replaces California Institute of Technology Social Science Working Paper 530

Soc Choice Welfare (1986) 3:199–211

Social Choice and Welfare
© Springer-Verlag 1986

Sequential Elections with Limited Information

A Formal Analysis

R. D. McKelvey[1] and P. C. Ordeshook[2]

[1] California Institute of Technology, Pasadena, CA 91125, USA
[2] University of Texas at Austin, Austin, TX 78712, USA

Received February 15, 1985 / Accepted June 9, 1986

Abstract. We develop an incomplete information model of a sequence of elections in a one-dimensional policy space, where voters have no contemporaneous information about candidate positions, and candidates have no information about voter preferences. The only source of information is contemporaneous endorsement data and historical data on the policy positions of previous winning candidates. We define a notion of "stationary rational expectations equilibrium", and show that such an equilibrium results in outcomes which are equivalent to those that would occur under full information.

1. Introduction

In this paper, we develop a model of a sequence of elections in a one-dimensional issue space, where voters have no contemporaneous information about candidate positions, and candidates have no information about voter preferences. The only source of information is contemporaneous endorsement data, and historical data on the policy positions of previous winning candidates.

We define a notion of "stationary rational expectations equilibrium" (SREE), to describe equilibrium behavior in this model. In an SREE, the candidates are assumed to adopt a stationary strategy – i.e., each period's strategy is an independent draw from the same distribution. Voters have beliefs of the candidates' positions which must be "rational". I.e., since the candidates' strategies are stationary, the distribution that voters believe a candidate's position will follow must agree with the observed historical distribution of his previous winning positions. Each voter follows a voting strategy that maximizes his expected utility, given his belief of the candidate positions and the observed endorsement (which tells him the left-right orientation of the two candidates). Candidates follow a strategy to maximize their utility, given the assumed stationarity of voter behavior.

* We acknowledge support from NSF Grants #SES 82-08184 and #SES 84-09654.

Specifically, they are assumed to adopt strategies which match the distribution of previous winning strategies. The unique stable SREE for the corresponding $n+2$ person election game is shown to be the same as the perfect information equilibrium, in which both candidates adopt the median voter's ideal point with probability one.

In earlier papers, we develop an alternative model of two-candidate, majority rule spatial elections, in which candidates have no information about voter preferences, and some voters have no contemporaneous information about candidate positions (McKelvey and Ordeshook 1984a, 1985a). In that model, we show that access to poll data is sufficient to yield a full information rational expectations equilibrium. This paper differs from our earlier work in that it concerns a sequence of elections rather than a single election, and the source of information is historical data rather than poll data. Further, while our earlier work assumes that there are some informed and some uninformed voters, here we assume that all voters are uninformed. Nevertheless, the results of both approaches are similar: equilibrium behavior in both models is similar to the behavior that would result were all voters informed.

This paper is a companion paper to McKelvey and Ordeshook (1985b), providing formal statement and proofs for the results reported in that paper. Further motivation and justifications of the assumptions, as well as experimental data relating to the model can be found in the companion paper.

2. The Formal Model

We assume that there are two *candidates*, designated by $K=\{1,2\}$, *n voters* (with n odd), designated by $N=\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$, and a one-dimensional convex *policy space*, $X\subseteq\mathbb{R}$. Each voter, $\alpha_i\in N$, has a *utility function* $u_i:X\to\mathbb{R}$, which is symmetric and single peaked about an *ideal point* y_i^*. We let K_0 be the three element set consisting of the two candidates, plus a third element, "0", which is used to represent "no endorsement" or "abstention". Thus, $K_0=\{1,2,0\}=K\cup\{0\}$. Let F be the set of functions from K_0 into K_0. (Elements of F will be used to represent voter strategies.) For any $k\in K$, we use $\bar{k}$ to denote the opposition candidate. I.e., $\bar{k}\in K-\{k\}$. We let $\mathscr{B}$ be the set of Borel measurable subsets of X.

We now define a sequence of identical games, where in each game, the players consist of the candidates, K, together with the voters, N. In each game, the strategy space for candidate k and voter α_i are denoted S_k and B_i, respectively. We assume $S_k=X$ and $B_i=F$ for all $k\in K$ and $\alpha_i\in N$. So candidate strategies are positions on the issue, while voter strategies are decisions as to who to vote for as a function of which candidate is endorsed (see below). We write $S=S_1\times S_2$, and $B=B_1\times\ldots\times B_n$, and we denote specific strategy choices of the candidates and voters, respectively, by $s=(s_1,s_2)\in S$ and $b=(b_1,\ldots,b_n)\in B$. For any s, $s'\in S$ and b, $b'\in B$, let $(s|s_k';b)$ denote the strategy $n+2$ tuple obtained by replacing the k^{th} candidate's strategy in s by s_k', and $(s,b|b_i')$ denote the strategy $n+2$ tuple resulting from replacing the i^{th} voter's strategy, b_i, by b_i'. Given a strategy $n+2$ tuple $(s,b)\in S\times B$, we now define the payoff function for the game.

