NEW DEFORMATIONS OF GROUP ALGEBRAS OF COXETER GROUPS

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Dedicated to the memory of Walter Feit

1. Introduction

The goal of this paper is to define new deformations of group algebras of Coxeter groups. Recall that a Coxeter group W is generated by elements $s_i, i \in I$ modulo two kinds of relations – the involutivity relations $s_i^2 = 1$ and the relations $(s_i s_j)^{m_{ij}} = 1$, where $2 \le m_{ij} = m_{ji} \le \infty$; in presence of the involutivity relations these are equivalent to the *braid relations* $s_i s_j s_i ... = s_j s_i s_j ...$ (m_{ij} factors). The traditional way to deform the group algebra $\mathbb{C}[W]$ is to deform the involutivity relations to the Hecke relations $(s_i - q)(s_i + q^{-1}) = 0$, and keep the braid relations unchanged. This yields the Hecke algebra $\mathbb{C}_q[W]$ of W, which is classically known to be a flat deformation of $\mathbb{C}[W]$.

On the contrary, the deformation A of $\mathbb{C}[W]$ that we study in this paper is obtained by keeping the involutivity relations fixed, and deforming the braid relations to

$$(s_i s_j - t_{ij1})...(s_i s_j - t_{ijm_{ij}}) = 0$$

when $m_{ij} < \infty$. Here t_{ijk}, $k \in \mathbb{Z}_{m_{ij}}$, are new commuting variables such that $t_{ijk} = t_{ji,-k}^{-1}$, and $s_p t_{ijk} = t_{jik} s_p$. We also consider the subalgebra A_+ of A generated by $s_i s_j$ and t_{ijk}. It is a deformation of the group algebra $\mathbb{C}[W_+]$ of the group W_+ of even elements of W. [1]

A priori, it is not clear that the algebras A, A_+ are "well behaved". We show that their "good" or "bad" behavior is determined completely by whether they are formally flat (i.e. whether the corresponding completed algebras over $\mathbb{C}[[\log t_{ijk}]]$ are flat deformations of $\mathbb{C}[W], \mathbb{C}[W_+]$). More specifically, we show that if A is formally flat, then it is algebraically flat (i.e., free as a module over $R := \mathbb{C}[t_{ijk}]$), and moreover any set of reduced words in s_i bijectively representing all elements of W defines a basis in A over R. In particular, this yields a canonical filtration $F^\bullet$ on A (by length of reduced words) such that $F^n A / F^{n-1} A$ is a finitely generated free R-module.

Unlike the Hecke algebra $\mathbb{C}_q[W]$, the algebras A, A_+ are not necessarily flat, and we determine when exactly this happens. First of all, it is easy to

[1] In view of the relation $s_p t_{ijk} = t_{jik} s_p$, A is not quite an honest deformation of $\mathbb{C}[W]$ (as the variables t_{ijk} are not central). However, the subalgebra A_+ is an honest deformation of $\mathbb{C}[W_+]$, and A is a semidirect product of $\mathbb{Z}_2$ with A_+.

see that flatness always holds for Coxeter groups of rank 1 and 2. Further, we show that for Coxeter groups of rank 3, the flatness of A, A_+ is equivalent to the condition that W is an infinite group. In other words, the (unordered) triple m_{12}, m_{13}, m_{23} must be different from the triples $(2, 2, m)$ $(m < \infty)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$, which correspond to finite Coxeter groups of rank 3: $A_1 \times I_{2m}$, A_3, B_3, H_3.

This implies that in any rank, a necessary condition for A, A_+ to be flat is that W does not contain finite parabolic subgroups of rank 3. Our main result asserts that this condition is also sufficient. The proof is based on consideration of constructible sheaves on the cell complex associated to the group W.

