**Jacobians and rank 1 perturbations relating to unitary Hessenberg matrices**

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In a recent work Killip and Nenciu gave random recurrences for the characteristic polynomials of certain unitary and real orthogonal upper Hessenberg matrices. The corresponding eigenvalue p.d.f.'s are $\beta$-generalizations of the classical groups. Left open was the direct calculation of certain Jacobians. We provide the sought direct calculation. Furthermore, we show how a multiplicative rank 1 perturbation of the unitary Hessenberg matrices provides a joint eigenvalue p.d.f. generalizing the circular $\beta$-ensemble, and we show how this joint density is related to known inter-relations between circular ensembles. Projecting the joint density onto the real line leads to the derivation of a random three-term recurrence for polynomials with zeros distributed according to the circular Jacobi $\beta$-ensemble.

# 1 Introduction

Consider the classical group $U(N)$ of $N \times N$ unitary matrices. There is a unique measure $d_H U$ — the Haar measure — which is invariant under both left and right multiplication by a fixed unitary matrix, thus giving a uniform distribution on the group. The corresponding eigenvalue probability density function (p.d.f.) has the explicit form (see e.g. [8])

$$\frac{1}{(2\pi)^N N!} \prod_{1 \le j < k \le N} |e^{i\theta_k} - e^{i\theta_j}|^2, \tag{1.1}$$

and this in turn is of fundamental importance in recent applications of random matrix theory to combinatorial models [20, 3], analytic number theory [15] and the quantum many body problem [9].

A basic question is how to best sample from (1.1). Until very recently, the only method available has been to first generate a member of $U(N)$ according to the Haar measure, by for example applying the Gram-Schmidt orthogonalization procedure to the columns of an $N \times N$ complex Gaussian matrix, then to calculate the eigenvalues of the resulting matrix. However, inspired by recent work of Dumitriu and Edelman [6], this situation has been dramatically improved upon by Killip and Nenciu [17]. Thus augmenting ideas from [6] with results from the theory of orthogonal polynomials on the unit circle, these authors have provided an explicit unitary Hessenberg matrix, with positive elements on the subdiagonal, which has for its eigenvalue p.d.f. the $\beta$-generation of (1.1),

$$\frac{1}{C_{\beta N}} \prod_{1 \le j < k \le N} |e^{i\theta_k} - e^{i\theta_j}|^\beta, \qquad C_{\beta N} = (2\pi)^N \frac{\Gamma(\beta N/2 + 1)}{(\Gamma(\beta/2))^N}. \tag{1.2}$$

In general the characteristic polynomial $\chi_N(\lambda)$ of such matrices can be calculated from the coupled recurrences

$$\begin{aligned} \chi_k(\lambda) &= \lambda \chi_{k-1}(\lambda) - \bar{\alpha}_{k-1} \tilde{\chi}_{k-1}(\lambda) \\ \tilde{\chi}_k(\lambda) &= \tilde{\chi}_{k-1}(\lambda) - \lambda \alpha_{k-1} \chi_{k-1}(\lambda) \end{aligned} \tag{1.3}$$

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$(k = 1, \ldots, N)$ where $\chi_0(\lambda) = \tilde{\chi}_0(\lambda) = 1$ and furthermore $\tilde{\chi}_k(\lambda) = \lambda^k \bar{\chi}_k(1/\lambda)$. For the unitary Hessenberg matrix relating to (1.2), the parameters $\{\alpha_j\}_{j=0,\ldots,N-1}$ are random variables with distributions specified in [17] (see (4.3) below). As a consequence of this result the joint distribution (1.1), or more generally (1.2), can be sampled by simply iterating (1.3) to generate $\chi_N(\lambda)$, then computing its roots.

The problem of efficiently sampling from the p.d.f.

$$\frac{1}{C_N(a,b;\beta)} \prod_{l=1}^N |1 - e^{i\theta_l}|^{2a+1} |1 + e^{i\theta_l}|^{2b+1} \prod_{1 \le j < k \le N} |e^{i\theta_j} - e^{i\theta_k}|^\beta |1 - e^{i(\theta_j + \theta_k)}|^\beta, \quad (0 \le \theta_l \le \pi) \tag{1.4}$$

was solved according to the same strategy in [17]. Here the underlying unitary Hessenberg matrix is real orthogonal with determinant +1, and thus the characteristic polynomial $\chi_{2N}(\lambda)$ has real coefficients. In this case $\tilde{\chi}_k(\lambda) = \lambda^k \chi_k(1/\lambda)$ and so only the first of the recurrences in (1.3) is required. Note that the eigenvalues of a real orthogonal matrix with determinant +1 come in complex conjugate pairs $e^{\pm i\theta}$; (1.4) is the joint distribution of those with $0 < \theta < \pi$. The case $\beta = 2$, $(a,b) = (\pm\frac{1}{2}, \pm\frac{1}{2})$ (the signs chosen appropriately) of (1.4) gives the eigenvalue p.d.f. for matrices from the real orthogonal and symplectic classical groups with Haar measure (see e.g. [8]). Like their counterparts from $U(N)$, such random matrices are of fundamental importance in applications of random matrix theory to combinatorial models [20, 3], analytic number theory [16] and the quantum many body problem [10].

In the work [17], Killip and Nenciu left open two questions concerning the direct computation of certain Jacobians, one relating to unitary Hessenberg matrices corresponding to (1.2), and the other to real orthogonal Hessenberg matrices corresponding to (1.4). Earlier, Dumitriu and Edelman [6] had left open an analogous question in the case of tridiagonal matrices corresponding to the Gaussian $\beta$-ensemble p.d.f.

$$\frac{1}{G_{\beta N}} \prod_{l=1}^N e^{-x_l^2/2} \prod_{1 \le j < k \le N} |x_k - x_j|^\beta.$$

It was remarked in [6] that one of the present authors (PJF) had communicated a direct derivation of the sought Jacobian for the change of variables from the elements of a tridiagonal matrix, to the eigenvalues and the first component of the eigenvectors. A primary purpose of this article is to show how a similar approach can be used to answer the two questions left open in [17]. We begin in Section 2 by presenting the calculation for the Jacobian in the case of a tridiagonal matrix. In Section 3 this calculation is extended to provide a direct calculation of Jacobians relating to unitary and real orthogonal Hessenberg matrices. Also shown is how portions of the working in [17] reliant on the theory of orthogonal polynomials on the unit circle, can alternatively be derived within a matrix setting. In Section 4, it is shown how a certain rank 1 multiplicative perturbation of unitary matrices leads to the derivation of a joint eigenvalue p.d.f. generalizing (1.2). An integration formula associated with this p.d.f. is discussed, which in turn is shown to include as special cases known inter-relations between circular ensembles. Furthermore, the multiplicative perturbation is used to give an alternative derivation of these inter-relations.

The Cayley transformation of the distributions obtained in Section 4, projecting the unit circle to the real line, are studied in Section 5. This leads to a random three-term recurrence for the projection onto the real line of polynomials with zeros distributed according to

$$\frac{1}{M_N(a;c)} \prod_{l=1}^N |1 - e^{i\theta_l}|^a \prod_{1 \le j < k \le N} |e^{i\theta_k} - e^{i\theta_j}|^{2c}, \tag{1.5}$$

which with $2c = \beta$ is known as the circular Jacobi $\beta$-ensemble [8]. In the case $a = 0$ this recurrence scheme is distinct from the scheme (1.3).

## 2 Calculation of a Jacobian for tridiagonal matrices

Let

$$T = \begin{bmatrix} a_n & b_{n-1} & & & & \\ b_{n-1} & a_{n-1} & b_{n-2} & & & \\ & b_{n-2} & a_{n-2} & b_{n-3} & & \\ & \ddots & \ddots & \ddots & & \\ & & b_2 & a_2 & b_1 \\ & & & b_1 & a_1 \end{bmatrix} \tag{2.1}$$

be a general real symmetric tridiagonal matrix. The problem posed in [6] is to compute the Jacobian for the change of variables from the description of $T$ in terms of its entries, to the description in terms of eigenvalues and variables relating to its eigenvectors.

As is well known, and easy to see by direct substitution, for each eigenvalue $\lambda_k$ and corresponding eigenvector $\vec{v}_k$, once the 1st component $\vec{v}_k^{(1)} =: q_k$, $q_k > 0$, of $\vec{v}_k$ is specified, the other components are then fully determined by $\{\lambda_k\}$ and the elements of $T$. However only $n-1$ of these components are independent due to the relation

$$\sum_{k=1}^{n} q_k^2 = 1, \tag{2.2}$$

which itself is a consequence of $T$ being symmetric and thus orthogonally diagonalizable. Thus the $2n-1$ variables

$$\vec{a} := (a_n, a_{n-1}, \dots, a_1), \qquad \vec{b} := (b_{n-1}, \dots, \vec{b}_1) \tag{2.3}$$

can be put into 1-to-1 correspondence with the $2n-1$ variables

$$\vec{\lambda} := (\lambda_1, \dots, \lambda_n), \qquad \vec{q} := (q_1, \dots, q_{n-1}) \tag{2.4}$$

where $\lambda_1 > \cdots > \lambda_n$ and $q_i > 0$. The Jacobian for the change of variables from (2.3) to (2.4) can be computed directly using the method of wedge products (for an introduction to the use of this technique in random matrix theory see [8]).

We will first isolate results required in the course of the calculation.

**Proposition 1.** *Let $(X)_{ij}$ denote the $ij$ entry of the matrix $X$. We have*

$$((T - \lambda I_n)^{-1})_{11} = \sum_{j=1}^{n} \frac{q_j^2}{\lambda_j - \lambda}. \tag{2.5}$$

*Also*

$$\prod_{1 \le i < j \le n} (\lambda_i - \lambda_j)^2 = \frac{\prod_{i=1}^{n-1} b_i^{2i}}{\prod_{i=1}^{n} q_i^2} \tag{2.6}$$

*and*

$$\det \Big[ [\lambda_k^j - \lambda_n^j]_{\substack{j=1,\dots,2n-1 \\ k=1,\dots,n-1}} \, [j\lambda_k^{j-1}]_{\substack{j=1,\dots,2n-1 \\ k=1,\dots,n}} \Big] = \prod_{1 \le j < k \le n} (\lambda_k - \lambda_j)^4. \tag{2.7}$$

Proof. The identity (2.5), which is well known, follows by writing the matrix entry as an inner product, and decomposing the vectors in this inner product as eigenvectors. The identity (2.6) is contained in [6]. It can be derived from (2.5) by using the fact that for a general $n \times n$ non-singular matrix

$$(X^{-1})_{11} = \frac{\det X_{n-1}}{\det X}, \tag{2.8}$$

where $X_{n-1}$ denotes the bottom right $n-1 \times n-1$ submatrix of $X$, introducing the corresponding characteristic polynomials $P_{n-1}(\lambda)$, $P_n(\lambda)$, and making use of the three term recurrence

$$P_k(\lambda) = (\lambda - a_k)P_{k-1}(\lambda) - b_{k-1}^2 P_{k-2}(\lambda), \quad P_0(\lambda) := 1.$$

For the identity (2.7), note that both sides are homogeneous symmetric polynomials of degree $2n(n-1)$. Furthermore, the determinant and its first three derivatives with respect to $\lambda_1$ vanish at $\lambda_1 = \lambda_2$. As a consequence, it follows that the determinant must in fact be proportional to the fourth power of the product of differences as given in the r.h.s. The fact that the proportionality constant is unity follows by comparing coefficients of $(\lambda_1^0 \lambda_2^1 \cdots \lambda_n^{n-1})^4$ on both sides. □

The Jacobian can now be computed according to the following result.