First, we define the *endorsement* to be a function, $e: S \to K$, of the candidates' positions on the issue. Specifically,

$$e(s)=\begin{cases} 1 & \text{if} \quad s_1 < s_2 \\ 2 & \text{if} \quad s_2 < s_1 \\ 0 & \textit{otherwise} \ . \end{cases} \tag{1}$$

Thus $e(s)$ tells the voters which candidate is to the left and which is to the right on the election issue. For any specific vector of votes by the n voters, $p \in K_0^n$, and for $k \in K_0$, we write

$$\mathbf{v}_k(p) = |\{\alpha_i \in N | p_i = k\}| \ , \tag{2}$$

and

$$\mathbf{w}(p)=\begin{cases} 1 & \text{if} \quad \mathbf{v}_1(p) > \mathbf{v}_2(p) \\ 2 & \text{if} \quad \mathbf{v}_2(p) > \mathbf{v}_1(p) \\ 0 & \textit{otherwise} \ . \end{cases} \tag{3}$$

(Here we use the notation $|A|$ to represent the number of elements of a set A.) Then, for any $(s, b) \in S \times B$, the *vote for candidate k* is given by

$$v_k(s,b) = \mathbf{v}_k(b(e(s))) \ . \tag{4}$$

The *winning candidate*, or *election outcome* is

$$w(s,b) = \mathbf{w}(b(e(s))) \ , \tag{5}$$

where $w(s,b) = 0$ means a fair coin is tossed to determine if 1 or 2 wins. We can then define the payoff to voter $\alpha_i \in N$ by

$$M_i(s,b) = u_i(s_{w(s,b)}) \ , \tag{6}$$

where s_0 represents the outcome resulting from a tie – namely a fair lottery between s_1 and s_2 – and we assume that the utility a voter associates with a tie is simply $u_i(s_0) = \frac{1}{2}u_i(s_1) + \frac{1}{2}u_i(s_2)$. Finally, the payoff to candidate $k \in K$ is

$$M_k(s,b)=\begin{cases} 1 & \text{if} \quad w(s,b) = k \\ -1 & \text{if} \quad w(s,b) = \bar{k} \\ 0 & \textit{otherwise} \ . \end{cases} \tag{7}$$

These definitions specify the normal form for one stage of the game. This normal form corresponds to an extensive form in which the candidates begin with a move in which they simultaneously choose policy positions. The voters are informed of the endorsement (which tells them which candidate is to the left and which to the right), and they then have a simultaneous move in which they vote for the candidate of their choice. While they know the endorsement before they vote, and can condition their strategy on this information, note that they do *not* know the candidate positions.

The game we analyze is an infinitely repeated version of the above single stage game. We assume that the candidates do not know the voter characteristics – i.e., they do not know the voter ideal points, y_i^*. However, all players observe the outcome $w(s,b)$ and the position $s_{w(s,b)}$, of winning candidates in previous plays of the game.

We now define an equilibrium for this repeated, incomplete information game. But first we need some further notation. Let Λ_k be a set of probability measures on S_k. So Λ_k represents the set of admissible mixed strategies of candidate k for a single stage of the game. We will only consider stationary strategies for candidates in the repeated game, i.e., we consider only strategies in which candidates adopt the same mixed strategy in each stage of the game. We assume throughout that elements of Λ_k are either absolutely continuous (with respect to Lebesque measure on X), or degenerate point masses (i.e., $\lambda_k(\{x\})=1$ for some $x \in X$). We let $\Lambda = \Lambda_1 \times \Lambda_2$. So for $\lambda = (\lambda_1, \lambda_2) \in \Lambda$, λ represents the product measure of λ_1 and λ_2.

Given $\lambda = (\lambda_1, \lambda_2) \in \Lambda$, we define several measures on $\mathscr{B}$. For $k \in K$, $j \in K_0$, and $b \in B$, define

$\lambda_{kj}(\cdot)$: The distribution of s_k given that j is endorsed, (i.e. given $e(s)=j$) , (8)

$\lambda_{kw}(\cdot|b)$: The distribution of s_k given that k wins (if k never wins, set $\lambda_{kw}(\cdot|b) = \lambda_{\bar{k}}(\cdot)$) . (9)

$\lambda_w(\cdot|b)$: The distribution of winning positions, i.e., of $s_{w(s,b)}$. (10)

Formally, for any $A \subseteq S$, and $k \in K$, we let $\pi_k(A) = \{x \in S_k | x = s_k \text{ for some } s \in S\}$ be the projection of A on coordinate k. For $k \in K_0$, we let $E_k = \{s \in S | e(s) = k\}$ be the set of strategy pairs where candidate k is endorsed, and for $b \in B$, $W_k(b) = \{s \in S | w(s,b) = k\}$ be the set of strategies where candidate k wins. Given $\lambda = (\lambda_1, \lambda_2) \in \Lambda$, we define some derived probability measures. For each $k \in K$ and $j \in K_0$, define the probability measures λ_{kj} as follows: For $C \in \mathscr{B}$,

$$\lambda_{kj}(C) = \lambda(\pi_k^{-1}(C)|E_j) = \frac{\lambda(\pi_k^{-1}(C) \cap E_j)}{\lambda(E_j)} \ . \tag{11}$$

So λ_{kj} represents the distribution of k's positions when he is endorsed. We also define, for any $b \in B$, the measure $\lambda_{kw}(\cdot|\mathrm{b})$ by, for all $C \in \mathscr{B}$,

$$\lambda_{kw}(C|b) = \begin{cases} \dfrac{\lambda(\pi_k^{-1}(C) \cap W_k(b)) + \frac{1}{2}\lambda(\pi_k^{-1}(C) \cap W_0(b))}{\lambda(W_k(b)) + \frac{1}{2}\lambda(W_0(b))} & \text{if} \quad \lambda(W_{\bar{k}}(b)) \neq 1 \\ \lambda_{\bar{k}}(C) & \textit{otherwise} \ . \end{cases} \tag{12}$$