The motivation for this paper comes from the fact that for rank 3 Coxeter groups W, the algebra A_+ coincides with the Hecke algebra corresponding to the orbifold H/F, where H is the sphere, Euclidean plane, or Lobachevsky plane, and F is the group generated by rotations by the angles $2\pi/m_{ij}$ around the vertices of a triangle in H with angles π/m_{ij}. Such Hecke algebras were introduced in [E], and it was shown in [E] (using the theory of Cherednik algebras and KZ functor) that they are formally flat if H is a Euclidean or Lobachevsky plane, but not flat if H is a sphere, which is our main result for rank 3. (In fact, the proof of the main result in arbitrary rank is based on similar ideas, the difference being that the category of D-modules used in [E] is replaced with the category of constructible sheaves.) Thus, as a by-product, we obtain the algebraic PBW theorem for the Hecke algebras of polygonal Fuchsian groups defined in [E].

The cases when H is the Euclidean plane and $m_{ij} < \infty$ (i.e., $(m_{12}, m_{13}, m_{23}) = (3, 3, 3), (2, 4, 4), (2, 3, 6)$, and W is the affine Weyl group of types A_2, B_2, G_2) were also discussed in [EOR], as in these cases the algebras A_+ are the generalized double affine Hecke algebras of rank 1 of types E_6, E_7, E_8 which provide quantizations of del Pezzo surfaces. Thus we obtain new PBW filtrations and bases on these algebras, which (unlike those in [EOR]) are constructed without using a computer. (These filtrations, however, have the flaw that the idempotent used to define the spherical subalgebra is not homogeneous).

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2. Definition of the algebras $A(M), A_+(M)$

2.1. **Coxeter groups.** Recall the basics of the theory of Coxeter groups (see e.g. [B]).

Let I be a finite set. Let $\mathbb{Z}_{\geq 2}$ denote the set of integers which are ≥ 2. A Coxeter matrix over I is a collection M of elements $m_{ij} \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$,

$i, j \in I$, $i \neq j$, such that $m_{ij} = m_{ji}$. The rank r of M is, by definition, the cardinality of the set I.

Let M be a Coxeter matrix. Then one defines the Coxeter group [2] $W(M)$ by generators s_i, $i \in I$, and defining relations

$$s_i^2 = 1,\ (s_i s_j)^{m_{ij}} = 1 \text{ if } m_{ij} \neq \infty.$$

The group $W(M)$ has a sign character $\xi : W(M) \to \{\pm 1\}$ given by $\xi(s_i) = -1$. Denote by $W_+(M)$ the kernel of ξ, i.e. the subgroup of even elements.

Let $I' \subset I$ be a subset, and M' the submatrix of M consisting of $m_{ij}, i, j \in I'$. Then we have a natural map $W(M') \to W(M)$. It is known ([B]) that this map is injective. Thus $W(M')$ is a subgroup of $W(M)$, which is called the parabolic subgroup corresponding to I'.

2.2. **Deformed Coxeter group algebras.** Define the algebra $A(M)$ by invertible generators $s_i, i \in I$, and t_{ijk}, $i, j \in I$, $k \in \mathbb{Z}_{m_{ij}}$ for (i, j) such that $m_{ij} < \infty$ with defining relations

$$t_{ijk} = t_{ji,-k}^{-1},\ s_i^2 = 1,$$

$$\prod_{k=1}^{m_{ij}} (s_i s_j - t_{ijk}) = 0 \text{ if } m_{ij} < \infty,$$

$$[t_{ijk}, t_{i'j'k'}] = 0,\ s_p t_{ijk} = t_{jik} s_p.$$

Define also the algebra $A_+(M)$ over $R := \mathbb{C}[t_{ijk}]$ ($t_{ijk} = t_{ji,-k}^{-1}$) by generators a_{ij}, $i \neq j$ ($a_{ij} = a_{ji}^{-1}$), and relations

$$\prod_{k=1}^{m_{ij}} (a_{ij} - t_{ijk}) = 0 \text{ if } m_{ij} < \infty,$$

$$a_{ij} a_{jp} a_{pi} = 1.$$

Let $0 \in I$ be an element. Then we can define an involutive automorphism σ_0 of $A_+(M)$ (as an algebra over $\mathbb{C}$) by the formulas $\sigma_0(t_{ijk}) = t_{jik}$, $\sigma_0(a_{ij}) = a_{ji}$ if $i = 0$ or $j = 0$, and $\sigma_0(a_{ij}) = a_{0i} a_{j0}$ if $i, j \neq 0$. It is easy to show that this automorphism is well defined.[3] Thus we can define the semidirect product $A_0(M) = \mathbb{C}\mathbb{Z}_2 \ltimes A_+(M)$ using the automorphism σ_0.

