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NECESSARY AND SUFFICIENT CONDITIONS FOR EFFICIENCY IN AN ECONOMY WITH AN INCOMPLETE SET OF MARKETS

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I. INTRODUCTION

Considerable attention has recently been focused on examining an economy in which there is not a complete set of markets. The nonexistence of certain markets is typically explained by either the presence of transaction costs or, in the case of contingent commodities, by a moral hazard argument. If, however, an efficient allocation of resources is achieved whenever certain markets are not present, it may be argued that the failure of these markets to exist occurs simply because they are not needed. Indeed, no trade would occur in these markets even if they were established since all individuals have imputed the same price to each nontraded commodity. The characterization of such an economy is the case with which this paper is concerned.

It should be recognized that if some commodities are not traded but may be consumed by each household, then it must be true that, in some sense, each household is capable of producing each of these commodities. For this reason, a convenient framework for studying this problem is the model of household production which has been proposed by Becker [2] and Lancaster [7]. In this formulation the household purchases goods and, through the use of a "household production function," transforms these goods into commodities. Although only the goods are traded, the commodities, and not the good themselves, are the arguments of the household's utility function. However, in contrast with many of the previous applications of the Becker-Lancaster model in which commodities are nonmarketable and may even be nonmeasurable, it should be emphasized that this paper will concern itself only with commodities which are measurable and potentially marketable.

The definition of efficiency which will be adopted here is that, independent of its utility function and endowment, each household must perceive the same set of commodity prices. When households possess arbitrary technologies and commodities are not traded, there is no reason to suspect that each household will impute the same set of commodity prices. It is the purpose of this paper to uncover technological restrictions which are necessary or sufficient for achieving an efficient allocation of resources. It should be noted that, as defined above, an implication of efficiency is that commodity prices must be independent of the commodity bundle consumed. Due to this, some of the results presented here are similar to those given by Pollak and Wachter [8] who, in the context of the Becker-Lancaster model, discuss technological restrictions which are needed to obtain commodity demand functions which exhibit the properties of traditional demand theory.

A direct application of the household production function model is in the analysis of the allocation of resources resulting in a two period model. Suppose that all households in the economy are sharecroppers who may work on a number of farms in return for a fraction of each farm's yield in the second period. Let each farm produce only one crop and the yield of each crop may vary with, say, the weather conditions that occur next period. Corresponding to the weather conditions that may prevail, the second period may be divided into states of the world. Within this framework, the fraction of each farm owned by a household may be considered as a good which, by means of the household's technology, is transformed into a commodity in the second period. There is an additional structure contained in this example since goods are not "used up" in any production process. In other words, the fraction of farm j used to produce consumption in state s may also be used to produce consumption in each other state. As examined, for example by Drèze [5], this structure is identical to that found in a securities model in which households use fractional holdings of firms to provide for future consumption.

The added structure which is contained in either of these two models provides one further result. As demonstrated by Arrow [1], a linear technology in which there is as many independent securities as future states is sufficient to ensure an efficient allocation. It will be shown here that this linear technology is also necessary for efficiency.

II. THE GENERAL MODEL OF HOUSEHOLD PRODUCTION

Suppose that there are N goods, denoted by $X = (x_1, \ldots, x_N)$, and S commodities, denoted by $C = (c_1, \ldots, c_S)$. Each household is endowed with a set of goods, $\bar{X}^h$, and a technology, T^h, for transforming the goods into commodities. There is a vector of prices, $\bar{P}$, at which the goods may be traded, but no markets exist in which the commodities may be traded.[1] Each household possesses a continuous, quasi-concave utility function, $U^h(C^h)$, and chooses commodities and goods so as to

$$\begin{aligned} &\text{maximize } U^h(C^h) \\ &\text{subject to} \\ &\qquad \bar{P}X^h \leqq \bar{P}\bar{X}^h \\ &\qquad (C^h, -X^h) \in T^h \\ &\qquad C^h \geqq 0 \;.^2 \end{aligned}$$

This problem may be rewritten as:[3]

$$\begin{aligned} &\text{maximize } U^h(C^h) \\ &\text{subject to} \qquad\qquad (U) \\ &\qquad e^h(\bar{P}, C^h) \leqq \bar{P}\bar{X}^h \end{aligned}$$

where $e^h(\bar{P}, \bar{C}^h)$ is a minimum expenditure function which is the solution to the problem:

$$\begin{aligned} &\text{minimize } \bar{P}X^h && \text{dual variables: } P \\ &\text{subject to} && \qquad (E) \\ &\qquad (C^h, -X^h) \in T^h && q^h \\ &\qquad C^h \geqq \bar{C}^h && \Pi^h \;. \end{aligned}$$

Regarding the technology available to each household, the following assumptions are made:[4]

T1 $0 \in T$: it is possible for the household to engage in no activity.

T2 T is closed: the limit of any convergent sequence of technologically feasible activities is itself feasible.

T3 If $(C, -X) \in T$ and $(C', -X') \leq (C, -X)$ then $(C', -X') \in T$: all feasible activities may be freely disposed.

T4 T is convex: the technology exhibits non-increasing returns.

T5 If $(C, -X) \in T$ and $X' \geq X$ then there is a $(C', -X') \in T$ such that $C' \geq C$: all inputs are productive.

T6 If $(C, -X) \in T$ and $C^+ = \{\max[0, c_1], \ldots, \max[0, c_S]\}$ then $(C^+, -X) \in T$ and $(C^+, -X) \geq 0$: the production of nonnegative commodities requires that all inputs are used at a nonnegative level.

T7 There is an $M < \infty$ such that if $(C, 0) \in T$, then each $c_s < M$, $s = 1, \ldots, S$: unlimited consumption is impossible.

Further assumptions will be given later to achieve specific purposes.

The first six of the above assumptions are fairly standard when considering this type of model. T2 is solely for mathematical convenience. As noted in Theorem 1, dispensing with this assumption would require that the result there be slightly, but not significantly, altered. Furthermore, many of the programming problems defined throughout this paper would have to be redefined in terms of supremums and infimums rather than maximums and minimums. As will be seen below, T4 allows the set dual to the technology to be represented as a closed convex cone and T1 and T3 guarantee that the dual variables (prices) are nonnegative. Whereas T3 and T7 guarantee that the negative orthant is contained in the technology, the nonnegativity restriction, T6, ensures that no activity in the orthant is of particular interest.

It can be seen that an assumption such as T7 is quite natural since if an unlimited amount of some commodity could be produced without using any goods then that commodity is not in any sense scarce and deserves no further consideration. Further, T7 is much weaker than the "no free lunch" postulate which is usually assumed.

Following the approach and interpretation provided by Cass [3], if assumptions T1-T4 are satisfied, the dual to the technology may be written as the closed convex cone

$$M^h = \{(\Pi^h, P, q^h): \Pi^h C^h - PX^h \leqq q^h \text{ for all } (C^h, -X^h) \in T^h\}. \quad (1)$$

M^h may be thought of as a market corresponding to the technology, T^h, where Π^h and P are commodity and good prices, respectively, and q^h is the implicit profit associated with producing C^h from X^h. The vector of good prices, P, is given and is the same for all households. From these prices the household imputes commodity prices, Π^h, and profit, q^h. As previously noted, these dual variables are nonnegative.

The set dual to M^h may be written as

$$T'^h = \{(C^h, -X^h, -y^h): \Pi^h C^h - PX^h - q^h y^h \leqq 0 \text{ for all } (\Pi^h, P, q^h) \in M\}. \quad (2)$$

T'^h is a closed convex cone with N + 1 goods, (X^h, y^h). The N + 1st input, y^h, may be considered as a fixed factor to which profits are imputed. For future notational convenience, let the vector Y^h be this N + 1 vector of good $Y^h = (X^h, y^h)$ and Q^h be the N + 1 vector of good prices, $Q^h = (P, q^h)$.

