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STABILITY OF PURE TRADE EQUILIBRIUM WITH EXTERNALITIES*

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ABSTRACT

Sufficient conditions for the stability of competitive equilibrium in a pure trade economy with externalities are developed in this paper. Externalities are introduced through the assumption that each individual's utility depends on the consumption of every other individual. A two-level adjustment process is postulated. At fixed prices, individual strategies must be made mutually consistent. Each individual's strategy is stated as a relation which maps prices and the demands of all other individuals into the demand of that individual. The equlibrium of the externality adjustment process is a demand allocation, depending on price, which is feasible and maximizes utility for each individual at given prices. Sufficient conditions for stability of the externality adjustment process are proved and interpreted.

The equilibrium demand functions are then used in a tâtonnement process to investigate the stability of competitive equilibrium. All the standard theorems on excess demand functions which give sufficient conditions for stability apply to the equilibrium demand functions of an economy with externalities. It is established that the stability properties of an economy without externalities possess a certain type of continuity. Any sequence of economies with externalities which converges in the proper sense to an economy without externalities characterized by gross substitutability has the property that for all $t > T$ the competitive equilibrium of the economy with externalities is stable. Weaker stability conditions on the limit economy can make this theorem fail.

STABILITY OF COMPETITIVE EQUILIBRIUM
WITH EXTERNALITIES

W. David Montgomery

1. INTRODUCTION AND NOTATION

In analyzing the stability of competitive equilibrium with externalities, notation itself can impose a heavy burden on the reader since large matrices of second derivatives appear. Therefore a self-contained section introducing notation appears appropriate. Individuals are indexed by the superscripts i and j, commodities by the subscripts k and ℓ. There are n individuals and m commodities. Vector inequalities are >>, >, and $\geq$ with their usual meaning. I_m is the m x m identity matrix, and the transpose of A is written A^T. All vectors start out as row vectors, but to simplify notation the inner product $a \cdot b^T$ will be written ab.

$x^i = (x^i_1 \ldots x^i_k \ldots x^i_m)$ is the consumption vector of the i^{th} individual.

$\overline{x}^i$ is the initial endowment of individual i.

$w = (x^1 \ldots x^i \ldots x^n)$ is a consumption allocation.

$w^i = (x^1 \ldots x^{i-1}, x^{i+1} \ldots x^n)$ represents the consumption allocation for all individuals other than i.

$U^i(w)$ is the utility function of individual i. It will sometimes be written $U^i(x^i, w^i)$ when distinguishing the i^{th} component vector of w is important.

$[\nabla_i U^i] = \left(\frac{\partial U^i}{\partial x^i_1} \ldots \frac{\partial U^i}{\partial x^i_m}\right)$ is the directional derivative of U^i taken in the direction of the x^i co-ordinate axes.

$$[\nabla^2_i U^i] = \begin{bmatrix} \frac{\partial^2 U^i}{\partial x^i_1 \partial x^i_1} & \cdots & \frac{\partial^2 U^i}{\partial x^i_1 \partial x^i_m} \\ \vdots & \cdots & \vdots \\ \frac{\partial^2 U^i}{\partial x^i_m \partial x^i_1} & \cdots & \frac{\partial^2 U^i}{\partial x^i_m \partial x^i_m} \end{bmatrix}$$

$$[\nabla_j \nabla_i U^i] = \begin{bmatrix} \frac{\partial^2 U^i}{\partial x^i_1 \partial x^j_1} & \cdots & \frac{\partial^2 U^i}{\partial x^i_1 \partial x^j_m} \\ \vdots & \cdots & \vdots \\ \frac{\partial^2 U^i}{\partial x^i_m \partial x^j_1} & \cdots & \frac{\partial^2 U^i}{\partial x^i_m \partial x^j_m} \end{bmatrix}$$

$$[\nabla\nabla_i U^i] = \left[\begin{array}{c|c|c|c} \nabla_1^2 U^1 & \nabla_2 \nabla_1 U^1 & \cdots & \nabla_n \nabla_1 U^1 \\ \hline \nabla_1 \nabla_2 U^2 & \nabla_2^2 U^2 & \cdots & \nabla_n \nabla_2 U^2 \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \nabla_1 \nabla_n U^n & \cdot & \cdots & \nabla_n^2 U^n \end{array}\right]$$

$$[DU] = \begin{bmatrix} \nabla_1 U^1 & 0 & \cdots & 0 \\ \cdot & \nabla_2 U^2 & \cdots & \cdot \\ 0 & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \nabla_n U^n \end{bmatrix}$$

$$A = \left[\begin{array}{c|c} [\nabla\nabla_i U^i] & [DU]^T \\ \hline [DU] & 0 \end{array}\right]$$

$$H^i = \left[\begin{array}{c|c} [\nabla_i^2 U^i] & [\nabla_i U^i]^T \\ \hline [\nabla_i U^i] & 0 \end{array}\right]$$

$$L^i = \left[\begin{array}{c|c} I_m & 0 \\ \hline 0 & -\frac{1}{\lambda_i} \end{array}\right]$$

$$L = \left[\begin{array}{c|ccc} I_{mn} & & & 0 \\ \hline & -\frac{1}{\lambda_1} & & 0 \\ 0 & 0 & \ddots & -\frac{1}{\lambda_n} \end{array}\right]$$

Consider a pure trade economy

$$E = (U^1(w), \ldots, U^n(w), \bar{x}^1, \ldots, \bar{x}^n)$$

in which the utility of each individual depends on his own consumption and on the consumption of everyone else. Competitive equilibrium of this economy is a price vector p^* and a consumption allocation w^* such that

(a) $p^* \geq 0$

(b) $\sum_i (x^{i*} - \bar{x}^i) \leq 0$

(c) x^{i*} maximizes $U^i(x^i, w^{i*})$ subject to $p^* x^i \leq p^* \bar{x}^i$.