Proposition 2.1. *The assignment $f(a_{ij}) = s_i s_j$, $f(\sigma_0) = s_0$ uniquely extends to an isomorphism $f : A_0(M) \to A(M)$.*

Proof. It is easy to check that f uniquely extends to a surjective homomorphism of algebras. Moreover, it is easy to construct the inverse of f: it is given by the formula $f^{-1}(s_i) = \sigma_0 a_{0i}$ for $i \neq 0$, and $f(s_0) = \sigma_0$. We are done. $\square$

[2] When we talk about Coxeter groups, we always assume that they are equipped with a fixed system of generators, which corresponds to the notion of a Coxeter system from [B].

[3] Note that up to inner automorphisms, σ_0 does not depend on the choice of the element $0 \in I$.

Thus, $A(M) = \mathbb{C}\mathbb{Z}_2 \ltimes A_+(M)$.

The following proposition explains the connection between the algebras $A(M)$, $A_+(M)$ and the groups $W(M)$, $W_+(M)$.

Let J be the ideal in R generated by the elements $t_{ijk} - \exp(2\pi \mathrm{i} k/m_{ij})$. It is easy to see that $JA(M) = A(M)J$, so $JA(M)$ is a two-sided ideal in $A(M)$.

Proposition 2.2. *One has* $A(M)/JA(M) = \mathbb{C}[W(M)]$, $A_+(M)/JA_+(M) = \mathbb{C}[W_+(M)]$.

Proof. Straightforward. □

2.3. **Spanning sets for $A(M), A_+(M)$.** If w is a word in letters s_i, let T_w be the corresponding element of $A(M)$. Choose a function $w(x)$ which attaches to every element $x \in W(M)$, a reduced word $w(x)$ representing x in $W(M)$.

We will now prove the following important result.

Theorem 2.3. *(i) The elements* $T_{w(x)}$, $x \in W$, *form a spanning set in* $A(M)$ *as a left* R*-module.*

(ii) The elements $T_{w(x)}$, $x \in W_+$, *form a spanning set in* $A_+(M)$ *as a left* R*-module.*

Proof. It is clear that (ii) follows from (i), so it suffices to prove (i).

Let us write the relation

$$\prod_{k=1}^{m_{ij}} (s_i s_j - t_{ijk}) = 0$$

as a deformed braid relation:

$$s_j s_i s_j ... + S.L.T. = t_{ij} s_i s_j s_i ... + S.L.T.,$$

where $t_{ij} = (-1)^{m_{ij}+1} t_{ij1} ... t_{ijm_{ij}}$, S.L.T. mean "smaller length terms", and the products on both sides have length m_{ij}. This can be done by multiplying the relation by $s_i s_j ...$ (m_{ij} factors).

Now let us show that $T_{w(x)}$ span $A(M)$ over R. Clearly, T_w for all words w span $A(M)$. So we just need to take any word w and express T_w via $T_{w(x)}$.

It is well known from the theory of Coxeter groups (see e.g. [B]) that using the braid relations, one can turn any non-reduced word into a word that is not square free, and any reduced expression of a given element of $W(M)$ into any other reduced expression of the same element. Thus, if w is non-reduced, then by using the deformed braid relations we can reduce T_w to a linear combination of T_u with words u of smaller length than w. On the other hand, if w is a reduced expression for some element $x \in W$, then using the deformed braid relations we can reduce T_w to a linear combination of T_u with u shorter than w, and $T_{w(x)}$. Thus $T_{w(x)}$ are a spanning set. The theorem is proved. □

Thus, $A_+(M)$ is a "deformation" of $\mathbb{C}[W_+(M)]$ over R, and similarly $A(M)$ is a "twisted deformation" of $\mathbb{C}[W(M)]$.

