J. Phys. C: Solid State Phys., Vol. 11, 1978. Printed in Great Britain. © 1978

Stability of the aligned state of ^{3}He-A in a superflow

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Received 1 November 1977

Abstract. The stability of the equilibrium orientation of $\hat{l}$ parallel or antiparallel to a counterflow is studied both dynamically and statically. In contrast to the recent work of Hall and Hook—who, we show, use incorrect dynamical equations—we find the equilibrium to be stable in the Ginzburg–Landau regime. At lower temperatures an instability should develop.

1. Introduction

Since very early work on ^{3}He-A (deGennes and Rainer 1974), it has been commonly thought that the equilibrium orientation of the orbital anisotropy vector $\hat{l}$ in a counterflow $\boldsymbol{v}_{so} = \boldsymbol{v}_s - \boldsymbol{v}_n$ is $\hat{l}$ parallel to $\boldsymbol{v}_{so}$.‡ In this way the smallest component of the anisotropic superfluid density tensor is involved in the kinetic energy of superflow and the energy is minimised. Recently Hall and Hook (1977) and Hook and Hall (1977)—to be referred to as HH—put forward the intriguing possibility that this equilibrium is in fact unstable to small perturbations, leading to a dynamic behaviour they proceed to study. Unfortunately, for the situation they consider, we find that this conclusion arises due to an error in the dynamic equations they use. Using the correct equations, in the Ginzburg–Landau region we find $\hat{l}$ parallel to $\boldsymbol{v}_{so}$ to be stable, confirming the intuitively plausible result. However, this arises only as a numerical coincidence of the coefficients in the energy expression, and at lower temperatures or for the hypothetical case in which spin–orbit coupling can be neglected, the solution $\hat{l}$ parallel to $\boldsymbol{v}_{so}$ would become unstable. Also we have only demonstrated local stability: There is no guarantee that an initial configuration very different than this will relax to this equilibrium under driven conditions.

Our calculations fall into two parts. In §2 we use what we believe are the correct dynamical equations. These have been derived from thermodynamic considerations, analogous to those employed by HH, by Ho (1977) and Hu and Saslow (1977) and from physical arguments by one of us (Brinkman and Cross 1978). Since this latter derivation is not yet readily available we briefly describe it here so that the rather complicated form of the dynamic equations used may be easily understood.

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‡ We will refer to the equilibrium as parallel. The antiparallel orientation is equally good, and the same analysis of the stability applies.

In § 3 we take advantage of the fact that the question is a *static* one (whether the energy is maximised or minimised by $\hat{\boldsymbol{l}}$ parallel to $\boldsymbol{v}_{\mathrm{so}}$) rather than dynamic, and study the stability by a method *independent of any assumptions on the form of the equations of motion.* This is in fact a conventional stability analysis. Both methods demonstrate the same stability of the equilibrium near T_{c}. We take this agreement to favour the dynamical equations used over those of Hall and Hook. In § 4 the stability under more general conditions is discussed. Finally we point out the error in the derivation of Hall and Hook.

2. Dynamical approach

The reactive driving terms in the equation of motion (and it is on these we differ from Hall and Hook) may be derived by straightforward arguments, analogous to those used in the well tested equations of spin dynamics. They are given simply by the torque $\boldsymbol{\tau}$, i.e., the derivative of the free energy F with respect to a rotation of $\hat{\boldsymbol{l}}$:

$$\boldsymbol{\tau} = -\hat{\boldsymbol{l}} \times \frac{\delta F}{\delta \hat{\boldsymbol{l}}} = \hat{\boldsymbol{l}} \times \boldsymbol{h}^0 \tag{1}$$

where $\boldsymbol{h}^0$ is a convenient 'molecular field'.