As shown in Lemma 1 of [3], the technology, T^h, may be written as

$$T^h = \{(C^h, X^h): (C^h, -X^h, -y^h) \in T'^h, y^h \leqq 1\}$$

and the problem (E) may be rewritten as

	dual variables
minimize $\overline{P}X^h$	P
subject to	
$(C^h, -X^h, -y^h) \in T'^h$	
$C^h \geqq \overline{C}^h$	Π^h
$y^h \leqq 1$	q^h,

(E)

which has as its corresponding dual problem

$$\text{maximize } \Pi^h \overline{C}^h - q^h$$
$$\text{subject to} \qquad \text{(D)}$$
$$(\Pi^h, P, q^h) \in M^h$$
$$P = \overline{P}.$$

Any feasible solution to (D) will yield values no greater than any feasible solution to (E) since

$$\Pi^h C^h - q^h \leqq PX^h \text{ for } (C^h, -X^h) \in T^h \text{ and } (\Pi^h, P, q^h) \in M^h$$

and, in particular,

$$\Pi^h \overline{C}^h - q^h \leqq \Pi^h C^h - q^h \leqq \overline{P}X^h \text{ for } (C^h, -X^h) \in T^h,\ C^h \geqq \overline{C}^h$$
$$\text{and } (\Pi^h, P, q^h) \in M^h,\ P = \overline{P}. \qquad (3)$$

To insure that (E) possesses a regular optimal solution it will be assumed that T^h satisfies Slater's condition, i.e.

T8 There exists $(C^h, -X^h) \in T$ with $c_s^h > \overline{c}_s^h$ for all h.

This condition guarantees that an optimal solution to (E), $(C^*, -X^*)$, is such that

$$\Pi^*(C^* - \overline{C}) = 0 \text{ and } \Pi^* C^* - \overline{P}X^* = q^* \text{ for some } (\Pi^*, \overline{P}, q^*) \in M,$$ [5]

and that both (E) and (D) have optimal solutions with equal optimal values. Hence, the minimum expenditure function may be written as

$$e(\overline{P}, \overline{C}) = \overline{P}X^* = \Pi^*\overline{C} - q^*. \qquad (4)$$

It should be noted that Slater's condition insures that:

For any $\overline{C} \geqq 0$, there is an X such that $(\overline{C}, -X) \in T$: any level of output is producible.

Otherwise, there would exist some utility function and endowment such that the solution to the utility maximization problem (U) would be a $\overline{C}$ which is not technologically feasible. In this instance (4) would need to be rewritten as

$$e(\overline{P}, \overline{C}) = \overline{P}\overline{X} > \Pi^*\overline{C} - q^*$$

III. IMPLICATIONS OF PARETO EFFICIENCY

In an allocation model such as the one considered here, it is well known that Pareto efficiency is attained only for the case in which all individuals perceive the same set of implicit commodity prices, i.e., $\Pi^h = \Pi^j = \Pi$ for all households h and j. In this paper, the concept of efficiency will be defined independently of any household's utility function and endowment. It is indeed possible that given all households' utility functions and endowments, efficiency may result for arbitrary technologies. In this instance efficiency results solely by chance, however, since for a different set of utility functions and endowments efficiency no longer is achieved. This situation is illustrated for two individuals in Figure 1(a) and (b). Although each individual has a different consumption possibilities set, both individuals impute the same commodities prices at the production point which maximizes their utility. These imputed commodity prices form the normal to the separating hyperplane at the utility-maximizing production point. If efficiency is to be examined on a utility-free and endowment-free basis, the focus may be placed entirely on the consumption possibilities set. This set, as defined by the minimum expenditure problem (E), may be defined as in (3)

$$\Pi^h C^h \leqq PX^h + q^h \leqq P\overline{X}^h + q^h \text{ for all } (C^h, -X^h) \in T^h.$$

In this framework, efficiency is equivalent to assuming that,

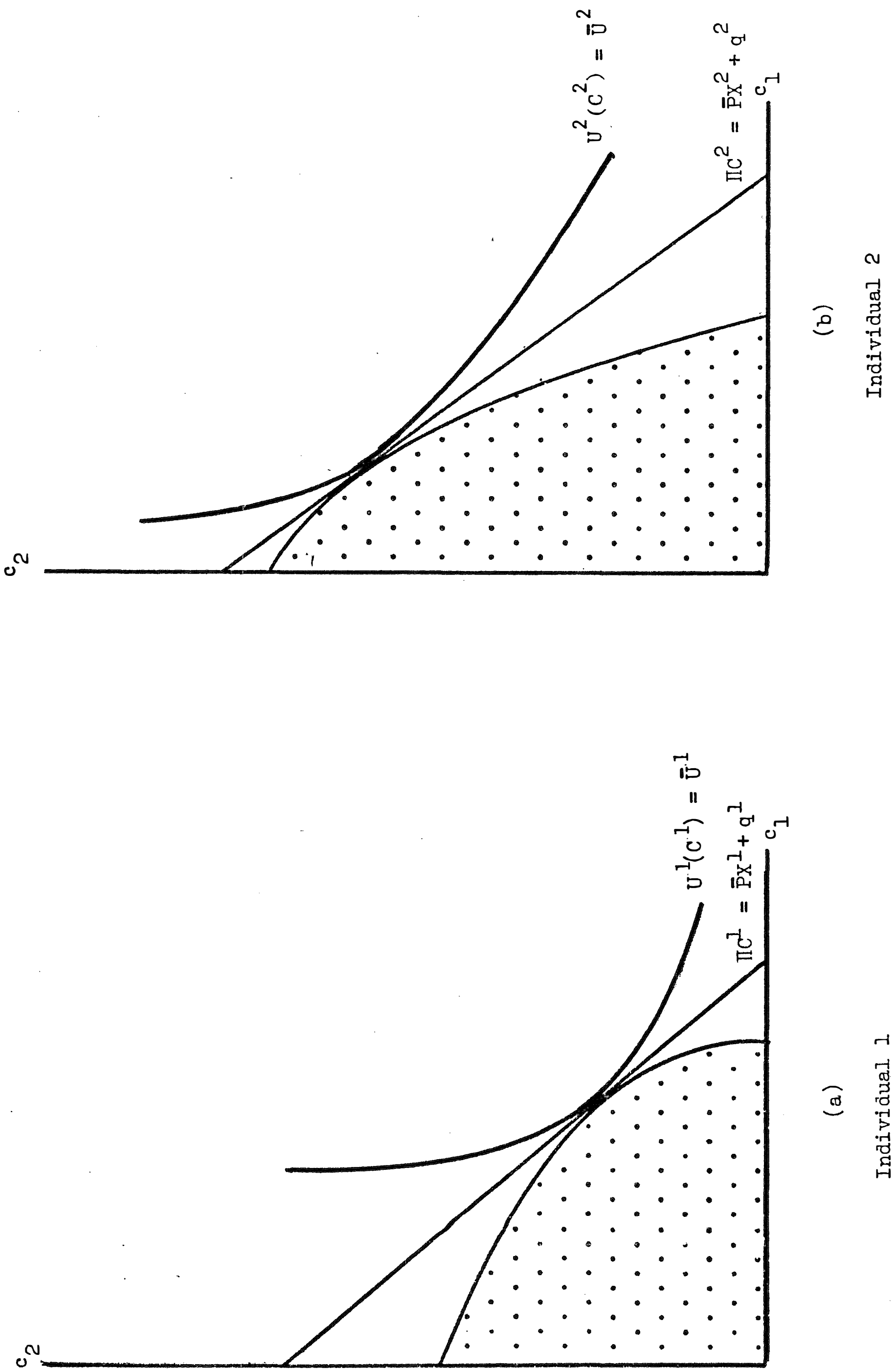

FIGURE 1

independent of each household's endowment and the commodities it consumes, the output prices each household perceives are identical. More simply, this requires that the boundary of each individual's consumption possibility set be defined by a hyperplane which has the same slope for each individual. This is shown in Figure 2.

Within this framework, each household's implicit profit must be independent of the commodity bundle consumed so that each household's perceived total income, $\bar{P}\bar{X}^h + q^h$, does not vary with its consumption.[6] There is no requirement, however, that each household's implicit profit be the same.

The definition of efficiency may now be formalized as:

Pareto efficiency occurs if and only if each household's dual set of prices, M^h, can be represented as

$$M^h = \{(\Pi, P, q^h): P \geqq 0, \ 0 \leqq \Pi \leqq \Pi(P), \ 0 \leqq q^h(P) \leqq q^h\}, \qquad (5)$$

where $\Pi(P)$ and $q^h(P)$ are functions solely of P.

The properties of the functions $\Pi(P)$ and $q(P)$ are examined in the following lemma.

Lemma 1: The implicit commodity prices, $\Pi(P)$, and implicit profit, $q(P)$, as specified in (5) satisfy the following conditions:

(a) $\Pi(0) = 0$ and $q(0) = 0$;

(b) $\Pi(P) > 0$ if $P > 0$;

(c) $\Pi(P)$ and $q(P)$ are linear homogeneous with respect to P;

(d) $\Pi(P)$ is a concave function with respect to P and $q(P)$ is a convex function with respect to P;

(e) $\Pi(P)$ and $q(P)$ are continuous with respect to P.

