SUPPLEMENTAL MATERIAL

I. THE HAMILTONIAN TERMS COMMUTE WITH EACH OTHER

The Hamiltonian in our model is a sum of commuting projectors. It is straightforward to see that all the H_{decorate} terms commute and that all the H_{tunnel} terms commute with all the H_{decorate} terms. In this section, we prove that all the H_{tunnel} terms commute with each other.

To prove that any pair of plaquette operators $\tau^x_{p_1}X_{p_1}$ and $\tau^x_{p_2}X_{p_2}$ commute, it is equivalent to prove that for any state in the Hilbert space, the final states are the same independent of the order of the plaquette operator action. Namely,

$$\tau^x_{p_1}X_{p_1}\tau^x_{p_2}X_{p_2}|\Psi\rangle = \tau^x_{p_2}X_{p_2}\tau^x_{p_1}X_{p_1}|\Psi\rangle. \tag{1}$$

For non-adjacent p_1 and p_2, these two terms involve different spins and Majoranas and act on the state independently, so they obviously commute. However, for adjacent p_1 and p_2, some of the Majorana modes that the two plaquette operators act on are the same, and it is not obvious whether they commute or not. Since X_p by construction, guarantees that the Majorana configurations match the plaquette spin configurations, and the plaquette spin configuration is independent of the order in which we apply the plaquette operators, the final configuration of the Majorana modes are actually the same, but the fermionic state can differ by a complex phase, i.e., the plaquette operators commute up to a complex phase. As we will argue below, such complex phases are actually all equal to zero, and the plaquette operators commute exactly.

Recall that $P_p^{\{\mu_{p,q}\}}$ projects onto the spin configuration of $\{\mu_{p,q}\}$ and Π_p projects onto the fermonic subspace that conforms to such spin configuration, so we only need to consider those states whose fermion parts match the spin configurations. We denote such states as $|\Psi_{\{\mu_{p,q}\}}\rangle \otimes |\Psi_{spin}\rangle$. For adjacent p_1 and p_2, it is sufficient to consider states of the form $|\Psi_{\{\mu_{p,q}\}}\rangle \otimes |\tau_1, \tau_2, ..., \tau_N\rangle$. We compute

$$\tau^x_{p_1}X_{p_1}|\Psi_{\{\mu_{p,q}\}}\rangle \otimes |\tau_1,\tau_2,...,\tau_N\rangle = V_{p_1}^{\{\mu_{p,q}\}}|\Psi_{\{\mu_{p,q}\}}\rangle \otimes |\tau_1',\tau_2,...,\tau_N\rangle \propto |\Psi_{\{\mu_p^1,\mu_q\}}\rangle \otimes |\tau_1',\tau_2,...,\tau_N\rangle. \tag{2}$$

$|\Psi_{\{\mu_{p,q}\}}\rangle$ and $|\Psi_{\{\mu_p^1,\mu_q\}}\rangle$ denote the same Majorana configurations except some, denoted by $\gamma_1^{\sigma_1}, \gamma_2^{\sigma_2}, ...\gamma_{2n}^{\sigma_{2n}}$, around the plaquette p_1. More explicitly, we arrange the Majorana modes so that $is_{2i-1,2i}\gamma_{2i-1}^{\sigma_{2i-1}}\gamma_{2i}^{\sigma_{2i}}|\Psi_{\{\mu_{p,q}\}}\rangle = |\Psi_{\{\mu_{p,q}\}}\rangle$, while $is_{2i,2i+1}\gamma_{2i}^{\sigma_{2i}}\gamma_{2i+1}^{\sigma_{2i+1}}|\Psi_{\{\mu_p^1,\mu_q\}}\rangle = |\Psi_{\{\mu_p^1,\mu_q\}}\rangle$. In this case, the operator $V_p^{\{\mu_{p,q}\}}$ is exactly of the form in Eq.(10) in the main text:

$$V_{p_1}^{\{\mu_{p_1,q}\}} = 2^{-\frac{n+1}{2}}(1 + is_{2,3}\gamma_2^{\sigma_2}\gamma_3^{\sigma_3})(1 + is_{4,5}\gamma_4^{\sigma_4}\gamma_5^{\sigma_5})\dots(1 + is_{2n,1}\gamma_{2n}^{\sigma_{2n}}\gamma_1^{\sigma_1}). \tag{3}$$