All attempts to derive microscopically the equation of motion either introduce or justify in some simple limit such a term (see for example the review article, Brinkman and Cross (1978)). The remaining controversy in understanding orbital dynamics comes essentially from the inertial and dissipative terms to which $\boldsymbol{\tau}$ is equated. This controversy is academic from the point of view of calculating the actual motion. In practice both theory and experiment suggest $\boldsymbol{\tau}$ may be equated to the viscous torque on $\hat{\boldsymbol{l}}$ (Cross 1977), giving

$$\mu\hat{\boldsymbol{l}} \times \dot{\hat{\boldsymbol{l}}} = \boldsymbol{\tau} = \hat{\boldsymbol{l}} \times \boldsymbol{h}^0 \tag{2}$$

where we have specialised to a uniform $\boldsymbol{v}_{\mathrm{n}}$ and perform the calculations in the frame with $\boldsymbol{v}_{\mathrm{n}} = 0$. A comparison with equation (27) of HH shows $\hat{\boldsymbol{l}} \times \boldsymbol{h}^0$ plays the role of their $-\boldsymbol{G}$. We re-emphasise that we differ from Hall and Hook on the evaluation of this static quantity.

To proceed we must specify the derivative $\delta F/\delta\hat{\boldsymbol{l}}$ in equation (1) in terms of known quantities.

For the situation under study the energy depending on the orientation of $\hat{\boldsymbol{l}}$ is simply the energy of spatial inhomogeneity of the order parameter—the 'bending energy':

$$F = \int f_{\mathrm{B}}(\hat{\boldsymbol{l}}, `\nabla\hat{\boldsymbol{l}}\text{'}, \boldsymbol{v}_{\mathrm{so}})\,\mathrm{d}V$$

where f_{B} is the bending energy density, known (Cross 1975) as a function of $\hat{\boldsymbol{l}}$ and its gradients, and $\boldsymbol{v}_{\mathrm{so}}$ the superfluid velocity. $\boldsymbol{\tau}$ is the change in energy when $\hat{\boldsymbol{l}}$ is rotated at one point only: thus the derivative in equation (1) is the functional derivative

$$\frac{\delta F}{\delta \hat{\boldsymbol{l}}} = \left(\frac{\partial f_{\mathrm{B}}}{\partial \hat{\boldsymbol{l}}} - \nabla_j \frac{\partial f_{\mathrm{B}}}{\partial \nabla_j \hat{\boldsymbol{l}}}\right)_{\delta\phi(r)=0} \tag{3}$$

It is very important that in the partial derivatives of f_{B} with $\hat{\boldsymbol{l}}$ it is *not* $\boldsymbol{v}_{\mathrm{so}}$ that is to be held constant†, since $\hat{\boldsymbol{l}}$ and $\boldsymbol{v}_{\mathrm{so}}$ are not independent variables (Mermin and Ho 1976).

† This point was omitted in Cross (1977).

Rather we must hold the third *independent* variable, the phase, constant: $\delta\phi(r) = 0$. We note that although a global phase variable cannot be defined in ^{3}He-A (where the broken gauge symmetry is part of the broken non-Abelian rotation symmetry), an infinitesimal phase change at each point is well defined, and corresponds exactly to an infinitesimal rotation about the local $\hat{l}$. Therefore $\delta\hat{l}$ and $\delta\phi$ describe infinitesimal rotations of the orbital order parameter about three orthogonal directions, and are independent variables sufficient to define the change in the order parameter. Using the chain rule of differentiation, and the relationship easily proved from the definitions (and closely related to the Mermin–Ho result)†

$$\delta v_{si} = \nabla_i \delta\phi - \hat{l}.(\nabla_i \hat{l} \times \delta\hat{l}) \tag{4}$$

we can write

$$\boldsymbol{h}^0(\boldsymbol{r}) = -\left[\frac{\partial f_B}{\partial \hat{l}(\boldsymbol{r})} - \nabla_j \frac{\partial f_B}{\partial(\nabla_j \hat{l}(\boldsymbol{r}))}\right]_{\delta v_s = 0} + \hat{l} \times (\boldsymbol{j}_s^0 . \nabla)\hat{l} \tag{5}$$

where $\boldsymbol{j}_s^0$, the supercurrent when $\boldsymbol{v}_n = 0$, arises as $(\partial f_B/\partial \boldsymbol{v}_{so})_{\delta l = 0}$. Now $\boldsymbol{h}^0$ may be immediately evaluated from the known f_B. The form of equation (5) has been derived independently by Ho and Hu and Saslow. Although the final form is complicated, our argument shows it is very easily understood.