**Theorem 1.** *The Jacobian for the change of variables (2.3) to (2.4) is equal to*

$$\frac{1}{q_n} \frac{\prod_{i=1}^{n-1} b_i}{\prod_{i=1}^{n} q_i}. \tag{2.9}$$

Proof. Rewriting (2.5) in the form

$$((I_n - \lambda T)^{-1})_{11} = \sum_{j=1}^{n} \frac{q_j^2}{1 - \lambda\lambda_j}, \tag{2.10}$$

recalling the explicit form of $T$ from (2.1), and equating successive powers of $\lambda$ on both sides gives

$$\begin{aligned}
1 &= \sum_{j=1}^{n} q_j^2 \\
a_n &= \sum_{j=1}^{n} q_j^2 \lambda_j \\
* + b_{n-1}^2 &= \sum_{j=1}^{n} q_j^2 \lambda_j^2 \\
* + a_{n-1} b_{n-1}^2 &= \sum_{j=1}^{n} q_j^2 \lambda_j^3 \\
* + b_{n-2}^2 b_{n-1}^2 &= \sum_{j=1}^{n} q_j^2 \lambda_j^4 \\
* + a_{n-2} b_{n-2}^2 b_{n-1}^2 &= \sum_{j=1}^{n} q_j^2 \lambda_j^5 \\
\vdots \quad & \quad \vdots \\
* + a_1 b_1^2 \cdots b_{n-2}^2 b_{n-1}^2 &= \sum_{j=1}^{n} q_j^2 \lambda_j^{2n-1}.
\end{aligned} \tag{2.11}$$

Here the $*$ denotes terms involving only variables already having appeared on the l.h.s. of preceding equations. Thus the variables $a_n, b_{n-1}, a_{n-1}, b_{n-2}, \ldots$ occur in a triangular structure. Upon taking differentials, the first of these equations implies

$$q_n dq_n = -\sum_{j=1}^{n-1} q_j dq_j.$$

For the differentials of the remaining equations, we use this to substitute for $dq_n$, and then take wedge products of both sides. On the l.h.s., the triangular structure gives

$$2^{n-1} \prod_{j=1}^{n-1} b_j^{4j-3} d\vec{a} \wedge d\vec{b} \tag{2.12}$$

where

$$d\vec{a} := \bigwedge_{j=1}^{n} da_j, \qquad d\vec{b} := \bigwedge_{j=1}^{n-1} db_j.$$

On the r.h.s. the wedge product operation yields

$$2^{n-1} q_n^2 \prod_{j=1}^{n-1} q_j^3 \det \Big[ [\lambda_k^j - \lambda_n^j]_{\substack{j=1,\dots,2n-1 \\ k=1,\dots,n-1}} [j\lambda_k^{j-1}]_{\substack{j=1,\dots,2n-1 \\ k=1,\dots,n}} \Big] d\vec{\lambda} \wedge d\vec{q} \tag{2.13}$$

where

$$d\vec{\lambda} := \bigwedge_{j=1}^{n} d\lambda_j, \qquad d\vec{q} := \bigwedge_{j=1}^{n-1} dq_j$$

(a common factor $2q_k$ has been removed from column $k$, $k = 1, \dots, n-1$ of the determinant, as has a common factor $q_k^2$ from columns $n-1+k$, $k = 1, \dots, n$).

By definition the Jacobian $J$ satisfies

$$d\vec{a} \wedge d\vec{b} = J d\vec{\lambda} \wedge d\vec{q}. \tag{2.14}$$

Equating (2.12) and (2.13), and using (2.7) to evaluate the determinant then shows

$$J = \frac{1}{q_n} \frac{\prod_{j=1}^{n-1} b_j}{\prod_{j=1}^{n} q_j} \Big( \frac{\prod_{j=1}^{n} q_j^2}{\prod_{j=1}^{n-1} b_j^{2j-1}} \Big)^2 \prod_{1 \le j < k \le N} (\lambda_k - \lambda_j)^4.$$

Recalling (2.6) gives the form of $J$ (2.9). □

In [6] indirect methods are used to derive (2.9) but with the factor $1/q_n$ not present. As noted in [17], the reasoning of [6] is most suited to working with the variables $\mu_j = q_j^2$, and doing so eliminates this apparent discrepancy.

# 3 Calculation of a Jacobian for unitary and real orthogonal Hessenberg matrices

## 3.1 Preliminaries

In general a unitary upper triangular Hessenberg matrix $H = [H_{i,j}]_{i,j=0,\dots,n-1}$ with positive elements along the sub-diagonal is parametrized by $n-1$ complex numbers $\alpha_0, \alpha_1, \dots, \alpha_{n-2}$ with $|\alpha_j| = 1$ and a further complex number $\alpha_{n-1}$ with $|\alpha_{n-1}| < 1$. Setting $\alpha_{-1} := -1$, $\rho_j := \sqrt{1 - |\alpha_j|^2}$ $(j = 0, \dots, n-2)$, one can check that if the diagonal entries are specified as $H_{i,i} = -\alpha_{i-1}\bar{\alpha}_i$, and subdiagonal entries as $H_{i+1,i} = \rho_i$, then the remaining non-zero entries are given by

$$H_{i,j} = -\alpha_{i-1}\bar{\alpha}_j \prod_{l=i}^{j-1} \rho_l, \qquad i < j. \tag{3.1}$$

Let $\lambda_j = e^{i\theta_j}$ $(j = 1, \dots, n)$ denote the eigenvalues of $H$ and let $q_j$ denote the modulus of the first component of the corresponding normalized eigenvectors (the $\{q_j\}$ thus satisfy (2.2)). With the $\{\theta_j\}$ ordered, there is an invertible 1-to-1 correspondence with the parameters $\{\alpha_j\}_{j=0,\dots,n-1}$. Our interest is to directly compute the Jacobian for the change of variables from $\{\alpha_j\}_{j=0,\dots,n-1}$ to $\{\theta_i\}_{i=1,\dots,n}$, $\{q_i\}_{i=1,\dots,n-1}$.

In preparation for the derivation of this result, note that with $H_k$ denoting the top $k \times k$ block of $H$,

$$\chi_k(\lambda) := \det(\lambda I_k - H_k)$$

satisfies (1.3) (see e.g. [13]). Also of interest is a variant of the characteristic polynomial of the bottom $k \times k$ submatrix. In relation to this, note that with the involution $\alpha_j \mapsto -\bar{\alpha}_j \alpha_{n-1}$ $(j = 0, \dots, n-2)$, the bottom $k \times k$ submatrix, after reflection in the anti-diagonal, becomes equal to the top $k \times k$ submatrix but with $\alpha_j \mapsto \alpha_{n-2-j}$ $(j = 0, \dots, n-2)$. Let the characteristic polynomial of the bottom $k \times k$ submatrix with the replacements $\alpha_j \mapsto -\bar{\alpha}_j \alpha_{n-1}$ $(j = 0, \dots, n-2)$ be denoted $\chi_k^b(\lambda)$. We see that this polynomial satisfies the recurrence (1.3) with $\alpha_j$ replaced by $-\bar{\alpha}_{n-2-j}\alpha_{n-1}$ in (1.3),

$$\begin{aligned} \chi_k^b(\lambda) &= \lambda \chi_{k-1}^b(\lambda) + \alpha_{n-1-k}\bar{\alpha}_{n-1}\tilde{\chi}_{k-1}^b(\lambda) \\ \tilde{\chi}_k^b(\lambda) &= \tilde{\chi}_{k-1}^b(\lambda) + \lambda \bar{\alpha}_{n-1-k}\alpha_{n-1}\chi_{k-1}^b(\lambda) \end{aligned} \tag{3.2}$$

$(k = 1, \dots, n)$ where $\chi_0^b(\lambda) = \tilde{\chi}_0^b(\lambda) = 1$ and $\tilde{\chi}_k^b(\lambda) = \lambda^k \bar{\chi}_k^b(1/\lambda)$. These recurrences can be used to derive the analogue of (2.6) [17]

**Proposition 2.** *For the unitary Hessenberg matrix specified by (3.1) and surrounding text, we have*

$$\prod_{1 \le i < j \le n} |\lambda_i - \lambda_j|^2 = \frac{\prod_{l=0}^{n-2}(1 - |\alpha_l|^2)^{n-1-l}}{\prod_{j=1}^n q_j^2}. \tag{3.3}$$

Proof. We follow the strategy sketched to prove (2.6) in the proof of Proposition 1. Analogous to (2.5) we have

$$((H - \lambda I_n)^{-1})_{11} = \sum_{j=1}^n \frac{q_j^2}{\lambda_j - \lambda} \tag{3.4}$$

where $q_j$ and $\lambda_j$ relate to $H$ as specified below (3.1). Using (2.8), (3.4) can be rewritten as

$$\frac{\chi_{n-1}^b(\lambda)|_{\alpha_j \mapsto -\bar{\alpha}_j\alpha_{n-1}}}{\prod_{i=1}^n(\lambda - \lambda_i)} = \sum_{j=1}^n \frac{q_j^2}{\lambda - \lambda_j}, \tag{3.5}$$

which in turn implies

$$\prod_{i=1}^n q_i^2 = \frac{\prod_{i=1}^n |\chi_{n-1}^b(\lambda_i)||_{\alpha_j \mapsto -\bar{\alpha}_j\alpha_{n-1}}}{\prod_{1 \le i < j \le n} |\lambda_i - \lambda_j|^2}.$$

