Actions of rigid groups on UHF-algebras

Eusebio Gardella, Martino Lupini

PII: S0022-1236(17)30458-5
DOI: https://doi.org/10.1016/j.jfa.2017.12.005
Reference: YJFAN 7929

To appear in: *Journal of Functional Analysis*

Received date: 28 August 2017
Accepted date: 13 December 2017

Please cite this article in press as: E. Gardella, M. Lupini, Actions of rigid groups on UHF-algebras, *J. Funct. Anal.* (2018), https://doi.org/10.1016/j.jfa.2017.12.005

ACTIONS OF RIGID GROUPS ON UHF-ALGEBRAS

EUSEBIO GARDELLA AND MARTINO LUPINI

Abstract. Let Λ be a countably infinite property (T) group, and let D be UHF-algebra of infinite type. We prove that there exists a continuum of pairwise non (weakly) cocycle conjugate, strongly outer actions of Λ on D. The proof consists in assigning, to any second countable abelian pro-p group G, a strongly outer action of Λ on D whose (weak) cocycle conjugacy class completely remembers the group G. The group G is reconstructed from the action through its (weak) 1-cohomology set endowed with a canonical pairing function.

Our construction also shows the following stronger statement: the relations of conjugacy, cocycle conjugacy, and weak cocycle conjugacy of strongly outer actions of Λ on D are complete analytic sets, and in particular not Borel. The same conclusions hold more generally when Λ is only assumed to contain an infinite subgroup with relative property (T), and for actions on (not necessarily simple) separable, nuclear, UHF-absorbing, self-absorbing C*-algebras with at least one trace.

Finally, we use the techniques of this paper to construct outer actions on R with prescribed cohomology. Precisely, for every infinite property (T) group Λ, and for every countable abelian group Γ, we construct an outer action of Λ on R whose 1-cohomology is isomorphic to Γ.

1. Introduction

Classification of group actions is a fundamental problem in operator algebras, and positive results are both scarce and useful. The subject is far more developed on the von Neumann algebra side, and it was started with Connes' classification of periodic automorphisms on the hyperfinite II_1 factor R; see [9]. Further generalizations to arbitrary automorphisms [8] and finite group actions [25] quickly followed, and these advances culminated in Ocneanu's work on amenable group actions on R [37]. A consequence of his results is that for any amenable group Λ, any two outer actions of Λ on R are cocycle conjugate. A converse to Ocneanu's theorem was proved by Jones in [26], and this result was considerably strengthened in a recent work by Brothier and Vaes in [4], building on [39]. We summarize these results in the following rather strong dichotomy for outer actions on R:

Theorem. *(Connes, Jones, Ocneanu, Brothier-Vaes). Let Λ be a countable group.*

(1) If Λ is amenable, then any two outer actions of Λ on R are cocycle conjugate.
(2) If Λ is not amenable, then there exist uncountably many non-cocycle conjugate outer actions of Λ on R. In fact, the relation of cocycle conjugacy of such actions is complete analytic.

Ocneanu's work served as a motivation for exploring analogs of the uniqueness statement in (1) in the context of C*-algebras. The first issue is to find the appropriate C*-analog of R. UHF-algebras of infinite type have historically played this role, as they can be regarded as "strong" C*-analogs of R. A "weak" analog is the Jiang-Su algebra $\mathcal{Z}$ (see [23]), which has also been studied in relation to uniqueness of actions of certain amenable groups [31, 33, 42]. This work focuses mostly on UHF-algebras. Even though the existence of plenty of projections makes their study easier, classification results for actions are relatively difficult to obtain because of K-theoretical restrictions; see [21].

In [3], Bratteli, Evans and Kishimoto studied a family of outer actions of $\mathbb{Z}$ on the CAR algebra. It follows from their results that no analog of Ocneanu's result can hold for outer actions. However, they provided evidence for the fact that a uniqueness result may hold if one assumes that not only the action is outer, but also its extension to the weak closure in the GNS representation is outer (this is called *strong outerness*).

Recall that a unital C*-algebra D is said to be *strongly self-absorbing* if it is infinite dimensional and there is an isomorphism $\varphi\colon D \to D \otimes_{\min} D$ which is approximately unitarily equivalent to the first tensor factor

Date: December 18, 2017.

2000 *Mathematics Subject Classification.* Primary 46L55, 54H05; Secondary 03E15, 37A55 .