From equations (2), (5) it is easy to prove the stability of $\hat{l}$ parallel to $\boldsymbol{v}_{so}$. Using the expression for f_B in the Ginzburg–Landau regime of Cross (1975), and following Hall and Hook in assuming the spin vector $\hat{d}$ to be parallel to $\hat{l}$, we find

$$\begin{aligned}\hat{l} \times \boldsymbol{h}^0 = \tfrac{1}{2}\rho_{\parallel}\hat{l} \times [&5\nabla^2\hat{l} - 2\,\mathrm{curl}\,\hat{l}(\hat{l}.\mathrm{curl}\,\hat{l}) + 4\hat{l} \times (\boldsymbol{v}_{so}.\nabla)\hat{l} - 2\hat{l} \times \nabla(\hat{l}.\boldsymbol{v}_{so}) \\ &+ 2\boldsymbol{v}_{so}(\hat{l}.\mathrm{curl}\,\hat{l}) + 2(\hat{l}.\boldsymbol{v}_{so})\,\mathrm{curl}\,\hat{l} + 2(\hat{l}.\boldsymbol{v}_{so})\,\boldsymbol{v}_{so}].\end{aligned} \tag{6}$$

$\rho_{\parallel}$ is the longitudinal superfluid density. Note $\boldsymbol{v}_{so}$ is $-\boldsymbol{w}$ of HH. It is sufficient to calculate $\boldsymbol{h}^0$ linear in small deviations $\delta\hat{l} = (\theta\cos\phi, \theta\sin\phi, 0)$ from the configuration $\hat{l} = (0, 0, 1)$, $\boldsymbol{v}_{so} = \boldsymbol{v}_{so}(0, 0, 1)$. To this order in θ, and assuming variations only in the z direction, current conservation is satisfied with $\delta\boldsymbol{v}_{so} = 0$. Then

$$(\mu/\rho_{\parallel})\,\dot{\theta} = \tfrac{5}{2}\theta'' - \theta[v_{so}^2 + 3v_{so}\phi' + \tfrac{5}{2}(\phi')^2] \tag{7}$$

where the prime denotes $\partial/\partial z$. This gives for the amplitude θ_0 of a disturbance $\theta = \theta_0 \cos qz$ a time dependence

$$(\mu/\rho_{\parallel})\,\dot{\theta}_0 = -\{\tfrac{5}{2}q^2 + [(v_{so} + \tfrac{3}{2}\phi')^2 + \tfrac{1}{4}(\phi')^2]\}\,\theta_0. \tag{8}$$

The quantity $\{\ \}$ is positive definite for any ϕ'. Any small deviation of $\hat{l}$ relaxes back to the initial configuration. The rate of relaxation depends on the form of the disturbance, but the fact of the relaxation does not.

3. Static approach

The standard attack on a stability question is to calculate the change in the total energy to second order in a small sinusoidal disturbance. A negative energy for any such disturbance indicates an instability. We have done this calculation with $\delta\boldsymbol{l}$ and $\delta\phi$ as variables, with the same result. An easier method, which should be generally useful in

† We choose units such that $\hbar/2m = 1$.

such stability questions, is to return to the fundamental description of the order parameter. Thus we write the orbital parameter

$$\boldsymbol{\Delta} = \mathbf{R} \cdot \boldsymbol{\Delta}_0 \tag{9}$$

with $\boldsymbol{\Delta}_0$ the initial order parameter $e^{ivz}(1, i, 0)$ and $\mathbf{R}(\boldsymbol{r})$ the rotation matrix describing the disturbance from this configuration. (For simplicity of notation we drop the suffices on v_{so}). $\mathbf{R}$ may be parametrised by the three components of a vector $\boldsymbol{\theta}$:

$$R_{ij} = \delta_{ij} + \epsilon_{ijk}\theta_k - \tfrac{1}{2}(\delta_{ij}\theta^2 - \theta_i\theta_j) \tag{10}$$

where terms to second order in $\boldsymbol{\theta}$ must be retained. Specialising to variations in the z direction is not necessary, but considerably simplifies the algebra. Then the contribution to the bending energy from variation of the orbital coordinates in the Ginzburg–Landau regime takes the form

$$f_B \propto (\nabla_z \Delta_i^*)(\nabla_z \Delta_i) + 2|\nabla_z \Delta_z|^2 \tag{11}$$

