of part **(b)** with $e = 1$. It is open whether solutions exist for other values of e, and part **(c)** is completely open.

The proposer derived the same two solutions, also without proving uniqueness.

Rational Cosine Ratios

10550 [1996, 809]. *Proposed by Raphael M. Robinson, University of California, Berkeley, CA.* What rational values are possible for $\cos\phi/\cos\theta$ when ϕ and θ are rational multiples of π?

Solution by John H. Lindsey II, Ft. Myers, FL. Suppose $r = \cos\phi/\cos\theta$ is rational, with ϕ and θ rational multiples of π. It is known (I. Niven, H. Zuckerman, and H. Montgomery, *An Introduction to the Theory of Numbers*, Wiley, 1991) that the only possible rational values for $\cos\phi$ are 0, $\pm 1/2$, and ± 1. Thus r can take the values 0, $\pm 1/2$, ± 1, and ± 2. We claim these are the only values r can take.

Suppose $\phi = a\pi/n$ and $\theta = b\pi/m$ with integers $a, b, n, m \geq 1$ and $\gcd(a, n) = \gcd(b, m) = 1$. We may assume that neither $\cos\phi$ nor $\cos\theta$ is rational.

If $m = n$, then there is a field automorphism σ of $\mathbb{Q}(e^{i\theta}) = \mathbb{Q}(e^{i\phi})$, the splitting field of $z^n - 1$, that fixes $\mathbb{Q}$ and takes $e^{i\theta}$ to $\pm e^{i\phi}$. Then

$$r = \frac{e^{i\phi} + e^{-i\phi}}{e^{i\theta} + e^{-i\theta}} = \frac{\pm\sigma(e^{i\theta} + e^{-i\theta})}{e^{i\theta} + e^{-i\theta}}.$$

Thus for $j \geq 1$,

$$r = \sigma^j r = \pm\sigma^{j+1}(e^{i\theta} + e^{-i\theta})/\sigma^j(e^{i\theta} + e^{-i\theta}). \qquad (*)$$

Now take the product of the equations $(*)$ as j ranges from 1 to the order of σ. The right-hand side telescopes to ± 1 since the numerators are, to within a factor of ± 1, a permutation of the denominators, while the left-hand side gives simply a power of r. Thus in this case $r = \pm 1$.

If $m < n$, let $k = \operatorname{lcm}(m, n)$. Then $\mathbb{Q}(e^{i\theta}, e^{i\phi}) = \mathbb{Q}(e^{i\pi/k})$. By assumption, $r = (e^{i\phi} + e^{-i\phi})/(e^{i\theta} + e^{-i\theta})$ is rational, so $e^{i\phi} + e^{-i\phi} \in \mathbb{Q}(e^{i\theta})$. But the index $[\mathbb{Q}(e^{i\phi}) : \mathbb{Q}(\cos\phi)] = 2$, since $e^{i\phi} \notin \mathbb{R}$, yet $e^{i\phi}$ satisfies a polynomial equation of degree 2 over $\mathbb{Q}(\cos\phi)$, namely, $(z - \cos\phi)^2 + 1 - (\cos\phi)^2 = 0$. Similarly $[\mathbb{Q}(e^{i\theta}) : \mathbb{Q}(\cos\theta)] = 2$. Now both these field extensions of $\mathbb{Q}(\cos\theta)$ are themselves subfields of $\mathbb{Q}(e^{i\pi/k}) = \mathbb{Q}(e^{i\phi}, e^{i\theta})$. But $[\mathbb{Q}(e^{i\phi}, e^{i\theta}) : \mathbb{Q}(e^{i\theta})] = 2$ because $\cos\phi \in \mathbb{Q}(\cos\theta)$ by assumption, and $i\sin\phi$ satisfies a polynomial of degree 2 with coefficients in $\mathbb{Q}(\cos\phi) = \mathbb{Q}(\cos\theta)$. On the other hand, in general $[\mathbb{Q}(e^{i\pi/N}) : \mathbb{Q}] = \varphi(2N)$, where φ is the Euler φ-function. Thus $\varphi(2k) = 2\varphi(2m)$ and similarly $\varphi(2k) = 2\varphi(2n)$.

From this it follows that there is some integer t such that $m = 2t$, $n = 3t$, and $3 \nmid t$. Let σ be the automorphism of $\mathbb{Q}(e^{i\pi/k})$ that takes $e^{i\pi/k}$ to $e^{i\pi j/k}$, where $j \equiv 2 \bmod 3$ and $j \equiv 1 \bmod 2m$. Then $\sigma(\cos\theta) = \cos\theta$, and of course $\sigma(r) = r$. Thus

$$\frac{\cos\phi}{\cos\theta} = \frac{e^{i\pi a/n} + e^{-i\pi a/n}}{2\cos\theta} = \frac{e^{i\pi aj/n} + e^{-i\pi aj/n}}{2\cos\theta},$$

so $\cos(j\phi) = \cos\phi$ and so $ja \equiv \pm a \bmod n$. But $n = 3t$, $j \equiv 1 \bmod t$, and $j \equiv 2 \bmod 3$, so this is impossible unless $t = 1$ or $t = 2$. So either $n = 3$ and $m = 2$, or $n = 6$ and $m = 4$. As neither case actually results in rational r, we are done.