In a more complete model [2] Arrow and Hahn prove the existence of competitive equilibrium for such an economy with assumptions weaker than the following, which we adopt.

(D) (Differentiability) $U^i(w)$ is continuously twice differentiable.

(Q) (Strict quasi-concavity) $[\nabla_i^2 U^i]$ is negative definite wherever evaluated under the constraint $p(x^i - \bar{x}^i) = 0$.

(N) (Nonsatiation) $U^i(x^i, w^i) > U^i(\hat{x}^i, w^i)$ if $x^i > \hat{x}^i$.

We will establish conditions for the stability of equilibrium of this economy, relying for existence of equilibrium on the proof in [2].

The stability of an economy without externalities can be analyzed without going behind excess demand functions which are assumed to be continuous and single valued. In the case of externalities individual demand for a good is not well defined as a function of p alone, so that simple summation of individual demand to obtain aggregate demand is not possible. Choices of all agents must be consistent in the sense of condition (c) of equilibrium. We formulate the problem as one in which two separate adjustment processes are at work. In the first, called the "externality adjustment process," prices are fixed. Given these prices, each individual maximizes utility in the belief that the current choices of all other individuals will remain unchanged. To illustrate the type of adjustment involved, consider an exchange economy with two individuals and two goods. Straightforward utility maximization gives individual one's demand for each good as a function of prices and individual two's demands:

$$x_1^1 = f_1(p_1, p_2, x_1^2, x_2^2)$$

$$x_2^1 = f_2(p_1, p_2, x_1^2, x_2^2)$$

If each individual satisfies his budget constraint with equality, Walras' Law implies that only one market, say for good one, need be considered, since demand for good two is uniquely determined by each individual's demand for good one. Since prices are fixed, we are left with a system

$$x_1^1 = \phi(x_1^2)$$

$$x_1^2 = \Psi(x_1^1).$$

Suppose that from any arbitrary starting point each individual adjusts his demand according to a gradient rule

$$\dot{x}_1^1 = g(\phi(x_1^2) - x_1^1)$$

$$\dot{x}_1^2 = h(\Psi(x_1^1) - x_1^2)$$

where g and h are arbitrary monotonic increasing functions. Olech's theorem [7] states that necessary and sufficient conditions for the global stability of this system are

$$g' + h' > 0$$

$$g'h'[1 - \phi'\Psi'] > 0$$

for all x_1^1, x_1^2.

Since $g' > 0$ and $h' > 0$, stability depends only on having

$$\phi'\Psi' < 1.$$

If the externality is symmetric, in the sense that ϕ' and Ψ' are of the same sign, then a magnitude restriction on the effect of the externality is needed to have stability. In particular, a restriction will be imposed on how one individual's marginal rate of substitution between goods one and two in his own consumption is changed by changes in the other individual's demand for good one.

Two special cases are illuminating. Suppose $|\phi'| > 1$ and $|\Psi'| > 1$ and $\text{sgn}\ \phi' = \text{sgn}\ \Psi'$. Then changing two's consumption of good one by one unit causes individual one to change his consumption by <u>more</u> than one unit, and similarly, for the obverse case. The explosive character of such a situation is clear. Conversely, suppose that individual one's demand is unaffected by individual two's choices. Then $\phi' \equiv 0$, and the system is always stable.

The equilibrium consumption levels reached by this externality adjustment process will not result in the clearing of all markets if prices are chosen arbitrarily. A second adjustment process in which p converges to an equilibrium value is also needed.

A tâtonnement process in the markets for commodities is assumed. When prices are announced, each individual acts as if he can buy and sell any quantity. Each individual announces to the auctioneer and to all other individuals his desired purchases and sales. Before another price is announced, the offers are allowed to converge to a Nash equilibrium by means of the adjustment process outlined above. If this process is stable, the tâtonnement can proceed. The stability of competitive equilibrium under tâtonnement will depend only on the behavior of equilibrium offers at each price vector. A general representation of such a two-level adjustment process can now be stated.

The externality adjustment process depends entirely on properties of individual utility functions. Three fundamental theorems are proved: the existence of consistent individual strategies, the uniqueness and continuity of equilibrium demand functions $w(p)$ for p in a neighborhood of p^*, and sufficient conditions for the stability of an adjustment process by which demand converges to $w(p)$ for fixed p. The excess demand functions are then used in defining stability of competitive equilibrium, which is examined exclusively in terms of excess demand.

2. STABILITY OF EXTERNALITY ADJUSTMENT

Each individual chooses x^i to satisfy the following first-order conditions (written in vector notation).

$$\nabla_i U^i(w) - \lambda_i p = 0 \tag{2.1}$$

$$p(x^i - \bar{x}^i) = 0$$

By hypothesis each of the first-order conditions is continuously differentiable, and from (N) the budget constraint is an equality.

Lemma 2.1: The system given in (2.1) has the property that for any w^i and for $p >> 0$ there exists an x^i which satisfies (2.1).