Note that the choice of $\{\sigma_i(\{\mu_{p_1,q}\})\}$ depends on the plaquette spin configuration. This point becomes important when considering two adjacent plaquettes. Now turn to two adjacent plaquttes p_1 and p_2 and we consider first flipping the p_1 spin and then the p_2 spin:

$$\tau^x_{p_2}X_{p_2}\tau^x_{p_1}X_{p_1}|\Psi_{\{\mu_{p,q}\}}\rangle \otimes |\tau_1,\tau_2,...\rangle = V_{p_2}^{\{\mu^1_{p,q}\}}V_{p_1}^{\{\mu_{p,q}\}}|\Psi_{\{\mu_{p,q}\}}\rangle \otimes |\tau_1',\tau_2',...\rangle, \tag{4}$$

versus first acting on p_2 and then p_1:

$$\tau^x_{p_1}X_{p_1}\tau^x_{p_2}X_{p_2}|\Psi_{\{\mu_{p,q}\}}\rangle \otimes |\tau_1,\tau_2,...\rangle = V_{p_1}^{\{\mu^2_{p,q}\}}V_{p_2}^{\{\mu_{p,q}\}}|\Psi_{\{\mu_{p,q}\}}\rangle \otimes |\tau_1',\tau_2',...\rangle \tag{5}$$

To prove that $\tau^x_{p_1}X_{p_1}$ and $\tau^x_{p_2}X_{p_2}$ commute is now equal to prove that the final states in (4) and (5) are exactly the same, not just the same up to a phase factor. To show this, we can use the following procedure. First, we notice the identity $P|\Psi_{\{\mu_{p,q}\}}\rangle = |\Psi_{\{\mu_{p,q}\}}\rangle$, where P is the projector onto the fermionic state $|\Psi_{\{\mu_{p,q}\}}\rangle$. Using this identity, we can prove that $\tau^x_{p_1}X_{p_1}$ and $\tau^x_{p_2}X_{p_2}$ commute by simply proving that

$$V_{p_1}^{\{\mu^2_{p,q}\}}V_{p_2}^{\{\mu_{p,q}\}}P = V_{p_2}^{\{\mu^1_{p,q}\}}V_{p_1}^{\{\mu_{p,q}\}}P, \tag{6}$$

where $\{\mu_{p,q}\}$ labels the initial spin configuration, and $\{\mu^1_{p,q}\}$ (resp. $\{\mu^2_{p,q}\}$) labels the spin configuration after the spin on plaquette p_1 (resp. p_2) is flipped. p_1 and p_2 share two triangles and one short bond, as seen in Fig.(1). In Eq.(6), projectors which do not act on the Majorana modes on the two triangles commute obviously. Projectors that do act on the shared triangles may fail to commute. Since the configuration of the Majorana modes on the shared triangles depend on the spin configurations of p_1, p_2, and the two plaquettes bordering both p_1 and p_2, we may enumerate all the possible $2^4 = 16$ spin configurations on these 4 plaquettes and explicitly check that Eq.(6) holds. We find that the 16 cases essentially reduce to the three cases listed in Fig.1 by symmetry arguments and similarity in proof techniques. A straightforward although lengthy calculation shows that Eq.(6) indeed holds for these three cases.