Proof: See Appendix.

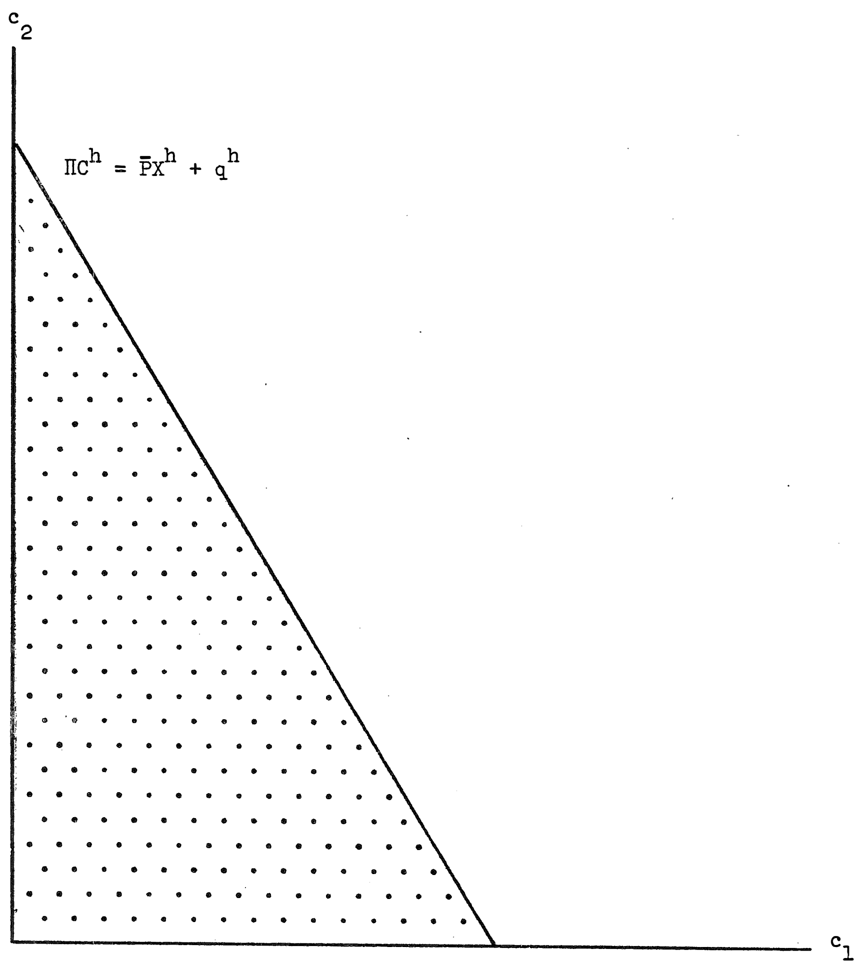

FIGURE 2

It will first be shown that a necessary condition for an efficient allocation is that no household's technology exhibits joint production. Prior to examining this concept, however, it will be useful to define production functions, $f^s(Y^s)$ over the augmented technology, T'. For nonnegative input levels, $Y \geqq 0$, let

$$f^s(Y^s) = \text{Max}\ \{c_s : (C^s, -Y^s) \in T',\ c_s \geqq 0\} \quad s = 1, \ldots, S, \tag{6}$$

where $C^s = (0, \ldots, 0, c_s, 0, \ldots, 0)$.[7] Lemma 2 details the properties which each production function must possess.

<u>Lemma 2</u>: Each production function, $f^s(Y^s)$, as specified by (6) is a continuous, nonnegative, linear homogeneous, and concave function with respect to Y^s.

<u>Proof</u>: See Appendix.

A technology, T', is said to exhibit no joint production or, in short, to be non-joint, if it can be represented as a set of production functions such that

$$T' = \{(C, -Y): C = \sum_s C^s,\ Y = \sum_s Y^s,\ c_s \leqq f^s(Y^s),\ Y^s \geqq 0,\ s = 1, \ldots, S\} \tag{7}$$

$$= \sum_s T'_s$$

where T'_s is a sub-technology in which c_s may be produced at any non-negative level, but all other commodities are not produced. Hence,

$$T'_s = \{(C^s, -Y^s): c_s \leqq f^s(Y^s),\ Y^s \geqq 0\} \quad s = 1, \ldots, S. \tag{8}$$

The definition of non-jointness provided in (7) may perhaps best be understood by considering a technology which exhibits joint production. Suppose only two commodities are producible in the technology and that they are joint products and are produced in the proportion $c_1 = ac_2$. If $\overline{C}$ may be produced from $\overline{Y}$ where $\overline{c}_1 = ac_2$ and $(\overline{C}, -\overline{Y}) \in \text{bd } T'$, then $(\overline{C}^1, -\overline{Y}) \in \text{bd } T_1'$ and $(\overline{C}^2, -\overline{Y}) \in \text{bd } T_2'$. Thus $(\overline{C}, -2\overline{Y}) \in \text{bd} \sum_s T_s'$, and it can be seen that $\sum_s T_s' \subset T'$, but $\sum_s T_s' \neq T'$. As will be seen in the proof of Theorem 1, it will always be true that $\sum_s T_s' \subset T'$ irregardless of whether or not the technology is non-joint.

The following lemma establishes the properties of each sub-technology.

Lemma 3: Each sub-technology, T_s', is a closed, convex cone and may be alternatively defined as

$$T_s' = \{(C^s, -Y^s): \Pi C^s - QY^s \leqq 0 \text{ for all } (\Pi, Q) \in M\} \tag{9}$$

Proof: See Appendix.

Lemma 4: T' exhibits no joint production if and only if there exist S sub-technologies of the form

$$T_s' = \{(C^s, -Y^s): \Pi C^s - QY^s \leqq 0 \text{ for all } (\Pi, Q) \in M\} \quad s = 1, \ldots, S,$$

such that $\sum_s T_s' = T'$.[8]

Proof: See Appendix.

Lemma 4 provides an alternate definition of no joint production in terms of being able to partition T' into S sub-technologies where all prices in the dual set, M, are also in the set dual to each sub-technology. Through the use of this lemma it may now be demonstrated that Pareto efficiency requires that the technology, T', must be non-joint.

Theorem 1: Suppose T satisfies T1-T8. If Pareto efficiency is achieved then T' exhibits no joint production.

Proof: Let T_s', s = 1, ..., S, be defined as in (8). By Lemma 4, T' exhibits no joint production if $\sum_s T_s' = T'$.

(i) To show $\sum_s T_s' \subset T'$, consider $(C^s, -Y^s) \in T_s'$, s = 1, ..., S. By (9),

$$\Pi C^s - QY^s \leqq 0 \text{ for all } (\Pi, Q) \in M, \quad s = 1, \ldots, S.$$

Summing over these S inequalities gives

$$\sum_s (\Pi C^s - QY^s) = \Pi C - QY \leqq \text{ for all } (\Pi, Q) \in M$$

where $C = \sum_s C^s$ and $Y = \sum_s Y^s$. Clearly, $(C, -Y) \in T'$. It should again be noted that this half of the proof is independent of the structure that efficiency imposes on M and is therefore not an implication of efficiency. As previously noted, $\sum_s T_s' \subset T'$ even if T' does exhibit joint production.

(ii) To show $T' \subset \sum_s T_s'$, suppose $(\overline{C}, -Y') \in T'$ but $(\overline{C}, -Y') \notin \sum_s T_s'$. There must be a $(\overline{C}, -\overline{Y}) \in \text{bd} \sum_s T_s'$ with $\overline{Y} > Y'$ or $\sum_s \overline{Y}^s > Y'$ for $(\overline{C}^s, -\overline{Y}^s) \in \text{bd } T_s'$. This can be seen by deriving $\overline{Y}$ from the optimal solution to the problem:[9]

$$\begin{aligned} &\text{minimize } \|Y\| \\ &\text{subject to} \\ &\quad (\overline{C}, -Y) \in \sum_s T_s' \\ &\quad Y = Y' + \lambda 1 \\ &\quad \lambda \geqq 0. \end{aligned}$$

Hence, $(\overline{C}^s, -\overline{Y}^s) \in \text{bd } T_s'$.