3. Flat Coxeter matrices

3.1. **Definition of a flat Coxeter matrix.** Denote by $\hat{A}(M)$, $\hat{A}_+(M)$ the formal versions of $A(M)$, $A_+(M)$, i.e., algebras generated (topologically) by the same generators and relations, but with $t_{ijk} = e^{2\pi i k/m_{ij}} e^{\tau_{ijk}}$, where τ_{ijk} are formal parameters. By virtue of Theorem 2.3, $\hat{A}_+(M)$ is an algebra over $\hat{R} := \mathbb{C}[[\tau_{ijk}]]$ which is a formal deformation of $\mathbb{C}[W_+]$ with deformation parameters τ_{ijk}.

Definition 3.1. We say that M is a flat Coxeter matrix if $\hat{A}_+(M)$ is a flat deformation of $\mathbb{C}[W_+]$, i.e. if $\hat{A}_+(M)$ is a topologically free left $\hat{R}$-module.

Since $\hat{A}(M) = \mathbb{C}\mathbb{Z}_2 \ltimes \hat{A}_+(M)$, for a flat Coxeter matrix we also have that $\hat{A}(M)$ is topologically free as a left $\hat{R}$-module.

3.2. **Bases of $A(M), A_+(M)$ for flat M.**

Proposition 3.2. *Let M be a flat Coxeter matrix. Then*

(i) The elements $T_{w(x)}$, $x \in W$, form a basis in $A(M)$ as a left R-module.

(ii) The elements $T_{w(x)}$, $x \in W_+$, form a basis in $A_+(M)$ as a left R-module.

Proof. It is sufficient to show that $T_{w(x)}$ are linearly independent. This follows from the fact that they are linearly independent in $\hat{A}(M)$, which is a consequence of the flatness of M. □

Corollary 3.3. *The R-modules $A(M)$ and $A_+(M)$ are free. Moreover, they carry a filtration $F^\bullet$, defined by the condition that $F^n A(M)$ is spanned by $T_{w(x)}$ for x of length $\leq n$. This filtration has the property that the R-modules $\mathrm{gr}_n A(M)$ and $\mathrm{gr}_n A_+(M)$ are finitely generated and free.*

We note that the filtration $F^\bullet$ is canonical, i.e., independent on the choice of the function $w(x)$.

Remark. Let $\Gamma(M)$ be the graph whose vertices are elements of I, and i, j are connected if m_{ij} is odd. Then the filtration F^n can be refined to a multi-filtration $F^{\mathbf{n}}$, where $\mathbf{n}$ is a nonnegative integer function on I which is constant on the connected components of $\Gamma(M)$. Namely, $F^{\mathbf{n}} A(M)$ is spanned by $T_{w(x)}$ with $w(x)$ involving $\leq \mathbf{n}(i)$ copies of s_i for each i. It is easy to see from the above that $\mathrm{gr}_{\mathbf{n}} A(M)$ is finitely generated and free over R.

3.3. **When is a Coxeter matrix flat?** Let us study the question when a given Coxeter matrix is flat (i.e., when the algebras $A(M), A_+(M)$ are "meaningful").

The case of rank 1 is trivial. In rank 2, the algebra $A_+(M)$ is $\mathbb{C}[a, a^{-1}]$ if $m_{12} = \infty$ and $\mathbb{C}[a, a^{-1}]/(P(a))$ where $P(a) = (a - t_{121})...(a - t_{12m_{12}})$ otherwise. Thus, any Coxeter matrix of rank 2 is flat.

However, in rank 3, we have a much more interesting situation. Namely, we have the following theorem.

Theorem 3.4. *A Coxeter matrix M of rank 3 is flat if and only if the group $W(M)$ is infinite.*

Coxeter matrices of rank 3 are conveniently written as triples of numbers m_{12}, m_{23}, m_{31} (the order does not matter). Recall that the Coxeter matrices of rank 3 producing a finite Coxeter group are the following:
1. $M = (2, 2, m)$, $m < \infty$ (type $A_1 \times I_{2m}$).
2. $M = (2, 3, 3)$ (type A_3)
3. $M = (2, 3, 4)$ (type B_3)
4. $M = (2, 3, 5)$ (type H_3)

Thus, the theorem claims that a Coxeter matrix is flat if and only if it does not fall into the four cases listed above.