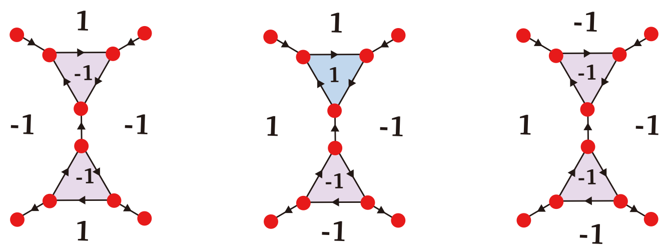

FIG. 1. The three relevant spin configurations when proving the commutativity of the plaquette operators.

II. TIME REVERSAL INVARIANCE OF THE HAMILTONIAN AND THE WAVE FUNCTION

Recall that we define the time reversal operation on the spins and fermions as $T = \prod \tau_x \otimes \prod (i\sigma_y) K$ with K the complex conjugation operator.

Under time reversal, both $D_{\overrightarrow{vw}}$ and $i(\gamma_v^\uparrow \gamma_w^\uparrow + \gamma_v^\downarrow \gamma_w^\downarrow)$ are even. $W_{vw}^+ i\gamma_v^\uparrow \gamma_w^\downarrow$ maps to $W_{vw}^- i\gamma_v^\downarrow \gamma_w^\uparrow$, and $(\frac{1+\tau_f^z}{2})\gamma_v^\downarrow \gamma_w^\downarrow$ maps to $(\frac{1-\tau_f^z}{2})\gamma_v^\uparrow \gamma_w^\uparrow$. Therefore H_{decorate} is time reversal invariant. It is not obvious that the tunneling term is also time reversal invariant, we need to check it explicitly. First, the spin term τ_p^x is invariant under time reversal. Similar to H_{decorate}, it is obvious that the Π_p's are even under time reversal. $P_p^{\{\mu_q, \mu_q\}}$ is mapped to its time reversal partner because $T P_p^{\{\mu_q, \mu_q\}} T^{-1} = P_p^{\{-\mu_q, -\mu_q\}}$. It can be explicitly checked that $V_p^{\{\mu_{p,q}\}}$ is also mapped to its time reversal partner under time reversal. Therefore, we see that

$$T V_p^{\{\mu_{p,q}\}} \Pi_p P_p^{\{\mu_{p,q}\}} T^{-1} = V_p^{\{-\mu_p, -\mu_q\}} \Pi_p P_p^{\{-\mu_p, -\mu_q\}}. \tag{7}$$

Although $X_p^{\{\mu_{p,q}\}} \Pi_p P_p^{\{\mu_{p,q}\}}$ alone is not time reversal invariant, the sum of all configurations of $\{\mu_p, \mu_q\}$ is invariant under time reversal.

Finally, let us come back to prove that the ground state wave function is time-reversal invariant. It suffices to prove that the weights of two configurations related by time reversal are complex conjugate of each other. Let us consider a fermionic state $|\Psi_f\rangle$ obtained by acting a sequence of plaquette operators on the initial fermionic state $|\Psi_i\rangle$ associated with the plaquette spin configuration where $\tau_p^z = 1$ for all p: $|\Psi_f\rangle = V_{p_1} V_{p_2} \dots V_{p_n} |\Psi_i\rangle$. The fermionic state $|\Psi_f^T\rangle$ associated with the time-reversal partner of this configuration can be obtained by acting another sequence of plaquette operators on the initial fermionic state: $|\Psi_f^T\rangle = V_{p_1'} V_{p_2'} \dots V_{p_m'} |\Psi_i\rangle$, where $p_1' \cup p_2' \cup \dots \cup p_m'$ form the complementary region of $p_1 \cup p_2 \cup \dots \cup p_n$. Note that the boundary of both regions agree. Using similar tricks as in Eq.(A11) of Ref.[1] for spinless fermions, we find that both $V_{p_1} V_{p_2} \dots V_{p_n}$ and $V_{p_1'} V_{p_2'} \dots V_{p_m'}$ can be reduced to the product of a sequence of projectors which act only on the Majoranas lying on the boundary of the region $p_1 \cup p_2 \cup \dots \cup p_n$:

$$V_{p_1} V_{p_2} \dots V_{p_n} = 2^{-\frac{n+1}{2}} (1 + i s_{2,3} \gamma_2^{\sigma_2} \gamma_3^{\sigma_3})(1 + i s_{4,5} \gamma_4^{\sigma_4} \gamma_5^{\sigma_5}) \dots (1 + i s_{2n,1} \gamma_{2n}^{\sigma_{2n}} \gamma_1^{\sigma_1}), \tag{8}$$

$$V_{p_1'} V_{p_2'} \dots V_{p_m'} = 2^{-\frac{n+1}{2}} (1 + i s_{2,3} \gamma_2^{\bar\sigma_2} \gamma_3^{\bar\sigma_3})(1 + i s_{4,5} \gamma_4^{\bar\sigma_4} \gamma_5^{\bar\sigma_5}) \dots (1 + i s_{2n,1} \gamma_{2n}^{\bar\sigma_{2n}} \gamma_1^{\bar\sigma_1}). \tag{9}$$

Furthermore, both $p_1 \cup p_2 \cup \dots \cup p_n$ and $p_1' \cup p_2' \cup \dots \cup p_m'$ are in the $\tau_p^z = -1$ configuration. Therefore, by the coupling rules we introduced earlier, σ_i and $\bar\sigma_i$ must be the opposite of each other for $i = 1, 2, \dots, 2n$. Hence Eq.(8) and Eq.(9) can be mapped into each other term by term under time reversal. Hence the weights associated with $|\Psi_f\rangle$ and $|\Psi_f^T\rangle$ are complex conjugate of each other.

III. WHY $T^2 = 1$ FERMION DOES NOT WORK

In the main text, we argued that the following decoration rules for the spinless Majorana modes breaks the time-reversal symmetry although it preserves the fermion parity invariance of the domain wall configurations: Away from the domain wall, we pair up Majorana modes that share a short bond $l = \langle \overrightarrow{vv'} \rangle$ as $i\gamma_v \gamma_v'$. On a domain wall, we pair up Majorana modes that share a long bond $\tilde{l} = \langle \overrightarrow{vw} \rangle$ as $i\gamma_v \gamma_w$. One may try to resolve this issue by adding a minus sign to the coupling when the left hand side of the long bond is in the $|1\rangle$ state. But this inevitably breaks the fermion parity invariance, as the following discussion shows.

Consider the two plaquette spin configurations in Fig.2. Due to the Kasteleyn orientation, the two configurations will have the same fermion parity if we stick to the original coupling rule which breaks time-reversal invariance. The modified coupling rule introduces some extra minus signs into the fermion parity of the second configuration and the number of minus signs is exactly equal to the number of clockwise oriented bonds on the domain wall, which is three in this case. Therefore, with the modified coupling rule, the two configurations have opposite fermion parity.

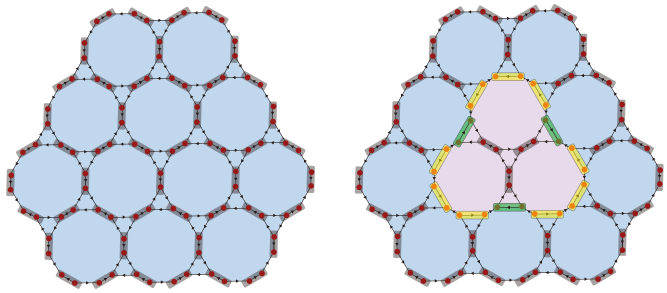

FIG. 2. Two configurations for spinless Majorana modes with opposite fermion parity. Extra minus signs are added to the coupling on the green bonds according to the modified coupling rule.

[1] Nicolas Tarantino and Lukasz Fidkowski, "Discrete spin structures and commuting projector models for two-dimensional fermionic symmetry-protected topological phases," Phys. Rev. B **94**, 115115 (2016).