Since U^i is jointly continuous in all its arguments, it is continuous in x^i for fixed $w^i \geq 0$. Since the budget set $\{x^i : px^i \leq p\bar{x}^i$ and $x^i \geq 0\}$ is compact, U^i attains its maximum for some x^i in the budget set for any fixed w^i and p. ||

Lemma 2.2: Let $X^i(w^i) = \{x^i : x^i$ maximizes $U^i(x^i, w^i)$ subject to $px^i \leq p\bar{x}^i\}$. Then for fixed $p >> 0$, $X^i(w^i)$ is an upper semi-continuous correspondence.

Proof: Debreu [5], p. 19 states the theorem that if $f : S \times T \longrightarrow R^1$ is continuous and $\varphi : S \longrightarrow T$ is closed and continuous on S, then the set $\mu(x)$ of elements of $\varphi(x)$ which maximize f on $\varphi(x)$ is upper semicontinuous on S. The function $U^i : R^{m+} \times R^{m(n-1)+} \longrightarrow R^1$ is continuous, and the closed budget correspondence $\varphi(w^i) = \{x^i : px^i \leq p\bar{x}^i$ and $x^i \geq 0\}$ is constant in (independent of) w^i and hence continuous. Hence the correspondence $X^i(w^i)$ is upper semicontinuous since it is the set of elements of $\varphi(w^i)$ which maximize U^i on $\varphi(w^i)$. ||

Since the Jacobian of 2.1 is $|L^iH^iL^{i}|$, which is nonvanishing by Q, we can use the implicit function theorem to prove that there exists an open neighborhood $N(\hat{w})$ such that for any $\hat{w}$, $X^i(w)$ is a single-valued function on $N(\hat{w})$. Since for a function upper semi-continuity is equivalent to continuity, this implies that there is one, and only one, continuous <u>function</u> $X^i(w^i)$ which satisfies (2.1).

These lemmas establish properties of each individual's optimal choice given arbitrary values of the choices of all other individuals. It remains to establish the existence of mutually consistent optimal choices when prices are set arbitrarily.

<u>Theorem 2.1</u>: There exists a w such that (2.1) is satisfied by w simultaneously for all i whenever $p >> 0$.

<u>Proof</u>: Let $T^i = \{x^i : px^i \le p\bar{x}^i$ and $x^i \ge 0\}$. Let $D^i(w) = \{x^i : x^i \in T^i$ and $U^i(x^i, w^i) \ge U^i(\hat{x}^i, w^i)$ for all $\hat{x}^i \in T^i\}$, where it is understood that $D^i(w)$ is independent of the values chosen for the i^{th} component vector of w. Note that $D^i(w) = X^i(w^i)$ for any choice of x^i. By Lemma 2.1 and (N) for any w^i and p there exists $x^i << \infty$ (since $p >> 0$) such that $x^i \in D^i(w)$, and by Lemma 2.2, D^i is upper semicontinuous.

Let $\tilde{x}^i_k = \dfrac{p \cdot \bar{x}^i}{p_k}$, $i = 1, \ldots, n$, $k = 1, \ldots, m$,

and let $\tilde{W} = \{w : 0 \le x^i_k \le \tilde{x}^i_k\}$. Clearly $\mathop{\mathrm{X}}\limits_{i=1}^{n} D^i \subset \tilde{W}$ for all w, so that the image of the restriction of $\mathop{\mathrm{X}}\limits_{i=1}^{n} D^i$ to $\tilde{W}$ is also contained in $\tilde{W}$. Let the restriction be represented by $\tilde{D} : \tilde{W} \longrightarrow \tilde{W}$. Since $\tilde{W}$ is closed, bounded, and convex, upper semicontinuity of $\tilde{D}$ is sufficient to satisfy the conditions of Kakutani's fixed-point theorem [2]. Therefore there exists a $w^* \in \tilde{W}$ such that $w^* \in \tilde{D}(w^*) = \sum\limits_{i=1}^{n} \tilde{D}^i(w^*)$. Since 2.1 are necessary conditions satisfied by any $w^* \in D(w^*)$, it follows that solutions of 2.1 exist. ||

To find conditions on U^i which imply the existence of a <u>unique</u> function $w(p)$ satisfying (2.1) we observe that the functions (2.1) are continuously differentiable in w, and apply the implicit function theorem again. We call a single-valued function $w(p)$ which maps p into the solutions of (2.1) an equilibrium demand function.

<u>Theorem 2.2</u>: If A is nonsingular then there exists a neighborhood of the equilibrium price vector $N(p^*)$ such that for all $p \in N(p^*)$ there exist single-valued, continuously differentiable equilibrium demand functions $w(p)$.

<u>Proof</u>: By assumption (N), p^* will be strictly positive. (If $p^*_k = 0$ for some k, any individual can increase his utility indefinitely by demanding more of good k.) Therefore in some small neighborhood of p^*, $N(p^*)$, p is strictly positive if $p \in N(p^*)$, so that Theorem 2.1 establishes the existence of solutions to 2.1.

The theorem now follows as a restatement of the implicit function theorem [1]. The determinant $|LAL|$ is the Jacobian of system (2.1), and LAL is nonsingular if and only if A is nonsingular. Since each function in (2.1) is continuously differentiable, and since by Theorem 2.2 we know that there exist solutions of 2.1), the hypothesis of the implicit function theorem is satisfied when $|A| \neq 0$. ||

The nonsingularity of A is a real restriction, in that it is not implied by other assumptions regarding the quasi-concavity of the U^i, which relate only to the diagonal blocks and associated bordering vectors, equal to $\nabla_i U^i$, of A.