So λ_{kw} represents the distribution of k's positions when he *wins* the election – except if k never wins, then λ_{kw} is defined as the distribution of k's winning positions. Note that in defining $\lambda_{kw}(\cdot|b)$, we must take account of the possibility that k wins outright as well as the probability that k wins the coin toss when there is a tie. Given $\lambda \in \Lambda$, and $b \in B$, we let $\lambda_w(\cdot|b)$ denote the overall distribution of winning positions. So for $C \in \mathscr{B}$,

$$\begin{aligned} \lambda_w(C|b) &= \sum_{k \in K} \lambda_{kw}(C|b)[\lambda(W_k(b)) + \tfrac{1}{2}\lambda(W_0(b))] \\ &= \sum_{k \in K} [\lambda(\pi_k^{-1}(C) \cap W_k(b)) + \tfrac{1}{2}\lambda(\pi_k^{-1}(C) \cap W_0(b))] \ . \end{aligned} \tag{13}$$

Finally, for any $\gamma \in \Lambda$, we define $B(\gamma) \subseteq B$ by

$$B(\gamma) = \{b^* \in B \mid \forall b \in B,\ b_i^* \in \arg\max_{b_i' \in B_i} \mathbf{E}_\gamma [M_i(s, b|b_i')]\} \ . \tag{14}$$

Here $\mathbf{E}_\gamma$ is the expectation under γ. Hence $B(\gamma)$ is the set of optimal voting strategies by the voters given beliefs $\gamma = (\gamma_1, \gamma_2)$ of the candidate strategies.

Definition: A *Stationary Rational Expectations Equilibrium* (SREE) for the game defined by (6)–(7) is a triple $(\lambda^*, \gamma^*, b^*) \in \Lambda \times \Lambda \times B$ satisfying

(V1) $b^* \in B(\gamma^*)$,

(V2) for all $k \in K$, $\gamma_k^* = \lambda_{kw}^*(\cdot|b^*)$,

(C1) for all $k \in K$, $\lambda_k^* = \lambda_{kw}^*(\cdot|b^*)$.

In this definition, (V1) requires that each voter adopt a voting strategy, b_i^*, to maximize his expected utility conditional on his belief of the candidate policy positions. Condition (V2) requires that voters have "rational expectations", i.e., their beliefs about the candidates must be consistent with the observed historical positions of that candidate, and (C1) requires that the candidate distribution must equal the distribution of their past winning positions. The rationale here is that if this condition fails, then some of the pure strategies adopted under λ_k are more successful than others, so the candidate strategy cannot be optimal.

3. Preliminary Lemmas

We establish three lemmas that yield our central results. For the first lemma, we need to define the notion of stochastic dominance. For any $c \in \mathbb{R}$, we define $L_c = \{t \in \mathbb{R} | t \leqq c\}$. Given two measures λ, μ on the Borel sets of $\mathbb{R}$, we say that λ *(weakly) stochastically dominates* μ, written $\mu \leqq \lambda$ iff $\mu(L_c) \geqq \lambda(L_c)$ for all $c \in \mathbb{R}$. We say λ *(strongly) stochastically dominates* μ, written $\mu < \lambda$ iff $\mu \leqq \lambda$ and it is not the case that $\lambda \leqq \mu$. Thus, λ weakly stochastically dominates μ whenever its cumulative density function is always less than or equal to that of μ. For strong domination, the two c.d.f.'s cannot be equal.

Lemma 1: *For each $k \in K$, $\lambda_{kk} < \lambda_k < \lambda_{k\bar{k}}$ whenever all measures are defined, unless $\lambda_k(\{t^*\}) = 1$ for some $t^* \in \mathbb{R}$. Further, for each $k \in K$, $\lambda_{kk} < \lambda_{\bar{k}k}$ whenever both measures are defined.*

Proof: For $k \in K$, and $t \in \mathbb{R}$ let $F_k(t) = \lambda_k(L_t)$ be the cumulative density function of λ_k. Then from the definition of λ_{kj}, for any $C \in \mathscr{B}$ we have

$$\lambda_{k\bar{k}}(C) = \frac{1}{\lambda(E_{\bar{k}})} \int_C F_{\bar{k}}(t) d\lambda_k(t) \ ,$$

and

$$\lambda_{kk}(C) = \frac{1}{\lambda(E_k)} \int_C (1 - F_{\bar{k}}(t)) d\lambda_k(t) \ .$$

Now, since $F_{\bar{k}}(t)$ is a monotonic increasing function of t and $(1-F_{\bar{k}}(t))$ is a monotonic decreasing function of t, the result that $\lambda_{kk}<\lambda_k<\lambda_{k\bar{k}}$ follows directly from Lemma 3.2 of McKelvey and Page (1984).