From this we see (3.3) follows if we can show

$$\prod_{i=1}^n |\chi_{n-1}^b(\lambda_i)| = \prod_{l=0}^{n-2}(1 - |\alpha_l|^2)^{n-1-l}. \tag{3.6}$$

To establish (3.6) we will use (3.2). With $\lambda_j^{(p)}$ denoting the $j$th zero of $\chi_p^b(\lambda)$, it follows from (3.2) that

$$\begin{aligned} \chi_k^b(1/\bar{\lambda}_j^{(k)}) &= \frac{1}{\bar{\lambda}_j^{(k)}}(1 - |\alpha_{n-k-1}|^2)\chi_{k-1}^b(1/\bar{\lambda}_j^{(k)}) \\ \tilde{\chi}_k^b(\lambda_j^{(k)}) &= (1 - |\alpha_{n-k-1}|^2)\tilde{\chi}_{k-1}^b(\lambda_j^{(k)}). \end{aligned} \tag{3.7}$$

Introducing the factorizations

$$\chi^b_{k-1}(x) = \prod_{i=1}^{k-1}(x - \lambda_i^{(k-1)}), \qquad \tilde{\chi}^b_k(x) = \prod_{i=1}^{k}(1 - x\bar{\lambda}_i^{(k)})$$

we deduce from (3.7) that

$$\begin{aligned}\prod_{i=1}^{k} \chi^b_k(1/\bar{\lambda}_i^{(k)}) &= (1 - |\alpha_{n-k-1}|^2)^k \prod_{i=1}^{k}(1/\bar{\lambda}_i^{(k)})^k \prod_{j=1}^{k-1} \tilde{\chi}^b_{k-1}(\lambda_j^{(k-1)}) \\ \prod_{i=1}^{k} \tilde{\chi}^b_k(\lambda_i^{(k)}) &= (1 - |\alpha_{n-k-1}|^2)^k \prod_{j=1}^{k-1}(\bar{\lambda}_j^{(k-1)})^{k-1} \chi^b_{k-1}(1/\bar{\lambda}_j^{(k-1)}).\end{aligned}$$

These latter two equations together imply

$$\prod_{i=1}^{k}(\bar{\lambda}_i^{(k)})^k \chi^b_k(1/\bar{\lambda}_i^{(k)}) = \prod_{l=0}^{k-1}(1 - |\alpha_{n-l}|^2)^{l+1}.$$

Making further use of the first equation in (3.7), setting $k = n$, and noting $|\lambda_i^{(n)}| = 1$ gives (3.6). □

In [17] (3.3) is derived using a different strategy relating to the determinant of the Toeplitz matrix formed from the moments of the underlying measure.

Also required is a determinant evaluation analogous to (2.7).

**Proposition 3.** *We have*

$$\det \begin{bmatrix} \begin{bmatrix} \lambda_k^j - \lambda_n^j \\ \lambda_k^{-j} - \lambda_n^{-j} \end{bmatrix}_{j,k=1,\dots,n-1} & \begin{bmatrix} j\lambda_k^j \\ -j\lambda_k^{-j} \end{bmatrix}_{\substack{j=1,\dots,n-1 \\ k=1,\dots,n}} \\ [\lambda_k^n - \lambda_n^n]_{k=1,\dots,n-1} & [n\lambda_k^n]_{k=1,\dots,n} \end{bmatrix} = \frac{\prod_{1\le j<k\le n}(\lambda_k - \lambda_j)^4}{\prod_{l=1}^{n} \lambda_l^{2n-3}}. \tag{3.8}$$

Proof. By inspection the determinant is a symmetric function of $\lambda_1, \dots, \lambda_n$ which is homogeneous of degree $n$. Upon multiplying columns 1 and columns $n$ by $\lambda_1^{2n-3}$ we see that the determinant becomes a polynomial in $\lambda_1$, so it must be of the form

$$\frac{1}{\prod_{l=1}^{n} \lambda_l^{2n-3}} p(\lambda_1, \dots, \lambda_n)$$

where $p$ is a symmetric polynomial of $\lambda_1, \dots, \lambda_n$ of degree $2n(n-1)$.

We see immediately that the determinant vanishes when $\lambda_1 = \lambda_2$. Furthermore, it is straightforward to verify that its derivatives $(\lambda_1 \frac{\partial}{\partial \lambda_1})^j$ $(j = 1, 2, 3)$ also vanish when $\lambda_1 = \lambda_2$. The polynomial $p$ must thus contain as a factor $\prod_{1\le j<k\le n}(\lambda_k - \lambda_j)^4$. As this is of degree $2n(n-1)$, it follows that the determinant must in fact be proportional to (3.8).

On the r.h.s. of (3.8), the coefficient of $\prod_{l=1}^{n} \lambda_l^{4(l-1)-2n+3}$ is unity. In the determinant, let us add $(n-1)$ times the first column to the $n$th column. Then we see that the coefficient of $\lambda_1^{-2n+3}$ is given by the cofactor coming from multiplying together the $(2n-2, 1)$ and $(2n-4, n)$ elements. In the cofactor we add $(n-2)$ times the first column to the $(n-1)$st column. The coefficient of $\lambda_1^{-2n+7}$ is given by the cofactor coming from multiplying together the $(2n-3, 1)$ and $(2n-5, n-1)$ elements. Proceeding in this manner we see that the coefficient of $\prod_{l=1}^{n} \lambda_l^{4(l-1)-2n+3}$ is also unity in the determinant. □

As remarked in the Introduction, the approach to the ensemble (1.4) given in [17] is via $2n \times 2n$ $(n = N)$ real orthogonal Hessenberg matrices with determinant $+1$. The elements being real implies

$\{\alpha_j\}_{j=0,\dots,2n-1}$ are real, while the determinant equalling $+1$ implies $\alpha_{2n-1} = -1$. Thus there are $2n-1$ independent real parameters $\alpha_0, \dots, \alpha_{2n-2}$. In the corresponding eigen-decomposition, there are $n$ independent eigenvalues $\lambda_j = e^{i\theta_j}$ $(j = 1, \dots, n,\ 0 \le \theta_j < \pi)$ and $n-1$ independent variables $q_j$ $(j = 1, \dots, n-1)$ where $\frac{1}{2}q_j^2$ is the square of the first component of both the eigenvalues $\lambda_j$ and $\bar{\lambda}_j$. Left open in [17] is the problem of a direct calculation of the corresponding Jacobian. For this the analogue of Proposition 2 is required.

**Proposition 4.** *[17] For a $2n \times 2n$ real orthogonal Hessenberg matrix of determinant $+1$, parametrized in terms of the real parameters $\{\alpha_i\}_{i=0,\dots,2n-2}$, $|\alpha_i| < 1$, we have*

$$\prod_{i=1}^{n} |\lambda_i - \frac{1}{\lambda_i}| \prod_{1 \le i < j \le n} |\lambda_i - \lambda_j|^2 |\lambda_i - 1/\lambda_j|^2 = 2^n \frac{\prod_{l=0}^{2n-2}(1-\alpha_l^2)^{(2n-1-l)/2}}{\prod_{i=1}^{n} q_i^2} \tag{3.9}$$

$$\prod_{j=1}^{n} |1 - \lambda_j|^2 = 2 \prod_{k=0}^{2n-2} (1 - \alpha_k), \qquad \prod_{j=1}^{n} |1 + \lambda_j|^2 = 2 \prod_{k=0}^{2n-2} (1 + (-1)^k \alpha_k). \tag{3.10}$$

Proof. Denoting the Hessenberg matrix in question by $H$, the analogue of (3.5) reads

$$((I_{2n} - \lambda H)^{-1})_{11} = \frac{1}{2} \sum_{j=1}^{n} q_j^2 \Big( \frac{1}{1 - \lambda \lambda_j} + \frac{1}{1 - \lambda \bar{\lambda}_j} \Big). \tag{3.11}$$

Analogous to the reasoning underlying (3.5), the l.h.s. is equal to $\chi_{2n-1}^b(1/\lambda)/\lambda\chi_{2n}^b(1/\lambda)$. We thus have

$$\Big| \frac{\chi_{2n-1}^b(\lambda_j)}{\chi_{2n}'(\lambda_j)} \Big| = \frac{1}{2} q_j^2 \qquad (j = 1, \dots, 2n)$$

where $\lambda_{j+n} = \bar{\lambda}_j$, $q_{j+n} = q_j$. Taking the product over $j = 1, \dots, 2n$, making use of (3.6), then taking the square root gives (3.9). For the results (3.10), one notes

$$\prod_{j=1}^{n} |1 - \lambda_j|^2 = \chi_{2n}(1), \qquad \prod_{j=1}^{n} |1 + \lambda_j|^2 = \chi_{2n}(-1),$$

while from (1.4) $\chi_{k+1}(\lambda)|_{\lambda=\pm1} = (\lambda - \alpha_k \lambda^k)\chi_k(\lambda)|_{\lambda=\pm1}$. □

We remark that in [17] (3.9) is deduced by making use of (3.3), which in turn is derived using formulas relating to the underlying measure. Our derivation of (3.10) is the same as that in [17].

We must make note too of a further determinant evaluation.

**Proposition 5.** *We have*

$$\det \Big[ [\lambda_k^j + \lambda_k^{-j} - (\lambda_k^j + \lambda_k^{-j})]_{\substack{j=1,\dots,2n-1 \\ k=1,\dots,n-1}} \quad [j(\lambda_k^j - \lambda_k^{-j})]_{\substack{j=1,\dots,2n-1 \\ k=1,\dots,n}} \Big]$$
$$= \prod_{j=1}^{n} (\lambda_j - 1/\lambda_j) \prod_{1 \le j < k \le n} (\lambda_k - \lambda_j)^2 (1/\lambda_k - 1/\lambda_j)^2 (\lambda_j - 1/\lambda_k)^2 (1/\lambda_j - \lambda_k)^2. \tag{3.12}$$

Proof. We see that the determinant is a symmetric rational function in $\lambda_1, \dots, \lambda_n$, and is antisymmetric under the mapping $\lambda_i \mapsto 1/\lambda_i$ for any $i = 1, \dots, n$. It must thus be of the form

$$\prod_{j=1}^{n} (\lambda_j - 1/\lambda_j)\, q(\lambda_1, \dots, \lambda_n) \tag{3.13}$$

where $q$ is symmetric and unchanged by the mapping $\lambda_i \mapsto 1/\lambda_i$. Noting too that the determinant vanishes when $\lambda_1 = \lambda_2$, we see that $q$ must contain as a factor

$$\prod_{1 \le j < k \le n} (\lambda_k - \lambda_j)^2 (1/\lambda_k - 1/\lambda_j)^2 (\lambda_j - 1/\lambda_k)^2 (1/\lambda_j - \lambda_k)^2. \tag{3.14}$$

The highest order term (in degree) of (3.14) multiplied by $\prod_{j=1}^n(\lambda_j - 1/\lambda_j)$ is $\prod_{j=1}^n \lambda_j \prod_{1\le j<k\le n}(\lambda_k - \lambda_j)^4$. On the other hand the highest order term in degree in the determinant is

$$\det\left[[\lambda_k^j - \lambda_n^j]_{\substack{j=1,\dots,2n-1\\k=1,\dots,n-1}} [j\lambda_k^j]_{\substack{j=1,\dots,2n-1\\k=1,\dots,n}}\right].$$

According to (2.7) this evaluates to the same expression, so in fact $q$ must be exactly equal to (3.14). □

## 3.2 The Jacobians

Using a similar strategy to that used to derive the Jacobian (2.9) in the proof of Theorem 1, the results of the previous subsection together with the method of wedge products allows the two Jacobians evaluated by indirect means in [17] to be derived directly.