Since unique equilibrium demand functions exist for all $p \in N(p^*)$, it is possible to ask if, for fixed p, the Nash equilibrium is stable. Consider the following general adjustment process:

$$\begin{aligned} \dot{x}^i &= \nabla_i U^i(w) - \lambda_i p \\ 0 &= p \cdot (x^i - \bar{x}^i) \end{aligned} \qquad (2.2)$$

where p is the fixed price vector. Note that if U^i depended only on x^i this would be a gradient process by which each individual could find his private utility maximum given p under the constraint that his budget constraint be satisfied exactly at every instant. In the case of externalities, however, the gradient $\nabla_i U^i$ is continuously altered by the changing consumption plans of other agents. If w^i is fixed, for each individual this process is equivalent to the modified gradient process of Arrow and Solow [3]. By further modifying their process, we can use their proof of local convergence of the gradient process to the maximum to prove the local stability of (2.2). We prove local stability by taking a Taylor series approximation to (2.2), expanding around the Nash equilibrium values. If this linear approximation is asymptotically stable, (2.2) is locally stable.

The linear approximation of (2.2)

$$\dot{x}^i = \nabla\nabla_i U^i(\hat{w}) \cdot (w^i - \hat{w}^i)^T - (\lambda_i - \hat{\lambda}_i)p^T \qquad (2.3)$$

$$0 = -p(x^i - \hat{x}^i). \qquad (2.4)$$

From (2.3), $p\dot{x}^i = 0$. Hence

$$p\dot{x}^i = p\nabla\nabla_i U^i(\hat{w})(w^i - \hat{w}^i)^T - (\lambda_i - \hat{\lambda}_i)pp^T = 0,$$

and we can solve for

$$\begin{aligned}(\lambda_i - \hat{\lambda}_i) &= (pp^T)^{-1} p\nabla\nabla_i U^i(\hat{w})(w - \hat{w})^T \\ &= Q_i(w - \hat{w})^T. \qquad (2.5)\end{aligned}$$

Each Q_i is a $1 \times m$ row vector. Define Q as the $n \times m$ matrix whose i^{th} row is Q_i. Then (2.3) and (2.4) can be written, when $\lambda - \hat{\lambda}_i$ is eliminated by 2.5, as

$$\dot{w} = ([\nabla\nabla_i U^i(\hat{w})] - \Pi Q)(w - \hat{w}). \qquad (2.6)$$

These mn differential equations have only mn - m degrees of freedom.

Let μ be a root of the equation

$$\text{Det}(\mu) = \left| \begin{array}{c|c} [\nabla\nabla_i U^i(\hat{w}) - \mu I_{mn}] & \Pi^T \\ \hline \Pi & 0 \end{array} \right| = 0. \qquad (2.6)$$

We can now state the fundamental stability theorem.

Theorem 2.3: If the real parts of the roots of Det(μ) are negative, then (2.2) is locally stable, i.e. it is stable in any region in which the linear approximation of (2.3) and (2.4) is valid.

Proof: We demon·trate that 2.6 is asymptotically stable. Choose as a candidate solution $w - \hat{w} = Ce^{\mu t}$. Then C must satisfy $\Pi C = 0$ if $w - \hat{w}$ is to satisfy the budget constraint (2.4). Also, to satisfy (2.6) we must have

$$e^{\mu t}([\nabla\nabla_i U^i(\hat{w})]C + \Pi QC) = \mu Ce^{\mu t}.$$

Putting these conditions together, C and μ must satisfy

$$\left[\begin{array}{c|c} [\nabla\nabla_i U^i(\hat{w}) - \mu I_{mn}] & \Pi^T \\ \hline \Pi & 0 \end{array} \right] \left[\begin{array}{c} C \\ \hline QC \end{array} \right] = 0.$$

This system of homogeneous linear equations has a solution if and only if the determinant of the left-hand side matrix, Det(μ), vanishes. If the real parts of all the roots of Det(μ) = 0 are negative, then (2.6) is asymptotically stable. ||

To give some economic sense to the condition that Det (μ) = 0 have roots with negative real parts we give two very restrictive

$\eta(V^i) < 0$, we can choose an $\varepsilon > 0$ such that the real part of $\mu(U^i)$ is also negative.

We have by this argument proved the following theorem:

<u>Theorem 2.5:</u> There exists a number $\varepsilon > 0$ such that if

$$0 < \frac{\partial^2 U^i(w)}{\partial x_k^i \partial x_\ell^j} < \varepsilon$$

for all w, for $i \neq j$, and for all k and ℓ, the Nash equilibrium is stable.