To see that $\lambda_{kk}<\lambda_{\bar{k}k}$, we note that, for $c\in\mathbb{R}$,

$$\lambda_{\bar{k}k}(L_c)=\frac{1}{\lambda(E_k)}\int_{-\infty}^{c} F_k(t)d\lambda_{\bar{k}}(t) \ ,$$

and

$$\lambda_{kk}(L_c)=\frac{1}{\lambda(E_k)}\int_{-\infty}^{c} (1-F_{\bar{k}}(t))d\lambda_k(t) \ .$$

But, writing $f_j(t)$ for the density function of λ_j, we can integrate by parts to obtain

$$\begin{aligned}\int_{-\infty}^{c} (1-F_{\bar{k}}(t))d\lambda_k(t) &= F_k(c)-\int_{-\infty}^{c} F_{\bar{k}}(t)f_k(t)dt \\ &= F_k(c)[1-F_{\bar{k}}(c)]+\int_{-\infty}^{c} f_{\bar{k}}(t)F_k(t)dt \\ &\geqq \int_{-\infty}^{c} F_k(t)d\lambda_{\bar{k}}(t) \ .\end{aligned}$$

Hence

$$\begin{aligned}\lambda_{kk}(L_c)&=\frac{1}{\lambda(E_k)}\int_{-\infty}^{c} (1-F_{\bar{k}}(t))d\lambda_k(t) \\ &\geqq\frac{1}{\lambda(E_k)}\int_{-\infty}^{c} F_k(t)d\lambda_{\bar{k}}(t)=\lambda_{\bar{k}k}(L_c) \ ,\end{aligned}$$

which proves $\lambda_{kk}<\lambda_{\bar{k}k}$.

Q.E.D.

Lemma 2: *If* $(\lambda^*, b^*)\in\Lambda\times B$ *characterizes an* SREE, *then it satisfies the following conditions*

(*a*) *for all* $\alpha_i\in N$, *all* $j\in K$ *and all* $k\in K_0$ *with* $\lambda^*(E_k)>0$, b_i^* *satisfies*

$$\mathbf{E}_{\lambda_{jk}^*}[u_i(x)]=\mathbf{E}_{\lambda_{jk}^*}[u_i(x)]\Rightarrow b_i^*(k)=j \ ,$$

(*b*) $w(s,b^*)$ *is constant for* λ^* a.e. $s\in S$,

(*c*) *for each* $k\in K_0$, *if* $\lambda^*(E_k)>0$, *then* $\exists\alpha_i\in N$ *for which*

$$\mathbf{E}_{\lambda_{1k}^*}[u_i(x)]=\mathbf{E}_{\lambda_{2k}^*}[u_i(x)] \ .$$

Proof:

(*a*) Note that $w(s,b)$ depends on s only through the dependence of b on s. Hence we write $w(s,b)=\mathbf{w}(b(e(s)))$. Then, by (V1) in the definition of an SREE,

$$b_i^*\in\arg\max_{b_i^*\in F}\mathbf{E}_{\lambda^*}[M_i(s,b)|b_i^*)] \quad \text{for all} \quad b\in B \ ,$$

But

$$\mathbf{E}_{\lambda^*}[M_i(s, b|b_i^*)] = \mathbf{E}_{\lambda^*}[u_i(s_{\mathbf{w}[(b|b_i^*)(e(s))]})] = \sum_{k\in K_0} \int_{E_k} u_i(s_{\mathbf{w}[(b|b_i^*)(k)]}d\lambda^*(s) \ . \tag{15}$$

So we maximize (15) by picking b_i^* so that for each $k\in K_0$, with $\lambda^*(E_k)>0$, then

$$b_i^*(k) \in \arg \max_{b_i^*(k)\in K_0} \int_{E_k} u_i(s_{\mathbf{w}[(b|b_i^*)(k)]}d\lambda^*(s) \ . \tag{16}$$

But since $\mathbf{w}$ is monotonic, and (16) must hold for all $b\in B$, this is equivalent to, for all $k\in K_0$, with $\lambda^*(E_k)>0$

$$b_i^*(k) \in \arg \max_{b_i^*(k)\in K_0} \int_{E_k} u_i(s_{b_i^*(k)})d\lambda^*(s) \ ,$$

or

$$b_i^*(k) \in \arg\max_{j\in K_0} \int_{E_k} u_i(s_j)d\lambda^*(s)$$

But for $j\in K$,

$$\int_{E_k} u_i(s_j)d\lambda^*(s) = \int u_i(x)d\lambda_{jk}^*(x) = \mathbf{E}_{\lambda_{jk}^*}[u_i(x)]$$

So the result follows.

(*b*) We write $w(s,b^*)=\mathbf{w}(b^*(e(s))$, and for any $k\in K_0$, write $w_k=\mathbf{w}(b^*(k))$. Also we write $E_k=\{s|e(s)=k\}$ for $k\in K_0$. Thus, $s\in E_k \Rightarrow w(s,b^*)=w_k$. If $\lambda^*(E_k)=1$ for any $k\in K_0$, then the result follows trivially, so we assume that $\lambda^*(E_k)<1$. But by the assumptions on Λ, $\lambda^*(E_0)=0$ or $\lambda^*(E_0)=1$, hence we have $\lambda^*(E_0)=0$ and $0<\lambda^*(E_k)<1$ for all $k\in K$. We now have two cases:

Case 1: $w_k \neq 0$ for any $k\in K$
Let $k\in K$, and let $w_k=j$. We then show that $w_{\bar{k}}=j$. Suppose not. Then $w_k=j$ and $w_{\bar{k}}=\bar{j}$. Now from (C1), we have, for all $C\in\mathscr{B}$,

$$\lambda_j^*(C)=\frac{\lambda^*(\pi_j^{-1}(C)\cap E_k)}{\lambda^*(E_k)}=\lambda_{jk}^*(C|b^*)$$

and

$$\lambda_{\bar{j}}^*(C)=\frac{\lambda^*(\pi_{\bar{j}}^{-1}(C)\cap E_{\bar{k}})}{\lambda^*(E_{\bar{k}})}=\lambda_{\bar{j}\bar{k}}^*(C|b^*)$$