**Theorem 2.** *Consider unitary Hessenberg matrices with entries specified by (3.1) and surrounding text. The Jacobian for the change of variables from* $\{\alpha_j\}_{j=0,\dots,n-1}$ *to* $\{\theta_i\}_{i=1,\dots,n}$, $\{q_i\}_{i=1,\dots,n-1}$ *is equal to*

$$\frac{\prod_{i=0}^{n-2}(1-|\alpha_i|^2)}{q_n \prod_{i=1}^n q_i}. \tag{3.15}$$

*Consider* $2n \times 2n$ *real orthogonal Hessenberg matrices as specified above Proposition 4. The Jacobian for the change of variables from* $\{\alpha_j\}_{j=0,\dots,2n-2}$ *to* $\{\theta_i\}_{i=1,\dots,n}$, $\{q_i\}_{i=1,\dots,n-1}$ *is equal to*

$$\frac{2^{n-1}}{q_n \prod_{i=1}^n q_i} \frac{\prod_{l=0}^{2n-2}(1-|\alpha_l|^2)}{\prod_{k=0}^{2n-2}(1-\alpha_k)^{1/2}(1+(-1)^k\alpha_k)^{1/2}}. \tag{3.16}$$

Proof. In relation to (3.15) we begin by equating successive powers of $\lambda$ on both sides of (3.4). Recalling the explicit form of $H$ given by (3.1) and surrounding text this gives

$$\begin{aligned}
1 &= \sum_{j=1}^n q_j^2 \\
\bar\alpha_0 &= \sum_{j=1}^n q_j^2 \lambda_j \\
* + \bar\alpha_1 \rho_0^2 &= \sum_{j=1}^n q_j^2 \lambda_j^2 \\
* + \bar\alpha_2 \rho_0^2 \rho_1^2 &= \sum_{j=1}^n q_j^2 \lambda_j^3 \\
&\vdots \\
* + \bar\alpha_{n-1} \rho_0^2 \rho_1^2 \cdots \rho_{n-2}^2 &= \sum_{j=1}^n q_j^2 \lambda_j^n
\end{aligned} \tag{3.17}$$

where the $*$ denotes terms involving only variables already having appeared on the l.h.s. of the preceding equation. Thus as in the corresponding equations (2.11) in the tridiagonal case a triangular structure results. We know that $\alpha_j$ $(j = 0,\dots,n-2)$ has an independent real and imaginary part, while $\alpha_{n-1} := e^{i\phi}$, $\lambda_j := e^{i\theta_j}$ $(j = 1,\dots,n)$ have unit modulus. Consequently the number of equations can be made equal to the number of variables by firstly using the first equation to eliminate $q_n^2$ in the subsequent equations, then appending to the list the complex conjugate of all but the last of the remaining equations.

Let us take differentials of these $2n-1$ equations, then take wedge products of both sides. Because of the triangular structure, we obtain on the l.h.s.

$$\rho_0^2 \rho_1^2 \cdots \rho_{n-2}^2 \prod_{l=1}^{n-2} \rho_{n-l-2}^{4l} d\vec{\alpha} \wedge d\phi, \tag{3.18}$$

while this operation on the r.h.s. yields

$$\begin{aligned} & q_n^2 \prod_{j=1}^{n-1} q_j^3 \left| \det \left[ \begin{array}{cc} \left[ \begin{array}{c} \lambda_k^j - \lambda_n^j \\ \lambda_k^{-j} - \lambda_n^{-j} \end{array} \right]_{j,k=1,\dots,n-1} & \left[ \begin{array}{c} j\lambda_k^j \\ -j\lambda_k^{-j} \end{array} \right]_{\substack{j=1,\dots,n-1 \\ k=1,\dots,n}} \\ [\lambda_k^n - \lambda_n^n]_{k=1,\dots,n-1} & [n\lambda_k^n]_{k=1,\dots,n} \end{array} \right] \right| d\vec{\theta} \wedge d\vec{q} \\ & = q_n^2 \prod_{j=1}^{n-1} q_j^3 \prod_{1 \le j < k \le n} |\lambda_k - \lambda_j|^4 d\vec{\theta} \wedge d\vec{q} \end{aligned} \tag{3.19}$$

where the equality follows upon using the determinant evaluation (3.8).

Analogous to (2.14), by definition the Jacobian $J$ satisfies

$$d\vec{\alpha} \wedge d\phi = J d\vec{\theta} \wedge d\vec{q}.$$

Equating (3.18) and (3.19) and making use of (3.3) gives (3.15).

Consider next the derivation of (3.16). Proceeding as in the derivation of (3.17), expanding (3.11) in powers of $\lambda$, we obtain

$$\begin{aligned} 1 &= \sum_{j=1}^{n} q_j^2 \\ \alpha_0 &= \frac{1}{2} \sum_{j=1}^{n} q_j^2 (\lambda_j + \bar{\lambda}_j) \\ * + \alpha_1 \rho_0^2 &= \frac{1}{2} \sum_{j=1}^{n} q_j^2 (\lambda_j^2 + \bar{\lambda}_j^2) \\ \vdots \quad & \quad \vdots \\ * + \alpha_{2n-2} \rho_0^2 \cdots \rho_{2n-3}^2 &= \frac{1}{2} \sum_{j=1}^{n} q_j^2 (\lambda_j^{2n-1} + \bar{\lambda}_j^{2n-1}). \end{aligned}$$

The l.h.s. again exhibits a triangular structure, and furthermore all quantities on the l.h.s. are real. Taking the differentials of both sides, and forming the wedge product of the l.h.s.'s of all but the first equation gives

$$\prod_{l=0}^{2n-3} \rho_l^{2(2n-2-l)} d\vec{\alpha}. \tag{3.20}$$

On the r.h.s., after substituting for $q_n dq_n$ using the differential of the first equation, this same procedure gives

$$2^{-n} q_n^2 \prod_{j=1}^{n-1} q_j^3 \left| \det \left[ [\lambda_k^j + \lambda_k^{-j} - (\lambda_k^j + \lambda_k^{-j})]_{\substack{j=1,\dots,2n-1 \\ k=1,\dots,n-1}} \quad [j(\lambda_k^j - \lambda_k^{-j})]_{\substack{j=1,\dots,2n-1 \\ k=1,\dots,n}} \right] \right| d\vec{\theta} \wedge d\vec{q}. \tag{3.21}$$

Here the Jacobian $J$ satisfies

$$d\vec{\alpha} = J d\vec{\theta} \wedge d\vec{q},$$

so (3.20) and (3.21) (with the determinant evaluated according to (3.12)) together give a formula for $J$ in terms of $\{q_i\}$, $\{\alpha_i\}$ and $\{\lambda_i\}$. The latter set of variables can be eliminated by making use of Proposition 4, and (3.16) results. □

# 4 A multiplicative rank 1 perturbation of unitary matrices

## 4.1 Circular analogue of the Dixon-Anderson density

Let $\vec{e}_1$ denote the $n \times 1$ unit vector $(1, 0, \dots, 0)^T$. Let $t$ be a complex number with $|t| = 1$. Then the matrix

$$I_n - (1-t)\vec{e}_1 \vec{e}_1^T$$

is a unitary matrix differing from the identity only in the top left entry which is $t$. Our interest in this section is the eigenvalue distribution of

$$\tilde{U} := (I_n - (1-t)\vec{e}_1 \vec{e}_1^T)U, \tag{4.1}$$

for $U$ a given unitary matrix. Such multiplicative rank 1 perturbations are discussed for example in [1]. The term multiplicative perturbation is used because $\tilde{U}$ is obtained from $U$ by multiplication of the first row by the unimodular complex number $t$, while the term rank 1 is used because the multiplicative perturbative factor differs from the identity by a rank 1 matrix. We will see that for $U$ a member of the circular $\beta$-ensemble, a joint eigenvalue p.d.f. generalizing (1.2) results.

First a rational function having as its zeros the eigenvalues of $\tilde{U}$ will be specified.

**Proposition 6.** *Let $U$ be an $n \times n$ unitary matrix with distinct eigenvalues $e^{i\theta_1}, \dots, e^{i\theta_n}$, and denote the corresponding matrix of eigenvectors by $V = [v_{jk}]_{j,k=1,\dots,n}$. The eigenvalues of $\tilde{U}$ as specified by (4.1) occur at the zeros of the rational function*

$$C_n(\lambda) = 1 + (t-1)\sum_{j=1}^n \frac{e^{i\theta_j}|v_{1j}|^2}{e^{i\theta_j} - \lambda}. \tag{4.2}$$

Proof. Noting from (4.1) that $\tilde{U} = U - (1-t)U'$, where $U'$ is the $n \times n$ matrix in which the first row is equal to the first row of $U$, and all other rows have all entries zero, we see

$$V^{-1}\tilde{U}V = \mathrm{diag}[e^{i\theta_1}, \dots, e^{i\theta_n}] + (t-1)[\bar{v}_{1j} v_{1k} e^{i\theta_k}]_{j,k=1,\dots,n}.$$

Thus $\tilde{U}$ has the same spectrum as a matrix which consists of a rank 1 multiplicative perturbation of a diagonalized unitary matrix. The characteristic polynomial of this matrix can be factorized as

$$\prod_{l=1}^n (e^{i\theta_l} - \lambda) \det\Big[\delta_{j,k} + (t-1)\bar{v}_{1j} v_{1k} e^{i\theta_k}/(e^{i\theta_j} - \lambda)\Big]_{j,k=1,\dots,n},$$

and the zeros must occur at the zeros of the determinant. Noting the simple determinant evaluation

$$\det[u_j \delta_{j,k} + 1]_{j,k=1,\dots,n} = \prod_{l=1}^n u_l \Big(1 + \sum_{j=1}^n \frac{1}{u_j}\Big),$$

the sought result follows. □

We remark that Proposition 6 can be extended to the case that each eigenvalue $e^{i\theta_j}$ has multiplicity $m_j$. Thus with $v_{1j}^{(s)}$ denoting the first component of the $s$th independent eigenvector corresponding to $e^{i\theta_j}$, we replace $|v_{1j}|^2$ in (4.2) by $\sum_{s=1}^{m_j} |v_{1j}^{(s)}|^2$.

