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Effect of Strain Hardening and Rate Sensitivity on the Dynamic Growth of a Void in a Plastic Material

The problem studied in this paper concerns the dynamic expansion of a spherical void in an unbounded solid under the action of remote hydrostatic tension. The void is assumed to remain spherical throughout the deformation and the matrix to be incompressible. The effects of inertia, strain hardening, and rate sensitivity on the short and long-term behavior of the void, as well as on its response to ramp loading, are investigated in detail.

1 Introduction

The problem studied in this paper concerns the dynamic expansion of a spherical void in a unbounded rate-dependent solid under the action of rapidly varying remote hydrostatic tension. Conditions such as those investigated here are prevalent in the vicinity of a crack running through a ductile metal, in porous solids subjected to blast loading, and in other situations of practical interest. In particular, void growth and coalescence is known to be a principal micromechanism of fracture in ductile metals. In dynamic crack growth, the microinertia associated with the rapid growth of the voids may influence, in ways as yet not fully understood, the conditions under which a ductile mode of fracture becomes possible.

Whereas much attention has been given to void growth under static conditions, the dynamic problem has remained relatively unexplored. Carroll and Holt (1972) investigated the static and dynamic collapse of voids using a hollow sphere model. The solid was assumed to be rate insensitive and ideally plastic. A notable outcome of their analysis is the small effect of elastic compressibility on the solution. Johnson (1981) applied Carroll's approach to voids expanding in a viscoplastic solid described by a simple linear overstress model. Glennie (1972) sought to extend the analysis of Rice and Johnson (1970) for a void growing in the vicinity of a blunted crack tip in a plastic material, by taking dynamic effects into consideration. However, his analysis is restricted to ideally plastic behavior and linear viscosity. More recently, Klöcker and Montheillet (1988) have conducted numerical calculations for a dynamically growing void in a plastic solid obeying a linear stress-strain rate

law. Their results exhibit the potentially stabilizing effect of inertia at the microscale.

The purpose of this investigation is to develop an understanding of the effect of inertia, strain hardening, and rate dependence on void growth in porous metals under conditions of rapid loading. To simplify the analysis, it is assumed that the material is incompressible and the void remains spherical at all times. These assumptions determine the velocity field throughout the body once the rate of expansion of the void is specified. Thus, the problem is reduced to the integration in time of an ODE for the void radius. Whereas a small compressibility is always introduced by the elastic response of the solid, Carroll and Holt's work (1972) suggests that this effect is negligible in cases where the plastic response dominates. The solid is assumed to obey J_2-flow theory of plasticity with power hardening and the elastic response is neglected. Rate sensitivity is introduced into the formulation through a standard power-law viscoplastic model. The details of the formulation of the constitutive model are given in Section 2.

The equations governing the expansion of a void are formulated in Section 3 for various constitutive assumptions. Although the majority of these equations cannot be solved in closed form, the short and long-term solutions can be obtained analytically with some generality. These solutions reveal useful insights into the behavior of the void as determined by the various constitutive descriptions. Finally, full numerical calculations are presented in Section 5 which confirm the short and long-term solutions and exhibit the nature of the intermediate transients.

2 Constitutive Model

Throughout the analysis, we shall assume that the matrix surrounding the void responds to monotonic stressing as a rigid-viscoplastic solid with flow rule

$$d_{ij} = \dot{\epsilon}_e \frac{3s_{ij}}{2\sigma_e} = \frac{\partial \phi(s)}{\partial s_{ij}} \tag{1}$$

where

$$\sigma_e = \sqrt{(3/2)s_{ij}s_{ij}} \tag{2}$$

$$s_{ij} = \sigma_{ij} - p\delta_{ij}, \quad p = \sigma_{kk}/3 \tag{3}$$

$$\dot{\epsilon}_e = \dot{\epsilon}_0 \left(\frac{\sigma_e}{\sigma_0}\right)^m \tag{4}$$

$$\phi(\mathbf{s}) = \frac{\sigma_0 \dot{\epsilon}_0}{m+1} \left(\frac{\sigma_e}{\sigma_0}\right)^{m+1}. \tag{5}$$

Here, σ_{ij} is the stress tensor, s_{ij} the deviatoric stresses, p the hydrostatic pressure, d_{ij} is the rate of deformation tensor, σ_e and ϵ_e the effective Mises stress and strain, respectively, σ_0 is a flow stress, and $\dot{\epsilon}_0$ and m are material constants. The flow stress is assumed to obey the hardening law

$$\sigma_0 = \sigma_y \left(\frac{\epsilon_e}{\epsilon_y}\right)^{1/n} \tag{6}$$

where σ_y, ϵ_y and n are material constants.