By lemma 1, it follows that λ_j^* and $\lambda_{\bar{j}}^*$are both degenerate point densities, implying that $\lambda^*(\{s\})=1$ for some $s\in S$. But this contradicts the fact that $\lambda^*(E_0)=0$ and $\lambda^*(E_k)>0$ for all $k\in K$. Hence we must have $w_k=w_{\bar{k}}$. So, since $\lambda^*(E_k\cup E_{\bar{k}})=1-\lambda^*(E_0)=1$, it follows that $w(s,b^*)=w_k$ for λ^* a.e. $s\in S$.

Case 2: $w_k=0$ for some $k\in K$.
Let $w_k=0$ and let $\bar{k}\in K-\{k\}$. We will show that $w_{\bar{k}}=0$. Assume $w_{\bar{k}}\neq 0$, say $w_{\bar{k}}=j\in K$, and let $\bar{j}\in K-\{j\}$. Then from (C1), we have, for all $C\in\mathscr{B}$,

$$\lambda_j^*(C)=\frac{\lambda^*(\pi_j^{-1}(C)\cap E_{\bar{k}})+\frac{1}{2}\lambda^*(\pi_j^{-1}(C)\cap E_k)}{\lambda^*(E_{\bar{k}})+\frac{1}{2}\lambda^*(E_k)}$$

$$=\frac{\lambda^*(\pi_j^{-1}(C))-\frac{1}{2}\lambda^*(\pi_j^{-1}(C)\cap E_k)}{1-\frac{1}{2}\lambda^*(E_k)}$$

$$=\frac{2\lambda_j^*(C)-\lambda^*(E)\lambda_{jk}^*(C)}{2-\lambda^*(E_k)} .$$

Solving this for $\lambda_j^*(C)$ yields

$$\lambda_j^*(C)=\lambda_{jk}^*(C) .$$

Also applying (C1) we have, for all $C\in\mathscr{B}$,

$$\lambda_{\bar{j}}^*(C)=\frac{\lambda^*(\pi_{\bar{j}}^{-1}(C)\cap E_k)}{\lambda^*(E_k)}=\lambda_{\bar{j}k}^*(C) .$$

By Lemma 1, it follows that λ_j^* and $\lambda_{\bar{j}}^*$ are both degenerate point densities. As in Case 1, this yields a contradiction, unless $w_{\bar{k}}=0$. But then $w(s,b^*)=0$ for λ^* a.e. $s\in S$.

(*c*) If $\lambda^*(E_0)=1$, then the result is trivial, so we assume $\lambda^*(E_0)=0$. There are two cases.

Case 1: $w(s,b^*)=k\neq 0$ for λ^* a.e. $s\in S$. In this case, it follows from (C1) and (12) that $\lambda_1^*=\lambda_2^*$. Hence $\lambda_{11}^*=\lambda_{22}^*$ and $\lambda_{12}^*=\lambda_{21}^*$. From Lemma 2a it follows that if $b_i^*(1)\neq 0$, then $b_i^*(1)=j\Rightarrow b_i^*(2)=\bar{j}$. So $\mathbf{w}(b^*(1))=k\Rightarrow \mathbf{w}(b^*(2))=\bar{k}$, a contradiction unless $b_i^*(1)=0$ for some $\alpha_i\in N$. But then, for this i, by Lemma 2a, we have that

$$\mathbf{E}_{\lambda_{11}^*}[u_i(x)]=\mathbf{E}_{\lambda_{21}^*}[u_i(x)] .$$

A similar argument shows that $b_i^*(2)=0$ for some $\alpha_i\in N$, from which it follows that $\mathbf{E}_{\lambda_{12}^*}[u_i(x)]=\mathbf{E}_{\lambda_{22}^*}[u_i(x)]$ for some $\alpha_i\in N$.

Case 2: $w(s,b^*)=0$ for λ^* a.e. $s\in S$. In this case let $k\in K$ satisfy $\lambda^*(E_k)>0$, and assume $b_i^*(k)\neq 0$ for all $\alpha_i\in N$. Then since there are an odd number of voters, we must have $v_1(s,b^*)\neq v_2(s,b^*)$ for any $s\in E_k$. But then $w(s,b^*)\neq 0$ for $s\in E_k$, a contradiction. So we must have $b_i^*(k)=0$ for some $\alpha_i\in N$. But this implies, by Lemma 2a, that $\mathbf{E}_{\lambda_{1k}^*}[u_i(x)]=\mathbf{E}_{\lambda_{2k}^*}[u_i(x)]$. Q.E.D.

We let $\Lambda^S\subseteq\Lambda$ be the set of measures satisfying, for all $\lambda=(\lambda_1,\lambda_2)\in\Lambda$,

(*a*) $\lambda_1=\lambda_2$

(*b*) λ_k is symmetric about t^* for some $t^*\in\mathbb{R}$. I.e., $\lambda_k(L_t)=1-\lambda_k(L_{2t^*-t})$ for all $t\in\mathbb{R}$, (where $L_c=\{x\in\mathbb{R}|x\leq c\}$ for any $c\in\mathbb{R}$).