Let us suppose now that the matrix $U$ is a unitary upper triangular Hessenberg matrix parametrized as specified by (3.1) and surrounding text. One of the main results of [17] is that the parameters $\{\alpha_j\}_{j=0,\dots,n-1}$ can be chosen from particular probability distributions so that the eigenvalue p.d.f. of $U$

is given by (1.2). The probability distributions in question are parametrized by a real number $\nu \geq 1$ and denoted by $\Theta_\nu$. For $\nu > 1$, the support of $\Theta_\nu$ is the open unit disk $|z| < 1$ in the complex plane, and the distribution is specified by the p.d.f.

$$\frac{\nu-1}{2\pi}(1-|z|^2)^{(\nu-3)/2}.$$

For $\nu = 1$, the support is the unit circle $|z| = 1$, and $\Theta_1$ denotes the uniform distribution. Proposition 4.2 of [17] tells us that if

$$\alpha_{n-j-1} \sim \Theta_{\beta j+1} \qquad (j = 0, \dots, n-1) \tag{4.3}$$

then the corresponding eigenvalue p.d.f. is given by (1.2). Furthermore, it tells us that the modulus squared of the first component of the eigenvectors $|v_{1j}|^2 := \mu_j$ have the distribution with measure

$$\frac{1}{C_{\beta N}} \prod_{i=1}^{n} \mu_i^{\beta/2-1} d\vec{\mu}, \tag{4.4}$$

where

$$C_{\beta N} = \frac{\Gamma^N(\beta/2)}{\Gamma(\beta N/2)}, \qquad d\vec{\mu} := d\mu_1 \dots d\mu_{n-1}.$$

This is an example of the Dirichlet distribution.

The latter fact motivates the study of the zeros of (4.2) with the $|v_{1j}|^2$ distributed according to the Dirichlet distribution. We will find that a conditional p.d.f. relating to (1.2) results provided the distribution of $t$ is appropriately chosen. First, some preliminary results must be established.

**Lemma 1.** *Suppose in (4.2) that*

$$0 < \theta_1 < \theta_2 < \dots < \theta_n \leq 2\pi. \tag{4.5}$$

*The function $C(\lambda)$ has exactly $n$ zeros occurring at $\lambda = e^{i\psi_1}, \dots, e^{i\psi_n}$, where*

$$\theta_{i-1} < \psi_i < \theta_i \qquad (i = 1, \dots, n, \quad \theta_0 := \theta_n \bmod 2\pi). \tag{4.6}$$

*Furthermore, with $\lambda_j := e^{i\theta_j}$, $\tilde{\lambda}_j := e^{i\psi_j}$, we have*

$$-(t-1)\lambda_j q_j = \frac{\prod_{l=1}^n (\lambda_j - \tilde{\lambda}_l)}{\prod_{l=1, l\neq j}^n (\lambda_j - \lambda_l)} \qquad (j = 1, \dots, n) \tag{4.7}$$

$$\prod_{l=1}^{n} \tilde{\lambda}_l = t \prod_{l=1}^{n} \lambda_l. \tag{4.8}$$

Proof. The fact that there are exactly $n$ zeros of unit modulus follows from the relationship of $C_n(\lambda)$ to the characteristic polynomial of a unitary matrix. The interlacing property is well known [1]. It can be seen graphically by writing (4.2) in the form

$$C_n(\lambda) = \frac{(t-1)}{2i}\Big(\cot\frac{\phi}{2} - \sum_{i=1}^{n} q_i \cot\Big(\frac{\psi-\theta_i}{2}\Big)\Big),$$

where we have set $t := e^{i\phi}$, $\lambda := e^{i\psi}$.

With the zeros so identified, and the poles as evident from (4.2), it follows that

$$C_n(\lambda) = \frac{\prod_{j=1}^n (\lambda - \tilde{\lambda}_j)}{\prod_{j=1}^n (\lambda - \lambda_j)}, \tag{4.9}$$

where use has also been made of the property $C_n(\lambda) \to 1$ as $\lambda \to \infty$. Comparing residues in (4.2) and (4.9) gives (4.7), while setting $\lambda = 0$ gives (4.8). □

The Jacobians for some change of variables are also required.

**Lemma 2.** *Let $J$ be the Jacobian for the change of variables $\{q_j\}_{j=1,\dots,n-1} \cup \{\phi\}$ to $\{\psi_j\}_{j=1,\dots,n}$. We have*

$$J == |1-t|^{-(n-1)} \prod_{1 \le j < k \le n} \Big| \frac{\tilde{\lambda}_k - \tilde{\lambda}_j}{\lambda_k - \lambda_j} \Big| \tag{4.10}$$

Proof. By definition $J$ is positive and satisfies

$$d\vec{q} \wedge dt = J d\vec{\psi}. \tag{4.11}$$

Now

$$\begin{aligned} d\vec{q} \wedge d\phi &= (t-1)^{-(n-1)} d(t-1)\vec{q} \wedge d\phi \\ &= (t-1)^{-(n-1)} \det \Big[ \Big[ \frac{\partial (t-1) q_l}{\partial \tilde{\lambda}_j} \Big]_{\substack{j=1,\dots,n \\ l=1,\dots,n-1}} \Big[ \frac{\partial t}{\partial \tilde{\lambda}_j} \Big]_{j=1,\dots,n} \Big] d\vec{\tilde{\lambda}}_j. \end{aligned}$$

But according to (4.7) and (4.8)

$$\frac{\partial (t-1) q_l}{\partial \tilde{\lambda}_j} = \frac{(t-1) q_l}{\lambda_l - \tilde{\lambda}_j}, \qquad \frac{\partial t}{\partial \tilde{\lambda}_j} = \frac{t}{\tilde{\lambda}_j},$$

and so with $\lambda_n = 0$ (temporarily as a notational convenience)

$$d\vec{q} \wedge d\phi = t \prod_{l=1}^{n-1} (-q_l) \det \Big[ \frac{1}{\tilde{\lambda}_j - \lambda_l} \Big]_{j,l=1,\dots,n} d\vec{\tilde{\lambda}}. \tag{4.12}$$

Since $J$ is positive and satisfies (4.11), it must be equal to the modulus of the terms multiplying $d\vec{\tilde{\lambda}}$ in this expression. Evaluating the determinant as a Cauchy double alternant, and evaluating $\prod_{l=1}^{n-1} q_l$ using (4.7) gives the stated result. □

The results of the above two lemmas allow a change of variables to be made from $\{q_j\}_{j=1,\dots,n-1} \cup \{t\}$ to $\{\lambda_l\}_{l=1,\dots,n}$.

**Theorem 3.** *With $|v_{1j}|^2 = q_j$ $(j = 1, \dots, n)$ in (4.2), let $\{q_j\}$ have the Dirichlet distribution with measure*

$$\frac{\Gamma((n-1)d + d_0)}{(\Gamma(d))^{n-1} \Gamma(d_0)} \Big( \prod_{j=1}^{n-1} q_j^{d-1} \Big) q_n^{d_0 - 1} d\vec{q}. \tag{4.13}$$

*Further, let the parameter $t$ in (4.2) be determined by the p.d.f. with measure*

$$\frac{\Gamma^2(\frac{1}{2}(d_0 + (n-1)d + 1))}{2\pi \Gamma((n-1)d + d_0)} |1-t|^{d_0 + (n-1)d - 1} d\phi. \tag{4.14}$$

*The conditional p.d.f. of $\{\tilde{\lambda}_j = e^{i\psi_j}\}_{j=1,\dots,n}$, given $\{\lambda_j = e^{i\theta_j}\}_{j=1,\dots,n}$, is equal to*

$$A \frac{\prod_{l=1}^{n} |e^{i\theta_n} - e^{i\psi_l}|^{d_0 - 1}}{\prod_{l=1}^{n-1} |e^{i\theta_n} - e^{i\theta_l}|^{d_0 + d - 1}} \frac{\prod_{j=1}^{n-1} \prod_{l=1}^{n} |e^{i\theta_j} - e^{i\psi_l}|^{d-1}}{\prod_{1 \le j < k \le n-1} |e^{i\theta_k} - e^{i\theta_j}|^{2d-1}} \prod_{1 \le j < k \le n} |e^{i\psi_k} - e^{i\psi_j}|, \tag{4.15}$$

$$A := \frac{\Gamma((n-1)d + d_0)}{(\Gamma(d))^{n-1} \Gamma(d_0)} \frac{\Gamma^2(\frac{1}{2}(d_0 + (n-1)d + 1))}{2\pi \Gamma((n-1)d + d_0)}.$$

Proof. Our proof, which at a technical level proceeds by making use of the results of Lemmas 1 and 2, is based on a strategy adopted for an analogous problem with real roots by Anderson [2], and many years before by Dixon [5].