A simpler argument establishes that the Nash equilibrium is stable if all externalities are uni-directional, that is, if U^1 is a function of x^1 alone, U^2 a function of x^1 and x^2, U^3 a function of x^1, x^2, x^3, and so on up to U^n a function of w. In this case Det(μ) has a block triangular form, since $[\nabla_j \nabla_i U^i] = 0$ for $j > i$. The determinant of a block triangular matrix is equal to the product of the determinants of the diagonal blocks [6]. That is

$$\left|\begin{array}{cc|cc|c|cc}
\nabla_1^2 U^1 - \mu I_n & p^T & & 0 & \cdots & & 0 \\
p & 0 & & & & & \\
\hline
\nabla_1 \nabla_2 U^2 & 0 & \nabla_2^2 U^2 - \mu I_m & p^T & \cdots & & \\
0 & & p & 0 & & & 0 \\
\hline
\cdot & & \cdot & & \cdots & & \cdot \\
\hline
\nabla_1 \nabla_n U^n & 0 & \nabla_2 \nabla_n U^n & 0 & \cdots & \nabla_n^2 U^n - \mu I_m & p^T \\
0 & & 0 & & & p & 0
\end{array}\right|$$

$$= \prod_{i=1}^{n} \left|\begin{array}{c|c} [\nabla_i^2 U^i - \mu^i I_m] & p^T \\ \hline p & 0 \end{array}\right| = 0$$

Each of these factors has only real negative roots by Q. Any root μ^i of each factor in the product is a root of the product. This gives us $n(m - 1)$ roots, which are all the roots μ, and establishes the following theorem.

<u>Theorem 2.6:</u> If $U^1 = U^1(x_1)$, $U^2 = U^2(x_1, x_2) \ldots U^n = U^n(x_1, \ldots, x_n)$, then the Nash equilibrium is locally stable.

Note that the roots of

$$\left|\begin{array}{c|c} [\nabla\nabla_i U^i - \mu I_m] & \Pi^T \\ \hline \Pi & 0 \end{array}\right| = 0$$

are invariant to a monotonic transformation of the utility function.

Transform each utility function U^i by a monotonic increasing function $F_i(U^i)$. The typical element of $[\nabla\nabla_i F^i(U_i)]$ then is

$$F_i' \frac{\partial^2 U^i}{\partial x_k^i \partial x_\ell^j} + F_i'' \frac{\partial U^i}{\partial x_k^i} \frac{\partial U^i}{\partial x_\ell^j},$$

$i, j = 1, \ldots n,\ \ k, \ell = 1, \ldots m.$

Because of the structure of the bordering matrix Π, we can use a bordering column multiplied by an appropriate scalar to eliminate all terms involving F_i'' in the usual fashion, and then factor all F_i' terms out of the determinant, obtaining the result that sufficient conditions. The first is that externalities be small relative to the curvature of the indifference surfaces of $U^i(x^i, w^i)$ for fixed w^i. If externalities are absent, then $U^i(w)$ is in fact independent of w^i and of the form $V^i(x_i)$. In this case stability of the Nash equilibrium is a direct consequence of strict quasiconcavity.

By assumption (Q), the roots η_i of

$$\left| \begin{array}{c:c} \nabla_i^2 V^i(\hat{x}_i) - \eta_i I_m & p^T \\ \hdashline p & 0 \end{array} \right| = 0$$

are real and negative. Therefore the roots η of

$$\left| \begin{array}{c:c:c:c} \begin{matrix} [\nabla_1^2 V^1(\hat{x}_1) - \eta I_m] & p^T \\ p & 0 \end{matrix} & 0 & \cdots & 0 \\ \hdashline 0 & \begin{matrix} [\nabla_2^2 V^2(\hat{x}_2) - \eta I_m] & p^T \\ p & 0 \end{matrix} & \cdots & 0 \\ \hdashline \vdots & \vdots & \ddots & \vdots \\ \hdashline 0 & 0 & \cdots & \begin{matrix} [\nabla_n^2 V^n(\hat{x}_n) - \eta I_m] & p^T \\ p & 0 \end{matrix} \end{array} \right| = 0$$

(2.

are also real and negative. By rearranging rows and columns the above determinent can be put in the form

$$\left| \begin{array}{c:c} [\nabla\nabla_i V^i(\hat{w}) - \eta I_{mn}] & \Pi^T \\ \hdashline \Pi & 0 \end{array} \right| ,$$

where all off diagonal blocks $[\nabla_i \nabla_j V^i(\hat{w})]$ are identically zero.

Now consider a utility function $U^i(x_i, w_i)$ such that U^i is not independent of w^i but such that

$$0 < \frac{\partial^2 U^i(w)}{\partial x_k^i \partial x_\ell^j} < \varepsilon$$

for all $i \neq j$, for all k, ℓ and for fixed $\varepsilon > 0$.

By construction $$\left| \frac{\partial^2 V^i(w)}{\partial x_k^i \partial x_\ell^j} - \frac{\partial^2 U^i(w)}{\partial x_k^i \partial x_\ell^j} \right| < \varepsilon \qquad (2.8)$$

for $i \neq j$. We can choose V^i such that the value of x^i which maximizes V^i subject to the budget constraint is equal to the $\hat{x}^i$ which is a Nash equilibrium value of the system with externalities. Let these V^i be such that

$$\left| \frac{\partial^2 U^i(\hat{w})}{\partial x_k^i \partial x_\ell^i} - \frac{\partial^2 V^i(\hat{w})}{\partial x_k^i \partial x_\ell^i} \right| < \varepsilon. \qquad (2.9)$$

Let $\mu(U^i)$ denote a root of (2.6) and let $\eta(V^i)$ denote a root of (2.7) when V^i is chosen as above. The modulus of a complex number is denoted $||\cdot||$. Since a characteristic root of a matrix is a continuous function of the elements of a matrix, it follows that for any $\delta > 0$ there exists an $\varepsilon > 0$ such that if (2.8) and (2.9) are satisfied with respect to that ε, $||\eta(V^i) - \mu(U^i)|| < \delta$. Hence, since

$$
\left|\begin{array}{c:c} [\nabla\nabla_i F_i(U^i - \mu I_m] & \Pi^T \\ \hdashline \Pi & 0 \end{array}\right|
$$

$$
= (F_1', \ldots, F_n')^{m-2} \left|\begin{array}{c:c} [\nabla\nabla_i U^i - \mu I_m] & \Pi^T \\ \hdashline \Pi & 0 \end{array}\right| .
$$

