

POINTWISE DOMINATION OF MATRICES AND COMPARISON OF \mathcal{S}_p NORMS

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Let p be a real number in $[1, \infty)$ which is not an even integer. Let $N = 2[p/2] + 5$. We give examples of $N \times N$ matrices A and B , so that $|a_{ij}| \leq b_{ij}$ but $\text{Tr}([A^*A]^{p/2}) > \text{Tr}([B^*B]^{p/2})$.

Let A and B be $N \times N$ matrices with

$$(1) \quad |a_{ij}| \leq b_{ij}.$$

If we define the p norm of a matrix by

$$(2) \quad \|A\|_p = \text{Tr}([A^*A]^{p/2})^{1/p}$$

then it is trivial that, if p is an even integer, then

$$(3) \quad \|A\|_p \leq \|B\|_p$$

when (1) holds. For one need only write out the trace explicitly in terms of matrix elements. In a more general context, we conjectured in [5] that (1) implies (3) whenever $p \geq 2$. The attractiveness of this conjecture is shown by the fact that I know of at least five people other than myself who have worked on proving it.

It was thus quite surprising that Peller [3] announced that (3) fails for some infinite matrices whenever p is not an even integer. In correspondence, Peller described his counterexample which relies on his beautiful but elaborate theory of \mathcal{S}_p Hankel operators (4) and on a paper of Boas (2). It follows from Peller's example that (3) must fail for some finite N but it is not clear for which N . Our purpose here is to give explicit N and to avoid the complications of Peller's \mathcal{S}_p -Hankel theory.

The idea of the construction is very simple. Boas [2] constructed polynomials $f(z)$, $g(z)$ with $\int |f(e^{i\theta})|^p d\theta > \int |g(e^{i\theta})|^p d\theta$ even though the coefficients, a_n , of f and coefficients, b_n , of g obey $|a_n| \leq b_n$. a and b should be thought of as Fourier coefficients of $f(e^{i\theta})$ and $g(e^{i\theta})$. It is obvious that for sufficiently large N , $\sum_{j=0}^{N-1} |f(e^{ij\theta_N})|^p \geq \sum_{j=0}^{N-1} |g(e^{ij\theta_N})|^p$ where $\theta_N = 2\pi/N$. Again f and g should be viewed as functions on Z_N and the coefficients of the polynomial (if N is larger than the degrees) as Z_N -Fourier components. But the functions on Z_N are naturally imbedded in $N \times N$ matrices in such a way $\|A\|_p^p$ is just $\sum |f(e^{ij\theta_N})|^p$ and so that the order (1) is equivalent to the order on Fourier coefficients.

To be explicit, given N and c_0, \dots, c_{N-1} let A be the matrix

$$\begin{pmatrix} c_0 & c_1 & \cdots & c_{N-2} & c_{N-1} \\ c_{N-1} & c_0 & \cdots & c_{N-3} & c_{N-2} \\ c_{N-2} & c_{N-1} & \cdots & c_{N-4} & c_{N-3} \\ \vdots & & & & \vdots \\ c_1 & c_2 & & & c_0 \end{pmatrix}$$

$z_N = \exp(i\theta_N)$ and let φ_j be the vector with components $(1, z_N^j, z_N^{2j}, \dots, z_N^{(N-1)j})$; $j = 0, \dots, N-1$ and observe that

$$(4) \quad A\varphi_j = f(j)\varphi_j$$

where

$$(5) \quad f(j) = \sum_{\ell=0}^{N-1} c_\ell \chi_\ell(j)$$

with

$$(6) \quad \chi_\ell(j) = z_N^{\ell j}.$$

We use (6) to define χ_ℓ for any integer ℓ although, of course, χ_ℓ is periodic in ℓ with period N .

Of course, we have just exploited the fact that if σ is the matrix which cyclicity permutes the coordinates by one component, then $A\sigma = \sigma A$ (indeed $A = \sum c_p \sigma^k$) and since $\sigma^N = 1$, σ is naturally diagonalized in terms of the group Z_N . The χ 's are just the characters of Z_N . (In Physicist's language, since A has periodic boundary conditions, one diagonalizes it in momentum space.)

Since the φ_j are orthogonal vectors, A is a normal operator. For such an operator $\|A\|_p^p$ is just the sum of the p th powers of the eigenvalues, i.e.,

$$(7) \quad \|A\|_p^p = \sum_{j=0}^{N-1} |f(j)|^p.$$

We take

$$(8a) \quad k = \left[\frac{1}{2}p \right] + 2$$

$$(8b) \quad N = 2k + 1 = 2 \left[\frac{1}{2}p \right] + 5.$$

Motivated by Boas' example, we choose

$$(9) \quad c_0 = 1; \quad c_1 = r; \quad c_k = \lambda r^k; \quad c_\ell = 0, \quad \text{if } \ell \neq 0, 1, k$$

where r is sufficiently small and

$$(10) \quad \lambda = \left(\frac{1}{2}p - 1\right)\left(\frac{1}{2}p - 2\right) \cdots \left(\frac{1}{2}p - k + 1\right)/k! .$$

Notice that since p is not an even integer and since $p/2 + 1 < k < p/2 + 2$, we have that $\lambda < 0$. Let $d_j = |c_j|$ and let B the corresponding matrix so (1) certainly holds.