If efficiency is achieved for some $(\Pi(P), Q(P)) \in M$, then for any $C \geqq 0$, there must be some $(C, -Y) \in \text{bd } T'$ such that $\Pi(P)C - Q(P)Y = 0$. In particular, consider C^s, then there is some $(C^s, -Y^s) \in \text{bd } T'$ such that $\Pi(P)C^s - Q(P)Y^s = 0$. By (8), T_s' may be defined as

$$T'_s = \{(C^s, -Y^s): (C^s, -Y^s) \in T', Y^s \geqq 0\}.$$

Since $(\overline{C}^s, -\overline{Y}^s) \in \text{bd } T'_s$, then $(\overline{C}^s, -\overline{Y}^s) \in \text{bd } T'$ and it has a non-trivial support of the form

$$\Pi(\overline{P})\overline{C}^s - Q(\overline{P})\overline{Y}^s = 0 \text{ for some } (\Pi(\overline{P}), Q(\overline{P})) \in M$$

and

$$\Pi(\overline{P})C - Q(\overline{P})Y \leqq \Pi(\overline{P})\overline{C}^s - Q(\overline{P})\overline{Y}^s = 0 \text{ for all } (C, -Y) \in T',$$

or in particular,

$$\Pi(\overline{P})C^s - Q(\overline{P})Y^s \leqq \Pi(\overline{P})\overline{C}^s - Q(\overline{P})\overline{Y}^s = 0 \text{ for all } (C^s, -Y^s) \in T'_s.$$

Summing over the S sub-technologies gives

$$\Pi(\overline{P})\sum_s \overline{C}^s - Q(\overline{P})\sum_s \overline{Y}^s = \Pi(\overline{P})\overline{C} - Q(\overline{P})\overline{Y} = 0$$

and since $(\overline{C}, -Y') \in T'$

$$\Pi(\overline{P})\overline{C} - Q(\overline{P})Y' \leqq \Pi(\overline{P})\overline{C} - Q(\overline{P})\overline{Y} = 0$$

or

$$Q(\overline{P})(\overline{Y} - Y') \geqq 0$$

which is a contradiction to $Q(\overline{P}) \geq 0$ and $\overline{Y} - Y' > 0$.

Hence $T' = \sum_s T'_s$.

It is instructive to consider how efficiency fails whenever the technology exhibits joint production. To do this, recall the example in which only two commodities are producible in the technology and may be produced only in fixed proportion. As shown in Figure 3, the consumption possibilities set in this example is not of the form which efficiency requires. Whereas this is an example of joint production in its strongest sense, all other technologies exhibiting some form of joint production may be shown to yield consumption possibilities sets which are also not of the form necessary for efficiency.

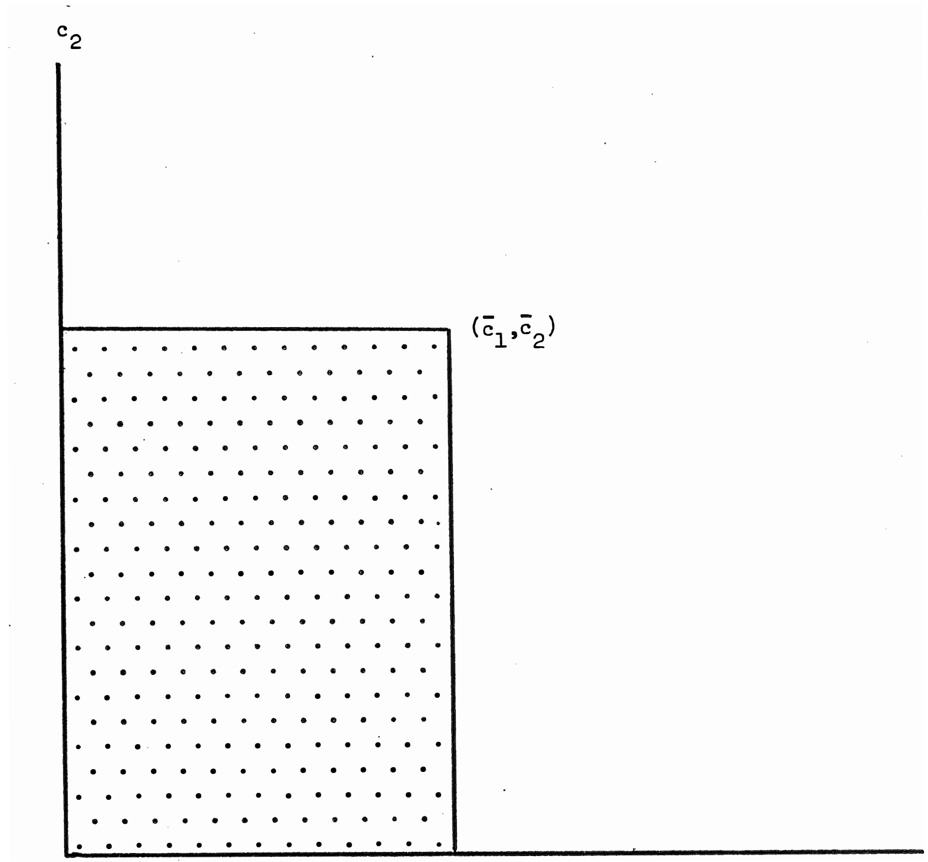

FIGURE 3

Eliminating the possibility of joint production from a household's technology does not preclude the possibility that there is some level of each commodity which the household may receive even if no goods are used in the production of that commodity. For each production function, $f^s(Y^s)$, the fixed factor associated with that function is y_s and, by T7, $f^s(0, y_s) = \bar{c}_s$ may be positive. In this instance, the household must consume at least $\bar{c}_s$ of each commodity. An example of such a consumption possibilities set is given in Figure 4. Clearly, there exist certain utility functions such that the household would be better off if trade were permitted. Souch would be the case if the household wished to consume less than $\bar{c}_s$ of some commodities at a corresponding increase in consumption of the other commodities. For this reason, the usual "no free lunch" postulate may be given as a necessary condition for efficiency.

<u>Theorem 2</u>: Suppose each household's technology, T^h, satisfies T1-T8. If Pareto efficiency is achieved then inputs are necessary for the production of any output.

Alternatively, Theorem 2 requires that:

$$\text{If } (C, -X) \in T \text{ and } C \geq 0, \text{ then } X \geq 0.$$

This theorem, when considered along with T1, guarantees that the origin is a boundary point of each household's technology. Further if X must be nonnegative, then $(0, -X) \in \text{bd } T$ only when $X = 0$.

It was seen in (2) that each household's augmented technology, T', exhibits constant returns to scale. An implication of efficiency is that a household's original technology, T, must also possess this property if production is impossible without goods.

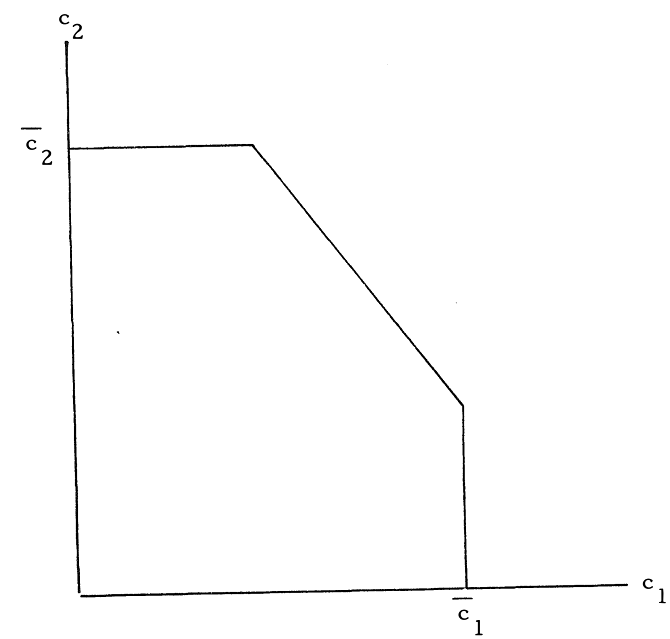

Figure 4

Furthermore, households must possess identical technologies. The following theorem proves this result.

Theorem 3: Suppose each household's technology, T^h, satisfies T1-T8. If Pareto efficiency is achieved then T^h is a cone and $T^h = T^j = T$ for all households h and j.