The rate-independent limit of the aforementioned model is attained by letting $m \to \infty$, whereupon the viscosity law (4) reduces to the condition

$$\sigma_e = \sigma_0. \tag{7}$$

The perfectly plastic limit, on the other hand, is attained by letting $n \to \infty$ in Eq. (4), which thus reduces to the identity

$$\sigma_0 = \sigma_y. \tag{8}$$

The inverse stress-strain relations are readily computed to be

$$s_{ij} = \sigma_e \frac{2d_{ij}}{3 \dot{\epsilon}_e} = \frac{\partial \psi(\mathbf{d})}{\partial d_{ij}} \tag{9}$$

where

$$\dot{\epsilon}_e = \sqrt{(2/3)d_{ij}d_{ij}} \tag{10}$$

$$\sigma_e = \sigma_0 \left(\frac{\dot{\epsilon}_e}{\dot{\epsilon}_0}\right)^{1/m} \tag{11}$$

$$\psi(\mathbf{d}) = \frac{m\sigma_0 \dot{\epsilon}_0}{m+1} \left(\frac{\dot{\epsilon}_e}{\dot{\epsilon}_0}\right)^{(m+1)/m} = s_{ij}d_{ij} - \phi(\mathbf{s}). \tag{12}$$

For use in subsequent developments, we note that the rate of plastic work per unit volume corresponding to the foregoing model is

$$\dot{w}^p = s_{ij}d_{ij} = \sigma_e \dot{\epsilon}_e = \sigma_0 \dot{\epsilon}_0 \left(\frac{\dot{\epsilon}_e}{\dot{\epsilon}_0}\right)^{(m+1)/m} \tag{13}$$

and that, in the rate-independent limit, the dissipated plastic work per unit volume is given by

$$w^p = \int_0^{\epsilon_e} \sigma_e d\epsilon_e = \frac{n\sigma_y \epsilon_y}{n+1} \left(\frac{\epsilon_e}{\epsilon_y}\right)^{(n+1)/n}. \tag{14}$$

3 Governing Equations

Next, consider a spherical void of radius a in an infinite body subjected to remote pressure p. Assume that the void remains spherical throughout the deformation. Then, it follows from the volume constraint that

$$\frac{d}{dt} \frac{4\pi}{3} (r^3 - a^3) = 0 \tag{15}$$

where r is the radius of a material sphere deforming with the solid. This condition uniquely determines the velocity field over the current configuration, with the result that

$$\dot{r} = v_r = \frac{a^2}{r^2} \dot{a}. \tag{16}$$

The corresponding rate of effective Mises strain is computed to be

$$\dot{\epsilon}_e = |\partial v_r / \partial r| = \frac{2a^2}{r^3} |\dot{a}|. \tag{17}$$

Using again the volume constraint one finds

$$r^3 = a^3 + R^3 - a_0^3 \tag{18}$$

where R is the undeformed radius of points currently at r. Inserting (18) in (17) we find

$$\dot{\epsilon}_e = \frac{2a^2}{a^3 + R^3 - a_0^3} |\dot{a}| \tag{19}$$

which gives the effective strain rate as a function of position on the undeformed configuration. This expression is readily integrated to yield

$$\epsilon_e = \frac{2}{3} \log\left(1 + \frac{a^3 - a_0^3}{R^3}\right) = \frac{2}{3} \log\left(1 + \frac{a^3 - a_0^3}{r^3 - (a^3 - a_0^3)}\right) \tag{20}$$

where use has been made once more of the deformation mapping (18).

From (16), the kinetic energy of the body is computed to be

$$K = \int_a^\infty \frac{1}{2} \rho v_r^2 4\pi r^2 dr = \frac{3}{2} \rho \dot{a}^2 \frac{4\pi a^3}{3}. \tag{21}$$

The rate of plastic work, on the other hand, follows by integration of (13) over the body, which together with (19) yields

$$\dot{W}^p = \int_a^\infty \sigma_y \left(\frac{\epsilon_e}{\epsilon_y}\right)^{1/n}$$

$$\dot{\epsilon}_0 \left(\frac{\dot{\epsilon}_e}{\dot{\epsilon}_0}\right)^{(m+1)/m} 4\pi r^2 dr$$

$$= \sigma_y \dot{\epsilon}_0 \left(\frac{2}{3\epsilon_y}\right)^{1/n} \left(\frac{2\dot{a}}{\dot{\epsilon}_0 a}\right)^{(m+1)/m} f(a/a_0, m, n) \frac{4\pi a^3}{3} \tag{22}$$

where

$$f(a/a_0, m, n) = \int_1^\infty \left[\log\left(\frac{x}{x - 1 + a_0^3/a^3}\right)\right]^{1/n} x^{-(m+1)/m} dx. \tag{23}$$

In the perfectly plastic case, one has $f = m$ and (22) reduces to

$$\dot{W}^p = m\sigma_y \dot{\epsilon}_0 \left(\frac{2\dot{a}}{\dot{\epsilon}_0 a}\right)^{(m+1)/m} \frac{4\pi a^3}{3}. \tag{24}$$