Proof. **If.** Let $M = (m_{12}, m_{23}, m_{31})$. The algebra $A_+(M)$ is defined over R by generators a_{12}, a_{23}, a_{31} with defining relations

$$(a_{ij} - t_{ij1})...(a_{ij} - t_{ijm_{ij}}) = 0, \; ij = 12, 23, 31,$$

and

$$a_{12}a_{23}a_{31} = 1.$$

This algebra is a deformation of the group algebra of the group $F = F_{m_{12},m_{23},m_{31}}$ generated by a_{12}, a_{23}, a_{31} with defining relations $a_{ij}^{m_{ij}} = 1$, $a_{12}a_{23}a_{31} = 1$. The group F is isomorphic to the group generated by rotations by angles $2\pi/m_{ij}$ around vertices of a triangle whose angles are π/m_{ij}. This triangle lies on the sphere, plane, and Lobachevsky plane if the quantity $S = \frac{1}{m_{12}} + \frac{1}{m_{23}} + \frac{1}{m_{31}}$ is $> 1, = 1$, and < 1, respectively. The cases 1,2,3,4 when $W(M)$ is finite correspond exactly to the case $S > 1$. Now, the flatness of the algebra $\hat{A}_+(M)$ for $S \leq 1$ follows from Theorem 3.3 of [E] (as $\hat{A}_+(M)$ is the Hecke algebra of the orbifold H/F, where H is the plane or Lobachevsky plane, depending on whether $S = 1$ or $S < 1$). Another proof of flatness of $\hat{A}_+(M)$ for $S = 1$ is given in [EOR] (in this case $\hat{A}_+(M)$ is a generalized double affine Hecke algebra). This proves the "if" direction of the theorem.

Only if. Suppose $W(M)$ is finite. Assume the contrary, i.e., that M is flat. Then the algebra $\hat{A}_+(M)$ is a free module over $\hat{R}$ of dimension $D = |W_+(M)|$. The eigenvalues of a_{ij} in the regular representation are equal to t_{ijk}, and occur with multiplicity D/m_{ij}. Thus in the regular representation we have $\det(a_{ij}) = (\prod_k t_{ijk})^{D/m_{ij}}$. So taking the determinant of the relation $a_{12}a_{23}a_{31} = 1$, we have

$$(\prod_{k=1}^{m_{12}} t_{12k})^{D/m_{12}} (\prod_{k=1}^{m_{23}} t_{23k})^{D/m_{23}} (\prod_{k=1}^{m_{31}} t_{31k})^{D/m_{31}} = 1.$$

This is a nontrivial relation on t_{ijk}, which contradicts to the flatness of M. The theorem is proved. ∎

Now we are ready to state our main result.

Theorem 3.5. *A Coxeter matrix M is flat if and only if for any 3-element subset $\{i,j,k\}$ of I, the Coxeter group generated by s_i, s_j, s_k is infinite.*

The theorem is proved in the next susbection. Note that in the process of proving Theorem 3.5, we give a new proof of the "if" part of Theorem 3.4, which relies on arguments from topology (constructible sheaves) rather than complex analysis (D-modules used in [E]).

3.4. **Proof of Theorem 3.5.** Let us first prove the easier "only if" direction. Let $I' = \{i,j,k\} \subset I$. Let M' be the corresponding Coxeter matrix. Since $W(M') \subset W(M)$, the flatness of $A_+(M)$ implies the flatness of $A_+(M')$. But if $A_+(M')$ is flat then by Theorem 3.4, $W(M')$ is an infinite group, as desired.