Proof: By (1), the dual to T^h is the closed, convex cone

$$M^h = \{(\Pi, P, q^h): \Pi C^h - PX^h \leqq q^h \text{ for all } (C^h, -X^h) \in T^h\}.$$

It will first be shown that Pareto efficiency requires that the implicit profit, q^h, is equal to zero. Using this it can be seen that each household's technology must be identical. Finally it will be shown that T is a cone.

For any given vector of good prices, say $\overline{P}$, Pareto efficiency obtains only if there is a vector of commodity prices, $\Pi(\overline{P})$, such that a hyperplane with the normal $(\Pi(\overline{P}), \overline{P})$ supports all boundary points in the technology with are solutions to each household's problem (U). By T1 and Theorem 2, $0 \in \text{bd } T^h$, and for the endowment $\overline{X} = 0$, $(C, -X) = (0, 0)$ is such a solution. By (1), $(\Pi(P), P, 0)$ are the corresponding dual variables on the boundary of M. Hence

$$\Pi(\overline{P})C - \overline{P}X \leqq 0 \text{ for all } (C, -X) \in T^h. \tag{10}$$

Any non-trivial $(C', -X') \in \text{bd } T^h$ which is a solution to (U) is supported by

$$\Pi(\overline{P})C' - PX' = q^h(\overline{P}).$$

Whereas (10) insures that $q^h(\overline{P}) \leqq 0$, (5) requires that q^h be non-negative. Thus $q^h(\overline{P}) = 0$.

The dual set of prices for each household may now be written as

$$M^h = \{(\Pi, P, q^h): P \geqq 0, \ 0 \leqq \Pi \leqq \Pi(P), \ q^h \geqq 0\},$$

and hence $M^h = M^j = M$ for all households h and j. This in turn implies that households possess identical technologies.

By setting q = 0, the dimensionality of M may be reduced by one, and M becomes

$$M = \{(\Pi, P): \Pi C - PX \leqq 0 \text{ for all } (C, -X) \in T\}.$$

This permits the dual cone of M to be written as

$$T' = \{(C, -X): \Pi C - PX \leqq 0 \text{ for all } (\Pi, P) \in M\} \tag{11}$$

and if T = T', then T is a cone.

(i) To show $T \subset T'$ suppose that there is a $(\overline{C}, -\overline{X}) \in T$ but $(\overline{C}, -\overline{X}) \notin T'$. By (11) this implies that $\Pi\overline{C} - P\overline{X} > 0$ for some $(\Pi, P) \in M$, which contradicts the definition of M, (1).

(ii) To show $T' \subset T$ suppose that there is a $(\overline{C}, -X') \in T'$ but $(\overline{C}, -X') \notin T$. Then there is a $(\overline{C}, -\overline{X}) \in \text{bd } T$ with $\overline{X} > X'$. To see this simply derive $\overline{X}$ from the optimal solution to the following programming problem:[10]

$$\begin{aligned} &\text{minimize } ||X|| \\ &\text{subject to} \\ &\quad (\overline{C}, -X) \in T \\ &\quad X = X' + \lambda 1 \\ &\quad \lambda \geqq 0. \end{aligned}$$

Again using (11) there exists a $(\Pi(\overline{P}), P) \geq 0$ such that

$$\Pi(\overline{P})\overline{C} - \overline{P}X' \leqq \Pi(\overline{P})\overline{C} - \overline{P}\overline{X}$$

or

$$\overline{P}(X' - \overline{X}) \geqq 0$$

which is a contradiction to $\overline{P} \geq 0$ and $X' - \overline{X} < 0$. Hence T = T'.

It may now be shown that the necessary conditions derived thus far are sufficient to ensure efficiency in an economy with an incomplete set of markets.

Theorem 4: Suppose T satisfies T1-T8. If all households possess the same technology, T, and T is a cone exhibiting no joint production, then a Pareto efficient allocation is achieved.

Proof: Since all households possess the same technology the dual set of prices, M, is the same for all households. It must be shown that M may be written as in (5),

$$M = \{(\Pi, P): P \geqq 0, \ 0 \geqq \Pi \geqq \Pi(P)\},$$

since, as previously shown, q may be deleted since T is a cone and $q = 0$ for all nonnegative P.

First, let $\overline{P} = 0$, then $\Pi = 0$ for all $(\Pi, P) \in M$. If this were not the case, $\Pi C - PX$ could be made arbitrarily large. Further, for $\overline{P} \geq 0$, recall the minimum expenditure problem

$$\min \{\overline{P}X: (C, -X) \in T, \ C \geqq C'\}$$

Choosing $C' > 0$, this problem has a regular optimal solution. Thus for $(C', -X')$ with $C' > 0$,

$$\Pi C - \overline{P}X \leqq \Pi' C' - \overline{P}X' \text{ for all } (C, -X) \in T \text{ and all } (\Pi, \overline{P}) \in M.$$

Since T exhibits no joint production, this problem may be examined for each commodity. Thus

$$\Pi C^{s\prime} - \overline{P}X^s \leqq \Pi' C^{s\prime} - \overline{P}X^s \leqq \Pi' C^{s\prime} - \overline{P}X^{s\prime} \text{ for all } (C^{s\prime}, -X^s) \in T_s' \text{ for all } (\Pi, \overline{P}) \in M.$$

The left-hand inequality further reduces to

$$\Pi C^{s\prime} \leqq \Pi' C^{s\prime} \text{ for all } (\Pi, \overline{P}) \in M$$

or

$$\pi_s c_s' \leqq \pi_s' c_s'$$

or that

$$\pi_s \leqq \pi_s' \text{ for all } (\Pi, \overline{P}) \in M.$$

Defining $\Pi(\overline{P}) = \Pi'$, M may be represented as in (5) and, therefore, efficiency is achieved.

IV. A SHARECROPPING MODEL

Consider an economy composed of N farms each of which produces a single crop. Let each household act as a sharecropper who, in return for his labor on farm j, receives a fraction of that farm's crop. The crop yield on each farm may vary with the state of the world that occurs, where states correspond to, say, weather conditions. The fractions of each farm j, x_j, to which a household has a claim may be though of as the goods in this model. With these goods the household produces consumption later if state of the world s obtains, c_s, $s = 1, \ldots, S$. It should be realized that the fraction of each farm j used in the production of some commodity, c_s, is the same fraction used in the production of any other commodity, $c_{s'}$. That is, goods are not diminished by any production process. Due to this property, the household's augmented technology may be defined as:

$$T' = \{(C, -X, -y): \sum_s C^s = C, \ \sum_s y^s = y, \ c_s \leqq f^s(X, y^s), \quad s = 1, \ldots, S\} \tag{12}$$

Further, this structure permits the following result to be obtained.

Theorem 5: Suppose that each household's technology, T, satisfies T1-T8, and that each augmented technology, T', is defined as in (12). If efficiency is achieved, then:

(a) household's possess identical, constant returns to scale technologies, and

(b) for each state, there must be a farm whose crop has a positive yield in that and only that state.

Proof: By application of Theorems 2 and 3, (a) follows immediately. To see that for any state there must be some farm whose crop has a positive yield in that state, assume that for some state there is no such farm. It is not possible to obtain any non-zero level of consumption in that state and, hence, efficiency is not achieved.

Finally, to show that a farm's crop may have a positive yiel d in one state, recall that for any $\overline{C}^s \geqq 0$, there must be a $(\overline{C}^s, -X^s) \in T$ such that

$$\overline{c}_s = f^s(X^s) \qquad (13)$$

$$0 = f^{s'}(X^s) \text{ for } s' \neq s. \qquad (14)$$

These two equations imply that for any $X^s \neq 0$, the goods used to produce c_s are not technologically feasible for the production of any other $c_{s'}$, $s' \neq s$. In other words, each farm must produce a crop which has a positive yield in only one state.

This result may be interpreted in the following way. Let there be two possible states: rain and drought. There must be some farm that plants its crop shallow and has a positive yield if there is a drought but a zero yield if there is rain. Alternatively, there must be some other farm that plants its crop deep and realizes a positive yield if there is rain but a zero yield if there is drought.

At this point it should be recognized that this sharecropping model is essentially equivalent to a securities model. In this model, households purchase fractions of firms, x_j, in order to provide for consumption if state s obtains, c_s, $s = 1, \ldots, S$. If security

holdings are required to be nonnegative, Theorem 5 states that there must be a complete set of "pure" Arrow-Debreu securities, i.e., for each state s, there is a firm s which pays a positive return, r_s, only in the event that state occurs. All other securities may then be represented as linear combinations of the "pure" Arrow-Debreu securities.