We must change variables in the product of (4.13), (4.14), and the Jacobian $J$. We know the latter is given by (4.10). To change variables in (4.13) we use (4.8), which gives

$$(\prod_{j=1}^{n-1} q_j^{d-1}) q_n^{d_0-1} = \frac{1}{|1-t|^{d_0+(n-1)d-n}} \frac{\prod_{j=1}^{n-1}\prod_{l=1}^{n} |\lambda_j - \tilde{\lambda}_l|^{d-1}}{\prod_{1\le j<l\le n-1} |\lambda_l - \lambda_j|^{2(d-1)}} \frac{\prod_{l=1}^{n} |\lambda_n - \tilde{\lambda}_l|^{d_0-1}}{\prod_{l=1}^{n-1} |\lambda_n - \lambda_l|^{d_0+d-2}}. \quad (4.16)$$

Multiplying (4.14), (4.10) and (4.16) we see that the dependence on $t$ cancels, and the expression (4.15) results. □

## 4.2 Properties of the corresponding joint density

Let us suppose $\{\theta_j\}_{j=1,\dots,n}$, assumed ordered as in (4.6) and with $\theta_n$ fixed, are distributed according to the p.d.f.

$$\frac{(n-1)!}{(2\pi)^{n-1} M_{n-1}((a_1+d_0+d-1)/2,(a_1+d_0+d-1)/2,d)} \prod_{l=1}^{n-1} |e^{i\theta_n} - e^{i\theta_l}|^{a_1+d_0+d-1} \prod_{1\le j<k\le n-1} |e^{i\theta_k} - e^{i\theta_j}|^{2d},$$

$$M_N(a,b,\lambda) = \prod_{j=0}^{N-1} \frac{\Gamma(\lambda j + a + b + 1)\Gamma(\lambda(j+1)+1)}{\Gamma(\lambda j + a + 1)\Gamma(\lambda j + b + 1)\Gamma(1+\lambda)}, \quad (4.17)$$

(for a discussion of this p.d.f. see [8]). Multiplying this with (4.15) gives the joint p.d.f.

$$C^{(n,n-1)}(\psi,\theta) := \frac{A(n-1)!}{(2\pi)^{n-1} M_{n-1}((a+a_1+d)/2,(a+a_1+d)/2,d)} \prod_{l=1}^{n} |e^{i\theta_n} - e^{i\psi_l}|^a$$
$$\times \prod_{1\le j<k\le n} |e^{i\psi_k} - e^{i\psi_j}| \prod_{l=1}^{n-1} |e^{i\theta_n} - e^{i\theta_l}|^{a_1} \prod_{1\le j<k\le n-1} |e^{i\theta_k} - e^{i\theta_j}| \prod_{j=1}^{n-1}\prod_{l=1}^{n} |e^{i\theta_j} - e^{i\psi_l}|^{d-1} \quad (4.18)$$

where $d_0 - 1 =: a$.

The case of (4.18) relevant to the circular $\beta$-ensemble of Killip and Nenciu is $a = d-1$, $a_1 = 1$ and $d = \beta/2$. Then (4.18) is symmetric in $\{\theta_l\}_{l=1,\dots,n}$ and in $\{\psi_l\}_{l=1,\dots,n}$, and $\theta_n$ may again be considered as variable ((4.18) should then be multiplied by $n/2\pi$ to get the correct normalization). It corresponds to the joint eigenvalue p.d.f. of a unitary Hessenberg matrix with parameters distributed according to (4.3), and thus with eigenvalue p.d.f. (1.2), and the same unitary Hessenberg matrix perturbed by multiplication of the first row by $t$. The factor $t$ is to be distributed according to (4.14) with $d_0 = d = \beta/2$. We know that the p.d.f. for $\{\theta_l\}_{l=1,\dots,n}$ can be sampled by computing the zeros of $\chi_n(\lambda)$ as calculated from (1.4) with $\{\alpha_j\}_{j=0,\dots,n-1}$ chosen as specified by (4.3). To sample from $\{\psi_l\}_{l=1,\dots,n-1}$ in the joint p.d.f., with the same $\{\alpha_j\}_{j=0,\dots,n-1}$ we again compute $\chi_n(\lambda)$ from the recurrences (1.4), but now with $\chi_0(\lambda) = \tilde{\chi}_0 = t$.

Next, let us turn our attention to integration formulas associated with (4.18). Since (4.15) is a conditional p.d.f. we must have

$$\int_R d\psi_1 \cdots d\psi_n \, C^{(n,n-1)}(\psi,\theta) = \frac{(n-1)!}{(2\pi)^{n-1} M_{n-1}((a+a_1+d)/2,(a+a_1+d)/2,d)} \quad (4.19)$$
$$\times \prod_{l=1}^{n-1} |e^{i\theta_n} - e^{i\theta_l}|^{a+a_1+d} \prod_{1\le j<k\le n-1} |e^{i\theta_k} - e^{i\theta_j}|^{2d} \quad (4.20)$$

where $R$ denotes the region specified by the inequalities (4.5). Special cases of (4.19) are two classical inter-relations between circular ensembles [7, 18] (for an extensive study of such formulas in random matrix theory see [11], and for application of the Dixon-Anderson density to the cases with real eigenvalues see [12]).

**Proposition 7.** *Let $COE_n$, $CUE_n$, $CSE_n$ — the circular ensembles with orthogonal, unitary and symplectic symmetry respectively — refer to the eigenvalue p.d.f. (1.2) with $\beta = 1,2,4$ respectively. Let alt refer to the operation of integrating over every second eigenvalue. Let $\mathrm{COE}_n \cup \mathrm{COE}_n$ denote the p.d.f. of $2n$ eigenvalues which results from superimposing two independent sequences of $n$ eigenvalues each with a $COE_n$ distribution. One has*

$$\mathrm{alt}(\mathrm{COE}_n \cup \mathrm{COE}_n) = \mathrm{CUE}_n \tag{4.21}$$

$$\mathrm{alt}(\mathrm{COE}_{2n}) = \mathrm{CSE}_n \tag{4.22}$$

Proof. For the first identity we require the fact that [14]

$$\mathrm{COE}_n \cup \mathrm{COE}_n \propto \prod_{1 \le j < k \le n} |e^{i\theta_{2k}} - e^{i\theta_{2j}}||e^{i\theta_{2k-1}} - e^{i\theta_{2j-1}}|. \tag{4.23}$$

We then see that (4.19) with $a_1 = d = 1$ is equivalent to the first identity. The second identity is immediately seen to correspond to (4.19) with $a_1 = 2$, $d = 2$. □

## 4.3 Matrix theoretic derivation of the COE, CUE, CSE inter-relations

The inter-relations (4.21), (4.22) were originally proved by establishing the same integration formulas as those noted in the proof of Proposition 7. In this subsection it will be shown how random matrices can be constructed in such a way that both (4.21) and (4.22) are immediate.

Let us consider first (4.21). This requires a different random matrix realization of the joint p.d.f. (4.18) in the case $d = a_1 = 1$, $a = 0$ to that given in the paragraph below (4.18). The theory underlying the construction is the following.

**Proposition 8.** *Let $M_1$ be a $2n \times 2n$ unitary matrix with real elements, which has a doubly degenerate spectrum with the independent eigenvalues distributed as $CUE_n$. Let the matrix of eigenvectors be $V = [v_{ij}]_{i,j=1,\dots,2n}$, and suppose the joint distribution of $\mu_j := (v_{1\,2j-1})^2 + (v_{1\,2j})^2$ $(j = 1,\dots,n)$ is equal to the Dirichlet distribution (4.4) with $\beta = 2$. Form the matrix $M_1'$ by multiplying the first row of $M_1$ by the complex number $t$, $|t| = 1$, where $t$ has distribution (4.14) with $d_0 = d = 1$. Then the perturbed matrix $M_1'$ has for its eigenvalue p.d.f. (4.18) with $a = 0$, $a_1 = d = 1$.*

Proof. Let $\{e^{i\theta_l}\}_{l=1,\dots,n}$ denote the independent eigenvalues of $M_1$. Proceeding as in the derivation of (4.2) shows that the characteristic polynomial of the perturbed matrix is equal to

$$\prod_{j=1}^{n} (e^{i\theta_l} - \lambda)^2 \Big(1 + (t-1) \sum_{j=1}^{n} \frac{e^{i\theta_j}\mu_j}{e^{i\theta_j} - \lambda}\Big). \tag{4.24}$$

Thus $M_1'$ has $n$ eigenvalues at $\{e^{i\theta_l}\}_{l=1,\dots,n}$, and $n$ eigenvalues given by the zeros of the rational function factor in (4.24). We are given that the former have p.d.f. $\mathrm{CUE}_n$, while Theorem 3 tells us that the latter have conditional p.d.f. (4.15) with $d_0 = d = 1$. Multiplying these together gives the stated joint distribution. □

To realize the matrix $M_1$, we begin with an element of $U(n)$ chosen according to the Haar measure. This gives the eigenvalue p.d.f. $\mathrm{CUE}_n$, with the eigenvectors such that the $\mu_j := |v_{1j}|^2$ $(j = 1,\dots,n)$ have joint distribution (4.4) with $\beta = 2$. To obtain a doubly degenerate spectrum, each element $x + iy$ is replaced by its $2 \times 2$ real matrix representation

$$\begin{bmatrix} x & y \\ -y & x \end{bmatrix},$$

so forming a $2n \times 2n$ matrix with real entries. Since the corresponding perturbed matrix $M_1'$ retains all distinct eigenvalues of $M_1$,

$$\mathrm{unpert}(M_1') = \mathrm{CUE}_n, \tag{4.25}$$

where the l.h.s. denotes the eigenvalue p.d.f. of $M_1'$ integrated over the perturbed eigenvalues. On the other hand Proposition 8 together with (4.23) tell us that with reference to the eigenvalue p.d.f., $M_1' = \mathrm{COE}_n \cup \mathrm{COE}_n$. Thus we have a matrix theoretic understanding of (4.21) in the sense that its validity is a consequence of spectral properties of $M_1'$ which avoid the need for explicit integration of the eigenvalue p.d.f.

We seek a similar understanding of (4.22). For this we must identify an ensemble of random matrices with a doubly degenerate spectrum, and their perturbations, which give rise to (4.18) in the case $d = 2$, $a = a_1 = 1$. In fact the very definition of the circular symplectic ensemble involves matrices with a doubly degenerate spectrum (see e.g. [8]). Thus, if for any $2n \times 2n$ matrix $X$ we set

$$X^D := Z_{2n} X^T Z_{2n}, \qquad \text{where} \quad Z_{2n} := I_n \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

and select $U \in U(2n)$ with Haar measure, then matrices of the form $U^D U$ make up the circular symplectic ensemble. Such matrices have a doubly degenerate spectrum, and the $n$ independent eigenvalues are distributed according to $\mathrm{CSE}_n$. Furthermore, with the matrix of eigenvectors denoted $V = [v_{ij}]_{i,j=1,\dots,2n}$, one has that the $\mu_j := |v_{1\,2j-1}|^2 + |v_{1\,2j}|^2$ $(j = 1, \dots, n)$ are distributed according to the Dirichlet distribution (4.4) with $\beta = 4$. Consideration of these facts, together with reasoning analogous to that used in the proof of Proposition 8, gives the sought realization.