In the rate-independent limit, it is possible to compute directly the dissipated plastic work from (14) and (20). Straightforward manipulations give

$$W^p = \int_a^\infty \frac{n\sigma_y \epsilon_y}{n+1} \left(\frac{\sigma_e}{\epsilon_y}\right)^{(n+1)/n} 4\pi r^2 dr \tag{25}$$

$$= \frac{n\sigma_y \epsilon_y}{n+1} \left(\frac{2}{3\epsilon_y}\right)^{(n+1)/n} g(a/a_0, n) \frac{4\pi a^3}{3} \tag{25}$$

where

$$g(a/a_0, n) = \int_1^\infty \left[\log\left(\frac{x}{x - 1 + a_0^3/a^3}\right)\right]^{(n+1)/n} dx. \tag{26}$$

Finally, the external power input into the system by the remotely applied pressure is

$$\dot{W} = 3p \frac{\dot{a}}{a} \frac{4\pi a^3}{3}. \tag{27}$$

If, for instance, the pressure history is of the type

$$p(t) = pH(t), \tag{28}$$

then Eq. (27) can be integrated to give

$$W = p \frac{a^3 - a_0^3}{a^3} \frac{4\pi a^3}{3}. \tag{29}$$

More generally, if p derives from a potential U through a relation of the type

$$p = U'(\Delta V) \tag{30}$$

where $\Delta V = 4\pi(a^3 - a_0^3)/3$ is the increase in volume of the void, then (29) generalizes to

$$W = U(4\pi(a^3 - a_0^3)/3). \tag{31}$$

The governing equation for a follows simply from the principle of conservation of energy, which in the present context states

$$\dot{W} = \dot{K} + \dot{W}^p. \tag{32}$$

Using (21), (22), and (27), Eq. (32) may be recast as

$$\frac{d}{dt}\left(\frac{3}{2}\rho \dot{a}^2 \frac{4\pi a^3}{3}\right) + \sigma_y \dot{\epsilon}_0 \left(\frac{2}{3\epsilon_y}\right)^{1/n}\left(\frac{2\dot{a}}{\dot{\epsilon}_0 a}\right)^{(m+1)/m}$$
$$\times f(a/a_0, m, n)\frac{4\pi a^3}{3} = 3p\,\frac{\dot{a}}{a}\frac{4\pi a^3}{3} \tag{33}$$

which defines a second-order, nonlinear ODE for $a(t)$, subject to the initial conditions

$$a(0) = a_0. \tag{34}$$

In addition, plastic irreversibility requires that the constraint

$$\dot{a} \geq 0 \tag{35}$$

be satisfied at all times.

In the rate-independent case with p deriving from a potential as in (30), a first integral of (33) can be obtained by recourse to the principle of conservation of energy in its form

$$W = K + W^p. \tag{36}$$

Using (21), (25), and (31), Eq. (36) becomes

$$\frac{3}{2}\rho \dot{a}^2 \frac{4\pi a^3}{3} + \frac{n\sigma_y \epsilon_y}{n+1}\left(\frac{2}{3\epsilon_y}\right)^{(n+1)/n} g(a/a_0, n)\frac{4\pi a^3}{3}$$
$$= U(4\pi(a^3 - a_0^3)/3). \tag{37}$$

This is a first-order ODE in the unknown a.

In subsequent developments it proves advantageous to use governing equations expressed in dimensionless form. To this end, introduce the material constant

$$k = \sigma_y \epsilon_y^{-1/n}\dot{\epsilon}_0^{-1/m}, \tag{38}$$

whereupon Eq. (4) becomes

$$\sigma_e = k\epsilon_e^{1/n}\dot{\epsilon}_e^{1/m}. \tag{39}$$

The scaling of the strain rates and stresses is effected by means of a suitably chosen reference strain rate $\dot{\epsilon}_{ref}$ and the associated reference stress

$$\sigma_{ref} = k(\dot{\epsilon}_{ref})^{1/m}. \tag{40}$$

The proper choice of $\dot{\epsilon}_{ref}$ depends on the problem under consideration. Dimensionless time $\tilde{t}$, radius $\tilde{a}$, velocity $\tilde{b}$, and pressure $\tilde{p}$ may be defined as

$$\begin{aligned}\tilde{t} &= \dot{\epsilon}_{ref}t \\ \tilde{a} &= a/a_0 \\ \tilde{b} &= d\tilde{a}/d\tilde{t} \\ \tilde{p} &= p/\sigma_{ref}.\end{aligned} \tag{41}$$

Using these definitions, the governing Eq. (33) may be reduced to the dimensionless form

$$\frac{d\tilde{b}}{dt} = -\frac{3}{2}\frac{\tilde{b}^2}{\tilde{a}} - \frac{C}{D}\frac{f(\tilde{a})}{\tilde{a}}\left(\frac{\tilde{b}}{\tilde{a}}\right)^{1-m} + \frac{1}{D}\frac{\tilde{p}}{\tilde{a}} \tag{42}$$

where f is given by Eq. (23) with the dependence on m and n omitted for simplicity of notation. The constants C and D are defined as

$$C = 3^{-1-1/n}2^{1+1/n+1/m} \tag{43}$$
$$D = \frac{\rho a_0^2 \dot{\epsilon}_{ref}^2}{\sigma_{ref}} = \frac{\rho a_0^2}{k}\dot{\epsilon}_{ref}^{2-1/m}.$$

Of particular interest here is the constant D. It is noted that D is proportional to the mass density ρ, and thus may be thought of as a measure of the effective inertia of the system. In particular, dynamic effects are absent when ρ and, hence, D vanish. It is also revealing that D increases with the initial void size a_0. Consequently, inertia effects may be expected to be more pronounced for large voids. Dynamic effects are also accentuated at higher strain rates $\dot{\epsilon}_{ref}$.