Solved also by R. J. Chapman (U. K.), J. E. Dawson (Australia), and the proposer.

Rationals To and Only To Rationals

10555 [1996, 809]. *Proposed by Jeffrey C. Lagarias, AT&T Research, Murray Hill, NJ, and Eric M. Rains, Harvard University, Cambridge, MA.*

(**a**) Suppose $f(t) \in \mathbb{R}[t]$ is a polynomial that maps rationals to rationals and irrationals to irrationals. Show that $f(t) = at + b$ with a and b rational.
(**b**) Does the same conclusion hold under the weaker assumption that $f: \mathbb{R} \to \mathbb{R}$ is an algebraic function (i.e., if there is a polynomial $P(x, y) \in \mathbb{R}[x, y]$ such that $P(t, f(t))$ is identically zero)?

Solution by Victor S. Miller, Center for Communications Research, Princeton, NJ.
(**a**) Using Lagrange interpolation we find that $f(t) \in \mathbb{Q}[t]$. Clearing denominators we may further arrange that $f(t) \in \mathbb{Z}[t]$. If $f(t)$ has degree n and leading coefficient c, then replacing f by $c^{n-1} f(t/c)$ yields a monic f. The following lemma completes part (**a**).

Lemma. *Let $f(t)$ be a monic polynomial with integer coefficients and with degree greater than 1. Then there exists an integer B such that $f(t) - B$ has a real irrational root.*
Proof. Choose a prime p with $p > f(1) - f(0)$ and with p larger than the largest root of $f(x) - f(0) - x$. Let $B = p + f(0)$. Then $f(1) - B = f(1) - f(0) - p < 0$ and $f(p) - B = f(p) - f(0) - p > 0$. By continuity, $f(t) - B$ has a real root in $(1, p)$. By the rational root theorem, the only possible positive rational roots are 1 and p. Thus $f(t) - B$ has a positive irrational root. $\square$
(**b**) The ideal $I = \{Q(x, y) \in \mathbb{R}(x)[y] : Q(t, f(t)) = 0 \text{ for all } t \in \mathbb{Q}\}$ is the principal ideal generated by $P(x, y)$, since $P \in I$ and P is irreducible. However, the linear equations defining I all have rational coefficients. Thus I is generated by polynomials defined over $\mathbb{Q}$. Thus P can be taken to have rational coefficients.

By the Hilbert Irreducibility Theorem, there is an infinite set of $x \in \mathbb{Q}$ such that $P(x, y)$ is irreducible over $\mathbb{Q}$ as a polynomial in y. If $\deg_y P(x, y) > 1$, this gives infinitely many rational values of x for which none of the roots of $P(x, y)$ are rational. So we must have $\deg_y P(x, y) = 1$. Thus there are relatively prime polynomials $g(x), h(x) \in \mathbb{Q}[x]$ with $P(x, y) = g(x) - yh(x)$ and $f(t) = g(t)/h(t)$. Again by the Hilbert Irreducibility Theorem, there is an infinite set of $y \in \mathbb{Q}$ such that $g(x) - yh(x)$ is irreducible over $\mathbb{Q}$. If $\max(\deg(g), \deg(h)) > 1$, this yields an $x \notin \mathbb{Q}$ such that $f(x) = y$. Thus f must be a fractional linear transformation with rational coefficients. Since f is defined on all of $\mathbb{R}$, it is a linear polynomial.

Editorial comment. The definition of "algebraic" given in part (**b**) of the problem statement is inaccurate. A function is algebraic if there is an *irreducible* polynomial $P(x, y) \in \mathbb{R}[x, y]$ such that $P(t, f(t))$ is identically zero. If P is not assumed to be irreducible, then f is only piecewise algebraic. There are nonlinear, piecewise algebraic functions that send rationals to rationals and irrationals to irrationals. The easiest example is probably $f(t) = |t|$, which satisfies $t^2 - f(t)^2 = 0$.

Miller points out that the lemma in part (**a**) of his solution follows immediately from the Hilbert Irreducibility Theorem, but is certainly weaker. For the two applications of Hilbert Irreducibility in part (**b**) of his solution, we require only one rational value of x such that $P(x.y)$ has an irrational root and only one rational value of y such that $g(x) - yh(x)$ has an irrational root. The latter result may be proved by an argument similar to the proof of part (**a**), using primes in arithmetic progressions. A more elementary proof of the former result would be interesting.

Solved also by the proposers. Part (a) was solved also by R. Barbara (France), E. Cavalli (Italy), J .H. Lindsey II, GCHQ Problems Group (U. K.), and NCCU Problems Group.

The Integrals Are Euler's Constant

10561 [1996, 903]. *Proposed by Jean Anglesio, Garches, France.* Show that

$$\gamma = \lim_{u\to\infty} \int_{1/u}^{u} \left(\frac{1}{2} - \cos x\right) \frac{dx}{x} \qquad \text{and} \qquad \gamma = \lim_{u\to\infty} \int_{1/u}^{u} \left(\frac{1}{1+x} - \frac{\cos x}{x}\right) \frac{dx}{x},$$