We compute $\|A\|_p^p$ using (7) and the binomial theorem which is certainly legitimate if r is sufficient small

$$\begin{aligned} |f(j)|^{p/2} &= \sum_{\ell=0}^{\infty} \binom{p/2}{\ell} \sum_{m=0}^{\ell} \binom{\ell}{m} r^{\ell+m(k-1)} \lambda^m \chi_{\ell+m(k-1)}(j) \\ &= f_1(j) + f_2(j) + f_3(j) + 0(r^{2k+1}) \end{aligned}$$

where

$$\begin{aligned} f_1 &= \sum_{\ell=0}^{k-1} \binom{p/2}{\ell} r^{\ell} \chi_{\ell} \\ f_2 &= \sum_{\ell=k}^{2k-1} \left[\binom{p/2}{\ell} + \lambda \binom{p/2}{\ell-k+1} \binom{\ell-k+1}{1} \right] r^{\ell} \chi_{\ell} \\ f_3 &= r^{2k} \chi_{2k} \left[\binom{p/2}{2k} + \lambda(k+1) \binom{p/2}{k+1} + \lambda^2 \binom{p/2}{2} \right] . \end{aligned}$$

Because $N = 2k + 1$, the characters χ_0, \dots, χ_{2k} are orthogonal so squaring and summing:

$$\|A\|_p^p = \sum_{j=0}^{k-1} \binom{p/2}{j}^2 r^{2j} + r^{2k} \left[\binom{p/2}{k} + \lambda \binom{p/2}{1} \right]^2 + 0(r^{2k+1}) .$$

The formula for $\|B\|_p^p$ is identical, except λ is replaced by $|\lambda| = -\lambda$. But λ is exactly chosen so that

$$\binom{p/2}{k} - \lambda \binom{p/2}{1} = 0 .$$

Thus, for r small, $\|A\|_p > \|B\|_p$.

It was necessary to take $N = 2k + 1$ rather than just $k + 1$ to avoid cross terms between the r_0 and r^{ℓ} ($\ell \leq 2k$) factors which have the wrong sign and only vanish because χ_0 and χ_{ℓ} are orthogonal for $\ell \leq 2k$.

We close this paper with a series of remarks:

(1) Peller constructs infinite matrices A, B which are matrices of compact operators on ℓ_2 with (1) holding, $B \in \mathcal{S}_p$ and $A \notin \mathcal{S}_p$. It is easy to get such operators from our examples as follows: normalize A, B so that $\|A\|_p > 1 > \|B\|_p \geq \|B\| \geq \|A\|$. Let us view ℓ_2 as the tensor algebra over C^N , i.e., as $C \oplus C^N \oplus C^{N^2} \oplus \dots$ and let $\Gamma(A) = 1 \oplus A \oplus (A \otimes A) \oplus \dots$. Then $|\Gamma(A)_{ij}| \leq |\Gamma(B)_{ij}|$ and $\Gamma(A), \Gamma(B)$ are compact, $\Gamma(B) \in \mathcal{S}_p$ but $\Gamma(A) \notin \mathcal{S}_p$.

(2) Given any measure space, (M, μ) with $L^2(M, \mu)$ infinite dimensional, we cannot have that $\|A\|_p \leq c\|B\|_p$ for some fixed c and all A, B with $|(Af)(m)| \leq (B|f|)(m)$. For one can always imbed C^N into $L^2(M, \mu)$ in a way preserving $\|A\|_p$ norms and order (map (a_1, \dots, a_n) into $\sum a_i f_i(m)$ with f_i multiples of characteristic functions of disjoint sets). If $\|A\|_p \leq c\|B\|_p$ held for $L^2(M)$ it would hold for any C^N . But by taking tensor products of our example one can arrange that $\|A\|_p/\|B\|_p$ is arbitrarily large. [It is interesting that this tensor product/operator theory version of Katznelson's remark (quoted in Bachelis [1]) is more natural than the function theoretic construction.]

(3) Let $N(p)$ be the smallest N for which there exist matrices for which (1) holds but (3) fails. Clearly we have shown

$$N(p) \leq 2 \left[\frac{1}{2} p \right] + 5$$

but equality is most unlikely for *any* p . Indeed for $1 \leq p < 2$, we have $N(p) = 2$ since if

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

then $\|B\|_p^p = 2^p$, $\|A\|_p^p = 2(\sqrt{2})^p > \|B\|_p^p$ if $p < 2$. Moreover, we owe to S. Friedland the following simple argument showing that $N(p) \geq 3$ if $p > 2$. If C, D are *positive* matrices with

$$(11) \quad |c_{ij}| \leq d_{ij}$$

then with $\mu_j(\cdot) =$ singular values, we trivially have

$$\mu_1(C) \leq \mu_1(D); \quad \mu_1(C) + \mu_2(C) \leq \mu_1(D) + \mu_2(D)$$

(since for 2×2 positive matrices $\mu_1(C) + \mu_2(C) = \text{Tr}(C)$). By general rearrangement inequalities [5]

$$\text{Tr}(C^p) \leq \text{Tr}(D^p)$$

for any $1 \leq p \leq \infty$. Given A, B obeying (1) and applying this remark to $C = A^*A, D = B^*B$, we see that (3) holds for any $p \geq 2$ if $N = 2$. It would be interesting to know the precise value of $N(p)$. Two natural guesses are $[p/2] + 1$ and $2[p/2]$.

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