It is interesting to note that the constant D may be expressed as $D = (a_0/l)^2$, where

$$l = \frac{1}{\dot{\epsilon}_{ref}}\left[\frac{\sigma_y \epsilon_y^{-1/n}}{\rho}\left(\frac{\dot{\epsilon}_{ref}}{\dot{\epsilon}_0}\right)^{1/m}\right]^{1/2} \tag{44}$$

which has the dimensions of length. As already stated, dynamic effects may be anticipated to be important when $a_0 \gg l$. Note, however, that l is defined in terms of the reference rate of strain $\dot{\epsilon}_{ref}$ and, hence, does not define a characteristic length of the solid. In fact, the sole length scale of the problem is set by the initial void radius a_0.

Equation (42) is the most general governing equation considered here. Several particular cases merit special attention. In the quasi-static limit $D \to 0$, Eq. (42) reduces to

$$\frac{d\tilde{a}}{d\tilde{t}} = \tilde{a}\left(\frac{\tilde{p}}{Cf(\tilde{a})}\right)^m. \tag{45}$$

This equation is separable and can therefore be solved by one simple quadrature, with the result

$$\int_1^{\tilde{a}} f^m(\tilde{a})\frac{d\tilde{a}}{\tilde{a}} = \int_0^{\tilde{t}}(\tilde{p}/C)^m d\tilde{t} \tag{46}$$

which implicitly determines $\tilde{a}$ as a function of $\tilde{t}$.

If, on the other hand, the perfectly plastic limit $n \to \infty$ is approached, f reduces to m and (42) becomes

$$\frac{d\tilde{b}}{d\tilde{t}} = -\frac{3}{2}\frac{\tilde{b}^2}{\tilde{a}} - \frac{C}{D}\frac{m}{\tilde{a}}\left(\frac{\tilde{b}}{\tilde{a}}\right)^{1/m} + \frac{1}{D}\frac{\tilde{p}}{\tilde{a}} \tag{47}$$

whereas Eq. (45) governing quasi-static processes further simplifies to

$$\frac{d\tilde{a}}{d\tilde{t}} = \tilde{a}\left(\frac{\tilde{p}}{Cm}\right)^m. \tag{48}$$

This equation governs the expansion of a void in an ideally viscous matrix under quasi-static conditions. In this case, solution (46) specializes to

$$\tilde{a} = \exp\left[\int_0^{\tilde{t}}[\tilde{p}(\tilde{t})/Cm]^m d\tilde{t}\right]. \tag{49}$$

The rate-independent case is also noteworthy. Using normalization (41), Eq. (37) reduces to

$$\left(\frac{d\tilde{a}}{d\tilde{t}}\right)^2 + \frac{n}{n+1}\frac{2C}{3D}g(\tilde{a}) = \frac{2\tilde{p}}{3D}\left(1 - \frac{1}{\tilde{a}^3}\right) \tag{50}$$

where g is given by Eq. (26), with its dependence on n suppressed for simplicity, and the constant C follows from (43a) by taking $m \to \infty$, with the result

$$C = 3^{-1-1/n}2^{1+1/n}. \tag{51}$$

Equation (50) is written for constant pressure, so that the potential U is given by (29). Other pressure potentials can be handled similarly.

As a final special case, we consider the quasi-static rate-independent problem. The equation governing this case may be obtained from (50) by taking the limit $D \to 0$, which yields

$$g(\tilde{a}) = \frac{n+1}{n} \frac{\tilde{p}}{C} \left(1 - \frac{1}{\tilde{a}^3}\right). \tag{52}$$

For monotonic loading, this establishes a one-to-one relation between the pressure $\tilde{p}$ and the radius of the void $\tilde{a}$.

4 Short and Long-Term Behavior

In general, neither Eq. (32) nor its rate-independent specialization (50) can be solved analytically in closed form. However, the short and long-term character of the solutions can be established with some generality. We start by investigating the short-term behavior of the solutions in the general case of a strain hardening, rate sensitive solid under dynamic loading. This case is governed by Eq. (42). For values of $\tilde{a}$ close to 1, a minor computation shows that the function $f(\tilde{a})$ behaves as

$$f(\tilde{a}) \sim \frac{mn}{m+n} 3^{1/n} (\tilde{a}-1)^{1/n} + O((\tilde{a}-1)^{1+1/n}). \tag{53}$$