If a condition which is not invariant to a monotonic increasing transformation is imposed on $[\nabla\nabla_i U^i]$, a theorem on global stability can be proved. The condition is that $[\nabla\nabla_i U^i]$ be negative quasidefinite wherever evaluated, that is, that $x([\nabla\nabla_i U^i] + [\nabla\nabla_i U^i]^T)x^T < 0$ for all $x \neq 0$. This assumption implies that $x[\nabla_i^2 U^i]x^1 < 0$ for all $x \neq 0$, or that U^i is strictly concave. A process in which the budget constraint is not satisfied instantaneously and both x^i and λ^i vary explicitly over time, will be considered.

Consider the following general adjustment process:

$$
\begin{aligned}
\dot{x}^i &= \nabla_i U^i(w) - \lambda_i p \\
\dot{\lambda}_i &= -p \cdot (x^i - \bar{x}^i)
\end{aligned} \tag{2.10}
$$

where p is the fixed price vector. We find a condition for stability by direct application of Lyapunov's second method.

<u>Theorem 2.7:</u> If $[\nabla\nabla_i U^i]$ is negative quasi-definite, then (2.10) is globally stable (i.e. w converges to $\hat{w}$ from any initial $w > 0$).

<u>Proof:</u> Form the function

$$
2V = \sum_i [\sum_k (x_k^i - \hat{x}_k^i)^2 + (\lambda_i - \hat{\lambda}_i)^2]
$$

where $\hat{x}_k^i$ and $\hat{\lambda}_i$ satisfy (2.1) (i.e. $\hat{x}^i$ and $\hat{\lambda}_i$ are equilibrium values). Clearly V is positive except when $x^i = \hat{x}^i$, $\lambda_i = \hat{\lambda}_i$, so that if $V' < 0$, V is a Lyapunov function, and the system 2.2 is globally asymptotically stable [8].

$$
\begin{aligned}
V' &= \sum_i [\sum_k (x_k^i - \hat{x}_k^i)\frac{dx_k^i}{dt} + (\lambda_i - \hat{\lambda}_i)\frac{d\lambda_i}{dt}] \\
&= \sum_i \left\{\sum_k \left[(x_k^i - \hat{x}_k^i)\left(\frac{\partial U^i(w)}{\partial x_k^i} - \lambda_i p_k\right) + (\lambda_i - \hat{\lambda}_i)p_k(x_k^i - \bar{x}_k^i)\right]\right\}
\end{aligned}
$$

Note that since

$$
\frac{\partial U^i(w)}{\partial x_k^i} - \lambda_i p_k = \left(\frac{\partial U^i(w)}{\partial x_k^i} - \frac{\partial U^i(\hat{w})}{\partial x_k^i}\right) + \hat{\lambda}_i p_k - \lambda_i p_k,
$$

$$
\begin{aligned}
V' = \sum_i & \left\{\sum_k (x_k^i - \hat{x}_k^i)\left(\frac{\partial U^i(w)}{\partial x_k^i} - \frac{\partial U^i(\hat{w})}{\partial x_k^i}\right)\right. \\
& \left. + \sum_k \left[(x_k^i - \hat{x}_k^i)(\hat{\lambda}_i p_k - \lambda_i p_k) + (p_k x_k^i - p_k \bar{x}_k^i)(\lambda_i - \hat{\lambda}_i)\right]\right\} .
\end{aligned}
$$

The expression in square brackets expands to

$$
\begin{aligned}
& \sum_k [\hat{\lambda}_i p_k x_k^i - \hat{\lambda}_i p_k \hat{x}_k^i - \lambda_i p_k x_k^i + \lambda_i p_k \hat{x}_k^i \\
& + p_k \lambda_i x_k^i - p_k \lambda_i \bar{x}_k^i - \hat{\lambda}_i p_k x_k^i + \hat{\lambda}_i p_k \bar{x}_k^i] \\
= & \sum_k (\hat{\lambda}_i (p_k \hat{x}_k^i - p_k \bar{x}_k^i) + \lambda_i (p_k \hat{x}_k^i - p_k \bar{x}_k^i))
\end{aligned}
$$

$= 0$ since the budget constraint must be satisfied with equality in equilibrium. Therefore

$$V' = \sum_i \Big[\sum_k (x_k^i - \hat{x}_k^i) \Big(\frac{\partial U^i(w)}{\partial x_k^i} - \frac{\partial U^i(\hat{w})}{\partial x_k^i} \Big) \Big],$$

or in vector notation

$$V' = \sum_i (x^i - \hat{x}^i)(\nabla_i U^i(w) - \nabla_i U^i(\hat{w})).$$

Let ξ lie on the line segment connecting w and $\hat{w}$. Then by the mean value theorem

$$[\nabla_i U^i(w) - \nabla_i U^i(\hat{w})]$$

$$= \nabla[\nabla_i U^i(\xi)] \cdot (w - \hat{w})$$

for some such ξ.