Lemma 3: *Let $u:X\rightarrow\mathbb{R}$ be symmetric and single peaked, with ideal point at* 0, *and let $\lambda=(\lambda_1,\lambda_2)\in\Lambda^S$, with λ not degenerate. For $k\in K$, define $\phi_k:X\rightarrow\mathbb{R}$ by*

$$\phi_k(y)=\mathbf{E}_{\lambda_{\bar{k}k}}[u(x-y)]-\mathbf{E}_{\lambda_{kk}}[u(x-y)] .$$

Then $\phi_k(y)$ *has exactly one root at* $y=x^*$, *where* $x^*=\mathbf{E}_{\lambda_1}(x)=\mathbf{E}_{\lambda_2}(x)$. *Further* $y<x^* \Rightarrow \phi_k(y)>0$ *and* $y>x^* \Rightarrow \phi_k(y)<0$.

Proof: We let f_1 and f_2 be the density functions of λ_1 and λ_2, respectively. Since $\lambda_1=\lambda_2$, these density functions are identical, so we can write $f=f_1=f_2$. We let g_1 and g_2 be the density functions for $\lambda_{\bar{k}k}$ and λ_{kk}. (Note that since $\lambda_1=\lambda_2$, we have $\lambda_{11}=\lambda_{22}$ and $\lambda_{12}=\lambda_{21}$.) Let $F(x)=\int_{-\infty}^{x} f(t)dt$ and $G_j(x)=\int_{-\infty}^{x} g_j(t)dt$ be the corresponding cumulative density functions. Then, writing u' for the first derivative of u,

$$\begin{aligned}\phi_k(y)&=\mathbf{E}_{\lambda_{\bar{k}k}}[u(x-y)]-\mathbf{E}_{\lambda_{kk}}[u(x-y)]\\&=\int u(x-y)[g_1(x)-g_2(x)]dx\\&=\int u'(x-y)[G_2(x)-G_1(x)]dx\ .\end{aligned}$$

(The last step follows from integration by parts.) We write $\psi(x)=G_2(x)-G_1(x)$. Then we can write G_2 and G_1 as

$$G_2(y)=\int_{-\infty}^{x}[1-F_1(t)]f_2(t)dt$$

$$G_1(y)=\int_{-\infty}^{x}F_2(t)f_1(t)dt\ ,$$

and then, using the fact that $F_1=F_2=F$, and $f_1=f_2=f$, we get

$$\psi(x)=G_2(x)-G_1(x)=\int_{-\infty}^{x}[1-2F(t)]f(t)dt\ .$$

Now from Lemma 2, it follows that $\lambda_{\bar{k}k}>\lambda_{kk}$, so $G_2(x)-G_1(x)\geqq 0$ for all x. Hence $\psi(x)$ is nonnegative. Next, using the symmetry of $f(t)$ about x^*, and the fact that $F(x^*)=\frac{1}{2}$, it follows easily that $\psi(x)$ is symmetric about x^*. I.e., $\psi(x)=\psi(2x^*-x)$ for all $x\in X$. Finally, since $\psi'(x)=[1-2F(x)]f(x)$ is positive for $x<x^*$ and negative for $x^*<x$, it follows that $\psi(x)$ is single peaked. Thus, we write

$$\phi_k(y)=\int u'(x-y)\psi(x)dx\ , \tag{17}$$

where ψ is nonnegative, symmetric and single peaked about x^*, and where $u'(t)=-u'(-t)$ for all t. But now we can rewrite (17) as

$$\begin{aligned}\phi_k(y)&=\int_{-\infty}^{y}u'(t-y)\psi(t)dt-\int_{y}^{\infty}u'(t-y)\psi(t)dt\\&=-\int_{y}^{\infty}u'(y-r)\psi(2y-r)dr-\int_{y}^{\infty}u'(t-y)\psi(t)dt\ .\\&=\int_{y}^{\infty}u'(t-y)[\psi(2y-t)-\psi(t)]dt\ .\end{aligned}$$

So if $x^*\leqq y$, then for $y\leqq t$, we have $2x^*-t\leqq 2y-t\leqq t$, so using the symmetry and single peakedness of ψ,

$$\psi(2y-t)\geqq\psi(2x^*-t)=\psi(t)\ .$$

And if $y \leqq x^*$, then for $y \leqq t$, we have $2y - t \leqq t \leqq 2x^* - (2y - t)$. So

$$\psi(t) \geqq \psi(2x^* - 2y + t) = \psi(2y - t) \ .$$

In both cases, these become strict inequalities if $y \neq x^*$, and are equalities when $y = x^*$, hence, since $u'(t - y) < 0$ for $y \leqq t$, we get

$$y < x^* \Rightarrow \phi_k(y) > 0 \ ,$$

$$y = x^* \Rightarrow \phi_k(y) = 0 \ ,$$

$$y > x^* \Rightarrow \phi_k(y) < 0 \ .$$

Hence, ϕ_k has a unique root at $y = x^*$, as we wished to show. Q.E.D.

4. Results

Several results follow from the lemmas of the previous sections. The first proposition shows that in an SREE, voter beliefs must correspond to the actual distribution adopted by the candidates. So $\lambda^* = \gamma^*$. Further, the equilibrium voting strategy for the voter is one which maximizes expected utility with respect to λ^*, *conditional* on the endorsement.