Using this approximation and setting $\tilde{a} \approx 1$, Eq. (42) reduces to

$$\frac{d\tilde{b}}{d\tilde{t}} = -\frac{3}{2} \tilde{b}^2 - \frac{C}{D} \frac{mn}{m+n} 3^{1/n} (\tilde{a}-1)^{1/n} \tilde{b}^{1/m} + \frac{\tilde{p}}{D}. \tag{54}$$

Assume that $\tilde{p}$ grows as $\tilde{t}^\gamma$ for small $\tilde{t}$, with $\gamma > 0$, and that, as a result, $\tilde{a} - 1$ grows as $\tilde{t}^\alpha$, for some α to be determined. Then, the first term of (54) grows as $\tilde{t}^{\alpha-2}$, the second as $\tilde{t}^{2\alpha-2}$, the third as $\tilde{t}^{\alpha/n+(\alpha-1)/m}$, and the fourth as $\tilde{t}^\gamma$. Evidently, the third term in (54) dominates the first two provided that

$$\alpha - 2 > \alpha/n + (\alpha-1)/m. \tag{55}$$

Then, the inertia terms in (54) are negligible and the short-term response is dominated by strain hardening and the rate sensitivity of the material. If condition (55) is satisfied, then from the third and fourth terms of (54) one computes

$$\alpha = \frac{(m\gamma+1)n}{n+m}. \tag{56}$$

Using this relation, (55) becomes

$$\gamma > \frac{2m+n}{mn-m-n}. \tag{57}$$

This condition places restrictions on the rate of growth of $\tilde{p}$ for the inertia terms to be negligible. If, for instance, we take $n = 10$ and $m = 100$, values which are of an order of magnitude commonly found in metals, then (57) necessitates $\gamma > 0.236$. However, this type of behavior is arguably unphysical since it requires the applied pressure to grow at an infinite rate initially.

Assume that (57) is indeed satisfied and that, consequently, the inertia terms are negligible for small times. Then Eq. (54) reduces to

$$(\tilde{a}-1)^{1/n} \left(\frac{d\tilde{a}}{d\tilde{t}}\right)^{1/m} = \frac{\tilde{p}}{A}, \quad A = \frac{mn}{m+n} 3^{1/n} C \tag{58}$$

valid for small $\tilde{t}$. This equation is separable and its general solution is

$$\tilde{a} \sim 1 + \left[\frac{m+n}{n} \int_0^{\tilde{t}} (\tilde{p}/A)^m d\tilde{t}\right]^{n/(m+n)}. \tag{59}$$

The rate-independent limit of (58) requires special handling. In the limit $m \to \infty$, (58) reduces to an algebraic relation. This can be solved for $\tilde{a}$ with the result

$$\tilde{a} \sim 1 + \frac{1}{3} \left(\frac{\tilde{p}}{Cn}\right)^n \tag{60}$$

valid for small times.

It is possible to derive an estimate of the time interval within which the short-term solution (59) dominates. Recall that (59) is obtained by neglecting the inertia terms in the governing Eq.

(54). Consequently, solution (59) ceases to be valid when the inertia terms become of the same order as the viscous term. As discussed earlier, the inertia term which grows fastest is $d\tilde{b}/d\tilde{t}$. Thus, the range of dominance of (59) may be estimated from the condition that

$$\left(\frac{d\tilde{b}}{d\tilde{t}}\right) \Big/ \left(\frac{C}{D} \frac{mn}{m+n} 3^{1/n} (\tilde{a}-1)^{1/n} \tilde{b}^{1/m}\right) \sim 1 \tag{61}$$

where the left-hand side is evaluated directly from (59). In the particular case in which the pressure history is of the form $\tilde{p} = P\tilde{t}^\gamma$, condition (61) reduces to

$$\tilde{t} \sim \left[\frac{D}{A} \left(\gamma \frac{mn}{m+n} + 1\right)^{1/n} \right. $$
$$\left. \times \left(\frac{n}{m+n}\right)^{(2m+n)/(m^2+mn)}\right]^{1/\{\gamma[1-mn/(m+n)]+(1+n)/n\}} \tag{62}$$

The interval of dominance of the rate-independent solution (60) follows simply as the limit of (62) as $m \to \infty$.

We next turn our attention to the long-term behavior of the solutions. For values of $\tilde{a} \gg 1$, the function $f(\tilde{a})$ behaves as

$$f \sim \int_1^\infty \left[\log\left(\frac{x}{x-1}\right)\right]^{1/n} x^{-(m+1)/m} dx \equiv f_\infty(m,n) \tag{63}$$

Using this approximation, Eq. (42) becomes

$$\frac{d\tilde{b}}{d\tilde{t}} = -\frac{3}{2} \frac{\tilde{b}^2}{\tilde{a}} - \frac{C}{D} \frac{f_\infty}{\tilde{a}} \left(\frac{\tilde{b}}{\tilde{a}}\right)^{1/m} + \frac{1}{D} \frac{\tilde{p}}{\tilde{a}}. \tag{64}$$