**Proposition 9.** *Let $M_2$ be a member of the circular symplectic ensemble as specified above. Form the matrix $M_2'$ by multiplying the first row of $M_2$ by the complex number $t$, $|t| = 1$, where $t$ has distribution (4.14) with $d_0 = d = 2$. The joint eigenvalue p.d.f. of $M_2'$ is then given by (4.18) with $a = a_1 = 1$, $d = 2$.*

Analogous to (4.25) it is immediate that

$$\mathrm{unpert}(M_2') = \mathrm{CSE}_n.$$

Because Proposition 9 tells us that $M_2'$ has a joint distribution formally equivalent to $\mathrm{COE}_{2n}$, (4.22) is reclaimed as a matrix theoretic identity.

# 5 Cayley transformation

## 5.1 Cauchy analogue of the Dixon-Anderson density

In general a unitary matrix $U$ is transformed to an Hermitian matrix $H$ by the Cayley transformation

$$H = i \frac{1_N - U}{1_N + U}. \tag{5.1}$$

At the level of the eigenvalues, the change of variables (5.1) in the workings of Sections 4.1 and 4.2 leads to a joint p.d.f. on interlacing variables on the real line, relating to the so called (generalized) Cauchy ensemble [21, 4]. Properties of this allow a random three term recurrence to be derived for the (projected) characteristic polynomial associated with the p.d.f. (1.5).

First we apply the change of variables implied by (5.1) to (4.2) with the l.h.s. written as (4.9).

**Proposition 10.** *Consider the rational function (4.2) with the lower terminal of summation extended to $j = 0$. Substitute for the l.h.s. (4.9) with the lower terminals in the products extended to $j = 0$. Under the change of variables*

$$\tilde{\lambda}_j = \frac{x_j - i}{x_j + i}, \qquad \lambda_j = \frac{y_j - i}{y_j + i} \quad (j \neq 0)$$
$$\lambda = \frac{z - i}{z + i}, \qquad t = \frac{c - i}{c + i} \tag{5.2}$$

*and with $\lambda_0 = 1$ we obtain*

$$\frac{\prod_{j=0}^{n}(z - x_j)}{(z^2 + 1)\prod_{j=1}^{n}(z - y_j)} = \frac{z - c}{q_0(z^2 + 1)} - \sum_{j=1}^{n} \frac{(q_j/q_0)}{z - y_j}, \tag{5.3}$$

*where*

$$x_0 > y_1 > x_1 > \cdots > y_n > x_{n+1}. \tag{5.4}$$

Proof. This follows from direct substitution, together with the formula

$$\frac{q_0}{c + i} = \frac{\prod_{l=1}^{n}(y_l + i)}{\prod_{l=0}^{n}(x_l + i)}, \tag{5.5}$$

which is a consequence of (4.7), with lower product terminals extended to $l = 0$, in the case $j = 0$. □

**Theorem 4.** *Consider the rational function (5.3). Let $\{q_j\}_{j=0,\dots,n-1}$ have the Dirichlet distribution with measure*

$$\frac{\Gamma(\sum_{j=0}^{n} d_j)}{\prod_{j=0}^{n} \Gamma(d_j)} \prod_{j=0}^{n} q_j^{d_j - 1}\, d\vec{q}. \tag{5.6}$$

*Let $c$ have the generalized Cauchy distribution with measure*

$$\frac{\Gamma(\gamma)\Gamma(\bar{\gamma})}{\pi 2^{2(1-\mathrm{Re}\,\gamma)}\Gamma(2\mathrm{Re}\,\gamma - 1)}(1 + ic)^{-\gamma}(1 - ic)^{-\bar{\gamma}}\, dc \tag{5.7}$$

*where*

$$\sum_{i=0}^{n} d_i + 1 = 2\mathrm{Re}\,\gamma. \tag{5.8}$$

*We have that the conditional p.d.f. of $\{x_j\}_{j=0,\dots,n}$ given $\{y_j\}_{j=1,\dots,n}$ is equal to*

$$\tilde{A} \prod_{j=0}^{n}(1 + ix_j)^{-\gamma}(1 - ix_j)^{-\bar{\gamma}} \prod_{j=1}^{n}(1 + iy_j)^{\gamma - d_j}(1 - iy_j)^{\bar{\gamma} - d_j}$$
$$\times \prod_{j=1}^{n}\prod_{l=0}^{n} |y_j - x_l|^{d_j - 1} \prod_{1 \le j < k \le n} |y_j - y_k|^{1 - d_j - d_k} \prod_{0 \le j < k \le n} |x_j - x_k| \tag{5.9}$$

*where*

$$\tilde{A} = \frac{\Gamma(\gamma)\Gamma(\bar{\gamma})}{\pi 2^{2(1-\mathrm{Re}\,\gamma)}} \frac{1}{\Gamma(2\mathrm{Re}\,\gamma - 1 - \sum_{i=1}^{n} d_i)\prod_{j=1}^{n}\Gamma(d_j)}. \tag{5.10}$$

Proof. The task is to change variables in the wedge product of (5.6) and (5.7) to $\{x_j\}_{j=0,\dots,n}$. We have

$$\begin{aligned}
(1 + ic)^{-\gamma}(1 - ic)^{-\bar{\gamma}} &= |(1 - ic)^{-\bar{\gamma}}|^2 \\
&= q_0^{-2\mathrm{Re}\,\gamma} \left| \left( \frac{\prod_{l=1}^{n}(1 - iy_l)}{\prod_{l=0}^{n}(1 - ix_l)} \right)^{\bar{\gamma}} \right|^2 \\
&= q_0^{-2\mathrm{Re}\,\gamma} \frac{\prod_{l=1}^{n}(1 + iy_l)^{\gamma}(1 - iy_l)^{\bar{\gamma}}}{\prod_{l=0}^{n}(1 + ix_l)^{\gamma}(1 - ix_l)^{\bar{\gamma}}}
\end{aligned} \tag{5.11}$$

where the second equality follows from (5.5), and the final equality uses the fact that since $x_l, y_l$ interlace according to (5.4),

$$\log\left(\frac{\prod_{j=0}^{n}(1+ix_j)}{\prod_{j=1}^{n}(1+iy_j)}\right)=\sum_{j=0}^{n}\log(1+ix_j)-\sum_{j=1}^{n}\log(1+iy_j).$$

Also, for $j=1,\dots,n$

$$q_j=\frac{q_0}{|1+iy_j|^2}\frac{\prod_{l=0}^{n}|y_j-x_l|}{\prod_{l=1,l\neq j}^{n}|y_j-y_l|}. \tag{5.12}$$

It remains to change variables in $d\vec{q}\wedge dc$. Since $x_j$ is related to $\tilde{\lambda}_j$ and $c$ to $t=e^{i\phi}$ as given in (5.2),

$$d\vec{q}\wedge dc=\frac{J}{2}|1+ic|^2\prod_{j=0}^{n}\frac{2}{(1+ix_j)(1-ix_j)}d\vec{x}$$

where $J$ is the Jacobian (4.10) (appropriately modified to account for the lower terminal being 0). In terms of the change of variables (5.2) the latter reads

$$J=2^{-n}q_0^n\frac{\prod_{0\le j<k\le n}|x_j-x_k|}{\prod_{1\le j<k\le n}|y_j-y_k|}$$

and thus we have

$$d\vec{q}\wedge dc=q_0^{n+2}\frac{1}{|1+ix_j|^2}\frac{\prod_{0\le j<k\le n}|x_j-x_k|}{\prod_{1\le j<k\le n}|y_j-y_k|}d\vec{x}. \tag{5.13}$$

Multiplying (5.11), (5.12) and (5.13) gives the stated result. □

We remark that the conditional p.d.f. (5.9) appears in [19] as a generalization of a conditional p.d.f. due to Dixon and Anderson (see (5.26) below). We remark too that the distribution (5.7) in the case $\gamma$ real is the classical $t$-distribution.

Integrating (5.9) over $\{x_j\}_{j=0,\dots,n}$ within the region (5.4) we must get unity. Using this allows us to derive for the multi-dimensional integral

$$I_n(\gamma;d)=\frac{1}{n!}\int_{-\infty}^{\infty}dx_1\cdots\int_{-\infty}^{\infty}dx_n\prod_{l=1}^{n}(1+ix_l)^{-\gamma}(1-ix_l)^{-\bar{\gamma}}\prod_{1\le j<k\le n}|x_j-x_k|^{2d} \tag{5.14}$$

a recurrence analogous to that obtained by Anderson [2] for the Selberg integral. Moreover, the intermediate workings will allow us to deduce a random three term recurrence for the characteristic polynomial associated with the p.d.f. (1.5).

**Corollary 1.** *We have*

$$I_{n+1}(\gamma;d)=\pi 2^{2-2\mathrm{Re}\,\gamma}\frac{\Gamma(2\mathrm{Re}\,\gamma-nd-1)\Gamma((n+1)d)}{\Gamma(d)|\Gamma(\gamma)|^2}I_n(\gamma-d;d). \tag{5.15}$$

Proof. Let us denote the region (5.4) by $R'$. As remarked, integrating (5.9) over $\{x_j\}_{j=0,\dots,n}$ within $R'$ must give unity. Setting

$$d_1=\cdots=d_n=d \tag{5.16}$$

this implies

$$\begin{aligned}\tilde{A}_d\int_{R'}dx_0\cdots dx_n\prod_{j=0}^{n}(1+ix_j)^{-\gamma}(1-ix_j)^{-\bar{\gamma}}\prod_{j=1}^{n}\prod_{l=0}^{n}|y_j-x_l|^{d-1}\prod_{0\le j<k\le n}|x_j-x_k|\\ =\prod_{j=1}^{n}(1+iy_j)^{-\gamma+d}(1-iy_j)^{-\bar{\gamma}+d}\prod_{1\le j<k\le n}|y_j-y_k|^{2d-1}\end{aligned} \tag{5.17}$$

where $\tilde{A}_d := \tilde{A}|_{d_1=\cdots d_n=d}$. Thus

$$\frac{1}{\tilde{A}_d} I_n(\gamma - d; d) = \int_{R'} dx_0 \cdots dx_n dy_1 \cdots dy_n \prod_{j=0}^{n} (1 + i x_j)^{-\gamma} (1 - i x_j)^{-\bar{\gamma}} \times \prod_{j=1}^{n} \prod_{l=0}^{n} |y_j - x_l|^{d-1} \prod_{1 \le j < k \le n} |y_j - y_k| \prod_{0 \le j < k \le n} |x_j - x_k|. \tag{5.18}$$

On the other hand, we know from [5, 2] that

$$\int_{R'} dy_1 \cdots dy_n \prod_{1 \le j < k \le n} |y_j - y_k| \prod_{j=1}^{n} \prod_{l=0}^{n} |y_j - x_l|^{d-1} = \frac{(\Gamma(d))^{n+1}}{\Gamma((n+1)d)} \prod_{0 \le j < k \le n} |x_j - x_k|^{2d-1}, \tag{5.19}$$

so the r.h.s. of (5.18) is also equal to

$$\frac{(\Gamma(d))^{n+1}}{\Gamma((n+1)d)} I_{n+1}(\gamma; d). \tag{5.20}$$

Equating (5.18) and (5.20) gives (5.15). □

Iterating (5.15) with $I_0(\gamma, d) = 1$ reclaims the gamma function evaluation [8]

$$n! I_n(\gamma; d) = 2^{dn(n-1) - 2(\mathrm{Re}\,\gamma - 1)} \pi^n M_n(\bar{\gamma} - d(n-1) - 1, \gamma - d(n-1) - 1, d) \tag{5.21}$$

where $M_N(a, b, \lambda)$ is given by (4.17).