(proportionality constants are not important, and have been dropped). With a sinusoidal dependence $\boldsymbol{\theta} = \boldsymbol{\theta}_0 e^{iqz} + \mathrm{cc}$, equations (9)–(11) give the total energy proportional to $\boldsymbol{\theta}_0^* \cdot \mathbf{H} \cdot \boldsymbol{\theta}_0$ with $\mathbf{H}$ the Hermitian matrix

$$\mathbf{H} = \begin{bmatrix} v^2 + \frac{5}{2}q^2 & 3viq & 0 \\ -3viq & v^2 + \frac{5}{2}q^2 & 0 \\ 0 & 0 & q^2 \end{bmatrix}. \tag{12}$$

(In this expression the contribution from the spin vector assumed parallel to $\hat{\boldsymbol{l}} = \frac{1}{2}i\, \boldsymbol{\Delta} \times \boldsymbol{\Delta}^*$ has been added. Without this contribution the coefficient $\frac{5}{2}$ in $\mathbf{H}$ becomes $\frac{3}{2}$). The eigenvalues of $\mathbf{H}$ are q^2 and $v^2 \pm 3vq + \frac{5}{2}q^2$, all positive. The energy of the distortion is positive definite and the stability of the equilibrium against small perturbations has been proven. The similarity to the result calculated in § 2 is evident. It is interesting that for the hypothetical case of no spin–orbit coupling one of the second pair of eigenvalues becomes negative for a window of wavevectors q around v. In practice critical velocities limit v to the range where spin–orbit coupling is dominant, except possibly in very small geometries $< 10\ \mu\mathrm{m}$.

4. Results

We have demonstrated by two different methods the stability in the Ginzburg–Landau regime of the equilibrium solution of $\hat{\boldsymbol{l}}$ parallel to a counterflow against the disturbances considered by Hall and Hook. We now consider the stability under more general conditions. This is important since the persistent motion observed by Paulson *et al* (1976) has been interpreted (Wheatley 1977) as confirming the suggestions of HH. We note, however, that no instability was observed—the system was always perturbed to set up the motion. It remains to be seen whether the dynamical equations of § 2 can predict a persistent driven motion for large amplitude excursions. Here we look at the question of whether deviations from the Ginzburg–Landau expressions may lead to an instability.

Even in the Ginzburg–Landau region equation (11) is not the most general form consistent with the symmetry of the system; it is a 'weak coupling' result. More generally there are (for nearly all purposes) just two independent coefficients in the Ginzburg–

Landau bending energy expressions. These may be chosen as the longitudinal and transverse superfluid densities $\rho_{\parallel}$, $\rho_{\perp}$. Repeating the calculation of § 3 with the general expression gives for unstability $\rho_{\perp} < \frac{3}{2}\rho_{\parallel}$. The weak-coupling values are $\rho_{\perp} = 2\rho_{\parallel}$. An instability near T_c *requires* strong-coupling corrections of at least 25% reducing the superfluid density anisotropy. This seems unlikely from the theoretical estimates (J Serene and D Rainer 1977, unpublished), and from experimental measurements (Berthold *et al* 1977).

Outside the Ginzburg–Landau regime the coefficients in equation (6) are no longer integrally related, and additional terms must be added to equation (11). The general form of f_B was written down by Cross (1975) and Mermin and Ho (1976). Using the notation of the latter authors, the stability requirement becomes