In many models, security holdings may be negative since short sales are permitted. Since this is inconsistent with T6, assumptions T6 and T7 may be replaced by

T6' The set $C(X) = \{C:(C, -X) \in T, C \geqq 0\}$ is compact.

The production functions defined previously for $Y \geqq 0$ now exist for any Y, since T6' guarantees the compactness of the set over which they are defined.

It should be noted that Theorem 2 needs no longer hold in the presence of short sales. Negative security holdings permit the trading off of consumption endowments in each state. With T6', the necessary conditions for efficiency become that each production function, $f^s(X, y^s)$ is linear in at least S securities. As in Theorem 5, for any $\overline{C}^s \geqq 0$, there must be a $(\overline{C}^s, -X^s, -y) \in T'$ such that

$$\overline{c}_s = f^s(X^s, y^s)$$
$$0 = f^{s'}(X^s, y^{s'}) \text{ for } s' \neq s$$

where $\sum_s y^s = y$. Furthermore, $\overline{C} = \sum_s \overline{C}^s$ may be produced from $X = \sum_s X^s$, or that

$$\overline{c}_s = f^s(X, y^s) = f^s(X^s, y^s)$$
$$= \sum_{s'} f^s(X^{s'}, y^s) \quad s = 1, \ldots, S \qquad (15)$$

since $f^s(X^{s'}, y^s) = 0$ for $s' \neq s$. Since $f^s(X, y^s)$ is linear homogeneous,

(15) requires that $f^s(X, y^s)$ is linear in all securities. Thus

$$f^s(X, y^s) = \sum_{j=1}^{S} x_j h_j^s(y^s) + h_o^s(y^s) \quad s = 1, \ldots, S,$$

where, by T1, $h_o^s(y^s) \geqq 0$, $j = 1, \ldots, S$.

The linear homogeneity of the production functions provides that

$$f^s(tX, ty^s) = t \sum_j x_j h_j^s(ty^s) + h_o^s(ty^s)$$

$$= t \sum_j x_j h_j^s(y^s) + th_o^s(y^s) \text{ for all } t \geqq 0.$$

By choosing X appropriately, it may be seen that

$$h_j^s(y^s) = r_j^s \geq 0, \; j = 1, \ldots, S,$$

and

$$h_o^s(y^s) = r_o^s y^s.$$

Thus the production function may be written as

$$f^s(X, y^s) = \sum_j r_j^s x_j + r_o^s y^s.$$

The term $r_o^s y^s$ may be thought of as exogeneous income the household receives if state s occurs.

V. FACTOR PRICE EQUALIZATION

At first glance, it may seem that the model presented in Section II is completely symmetric to an international trade model in which countries trade outputs but not inputs. The question then asked is: What are the necessary and sufficient conditions for countries to equalize factor prices? In attempting to sketch proofs to the theorems which, in this framework, are the analogs to those

presented in Section III, it will become apparent that there is some asymmetry between the two cases.

Suppose each country is endowed with a set of factors, $\overline{X}^c$, and a technology T^c, for transforming inputs, X, into outputs, C. There is a vector of prices, $\overline{\Pi}$, at which outputs may be traded, but no markets exist for trading factors. Each country chooses outputs and inputs so as to

	dual variables
maximize $\overline{\Pi} C^c$	Π
subject to	
$(C^c, -X^c) \in T^c$	q^c
$X^c \leqq \overline{X}^c$	P^c,

which has as its corresponding dual problem

minimize $P^c \overline{X}^c + q^c$

subject to

$$(\Pi, P^c, q^c) \in M$$

$$\Pi = \overline{\Pi}.$$

To continue with the analogy, factor price equalization requires that, independent of a country's endowment, each country must impute the same set of factor prices. The definition of factor price equalization may be formalized as:

Factor price equalization occurs if and only if each country's dual set of prices, M^c, can be represented as

$$M^c = \{(\Pi, P, q^c): \Pi \geqq 0, \; 0 \leqq P(\Pi) \leqq P, \; 0 \leqq q^c(\Pi) \leqq q^c\}$$

where $P(\Pi)$ and $q^c(\Pi)$ are functions solely of Π.

In examining the necessary conditions for factor price equalization it is easy to show that Theorems 2 and 3 continue to hold. If $0 \in$ bd T, then each country's implicit profit, q^c, may be set equal to zero and the dimensionality of the dual set of prices, M^c, may be reduced by one. Then $M^c = M^d = M$ for all countries c and d. As before, the dual cone to M, T', may be shown to equal the original technology, T. Hence, if each country's technology satisfies T1-T8, then a necessary condition for factor price equalization is that all countries must possess identical, constant returns to scale technologies.

However, in attempting to prove that Theorem 4 continues to hold, the asymmetry between inputs and outputs becomes apparent. Indeed, the absence of joint production is not a necessary condition for factor prices equalization. Consider a technology in which some outputs are produced jointly. For given output prices, there is no reason that imputed input prices are not the same for every level of factor endowment. Factor price equalization requires only that for every set of factors, Y, for which the inputed level of expenditure is fixed, $Q(\overline{\Pi})\, Y = k$, there must be some feasible level of output, $(C, -Y) \in T'$, such that the value of output is constant, i.e., $\overline{\Pi} C = k$.

VI. CONCLUDING COMMENTS

It should be noted that, as defined by (5), efficiency is tantamount to assuming that a nonsubstitution theorem holds. Reexamining Figure 2 should yield this equivalence transparent. Nonsubstitution is the same as assuming that the dual problem, (D), has an optimal solution $(\Pi^*, \overline{P}, q^*)$ which is independent of its objective vector $(\overline{C}, -1)$. In other words, for fixed input prices, $\overline{P}$, there is essentially a single price system which will be observed

in a competitive equilibrium. Thus, the theorems in Section III may also be thought of as necessary and sufficient conditions for non-substitution to obtain in an economy.

Although the results presented in this paper deal only with household behavior there is a straightforward adaption for dealing with the problem a firm faces. By discarding problem (U), the problem (E) may be interpreted as a model of firm behavior. Suppose that each firm possesses a technology, T^f, with which it produces output, C^f, from a set of inputs, X^f, and let the firm minimize its cost of production subject to producing some given level of output, $\overline{C}^f$. Pareto efficiency requires that, independent of the level of output the firm wishes to produce, the implicit price of each output must be the same for all firms. In other words, the "northeast" boundary of the production possibilities set must be a hyperplane. The results of this paper are then applicable in the analysis of the firm's technology.

APPENDIX

Prior to proceeding to Lemma 1, the following lemma must first be proved for future use.

Lemma A1: Every convergent sequence of boundary points of a closed set converges to a boundary point.

Proof: Let $\{x_n\}$ be a sequence of boundary points of a closed set X which converges to some x_0. Since X is closed, $x_0 \in X$. Thus, if x_0 is not a boundary point of X it must be an interior point. Define $B(x_0, \epsilon)$ as an n-ball of radius ϵ about x_0, such that any point in $B(x_0, \epsilon)$ is contained in X. For some n, $x_n \in B(x_0, \epsilon)$. Choose N^* sufficiently large such that for $n \geq N^*$, $d(x_0, x_n) = \delta < \frac{\epsilon}{2}$ where $d(x_0, x_n)$ is the distance between x_0 and x_n. Thus, $B(x_n, \delta) \subset B(x_0, \epsilon)$ and if x_0 is an interior point of X, then so must be x_n, for $n \geq N^*$, contradiction. Hence, x_0 must be a boundary point.

Lemma 1: The implicit commodity prices, $\Pi(P)$, and implicit profit, $q(P)$, as specified in (5) satisfy the following conditions:

(a) $\Pi(0) = 0$ and $q(0) = 0$:

(b) $\Pi(P) > 0$ if $P > 0$;

(c) $\Pi(P)$ and $q(P)$ are linear homogeneous with respect to P;

(d) $\Pi(P)$ is a concave function with respect to P and $q(P)$ is a convex function with respect to P;

(e) $\Pi(P)$ and $q(P)$ are continuous with respect to P.

Proof: (a) Let P = 0, and suppose $\Pi(0) \geq 0$ and $q(0) > 0$. Since for any $\overline{C} \geqq 0$ there is a $(\overline{C}, -X) \in T$, $\overline{C}$ may be increased to any level desired, and hence if $\Pi(0) \geq 0$, $\Pi(0)\overline{C}$ is unbounded. Thus $\Pi(0) = 0$.