Therefore

$$V' = \sum_i (x^i - \hat{x}^i) \nabla\nabla_i U^i(\xi) \cdot (w - \hat{w})$$

$$= (w - \hat{w}) [\nabla\nabla_i U^i(\xi)] (w - \hat{w})$$

$$< 0, \text{ since } \nabla\nabla_i U^i(\xi) \text{ is quasinegative definite.}$$

3. STABILITY OF TATONNEMENT

The demand functions derived in Section 2 are sufficient to characterize the tâtonnement process. As usual, let

$$\dot{p}_k = g_i(\sum_i (x_k^i(p) - \bar{x}_k^i)). \qquad (3.1)$$

We know that an equilibrium exists with $\dot{p}_k = 0$ for all k. The function g_i serves to define speeds of adjustment of prices. The equilibrium p^* will be stable for all speeds of adjustment if and only if the matrix

$$M = \begin{bmatrix} \sum_i \frac{\partial x_1^i}{\partial p_1} & \cdots & \sum_i \frac{\partial x_1^i}{\partial p_n} \\ \cdot & \cdots & \cdot \\ \sum_i \frac{\partial x_n^i}{\partial p_1} & \cdots & \sum_i \frac{\partial x_n^i}{\partial p_n} \end{bmatrix}$$

is D-stable, i.e. if and only if DM is stable for all $D = \text{diag}\{d_1, \ldots, d_n\}$ with $d_i > 0$. Since we are dealing only with excess demand functions at this point, all the general theorems regarding sufficient (or necessary) conditions for D-stability of a matrix apply (see [8] or [9]). Thus, once the stability of the externality adjustment process is established, there is nothing special about the case of externalities in terms of restrictions on the partial derivatives of excess demand functions. We can extend the theorem that when externalities are small the Nash equilibrium is stable to show that under certain conditions the stability properties of an economy without externalities are preserved when small externalities are present.

Perturbing p in the first-order conditions (2.1) and rearranging differential coefficients gives a system of equations

$$LAL^T \cdot d\hat{X} = d\hat{P} \tag{3.2}$$

where $d\hat{X} = (dx^1 \ldots dx^n, \langle d\lambda_1 \ldots d\lambda_n \rangle)$

and $d\hat{P} = (\lambda_1 dP, \ldots, \lambda_n dP, \langle d(p(x^i - \bar{x}^i)) \rangle)$.

This system of equations can be solved for $\frac{\partial x_k^i}{\partial p_j}$ in the usual fashion. Let M be the matrix of derivatives $\left[\Sigma_i \frac{\partial x_k^i}{\partial p_j} \right]$ thus obtained.

A continuity theorem regarding stability of competitive equilibrium with externalities can be proved by observing what happens to $[LAL^T]^{-1}$ as the magnitude of external effects varies. An economy without externalities is denoted E^0, and consists of a set of individuals each with a utility function $V^i(x^i)$ and endowment $\bar{x}^i$. We relate this economy to the economy with externalities, E, by assuming that $U^i(x^i, w^i) = V^i(x^i)$ for all x^i when externalities are absent.

An economy with externalities, E^t, will in general be an $n + mn$ tuple $[{}^tU^1 \ldots {}^tU^n, {}^t\bar{x}^1 \ldots {}^t\bar{x}^n]$ We will consider only economies in which initial endowments are identical, so that an economy E^t is completely described as a collection of n utility functions. Consider a sequence of economies $\{E_t\}_{t=1}^{\infty}$ which converges to E^0 in the sense that there exists a T_0 such that if $t > T_0$,

$\max_{w \in \tilde{W}} (|{}^tU^i(w) - V^i(w)|) < \epsilon$ for any $\epsilon > 0$, and for all i.

We will consider a specific class, $\mathcal{E}(E^0)$, of sequences $\{E^t\}$ which converge to E^0 with the following properties for each ${}^tU^i$.

A) ${}^tH^i \equiv \bar{H}^i$ for all $t \in \{1, \ldots, \infty\}$, where ${}^tH^i$ is the bordered Hessian of the utility function ${}^tU^i$ and $\bar{H}^i$ the bordered Hessian of V^i.

B) Let $[\phi_{i,j}(t)]$ be a matrix whose elements are functions $\phi^{ij}_{k\ell}(t)$ such that $\phi^{ij}_{k\ell}(t)$ converges monotonically to zero as $t \to \infty$, and let A_{ij} be some $m \times m$ matrix. Then for $i \neq j$ let $\nabla_j \nabla_i {}^tU^i = [\phi_{ij}(t)] \otimes A_{ij}$ where $\otimes$ denotes the Hadamard (entry-wise) product.

Under (A) and (B) the matrix

$$A^t = \left[\begin{array}{c|c} [\nabla\nabla_i {}^tU^i] & \Pi^T \\ \hline \Pi & 0 \end{array} \right]$$

converges in the maximum element norm[1] to

$$A^0 = \left[\begin{array}{ccc|c} \nabla_1^2 V^1 & & 0 & \\ & \ddots & & \Pi^T \\ 0 & & \nabla_n^2 V^n & \\ \hline & \Pi & & 0 \end{array} \right] \quad \text{as } t \to \infty.$$

Choose a permutation matrix B such that

$$BA^0B^T = \text{diag}\,\{\bar{H}^1, \ldots, \bar{H}^n\} = \tilde{A}^0$$

and let $\tilde{A}^t = BA^tB^T$. Let M^t be the matrix of derivatives of excess demand functions found by solving 4.1 with $A = A^t$, and M^0 be a solution of 4.1 with $A = A^0$.