Now let us prove the "if" direction. To do so, we introduce a 2-dimensional cell complex Σ attached to M as follows. The zero-dimensional cells of Σ are the elements of $W := W(M)$. The 1-dimensional cells are edges connecting w and $s_i w$ for each $i \in I$. Then we have cycles $w, s_i w, s_j s_i w, s_i s_j s_i w ..., s_j w$ of length $2m_{ij}$ (if $m_{ij} < \infty$), and the 2-dimensional cells of Σ are $2m_{ij}$-gons attached to these cycles.

It is easy to see that W acts properly discontinuously on Σ, hence so does its subgroup W_+. Moreover, it is clear that the only fixed points of the W_+-action on Σ are the centers of the $2m_{ij}$-gons, with stabilizer $\mathbb{Z}_{m_{ij}}$. Thus we can define an orbifold cell complex $Y := \Sigma/W_+$. It has two vertices (the north and south pole, N, S), edges e_i between them corresponding to s_i, and for each 2-element set $\{i,j\} \subset I$, a disk D_{ij} whose boundary is identified with the circle made up by e_i and e_j. The disk D_{ij} has an orbifold point z_{ij} in the center, whose isotropy group is $\mathbb{Z}_{m_{ij}}$.

We will need the following theorem (see e.g. [DM]).

Theorem 3.6. *If W has no finite parabolic subgroups of rank 3 then Σ is contractible.*

Let $\mathcal{C}$ be the category of constructible sheaves of complex vector spaces on Σ with respect to the stratification into cells (see [Sch] for definitions). Let $\mathbb{C}$ be the constant sheaf on Σ.

Lemma 3.7. *For any Coxeter matrix we have* $\mathrm{Ext}^1_{\mathcal{C}}(\mathbb{C},\mathbb{C}) = H^1(\Sigma,\mathbb{C})$*, and* $\mathrm{Ext}^2_{\mathcal{C}}(\mathbb{C},\mathbb{C}) \subset H^2(\Sigma,\mathbb{C})$.

Proof. [4] Let $\widetilde{\mathcal{C}}$ be the abelian category of all sheaves of complex vector spaces on Σ. Then $\mathcal{C}$ is full abelian subcategory closed under extensions. In this situation for any $F, G \in \mathcal{C}$, $\mathrm{Ext}^1_{\mathcal{C}}(F,G) = \mathrm{Ext}^1_{\widetilde{\mathcal{C}}}(F,G)$, and the natural map $\mathrm{Ext}^2_{\mathcal{C}}(F,G) \to \mathrm{Ext}^2_{\widetilde{\mathcal{C}}}(F,G)$ is injective. But it is well known that $\mathrm{Ext}^i_{\widetilde{\mathcal{C}}}(\mathbb{C},\mathbb{C}) = H^i(\Sigma,\mathbb{C})$. This implies the statement. □

[4] This argument was provided to us by A. Polishchuk.

Remark. As was explained to us by A. Polishchuk, the inclusion $\mathrm{Ext}^2_{\mathcal{C}}(F,G) \to \mathrm{Ext}^2_{\tilde{\mathcal{C}}}(F,G)$ is in fact an isomorphism. It follows from the fact that our stratification satisfies the property that the closure of every cell is homeomorphic to a closed ball (in a way compatible with boundaries). Indeed, it suffices to prove this when F and G are simple. Then it follows from the fact that the algebra of $\mathrm{Ext}^*_{\tilde{\mathcal{C}}}$ between simple objects of $\mathcal{C}$ is generated in degree 1 (apply Corollary 2.2 of [P] for zero perversity).

Lemma 3.8. *If W has no finite parabolic subgroups of rank 3, one has $\mathrm{Ext}^j_{\mathcal{C}}(\mathbb{C},\mathbb{C}) = 0$ for $j = 1, 2$.*

Proof. The result follows from Theorem 3.6 and Lemma 3.7. □

Let $\mathcal{D}$ be the category of constructible sheaves on the orbifold Y with respect to stratification into cells. Thus, an object of $\mathcal{D}$ is a constructible sheaf on the complement $Y' \subset Y$ of the points z_{ij} with respect to the same stratification (intersected with Y'), such that the monodromy g_{ij} around z_{ij} satisfies the equation $g_{ij}^{m_{ij}} = 1$.