Clearly, if P = 0 and $\Pi(0) = 0$, then $q(0) = 0$.

(b) Let $P > 0$, and suppose that some $\pi_s(P) = 0$. For the commodity vector C^s, where $c_s > 0$ and $c_{s'} = 0$ for $s' \neq s$,

$$\Pi(P)C^s - PX < 0 \text{ for all } (C^s, -X) \in T.$$

Hence $q(P) < 0$, a contradiction.

(c) This follows immediately since M is a cone.

(d) Consider $(\Pi(P^0), P^0, q(P^0)) \in M$ and $(\Pi(P^1), P^1, q(P^1)) \in M$. Since M is a convex set, for $0 \leqq t \leqq 1$, $(\Pi^t, P^t, q^t) \in M$, where

$$P^t = tP^0 + (1 - t)P^1 ,$$

$$\Pi^t = t\Pi(P^0) + (1 - t)\Pi(P^1) ,$$

and

$$q^t = tq(P^0) + (1 - t)q(P^1) .$$

By (5),

$$\Pi^t \leqq \Pi(P^t) \text{ and } q^t \geqq q(P^t) .$$

(e) To show that $\Pi(P)$ and $q(P)$ are continuous with respect to P, consider the set

$$M(\overline{P}) = \{\Pi(P), P, q(P)):(\Pi(P), P, q(P)) \in M, \ 0 \leqq P \leqq \overline{P}\}$$

To show that $M(\overline{P})$ is bounded, suppose that there is a sequence $\{(\Pi^i, \overline{P}, q^i)\} \in M$ such that at least one $\pi_s^i \to \infty$. (Without loss of generality, assume that $\pi_1^i \to \infty$.) Set $\pi_s^i = 0$ for $s \neq 1$ and $q^i = q(\overline{P})$. Then the sequence $\{((\pi_1^i, 0, \ldots, 0), \overline{P}, q(\overline{P}))\} \in M$ and $\pi_1^i \to \infty$. Let $t = \frac{1}{\frac{1}{T}\pi_1^i}$, and since M is a cone, $((t\pi_1^i, 0, \ldots, 0), t\overline{P}, tq(\overline{P})) \in M$.

Let $\pi_1^i \to \infty$ and, since M is closed,

$$((t\pi_1^i, 0, \ldots, 0), t\overline{P}, tq(\overline{P})) \to ((T, 0, \ldots, 0), 0, 0) \in M .$$

Since T may be made arbitrarily large, then there is a contradiction to (5), $0 \leqq \pi_1 \leqq \pi_1(P)$, since by (a) of this lemma, $\pi_1(0) = 0$. Thus $\Pi(P)$ is bounded. To show that q(P) is also bounded, recall, by T7, for $(C, 0) \in T$, C is bounded from above, or that

$$q(P) = \Pi(P)C$$

is bounded since $\Pi(P)$ is bounded. Since q(P) and $\Pi(P)$ are bounded, then $M(\overline{P})$ is bounded.

Since $M(\overline{P})$ is bounded, then any sequence in $M(\overline{P})$, say $\{\Pi(P^v), P^v, q(P^v)\}$ must be bounded. That sequence has a convergent subsequence, without loss of generality the original sequence itself, such that

$$\{\Pi(P^v), P^v, q(P^v)\} \rightarrow (\Pi^0, P^0, q^0)$$

Since $\{\Pi(P^v), P^v, q(P^v)\}$ is a sequence of boundary points of M, then by Lemma A1, it must converge to a boundary point, (Π^0, P^0, q^0). But by (5), $(\Pi(P^0), P^0, q(P^0))$ is also a boundary point of M, and since $\Pi(P)$ and q(P) are functions, then

$$\{\Pi(P^v)\} \rightarrow \Pi^0 = \Pi(P^0)$$

and

$$\{q(P^v)\} \rightarrow q^0 = q(P^0) .$$

Hence, $\Pi(P)$ and q(P) are continuous functions with respect to P.

<u>Lemma 2</u>: Each production function, $f^s(Y^s)$, as specified by (8), is a continuous, nonnegative, linear homogeneous, and concave function with respect to Y^s.

<u>Proof</u>: The nonnegativity of $f^s(Y^s)$ follows trivially from the definition of the production function in (6). Further, the linear homogeneity of each $f^s(Y^s)$ results since T' is a cone.

In order to see that $f^s(Y^s)$ is a concave function with respect to Y^s, let $\overline{Y}^s$ and $\overline{\overline{Y}}^s$ be two feasible input vectors, and

$$\hat{Y}^s = (1 - t)\overline{Y}^s + t\overline{\overline{Y}}^s \text{ for } 0 \leqq t \leqq 1 .$$

By (8),

$$f^s(\overline{Y}^s) = \overline{c}_s$$

and

$$f^s(\overline{\overline{Y}}^s) = \overline{\overline{c}}_s .$$

Since T' is convex

$$(\hat{C}^s, - \hat{Y}^s) \in T$$

where

$$\hat{c}_s = (1 - t)\overline{c}_s + t\overline{\overline{c}}_s .$$

Now

$$f^s(\hat{Y}^s) \geqq \hat{c}_s$$

and therefore

$$(1 - t)f^s(\overline{Y}^s) + tf^s(\overline{\overline{Y}}^s) = \hat{c}_s \leqq f^s(\hat{Y}^s) .$$

To demonstrate that $f^s(Y^s)$ is continuous with respect to Y^s, recall that the set

$$C(Y) = \{c_s:(C^s, - Y) \in T', \ c_s \geqq 0\}$$

is compact and non-empty. Let $\{Y^v\}$ be a sequence of input vectors converging to Y^o. Since C(Y) is compact, a subsequence can be chosen, without loss of generality the original sequence itself, with an associated sequence $\{c_s^v\}$ such that $c_s^v = f^s(Y^v)$ and $\lim_{v\to\infty} c_s^v = c_s^o$. If c_s^* is an optimal solution to the problem

$$\max \{c_s:(C^s, - Y^o) \in T', \ c_s \geqq 0\},$$

then

$$\lim_{v\to\infty} f^s(Y^v) = \lim_{v\to\infty} c_s^v = c_s^o \leqq c_s^* = f^s(Y^o)$$

and it remains to be shown that a strict inequality is impossible. Now, since by T1, the origin is in T', and by free disposal, T3, it is known that any (N + 1)-vector in the i^{th} direction, e^i, there is some C^{si} such that $(C^{si}, -e^i) \in T'$ for i = 1, ..., N + 1 where $C^{si} = (0, \ldots, 0, c_s^i, 0, \ldots, 0)$. Further, since T' is a convex cone, by determining the scalar

$$\alpha^v = \min \left[1, \frac{y_1^v}{y_1^o}, \frac{y_2^v}{y_2^o}, \ldots, \frac{y_{N+1}^v}{y_{N+1}^o}\right]$$

(if necessary, employing the convention $\frac{0}{0} = 1$), it is seen that

$$(\alpha^v C^{s*} + \sum_{i=1}^{N+1} (y_i^v - \alpha^v y_i^o) C^{si}, -[\alpha^v Y^o + \sum_{i=1}^{N+1} (y_i^v - \alpha^v y_i^o) e^i]) =$$

$$(\alpha^v C^{s*} + \sum_{i=1}^{N+1} (y_i^v - \alpha^v y_i^o) C^{si}, -Y^v) \in T' .$$

Hence,

$$c_s^v = f^s(Y^v) \geqq \alpha^v c_s^* + \sum_{i=1}^{N+1} (y_i^v - \alpha^v y_i^o) c_s^i$$

which implies (if necessary, picking an appropriate subsequence)

$$c_s^o = \lim_{v\to\infty} c_s^v \geqq \lim_{v\to\infty} [\alpha^v c_s^* + \sum_{i=1}^{N+1} (y_i^v - \alpha^v y_i^o) c_s^i] = c_s^* .$$

Hence, $f^s(Y^v) \to f^s(Y^o)$ and, therefore, $f^s(Y)$ is a continuous function with respect to Y.

Lemma 3: Each sub-technology, T_s', is a closed, convex cone and may be alternatively defined as

$$T_s' = \{(C^s, -Y^s): \Pi C^s - QY^s \leqq 0 \text{ for all } (\Pi, Q) \in M\}.$$

Proof: Recall from Lemma 2 that $f^s(Y^s)$ is continuous, concave and linear homogeneous with respect to Y^s. Since $f^s(Y^s)$ defines the "northeast" boundary of T_s', the following properties of T_s' are trivial to show.