Next we proceed to establish the order of the terms involved in this equation. To this end, assume that $\tilde{p} \sim \tilde{t}^\gamma$ and $\tilde{a} \sim \tilde{t}^\alpha$ for large $\tilde{t}$. Then the first two terms in Eq. (64) go as $\tilde{t}^{\alpha-2}$, the third as $\tilde{t}^{-\alpha-1/m}$, and the fourth as $\tilde{t}^{\gamma-\alpha}$. Of the first two groups of terms, it is clear that the first dominates provided that

$$\alpha - 2 > -\alpha - 1/m. \tag{65}$$

Under these conditions, the dissipative term in (64) is negligible and the long-term response is dominated by inertia. Then, matching the dominant terms in Eq. (64), one finds

$$\alpha = 1 + \gamma/2. \tag{66}$$

Using this result, condition (65) becomes

$$\gamma > -1/m \tag{67}$$

If attention is restricted to nondecreasing pressure histories, for which $\gamma \geq 0$, then (67) is satisfied for all finite values of $m > 0$. Under these conditions, one concludes that the long-term response of the void is dominated by inertia.

Note, however, that inequality (67) is violated if $m \to \infty$ and $\gamma = 0$, i.e., for a rate-insensitive solid subjected to constant pressure. Then, all terms in (64) are of the same order in $\tilde{t}$. This case needs to be handled separately from those satisfying restriction (67). Start by noting that, for large $\tilde{a}$, the function $g(\tilde{a})$ behaves as

$$\lim_{\tilde{a} \to \infty} g(\tilde{a}) = \int_1^\infty \left[\log\left(\frac{x}{x-1}\right)\right]^{(n+1)/n} dx \equiv g_\infty. \tag{68}$$

Inserting this approximation into Eq. (50) specialized to the case $\tilde{a} \to \infty$, one finds

$$\left(\frac{d\tilde{a}}{d\tilde{t}}\right)^2 = \frac{2}{3D} (\tilde{p} - \tilde{p}_{cr}) \tag{69}$$

where the critical pressure $\tilde{p}$ is defined as

$$\tilde{p}_{cr} = \frac{n}{n+1} Cg_\infty. \tag{70}$$

Assume for now that the pressure is supercritical, i.e., $\tilde{p} \geq \tilde{p}_{cr}$. Then, solutions of (70) exist and are given by

$$\tilde{a} \sim \left[\frac{2}{3D}(\tilde{p} - \tilde{p}_{cr})\right]^{1/2} \tilde{t}. \tag{71}$$

It is observed that the growth of the void radius is linear in time. In particular, the void grows unboundedly under the effect of a supercritical pressure. The rate of growth becomes increasingly smaller as $\tilde{p} \to \tilde{p}_{cr}$. Beyond this limit, the plastic irreversibility constraint (35) is activated to prevent shrinkage of the void. Thus, for undercritical pressures $\tilde{p} < \tilde{p}_{cr}$, the void undergoes bounded growth, eventually attaining a stationary configuration.

Next, consider the case in which restriction (67) is satisfied. Eliminating the negligibly small dissipative term from (64), the equation governing the long-term behavior of the void is found to be

$$\frac{d\bar{b}}{d\tilde{t}} = -\frac{3}{2}\frac{\bar{b}^2}{\tilde{a}} + \frac{1}{D}\frac{\tilde{p}}{\tilde{a}}. \tag{72}$$

Define an effective time τ as

$$\tau = \int_0^{\tilde{t}} \sqrt{\tilde{p}}\,d\tilde{t}. \tag{73}$$

With the aid of this definition, Eq. (72) may be expressed as

$$\frac{d^2\tilde{a}}{d\tau^2} = -\frac{3}{2\tilde{a}}\left(\frac{d\tilde{a}}{d\tau}\right)^2 + \frac{1}{D\tilde{a}}. \tag{74}$$

A first integral of this equation is found to be

$$\left(\frac{d\tilde{a}}{d\tau}\right)^2 = \frac{2}{3D}\left(1 - \frac{1}{\tilde{a}^3}\right). \tag{75}$$

Specializing this expression to the case $\tilde{a} \to \infty$, it reduces to

$$\frac{d\tilde{a}}{d\tau} = \sqrt{\frac{2}{3D}} \tag{76}$$

which integrates to

$$\tilde{a} \sim \tau\sqrt{\frac{2}{3D}} = \sqrt{\frac{2}{3D}}\int_0^{\tilde{t}} \sqrt{\tilde{p}}\,d\tilde{t}. \tag{77}$$

Thus, the long-term growth of the void is linear in the effective time τ.

The result (77) ceases to apply in the quasi-static limit, i.e., when the constant $D \to 0$. The long-term behavior for this case, however, may be deduced directly from the exact solution (46). Replacing f by the long-term approximation f_∞ defined by (63), solution (46) reduces to

$$\tilde{a} \sim \exp\left[\int(\tilde{p}/Cf_\infty)^m d\tilde{t}\right]. \tag{78}$$

If, for instance, we take the pressure $\tilde{p}$ to be constant, then the long-term behavior of the void is one of exponential growth.