Proposition 1: *An* SREE $(\lambda^*, \gamma^*, b^*)$ *can be characterized by a pair* $(\lambda^*, b^*) \in \Lambda \times B$, *where* $\gamma^* = \lambda^*$, *and* (λ^*, b^*) *satisfies:*

(V1′) $b^* \in B(\lambda^*)$,

(C1′) for all $k \in K$, $\lambda_k^* = \lambda_{kw}^*(\cdot | b^*)$.

Further, for all $\alpha_i \in N$ *and all* $j \in K$, b_i^* *satisfies*

$$b_i^*(k) = j \quad \text{if} \quad \mathbf{E}_{\lambda_{jk}^*}[u_i(x)] > \mathbf{E}_{\lambda_{\bar{j}k}^*}[u_i(x)]$$

for all $k \in K$ *with* $\lambda^*(\{s \in S | e(s) = k\}) \neq 0$.

Proof: That $\gamma^* = \lambda^*$ follows directly from (V2) and (C1) of the definition of an SREE. Then (V1′) and (C1′) are immediate consequences of the fact that $\gamma^* = \lambda^*$. The last assertion of the proposition follows directly from Lemma 2a. Q.E.D.

Assume now that $\lambda_k \in \Lambda_k$ is symmetric about $x^* \in \mathbb{R}$, i.e., its cumulative density $F_k : \mathbb{R} \to [0, 1]$ satisfies $F_k(x^* + t) + F_k(x^* - t) = 1 \ \forall t \in \mathbb{R}$. Let $\Lambda^S \subseteq \Lambda$ be defined, as in the previous section, to be the set of distributions such that if $\lambda = (\lambda_1, \lambda_2) \in \Lambda^S$, then $\lambda_1 = \lambda_2$ and λ_k is symmetric around x^* for some $x^* \in X$.

The following theorem shows that if candidate strategies are symmetric (an assumption required only in Theorem 1, not in Theorem 2), then, either candidate strategies must be identical pure strategies, or the expected policy position must equal the ideal point of the median voter.

Theorem 1: *If n is odd, and* $(\lambda^*, b^*) \in \Lambda \times B$ *characterizes an* SREE, *with* $\lambda^* \in \Lambda^S$, *then either*

$(a)\ \lambda_1^*(\{x\}) = \lambda_2^*(\{x\}) = 1 \quad \textit{for some} \quad x \in X$ or

$(b)\ \mathbf{E}_{\lambda_1^*}(x) = \mathbf{E}_{\lambda_2^*}(x) = y^*$,

where y^* *is the median of the ideal points,* $\{y_i^* | \alpha_i \in N\}$.

Proof: We show that not (a) implies (b).

If (a) does not hold, then $\lambda_1 = \lambda_2$, and neither is degenerate, hence $\lambda(E_k) \neq 0$ for $k \in K$. For each $\alpha_i \in N$, define $v_i(y) = u_i(y + y_i^*)$. So $u_i(x) = v_i(x - y_i^*)$, where v_i is symmetric and single peaked about 0. Write

$$\phi_i(y) = \mathbf{E}_{\lambda_{\bar{k}k}^*}[v_i(x-y)] - \mathbf{E}_{\lambda_{kk}^*}[v_i(x-y)] \ .$$

By Lemma 3b, it follows that

$$y < x^* \Rightarrow \phi_i(y) > 0$$
$$y = x^* \Rightarrow \phi_i(y) = 0$$
$$y > x^* \Rightarrow \phi_i(y) < 0 \ .$$

But $\phi_i(y_i^*) = \mathbf{E}_{\lambda_{\bar{k}k}^*}[u_i(x)] - \mathbf{E}_{\lambda_{kk}^*}[u_i(x)]$, and by Lemma 1a, it follows that

$$\phi_i(y_i^*) > 0 \Rightarrow b_i^*(k) = k$$
$$\phi_i(y_i^*) < 0 \Rightarrow b_i^*(k) = \bar{k} \ .$$

Hence

$$y_i^* < x^* \Rightarrow b_i^*(k) = k \quad \text{and} \quad b_i^*(\bar{k}) = \bar{k}$$
$$y_i^* > x^* \Rightarrow b_i^*(k) = \bar{k} \quad \text{and} \quad b_i^*(\bar{k}) = k \ .$$

Since in equilibrium, we need $w(s, b^*)$ constant for λ^* a.e. $s \in S$ (see Lemma 1b), it follows that we need $x^* = y_m^*$, where y_m^* is the median of the y_i^*. Q.E.D.

By introducting an additional stability condition, we can narrow the class of admissible SREE's further. For this result, we endow Λ_k with the weak topology.

Definition: *If* (λ^*, b^*) characterizes an SREE, then it is *stable* if, for each $k \in K$, there is a neighborhood $N(\lambda_k^*)$ of λ_k^* such that whenever $(\lambda', b') \in \Lambda \times B$ satisfies $\lambda_k' \in N(\lambda_k^*)$, $\lambda_{\bar{k}}' = \lambda_{\bar{k}}^*$ and $b' \in B(\lambda')$, then

$$\lambda_w'(\cdot | b') = \lambda_w^*(\cdot | b^*) \ .$$

So an SREE is *stable* if and only if, whenever one candidate changes his distribution, and voters vote optimally (given the new distribution), the distribution of winning positions does not change. I.e., small errors by a candidate do not change his recommended best response. The following theorem shows that the only stable SREE occurs when both candidates adopt the median voter's ideal point with probability one. Thus the stable SREE results in voters and candidates behaving as if they had complete information.