Let $\pi : \Sigma \to Y$ be the natural projection, $\pi_! : \mathcal{C} \to \mathcal{D}$ be the direct image functor with compact supports, and $\mathbf{M} = \pi_!(\mathbb{C})$. Thus $\mathbf{M}$ is a local system on the orbifold Y. The monodromy representation of $\mathbf{M}$ over Y' is the regular representation of W_+.

Lemma 3.9. *One has $\mathrm{Ext}^j_{\mathcal{D}}(\mathbf{M},\mathbf{M}) = 0$ for $j = 1, 2$.*

Proof.

$$\mathrm{Ext}^j_{\mathcal{D}}(\mathbf{M},\mathbf{M}) = \mathrm{Ext}^j_{\mathcal{D}}(\pi_!\mathbb{C},\pi_!\mathbb{C}) =$$
$$\mathrm{Ext}^j_{\mathcal{C}}(\mathbb{C},\pi^*\pi_!\mathbb{C}) = \mathbb{C}[W_+] \otimes \mathrm{Ext}^j_{\mathcal{C}}(\mathbb{C},\mathbb{C}).$$

But $\mathrm{Ext}^j_{\mathcal{C}}(\mathbb{C},\mathbb{C}) = 0$ by Lemma 3.8. We are done. □

A basic fact about constructible sheaves, going back to Fulton, Goresky, McCrory, and MacPherson ([Sh]; see also [Vy]), is that the category of constructible sheaves on a cell complex with respect to the stratification into cells is equivalent to the category of "cellular sheaves", i.e., modules over a certain algebra B (path algebra of a certain quiver with relations).

Let us construct B in the case of the orbifold cell complex Y. Let us fix an ordering on the set I. Let $\mathcal{S} \in \mathcal{D}$. Let V_N, V_S, V_i, V_{ij} $(i \neq j)$ be the spaces of sections of $\mathcal{S}$ over small neighborhoods of N, S, the midpoints of e_i, and the points of D_{ij} close to the midpoints of e_i, respectively. We then have natural restriction maps $f_{Ni} : V_N \to V_i$, $f_{Si} : V_S \to V_i$, $h_{ij} : V_i \to V_{ij}$, and monodromy maps $g_{ij} : V_{ij} \to V_{ji}$, with relations

$$g_{ij}h_{ij}f_{Ni} = g_{ji}f_{Nj}, \quad \text{for } i < j,$$
$$g_{ij}h_{ij}f_{Si} = g_{ji}f_{Sj}, \quad \text{for } i > j,$$

and

$$(g_{ij}g_{ji})^{m_{ij}} = 1.$$

Thus, $\mathcal{S}$ defines a representation of a certain quiver with relations. We define the algebra B to be the path algebra of this quiver (modulo the relations). It is then well known that category $\mathcal{D}$ is equivalent to the category of B-modules: the equivalence sends $\mathcal{S}$ to the B-module

$$V := V_N \oplus V_S \oplus \oplus_i V_i \oplus \oplus_{i,j} V_{ij}.$$

Now let $\tau = (\tau_{ijk}), k \in \mathbb{Z}_{m_{ij}}$, be a collection of formal parameters. Define the algebra $B(\tau)$ by the same generators as B, and the same relations except for one modification: the relation $(g_{ij}g_{ji})^{m_{ij}} = 1$ is replaced with

$$(g - t_{ij1})...(g - t_{ijm_{ij}}) = 0,$$

where $g := g_{ij}g_{ji}$. (we recall that $t_{ijk} := e^{2\pi \mathrm{i}k/m_{ij}} e^{\tau_{ijk}}$). It is clear that $B(\tau)/(\tau = 0) = B$.

We will need the following flatness result.