As in the short-term analysis, it is possible to estimate the time beyond which the long-term solutions start to dominate. Particularly revealing is the estimate for solution (77). This solution pertains to the dynamic rate-dependent case and is obtained by neglecting the viscous term in (64). Consequently, a lower bound for the range of validity of (77) may be derived by estimating the time at which inertia and viscous terms are of the same order, i.e.,

$$\left(\frac{d\bar{b}}{d\tilde{t}} + \frac{3}{2}\frac{\bar{b}^2}{\tilde{a}}\right)\bigg/\left[\frac{C}{D}\frac{f_\infty}{\tilde{a}}\left(\frac{\bar{b}}{\tilde{a}}\right)^{1/m}\right] \sim 1 \tag{79}$$

where the left-hand side is evaluated from (77). Setting, for simplicity, $\tilde{p} = P\tilde{t}^\gamma$, condition (79) reduces to

$$\tilde{t} \sim \left[\frac{3C}{2P}\frac{(\gamma+1)^{(m+1)/m}}{\gamma + 3(\gamma+1)/2}\right]^{1/(2\gamma+1/m)} \tag{80}$$

For the case of constant pressure, $\gamma = 0$, estimate (80) specializes to

$$\tilde{t} \sim \left(\frac{Cf_\infty}{P}\right)^m \tag{81}$$

This estimate has some revealing consequences. For n large, f_∞ is well approximated by the ideally plastic limit $f_\infty \approx m$. Inserting this approximation into (81), we obtain

$$\tilde{t} \sim \left(\frac{Cm}{P}\right)^m \tag{82}$$

Consider further the case of a body whose behavior is nearly rate independent, i.e., for which m is large. As m is increased, the right-hand side of (82) grows as m^m and soon becomes extremely large. Under these conditions, the long-term solution (77) may fail to become dominant within any physically reasonable time scale. Instead, following the initial transients, the response of the void may be expected to be close to that predicted by the *rate-independent* long-term solution (71). This supposition is indeed born out by the numerical simulations presented in the next section.

5 Response to Impulse Loading

In this section, we present full numerical solutions which confirm the trends revealed by the preceding short and long-term analyses, and provide useful insights into the nature of the intermediate transients. We consider a ramp variation in time of the applied pressure of the form

$$p(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ p_\infty t/t_1, & \text{if } 0 \leq t \leq t_1; \\ p_\infty, & \text{if } t_1 \leq t. \end{cases} \tag{83}$$

Thus, the remote pressure is assumed to increase linearly from $p = 0$ at $t = 0$ to a final value of p_∞ at t_1 and to remain constant thereafter.

The calculations are carried out for the following choice of material constants: $\rho = 7800$ kg/m^3, $k = 10^8$ SI, $m = 100$, $n = 10$, and $a_0 = 10$ μm. These parameters are roughly representative of some types of steels. The terminal pressure is taken to be $p_\infty = 750$ MPa and the rise time $t_1 = 10^{-2}$ μs. If we set the reference strain rate $\dot{\epsilon}_{ref} = 1/t_1$, then the effective inertia constant (43b) takes the value $D = 64.88$, whereas the characteristic length (44) becomes $l = 1.24$ μm. Since $a_0 \gg l$, inertia effects are expected to be significant. Also note that the critical pressure is in this case $p_{cr} = 705$ MPa. Thus, the applied terminal pressure p_∞ is supercritical and unbounded void growth takes place.

Time histories of void expansion are shown in Figs. 1 and 2 for the following cases: (i) dynamic rate dependent, (ii) quasi-static rate dependent, and (iii) dynamic rate independent. This comparison brings out differences between the effects of inertia and rate dependence. Thus, in the quasi-static rate-dependent case, inertia is neglected altogether, which brings rate dependency effects to the forefront. In the dynamic rate-independent analysis, the individual effect of inertia is singled out. Finally, the dynamic rate-dependent solution exhibits the combined effect of inertia and rate dependence.

The computed short-term solutions are depicted in Fig. 1. In the dynamic rate-dependent case, the short-term response is given by Eq. (59). For the present choice of parameters, this reduces to

$$\tilde{a} \sim 1 + 0.531\tilde{t}^{9.18}. \tag{84}$$

The interval of dominance of this solution as estimated from (62) is $\tilde{t} \sim 0.701$, or, in absolute time, $t \sim 0.701 \times 10^{-2}$ μs. In the quasi-static rate-dependent case, the short-term response is also given by (84). The short-term dynamic rate-independent solution, on the other hand, is given by Eq. (60). For the values of the parameters adopted in the calculations, (60) specializes to
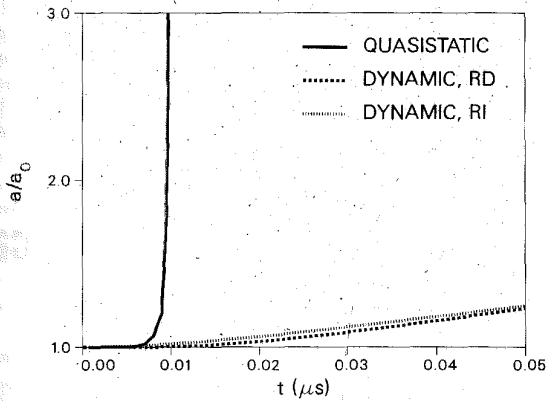
VOID RADIUS HISTORIES