Theorem 2: *There exists a stable* SREE. *Further, if n is odd and* $(\lambda^*, b^*) \in \Lambda \times B$ *characterizes a stable* SREE, *then* $\lambda_1^*(\{y^*\}) = \lambda_2^*(\{y^*\}) = 1$ for $y^* \in X$, *where* y^* *is a median ideal point of the electorate.*

Proof: Let (λ^*, b^*) characterize a stable SREE. We note, first, from the definition of stability, that for any $b' \in B(\lambda^*)$, $\lambda_w^*(\cdot|b') = \lambda_w^*(\cdot|b^*)$, so (λ^*, b') characterizes a SREE. But now, define $b' \in B(\lambda^*)$ as follows:

$$b_i'(k) = \begin{cases} b_i^*(k) & \text{if} \quad \mathbf{E}_{\lambda_{kk}^*}(u_i(x)) \neq \mathbf{E}_{\lambda_{\bar{k}k}^*}(u_i(x)) \\ e(k) & \textit{otherwise} \end{cases} .$$

Clearly, $b' \in B(\lambda^*)$. Further, we have $w(s, b') = e(s)$ for all $s \in S$. Hence

$$\lambda_w^*(C|b') = \lambda^*(E_k)\lambda_{kk}^*(C) + \lambda^*(E_{\bar{k}})\lambda_{\bar{k}k}^*(C) .$$

But, for any $c \in \mathbb{R}$,

$$\begin{aligned} \lambda_w^*(L_c|b') &= \lambda^*(E_1)\lambda_{11}^*(L_c) + \lambda^*(E_2)\lambda_{22}^*(L_c) \\ &= \int_{-\infty}^{c} (1 - F_2(x))f_1(x)dx + \int_{-\infty}^{c} (1 - F_1(x))f_2(x)dx \\ &= F_1(c) + F_2(c) - \left[\int_{-\infty}^{c} F_2(x)f_1(x)dx + \int_{-\infty}^{c} F_1(x)f_2(x)dx \right] \\ &= F_1(x) + F_2(c) - F_1(c)F_2(c) \\ &= 1 - (1 - F_1(c))(1 - F_2(c)) \geqq F_1(c) \geqq F_2(c) \end{aligned}$$

with strict inequality when $F_1(c) \neq F_2(c)$ or $0 < F_k(c) < 1$. Hence $\lambda_w < \lambda_1$ and $\lambda_w < \lambda_2$. But then $\lambda_w < \frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2 = \lambda^*(w|b^*)$. Hence $\lambda^*(w|b') \neq \lambda^*(w|b^*)$, a contradiction, unless $\lambda_1^*(\{y\}) = \lambda_2^*(\{y\}) = 1$ for some $y \in X$.

We must now only show that $y = y^*$. Suppose not, assume w.l.o.g. that $y < y^*$. Then for any neighborhood $N(\lambda_k^*)$ of λ_k^*, we pick y' such that $y < y' < y^*$ and set $\lambda_k'(\{y'\}) = 1$. Then if y' is chosen so that $\lambda_k' \in N(\lambda_k^*)$, we have $b' \in B(\lambda') \Leftrightarrow$

$$y_i^* < \frac{y + y'}{2} \Rightarrow b'(k) = k$$

$$y_i^* > \frac{y + y'}{2} \Rightarrow b'(k) = \bar{k} .$$

but since $\frac{y + y'}{2} < y^*$, it follows that $w(s, b') = \bar{e}(s)$, where $\bar{e}(s) \in K - \{e(s)\}$. Hence

$$\lambda_w'(\{y'\}|b') = 1$$

$$\lambda_w^*(\{y\}|b^*) = 1$$

so the two are not equal, hence (λ^*, b^*) is not stable.

Finally, to prove existence, we let $(\lambda^*, b^*) \in \Lambda \times B$ satisfy $\lambda_k^*(\{y^*\}) = 1$. Let $N(\lambda_k^*)$ be any neighborhood of λ_k^*, and let $\lambda_k' \in N(\lambda_k^*)$, $\lambda_{\bar{k}}^* = \lambda_{\bar{k}}^*$, and $b' \in B(\lambda')$. Then we must have, for $k \in K$, if $\lambda'(E_k) \neq 0$,

$$y_i^* \geqq y_i \Rightarrow b_i'(k) = \bar{k}$$

$$y_i^* \leqq y_i \Rightarrow b_i'(k) = k$$

But then $w(s, b) = \bar{k}$ for all $s \in S$, hence $\lambda'_w(\{y^*\})$ so $\lambda'(w|b') = \lambda^*(w|b^*)$.

Q.E.D.

References

McKelvey RD, Ordeshook PC (1985a) Elections with limited information: A fulfilled expectations model using contemporaneous poll and endorsement data as information sources. J Econ Theory 36:55–85

McKelvey RD, Ordeshook PC (1985b) Sequential elections with limited information. Am J Polit Sci 29:480–512

McKelvey RD, Ordeshook PC (1986) Elections with limited information: A multidimensional model. Math Soc Sci (in press)

McKelvey RD, Page T (1986) Common knowledge, consensus, and aggregate information. Econometrica 54:109–127