## 5.2 A random three term recurrence

Consider the rational function (5.3). Suppose $\{q_j\}_{j=0,\dots,n}$ have the Dirichlet distribution (5.6) with equal parameters (5.16), and suppose $c$ has the distribution (5.7). Suppose furthermore that $\{y_j\}_{j=1,\dots,n}$ have distribution with measure

$$\frac{1}{I_n(\gamma - d; d)} \prod_{j=1}^{n} (1 + i y_j)^{-\gamma + d} (1 - i y_j)^{-\bar{\gamma} + d} \prod_{1 \le j < k \le n} |y_k - y_j|^{2d}. \tag{5.22}$$

The marginal distribution of $\{x_j\}$ is then given by multiplying this with (5.9) and integrating $\{y_j\}$ over the region $R'$ (5.4). Using (5.19) gives

$$\frac{1}{I_{n+1}(\gamma; d)} \prod_{j=0}^{n} (1 + i x_j)^{-\gamma} (1 - i x_j)^{-\bar{\gamma}} \prod_{0 \le j < k \le n} |x_j - x_k|^{2d}. \tag{5.23}$$

Hence with $p_{n+1}(z; \gamma; d)$ denoting the random monic polynomial of degree $n+1$ with zeros at $\{x_j\}_{j=0,\dots,n}$ having distribution (5.23), we see that (5.3) can be written

$$\frac{p_{n+1}(z; \gamma; d)}{(z^2 + 1) p_n(z; \gamma - d; d)} = \frac{z - c}{q_0 (z^2 + 1)} - \sum_{j=1}^{n} \frac{(q_j / q_0)}{z - y_j}. \tag{5.24}$$

A companion identity to (5.24) is also required. For this purpose we introduce the random rational function

$$\frac{\prod_{k=1}^{n-1} (z - u_k)}{\prod_{j=1}^{n} (z - y_j)} = \sum_{j=1}^{n} \frac{\mu_j}{z - y_j} \tag{5.25}$$

where $\{\mu_j\}$ have Dirichlet distribution

$$\frac{\Gamma(nd)}{(\Gamma(d))^n}\prod_{j=1}^n \mu_j^{d-1}.$$

We know from the work of Dixon [5] and Anderson [2] that the conditional p.d.f. of $\{u_k\}$ given $\{y_j\}$ is equal to

$$\frac{\Gamma(nd)}{(\Gamma(d))^n}\frac{\prod_{1\le j<k\le n-1}(u_j-u_k)}{\prod_{1\le j<k\le n}(y_j-y_k)^{2d-1}}\prod_{j=1}^{n-1}\prod_{k=1}^{n}|u_j-y_k|^{d-1}, \tag{5.26}$$

provided

$$y_1>u_1>\cdots>y_{n-1}>u_{n-1}>y_n. \tag{5.27}$$

It follows from (5.26) that if $\{y_j\}$ have distribution (5.22), then the marginal distribution of $\{u_j\}$ is equal to

$$\frac{\Gamma(nd)}{(\Gamma(d))^n}\frac{1}{I_n(\gamma-d;d)}\prod_{1\le j<k\le n-1}(u_j-u_k)\int_{\tilde R}dy_1\cdots dy_n$$
$$\times\prod_{j=1}^n(1+iy_j)^{-\gamma+d}(1-iy_j)^{-\bar\gamma+d}\prod_{j=1}^{n-1}\prod_{k=1}^{n}|u_j-y_k|^{d-1}$$

where $\tilde R$ is the region (5.27). According to (5.17) this can be evaluated as

$$\frac{1}{I_{n-1}(\gamma-2d,d)}\prod_{j=1}^{n-1}(1+iu_j)^{-\gamma+2d}(1-iu_j)^{-\bar\gamma+2d}\prod_{1\le j<k\le n-1}|u_j-u_k|^{2d}.$$

We therefore conclude that (5.25) can be written

$$\frac{p_{n-1}(z;\gamma-2d;d)}{p_n(z;\gamma-d;d)}=\sum_{j=1}^n\frac{\mu_j}{z-y_j}. \tag{5.28}$$

Comparison of (5.24) and (5.28) implies $\{p_n(z;\gamma+(n-1)d;d)\}$ satisfy a random three term recurrence.

**Theorem 5.** *With $B[\alpha,\beta]$ denoting the classical beta distribution, let*

$$b_n\sim B[2\mathrm{Re}\,\gamma+nd-1,nd]\quad(n\ne0),\qquad b_0=1, \tag{5.29}$$

*and let $c_n$ have the Cauchy distribution*

$$\frac{\Gamma(\gamma+nd)\Gamma(\bar\gamma+nd)}{\pi 2^{2(1-nd-\mathrm{Re}\,\gamma)}\Gamma(2(\mathrm{Re}\,\gamma+nd)-1)}(1+ic)^{-(\gamma+nd)}(1-ic)^{-(\bar\gamma+nd)} \tag{5.30}$$

*(this is (5.7) with $\gamma\mapsto\gamma+nd$). We have that for $n=0,1,\dots,$*

$$p_{n+1}(z;\gamma+nd;d)$$
$$=\frac{(z-c_n)}{b_n}p_n(z;\gamma+(n-1)d;d)+\Big(1-\frac{1}{b_n}\Big)(1+z^2)p_{n-1}(z;\gamma+(n-2)d;d), \tag{5.31}$$

*where $p_0:=1$.*

Proof. In (5.24) and (5.28) replace $\gamma\mapsto\gamma+nd$. We know that in general if $\{d_j\}_{j=0,\dots,n}$ have Dirichlet distribution (5.6), then each $d_j$ has beta distribution $\mathrm{B}[d_j,\sum_{l=0,l\ne j}^n d_j]$. Using this fact it follows that in (5.24) we now have

$$q_j\sim\mathrm{B}[d,(n-1)d]\ \ (j\ne0),\qquad q_0\sim\mathrm{B}[2\mathrm{Re}\,\gamma+nd-1,nd],$$

where in deriving the former use has also been made of (5.8), while in (5.28)

$$\mu_j \sim B[d, (n-1)d].$$

The quantities are constrained by $\sum_{j=0}^n q_j = 1$, $\sum_{j=1}^n \mu_j = 1$. Substituting (5.31) in (5.24) we thus see that (5.28) results, thereby verifying the correctness of (5.31). □

To relate this to the circular Jacobi $\beta$-ensemble (1.5), we note that with

$$x_j = i\frac{1-e^{i\theta_j}}{1+e^{i\theta_j}} \ (j \neq 0), \qquad x_0 = 0,$$

the p.d.f. (5.23) with $\gamma \mapsto \gamma + 2d$ ($\gamma$ real) extended to a measure via the multiplication by $dx_1 \cdots dx_n$, becomes equal to (1.5) with $a = 2\gamma - 2$ and extended to a measure via the multiplication by $d\theta_1 \cdots d\theta_n$. Thus the zeros of the polynomial $p_n(z; \gamma + (n-1)d; d)$, with $\gamma$ real, $x_1, \dots, x_n$ say, under the mapping

$$\frac{x_j - i}{x_j + i} = e^{i\theta_j} \quad (j = 1, \dots, n) \tag{5.32}$$

give for $\{\theta_j\}$ the distribution (1.5) with $a = 2\gamma - 2$.

As an illustration, let us consider the case $\gamma = 1$, $d = 1$, which relates to averaging over $U(N)$. There are a number of averages over $U(N)$ which are known analytically. For example, with $p \in \mathbb{Z}_{>0}$, [7]

$$\langle |\mathrm{Tr}\, U^p|^2 \rangle_{U \in U(N)} = \begin{cases} p, & 0 < p \le N \\ N, & p \ge N \end{cases}$$

Since

$$\langle |\mathrm{Tr}\, U^p|^2 \rangle_{U \in U(N)} = \Big\langle \Big| \sum_{j=1}^N e^{ip\theta_j} \Big|^2 \Big\rangle_{U(N)}$$

we can compute the Monte Carlo approximation

$$\langle |\mathrm{Tr}\, U^p|^2 \rangle_{U \in U(N)} = \frac{1}{M} \sum_{k=1}^M \Big| \sum_{j=1}^N \Big( \frac{x_j^{(k)} - i}{x_j^{(k)} + i} \Big)^p \Big|^2 + O\Big( \frac{1}{\sqrt{M}} \Big) \tag{5.33}$$

where use has been made of (5.32) and $x_j^{(k)}$ refers to the $j$th generation of $p_N(z; N; 1)$ from (5.31).

For the latter task, we read off from (5.29) that

$$b_n \sim \mathrm{B}[1+n, n].$$

Also, by definition the Student $t$-distribution $T_\nu$ say, has p.d.f. proportional to $(1 + t^2/\nu)^{-(\nu+1)/2}$ so

$$c_n \sim \frac{1}{\sqrt{\nu}} T_\nu \Big|_{\nu = 2n+1}.$$

Significantly, the zeros of the lower order polynomials in the sequence $\{p_j(z; j; 1)\}_{j=0,1,\dots,N}$ themselves allow us, via (5.32), to sample from $U(j)$. Hence (5.33) can be calculated for all values of $N$ less than the sought value within the same calculation. Monte Carlo results obtained this way are presented in Table 5.2. The consistency of these results is evident.

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| $p\backslash N$ | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| 1 | 0.98 | 0.99 | 0.99 | 0.98 |
| 2 | 2.02 | 2.00 | 2.05 | 1.99 |
| 3 | 2.00 | 3.00 | 2.95 | 3.00 |
| 4 | 1.96 | 3.00 | 3.97 | 4.01 |
| 5 | 2.03 | 2.98 | 4.00 | 5.05 |

Table 1: Computation of (5.33) with $M = 5,000$ and $p$ and $N$ as indicated


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