(1) T_s' is closed by the continuity of $f^s(Y^s)$,

(2) T_s' is convex by the concavity of $f^s(Y^s)$, and

(3) T_s' is a cone by the linear homogeneity of $f^s(Y^s)$.

Further recall that

$$\Pi C - QY \leqq 0 \text{ for all } (\Pi, Q) \in M \text{ and all } (C, -Y) \in T',$$

and since T_s' is a subset of T', then

$$\Pi C^s - QY^s \leqq 0 \text{ for all } (\Pi, Q) \in M \text{ and all } (C^s, -Y^s) \in T_s' .$$

Hence, T_s' may be alternatively defined as

$$T_s' = \{(C^s, -Y^s): \Pi C^s - QY^s \leqq 0 \text{ for all } (\Pi, Q) \in M\}.$$

Lemma 4: T' exhibits no joint production if and only if there exists S sub-technologies of the form

$$T_s' = \{(C^s, -Y^s): \Pi C^s - QY^s \leqq \text{ for all } (\Pi, Q) \in M\} \quad s = 1, \ldots, S,$$

such that

$$\sum_s T_s' = T'.$$

Proof: (i) The sufficiency half of this proof follows trivially since it has already been asserted that T' is non-joint if $T' = \sum_s T_s'$ where

$$T_s' = \{(C^s, -Y^s): c_s \leqq f^s(Y^s), Y^s \geqq 0\} \quad s = 1, \ldots, S$$

and Lemma 3 shows that T_s' may alternatively be written as in (11).

(ii) Suppose that $T' = \sum_s T_s'$ where T_s' is defined as in (9). For $Y^s \geqq 0$, define a production function over each T_s' as:

$$f^s(Y^s) = \max\{c_s : (C^s, -Y^s) \in T_s', \; c_s \geqq 0\} \quad s = 1, \ldots, S \qquad \text{(A1)}$$

In order to prove that this maximum exists, it will be shown that the constraint set is compact. This set is closed since, by Lemma 3, T_s' is closed. To show that this set is bounded consider the programming problem

$$\max \{\Pi C : (C, -Y) \in T', \; 0 \leqq Y \leqq \overline{Y}\}.$$

Let $\overline{Y} > 0$, then for $\Pi > 0$ this problem has a regular optimal solution since it satisfies Slater's condition. This in turn implies that there is some $(\Pi, P) \in M$ with $\Pi > 0$. By (9)

$$\Pi C^s - QY^s \leqq \text{ for all } (\Pi, P) \in M,$$

and since $\Pi > 0$ for some $(\Pi, P) \in M$, c_s must be bounded.

Next consider the set V_s, where

$$V_s = \{(C^s, -Y^s) : 0 \leqq Y^{s\prime} \leqq Y^s, \; c_s \leqq f^s(Y^{s\prime})\} \quad s = 1, \ldots, S.$$

Now if $V_s = T_s'$ for all s then $T' = \sum_s V_s$ and is of the form of (9). T' may then be said to be non-joint.

To show that $V_s = T_s'$, consider a point $(C^s, -\overline{Y}^s) \in T_s'$. By (A1), $c_s \leqq f^s(\overline{Y}^s)$, and as T_s' is defined in (11), if $(C^s, -\overline{Y}^s) \in T_s'$ and $Y^s \geqq \overline{Y}^s$ then $(C^s, -Y^s) \in T_s'$. Thus $(C^s, -Y^s) \in V_s$ for $0 \leqq \overline{Y}^s \leqq Y^s$.

In a similar manner consider a point $(C^s, -\overline{Y}^s) \in V_s$. By the definition of $f^s(\overline{Y}^s)$ in (A1), $(C^s, -\overline{Y}^s) \in T_s'$. Therefore, $T_s' = V_s$ and T' may be said to be non-joint.

FOOTNOTES

1. This assumption is overly strong. Indeed, if some of the commodities are traded, say the first S_1 of them, then the technology may be written as

$$\left.\begin{array}{c} C_1^h \leqq X_1^h \\ (C_2^h, -X_2^h) \in T^h \end{array}\right\} T^h$$

where C_1^h and X_1^h are $S_1 \times 1$ vectors and, corresponding to the nontraded commodities, C_2^h is of dimension $(S - S_1) \times 1$ which is produced from X_2^h of dimensionality $(N - S_1) \times 1$.

2. The sign convention concerning X follows that adopted by Cass [3]. This allows for both inputs and outputs to be measured in nonnegative quantities and along with two further assumptions regarding T^h given later in this section (in particular, T1 and T3) this convention ensures the nonnegativity of the associated dual variables.

3. Throughout this paper (i) $X \geqq Y$ means $x_s \geqq y_s$ for all s, (ii) $X \geq Y$ means $x_s \geqq y_s$ for all s and $x_s > y_s$ for some s, and (iii) 0 denotes the zero vector of appropriate dimensionality.

4. Except where it is needed for clarification, the superscript h, denoting household, will be deleted.

5. See [3], pp. 276-278.

6. It is at this point that the analysis substantially differs from that of Pollak and Wachter [8]. For their purposes it is critical

that implicit profit, $q^h(P)$, be zero for all households. If this were not the case, income will depend upon good prices and the commodity demand functions they derive may fail to exhibit traditional properties.

7. To show that this maximum exists, choose an arbitrary non-negative level of Y^s, say $\overline{Y}^s$, and examine $C(\overline{Y}^s) = \{c_s: (C^s, -\overline{Y}^s) \in T', c_s \geqq 0\}$. Since T' is closed, then $C(\overline{Y}^s)$ is closed. Since $0 \in T'$, then $(0, -\overline{Y}^s) \in T'$. This implies that $C(\overline{Y}^s)$ is non-empty. To show that $C(\overline{Y}^s)$ is bounded, suppose that there is a sequence $(C^{si}, -\overline{Y}^s) \in T'$ such that $c_s^i \to \infty$. Let $t = \dfrac{1}{\frac{1}{M}||C^{si}||} = \dfrac{1}{\frac{1}{M}c_s^i}$, and since T' is a cone, $(tC^{si}, -t\overline{Y}^s) \in T'$. Let $c_s^i \to \infty$ and, since T' is closed,

$$\left(\frac{C^{si}}{\frac{1}{M}c_s^i}, -\frac{\overline{Y}^s}{\frac{1}{M}c_s^i}\right) \to ((0, \ldots, 0, M, 0, \ldots, 0), 0) \in T'.$$

Since M can be made arbitrary large, this point in T' can be seen to violate T7. Thus $C(\overline{Y}^s)$ is bounded and the production function $f^s(Y^s)$ exists.

8. This lemma is very similar to a theorem found in Hall [6] who shows that non-jointness is equivalent to being able to sum input requirement sets.

9. This minimum can be shown to exist by arguing that the constraint set may be made compact without affecting the solution. Since each T_s' is a closed cone with vertex 0, then $\sum_s T_s'$ is closed. This guarantees that the constraint set is closed. Since T_s' is a subset of T', then, by T8, there is some Y^s such that $(\overline{C}^s, -Y^s) \in T'$ for $s = 1, \ldots, S$. It is

necessarily true that $(\overline{C}^s, -Y^s) \in T_s'$ or that $(\overline{C}, -Y) \in \sum_s T_s'$ where $Y = \sum_s Y^s$. Since $(\overline{C}, -Y') \notin \sum_s T_s'$, at least one component of Y' must be increased in order to produce $\overline{C}$ in $\sum_s T_s'$. Thus Y may be made strictly greater than Y' and $\overline{Y}$ may be chosen such that $Y \geqq \overline{Y} > Y' \geqq 0$. Adding the restriction $Y \geqq \overline{Y} > Y'$ will bound the set and not alter the solution.

10. To show that this minimum exists, the constraint set may be shown to be compact. This set is closed since, by T2, T is closed. By T8, there is some X such that $(\overline{C}, -X) \in T$. Since $(\overline{C}, -X') \notin T$, at least one component of X' must be increased in order to produce $\overline{C}$ in T. Thus, by free disposal, X may be made strictly greater than X' and $\overline{X}$ may be chosen such that $X \geqq \overline{X} > X' \geqq 0$. Adding the restriction $X \geqq \overline{X} > X'$ to the constraint set will bound the set and not affect the solution.

11. It is in this proof that assumption T2 is critical to the analysis. If T were not closed then it is the closure of T that is dual to M.

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