$$\tfrac{1}{4}(\rho_{\parallel} + 2C_0)^2 K_3^{-1} \rho_0^{-1} < 1 \tag{13}$$

with $\rho_{\parallel} = \rho_s - \rho_0$. We have evaluated these coefficients at all temperatures using the expressions of Cross (1975), who used a weak-coupling calculation but included the large Fermi liquid corrections, and following the computational procedure of Combescot (1975). As the temperature decreases from T_c the left-hand side of equation (13) increases monotonically, and becomes larger than 1 below an 'instability temperature' T_i. To compare with experiment, rather than quoting the calculated T_i which depends, for example, on the strong-coupling renormalisation of the gap magnitude, it is more useful to quote the calculated value of $(\rho_{\perp}/\rho)(1 + \frac{1}{3}F_1^s)$ evaluated at T_i. This is independent of the renormalisation of the gap magnitude and turns out to be rather independent of F_1^s (to a few per cent) for pressures 21–34 bar. It is also easily measured experimentally. The calculation of T_i depends somewhat on the value used for the spin antisymmetric Fermi liquid coefficient F_1^a, not well known for ^{3}He. With F_1^a small the value of $(\rho_{\perp}/\rho)(1 + \frac{1}{3}F_1^s)$ at T_i is 0·55; for $-F_1^a$ as large as 1 it is 0·40. Both these estimates give a T_i well within the range of temperatures for which the A phase is stable at melting pressure. The estimation of T_i may, however, depend quite significantly on the smaller strong-coupling corrections to the coefficients, that is those that cannot be expressed as a renormalisation of the gap magnitude.

Finally, the method of § 3 has been generalised to disturbances varying in arbitrary directions: again we find stability in the Ginzburg–Landau regime.

5. Conclusions

We have shown, from both dynamic and static considerations, that the equilibrium orientation of $\hat{l}$ parallel to a counterflow is stable in the Ginzburg–Landau regime *unless* $\rho_{\perp} < \frac{3}{2}\rho_{\parallel}$, which is unlikely. This calculation is sufficient to cover the range of temperatures of the experiments of Paulson *et al* (1976). Extending the calculations suggests an instability should develop at low temperatures.

The equations of HH, which incorrectly predict an instability with the weak coupling Ginzburg–Landau coefficients clearly must be modified. J R Hook and H E Hall (private communication) compared the detailed form of their equations with the ones used in § 2. They show a formal identity between the two sets proves the differences lie in the evaluation of ψ of Hu and Saslow—which is $(\partial f_B/\partial \hat{l})_{v_s}$ in our notation. These differences can clearly be seen to arise from the incorrect treatment of the dependence of the bending energy on $\hat{l}$ in deriving equations (12) and (14) of Hall and Hook (1977). In particular the $\hat{l}$ dependence of the tensors $\mathbf{T}$ and $\mathbf{T}_n$ has been neglected. We have not tried to prove that this correction then makes the two sets equivalent. With the agreement of the results of the formal deviation of Ho and Hu and Saslow with those of the physical derivation of § 2, and the agreement of the dynamic and static calculations in §§ 2 and 3, we may have confidence that the dynamical equations used are indeed correct.

Acknowledgments

This work was stimulated by correspondence or conversations with H E Hall, A L Fetter, N D Mermin, T-L Ho and W F Brinkman.

Note. On completion of this work we received a preprint from P Bhattacharya, T-L Ho and N D Mermin who also prove the stability of the equilibrium in the Ginzburg–Landau regime. The emphasis of their paper is on the behaviour with an instability. Our results complement theirs on the existence of an instability.

We have since heard that the preprint Hook and Hall (1977) has been withdrawn, following the criticisms detailed above. Apparently the controversy over the correct orbital dynamic equations has now been settled.

References

Brinkman W F and Cross M C 1978 *Progress in Low Temperature Physics* vol 7 ed J Brewer (Amsterdam: North Holland)
Berthold J E, Giannetta R W, Smith E N and Reppy J D 1977 *Phys. Rev. Lett.* **37** 1138
Combescot R 1975 *J. Low Temp. Phys.* **18** 537
Cross M C 1975 *J. Low Temp. Phys.* **21** 525
—— 1977 *J. Low Temp. Phys.* **26** 165
de Gennes P G and Rainer D 1974 *Phys. Lett.* **46**A 429
Hall H E and Hook J R 1977 *J. Phys. C: Solid St. Phys.* **10** L91
Hook J R and Hall H E 1977 preprint
Ho T-L 1977 in *Quantum Fluids and Solids* eds S B Trickey, E D Adams and J W Duffy (New York: Plenum)
Hu C R and Saslow W M 1977 *Phys. Rev. Lett.* **38** 605
Mermin N D and Ho T-L 1976 *Phys. Rev. Lett.* **36** 594
Paulson D N, Krusius M and Wheatley J C 1976 *Phys. Rev. Lett.* **37** 599
Wheatley J C 1977 in *Quantum Fluids and Solids* eds S B Trickey, E D Adams and J W Duffy (New York: Plenum)