---

1. By the maximum element norm, we mean a scalar function of a matrix A, f(A), such that $f(A) = \max_{i,j} |a_{ij}|$ (see [4]).

Since a characterization of D-stability by necessary and sufficient conditions is lacking in the literature, and not apparently in the offing, continuity of the stability properties of a sequence of economies must be done in terms of some specific sufficient conditions for stability.

Theorem 3.1: If a sequence of economies $\{E^t\}_{t=1}^{\infty}$ is in the class $\mathcal{E}(E^0)$ for an economy E^0 which has gross substitutability (GS), then there exists $T_G < \infty$ such that E^t has GS for all $t > T_G$.

Proof: Note that since the off-diagonal blocks of $L\tilde{A}^0 L^T$ are identically zero, we obtain M^0 from solving 3.1, with

$$(L\tilde{A}^0 L^T)^{-1} = \begin{bmatrix} (L^1\bar{H}^1 L^1)^{-1} & & 0 \\ & \ddots & \\ 0 & & (L^n\bar{H}^n L^n)^{-1} \end{bmatrix}$$

Since E^0 has GS, M^0 is quasi-dominant diagonal [9].

Each element in the inverse of a matrix M is a continuous function of the elements of that matrix as long as M is nonsingular, since each element is the ratio of two nonzero polynomials, viz. the determinant of cofactor of M and the determinant of M.

By the same argument as that used in the previous section we can establish that there exists an ε such that if $f(A^t - A^0) < \varepsilon$, then the nonsingularity of the matrix A^0 implies the nonsingularity of A^t. Hence there exists a T such that A^t is nonsingular if $t > T$. By construction $[L\tilde{A}^t L^T]$ converges to diag $\{L^i\bar{H}^i L^i\}$ in the maximum element norm, and the inverse of diag $\{L^i\bar{H}^i L^i\}$ is diag $\{[L^i\bar{H}^i L^i]^{-1}\}$. Since all elements of A^t are close to the elements of A^0 in absolute value, by continuity each element of $[A^t]^{-1}$ is close to the corresponding element of $[A^0]^{-1}$ in absolute value. Let A^t_{ij} be the i, j block of the matrix A^t. In particular, for any $\delta > 0$, there exists an ε' such that by continuity of the inverse, $f([\tilde{A}^t]^{-1}_{ij}) < \delta$ when $f(\tilde{A}^t_{ij}) < \varepsilon'$ for all $i \neq j$. Since $f(\tilde{A}^t_{ij}) < \varepsilon'$ for all $i \neq j$, $f(\tilde{A}^t - \tilde{A}^0) < \varepsilon'$ since $A^t_{ii} = A^0_{ii}$. Hence for any $\delta > 0$ there exists an $\varepsilon' > 0$ such that for all $t > T(\varepsilon')$, $f([\tilde{A}^t]^{-1}_{ij}) < \delta$ for $i \neq j$.

M is a continuous function of $(LAL)^{-1}$ by (3.2). Therefore there exists a $T(\varepsilon)$ such that for all $t > T(\varepsilon)$, no element of M^t differs from the corresponding element of M^0 by more than ε for any positive ε. This is all we need to establish continuity of equilibrium.

Since M^c is quasi-dominant diagonal, there exist positive weights c_j such that

$$c_i |m^0_{ii}| > \sum_{i \neq j} c_j |m^0_{ij}|$$

Choose $\hat{\varepsilon}$ such that $\sum_{i \neq j} c_j |\hat{\varepsilon}| < \sum_i c_i |m^0_{ij}| - \sum_{i \neq j} c_j |m^0_{ij}|$.

Then M^t is quasi-dominant diagonal, and E^t is locally stable for all speeds of adjustment for all $t > T(\hat{\varepsilon}) = T_G$. ||

Remark: If a sequence $\{E^t\}_{t=1}^{\infty}$ in the class $\mathcal{E}(E^0)$ converges to an E which has sign stability, there may not be a T such that E^t is sign-stable for all $t \in T$. The same remark holds if sign-stable is replaced with weak GS. The same example establishes both. Let the economy E have the matrix of partial derivatives of excess demand functions with sign pattern

$$\begin{bmatrix} - & + & 0 \\ 0 & - & 0 \\ + & + & - \end{bmatrix}$$

Then E is weak GS and sign stable.. Let E^t have matrix

$$M^t = \begin{bmatrix} m_{11} & m_{12} & \frac{1}{s_t} m_{13} \\ \frac{1}{r_t} m_{21} & m_{22} & \frac{1}{u_t} m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

with $m_{ij} > 0$, all i, j.

Take $\{\frac{1}{r_t}\} \longrightarrow 0$, $\{\frac{1}{s_t}\} \longrightarrow 0$, and $\{\frac{1}{u_t}\} \dashrightarrow 0$, and let each sequence be strictly increasing. Then for any finite t, $r_t < 0$, $s_t < 0$, and $u_t < 0$, M^t is neither sign stable nor weak G S.

A classical theorem on stability of competitive equilibrium is that no trade at equilibrium implies stability of equilibrium. The theorem is not true unless further conditions are imposed when there are externalities. In the classical case the theorem follows from the observation that the matrix of substitution terms is negative-definite and that with no trade at equilibrium income effects vanish [9]. It can be seen from (3.2) that the substitution terms of the equilibrium demand functions depend not on H^i but on A, and that in general the matrix of substitution terms is negative definite with externalities only if the off-diagonal blocks of A are small. With this assumption, the theorem can be proved.

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