Proposition 3.10. *For any Coxeter matrix M, the algebra $B(\tau)$ is a flat deformation of B.*

Proof. For any $i < j$ let p_{ij} be the idempotent of B which acts by 1 on V_N, V_S, V_i, V_j and on V_{ij}, V_{ji} if $m_{ij} < \infty$, and acts by 0 on all the other spaces. Then the direct sum of right modules $p_{ij}B$ over B is faithful, so it suffices to show that they can be deformed to right $B(\tau)$-modules. Clearly, B preserves the kernel of p_{ij}, so $p_{ij}BB = p_{ij}Bp_{ij}$, and thus $B_{ij} := p_{ij}B$ is a unital algebra with unit p_{ij}. Replacing relations as above, we can define a deformation $B_{ij}(\tau) = p_{ij}B(\tau)p_{ij}$ of B_{ij}, and it suffices to show that this deformation is flat.

We clearly only need to consider the case $m_{ij} < \infty$. Let us consider the category $\mathcal{E}_{ij}$ of representations of B_{ij} in which all the quiver arrows are isomorphisms. It is easy to see (by explicitly writing a basis of B_{ij}) that the direct sum of these representations is faithful. Thus it suffices to show that any such representation can be deformed to a representation of $B_{ij}(\tau)$. But it is easy to see that the category $\mathcal{E}_{ij}$ is equivalent to the category of vector spaces U with a linear map $g : U \to U$ such that $g^{m_{ij}} = 1$. This shows that any object of this category can be easily deformed to a module over $B_{ij}(\tau)$ (by deforming g to an operator satisfying the equation $(g - t_{ij1})...(g - t_{ijm_{ij}}) = 0$), as desired. □

Now we can finish the proof of Theorem 3.5. We can regard $\mathbf{M}$ as a B-module (in which all the arrows are isomorphisms). By Lemma 3.9, $\mathrm{Ext}^1_B(\mathbf{M}, \mathbf{M}) = \mathrm{Ext}^2_B(\mathbf{M}, \mathbf{M}) = 0$. This implies that $\mathbf{M}$ can be uniquely deformed to a module $\mathbf{M}_\tau$ over B_τ. The module $\mathbf{M}_\tau$ is a quiver representation where all arrows are isomorphisms, so it represents a local system on Y' (over $\mathbb{C}[[\tau]]$). The monodromy of this local system is a representation of $A_+(M)$, deforming the regular representation of $\mathbb{C}[W_+]$. The existence of such deformation implies the flatness of $A_+(M)$. Theorem 3.5 is proved.

3.5. **Flatness of Hecke algebras of polygonal Fuchsian groups.** Let $\Gamma = \Gamma(m_1, ..., m_r)$, $r \geq 3$, be the Fuchsian group defined by generators c_j, $j = 1, ..., r$, with defining relations

$$c_j^{m_j} = 1, \prod_{j=1}^{r} c_j = 1.$$

Here $2 \leq m_j \leq \infty$.

In [E] the first author defined the Hecke algebra of Γ, $\mathcal{H}(\Gamma)$, by the same (invertible) generators and relations

$$(c_j - t_{j1})...(c_j - t_{jm_j}) = 0, \text{ if } m_j < \infty, \prod_{j=1}^{r} c_j = 1,$$

where t_{jk} are invertible variables.

Theorem 3.11. *The algebra $\mathcal{H}(\Gamma)$ is free as a left module over $R := \mathbb{C}[t, t^{-1}]$ if and only if $\sum_j (1 - 1/m_j) \geq 2$ (i.e., Γ is Euclidean or hyperbolic).*

The formal version of this theorem is proved in [E].

Proof. Let M be the Coxeter matrix of rank r such that $m_{i,i+1} := m_i$ for $i \in \mathbb{Z}_r$, and $m_{ij} = \infty$ otherwise. It is easy to deduce from Theorem 3.5 that M is flat if and only if $\sum_j (1 - 1/m_j) \geq 2$. But for such matrix $A_+(M) = \mathcal{H}(\Gamma)$. We are done. □

Note that since $\mathcal{H}(\Gamma) = A_+(M)$, we actually obtain a basis of $\mathcal{H}(\Gamma)$, given by $T_{w(x)}$ for even x, and also a canonical filtration on $\mathcal{H}(\Gamma)$ with free finitely generated quotients.

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