Fig. 1 Initial stages of the numerical solutions for ramp loading and the following choice of parameters: $\rho = 7800$ kg/m^3, $k = 10^8$ SI, $m = 100$, $n = 10$, and $a_0 = 10$ μm, $p_\infty = 750$ MPa, and $t_1 = 10^{-2}$ μs. The critical pressure corresponding to these material constants is $p_{cr} = 705$ MPa. Solutions are obtained for the following material cases: (a) quasi-static, rate dependent, (b) dynamic, rate dependent, and (c) dynamic, rate independent.
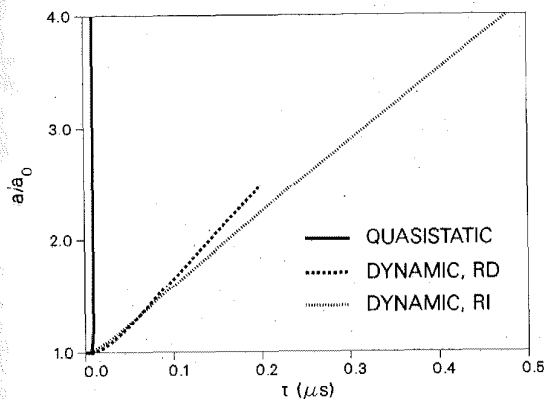
VOID RADIUS HISTORIES

Fig. 2 Long-term solutions for the problem of Fig. 1.

$$\tilde{a} \sim 1 + 1.62\tilde{t}^{10} \tag{85}$$

which, according to estimate (62), dominates up to $\tilde{t} \sim 0.744$, or, in absolute time, $t \sim 0.744 \times 10^{-2}$ μs.

Solutions (84) and (85) have a crossover point at $\tilde{t} = 0.255$. Prior to this time, the void radius is larger in the rate-dependent solution. Beyond the crossover time, the rate-independent solution gives the larger expansion, Fig. 1. Thus, as expected, the rate sensitivity of the material has the effect of retarding the expansion of the void.

Figure 2 shows the long-term behavior of the void. The long term response for the dynamic rate-dependent case is given by Eq. (77), which here reduces to

$$\tilde{a} \sim 0.253\tilde{t}. \tag{86}$$

However, estimate (82) gives $\tilde{t} \sim 22 \times 10^{100}$ as the onset of dominance of (86). Not surprisingly, the computed slope $d\tilde{a}/d\tilde{t} \approx 0.085$ does not match the analytical value of 0.253 derived from (86). Instead, as surmised in the preceding section, the computed growth rate is closer to that predicted by the long-

term rate-independent solution (71), which for the present example particularizes to

$$\tilde{a} \sim 0.0620\tilde{t}. \tag{87}$$

The computed growth rate for the actual rate-independent case, $d\tilde{a}/d\tilde{t} \approx 0.0625$, closely matches the analytical solution (87).

Both the rate-dependent and rate-independent dynamic solutions exhibit linear growth following the initial transient, Fig. 2. Here, however, we encounter the seemingly paradoxical situation that the rate-dependent solution grows faster than the rate-independent one. This behavior is counterintuitive, since one would reasonably expect the presence of rate sensitivity to retard the growth of the void relative to the rate insensitive solution.

Finally, the long-term behavior of the quasi-static rate-dependent solution is given by (78), and corresponds to a exponential growth of the void. Indeed, the numerical solution exhibits a sharp upturn as the critical pressure is exceeded, Fig. 2, in contrast with the slower growth rate predicted by the dynamic solutions. This comparison underscores the potentially stabilizing effect of microinertia on void growth, both in rate sensitive and rate insensitive solids.

6 Discussion

A striking feature of the solutions presented above is the widely varying predictions which are obtained depending on the level of description adopted in the analysis. Both inertia, strain hardening, and rate sensitivity have a marked influence on the short and long-term solutions, as well as on the intermediate transients, sometimes with counterintuitive consequences. For instance, under dynamic conditions, a rate-dependent description of the solid may sometimes result in faster void growth than predicted by the corresponding rate independent solution.

While the equations governing the expansion of a void cannot be generally solved in closed form, even with the simplifying assumption adopted in the present analysis, a good deal of progress can be made toward characterizing the short and long term responses. Our analysis shows that microinertia effects are particularly significant for voids which are larger than a characteristic dimension which depends on both the mechanical properties of the solid and the rate of expansion. We have also shown that, whereas the early stages of deformation are dominated by viscous effects, inertia tends to dominate the long-term response of the void. This sets limits on the applicability of phenomenological theories of voided elastic-plastic solids which neglect microinertia. The solutions given here provide possible avenues for extending these theories so as to include material inertia on the physical scale of the ductile void growth mechanism.

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