guarantee that aggregate expected consumer surplus will work in all possible cases. The same comments apply to income and natural events if they vary randomly (and if, in the latter case, they directly enter the utility function).

FOOTNOTES

1. Readers unfamiliar with the literature might refer to Massel [2] or Turnovsky [6,7].

2. Assume that

(i) d_1 and s_1 are defined and have continuous first derivatives for positive prices and incomes and for the ranges of W and V.

(ii) For every positive (p,m) and every (w,v) in the range of (W,V), there exists a unique positive p_1 which satisfies

$$d_1 = s_1$$

(iii) When evaluated at the points described in (ii),

$$\frac{Dd_1}{Dp_1} \neq \frac{DS_1}{Dp_1}$$

That is, we simply assume that demand and supply are continuously differentiable, always intersect once and only once and never have the same slope at the point where they intersect. This last assumption can of course be guaranteed by assuming that demand slopes downward and supply slopes upward.

3. The evaluation is of course carried out using the individuals' utility functions.

4. Although G is fixed, the welfare index should work for any fixed G.

5. If just consumer i's income is shifting we might not expect this to cause a significant shift of aggregate demand. Therefore the following discussion is most relevant in a case where all consumers incomes are varying together, possibly due to the business cycle.

6. That is, we use a general equilibrium argument to infer what partial equilibrium situations are plausible.

7. See DeGroot [1] for a treatment of the expected utility theorem.

8. Recall this term was defined on page 5.

9. See Silberburg [5].

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