Key words and phrases. Property (T), 1-cohomology, profinite group, Rokhlin property, model action, Borel complexity, conjugacy, cocycle conjugacy, complete analytic set.

The first-named author was partially funded by SFB 878 *Groups, Geometry and Actions*, and by a postdoctoral fellowship from the Humboldt Foundation. The second-named author was partially supported by the NSF Grant DMS-1600186. This work was initiated during a visit of the authors at the Mathematisches Forschungsinstitut Oberwolfach in August 2016 supported by an Oberwolfach Leibnitz Fellowship. The authors gratefully acknowledge the hospitality of the Institute.

embedding. The only known examples of stably finite strongly self-absorbing C*-algebras are the UHF-algebras of infinite type, and the Jiang-Su algebra $\mathcal{Z}$, and in fact it is conjectured that the list is complete.

Several results in the literature, which are reviewed below, suggest that the following may be true (part (1) below has also been independently conjectured by Szabo in [42]):

Conjecture A. *Let D be a stably finite strongly self-absorbing C*-algebra and let Λ be a torsion-free countable group.*

(1) If Λ is amenable, then any two strongly outer actions of Λ on D are cocycle conjugate.
(2) If Λ is not amenable, then there exist uncountably many non-cocycle conjugate strongly outer actions of Λ on D. Even more, the relation of cocycle conjugacy of such actions is complete analytic.

The reason for excluding groups with torsion is the fact that automorphisms of finite order, even when they are strongly outer, generate unexpected phenomena at the level of K-theory which obstruct any uniqueness-type result as in (1). For instance, it is easy to construct $\mathbb{Z}_2$-actions on the CAR algebra $\otimes_{n\in\mathbb{N}} M_2$, which are strongly outer but not cocycle conjugate. As an example, one can take the nontrivial group element to act as the following infinite tensor products:

$$\bigotimes_{n\in\mathbb{N}} \mathrm{Ad}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \quad \text{and} \quad \bigotimes_{n\in\mathbb{N}} \mathrm{Ad}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

As mentioned before, the cases of D being a UHF-algebra of infinite type or the Jiang-Su algebra are the most relevant ones. Part (1) of the conjecture above has been confirmed in a number of particular cases: for UHF-algebras, the case $\Lambda = \mathbb{Z}$ was proved by Kishimoto in [31], while the case $\Lambda = \mathbb{Z}^N$ was obtained by Matui in [33]. For the Jiang-Su algebra $\mathcal{Z}$, the case of $\Lambda = \mathbb{Z}$ was considered by Sato in [41], while Matui-Sato proved the case $\Lambda = \mathbb{Z}^2$ and $\Lambda = \mathbb{Z} \rtimes_{-1} \mathbb{Z}$ in [34] and [35]. More recently, and inspired by the work of Winter on $\mathcal{Z}$- and UHF-stable classification of C*-algebras [48], and for *elementary* amenable groups, Szabo [42] reduced the case $D = \mathcal{Z}$ to the case when D is an infinite type UHF algebra. He also showed that part (1) of Conjecture A holds for a group Λ if and only if holds for all the finitely-generated subgroups of Λ. In particular, it follows from this and Matui's result that part (1) of Conjecture A holds when D is either a UHF-algebra or $\mathcal{Z}$, and when Λ is a torsion-free *abelian* group.

We now turn to part (2) in the above conjecture. It should be mentioned that it easy to see using Jones' argument from [26] that, for any nonamenable group Λ, there exist at least two strongly outer actions of Λ on any finite strongly self-absorbing C*-algebra. Beyond this, nothing was known until now concerning the number of cocycle conjugacy classes (or the complexity of the cocycle conjugacy relation) for strongly outer actions of nonamenable groups on finite strongly self-absorbing C*-algebras.

In the present paper, we initiate the study of actions of nonamenable groups on UHF-algebras, and we make the first contributions to part (2) in the above conjecture. Our main result is as follows:

Theorem B. *(See Corollary 4.8 and Corollary 5.11). Let D be a UHF-algebra of infinite type, and let Λ be a countable group containing an infinite subgroup with relative property (T). Then there exist uncountably many non-cocycle conjugate strongly outer actions of Λ on D. Indeed, the relation of cocycle conjugacy of such actions is complete analytic.*

(Our result holds for a more general class of not necessarily simple C*-algebras; see Theorem 4.7 for the precise statement.)

It is worth mentioning that our results cannot be derived from those of Brothier-Vaes. First, there is no general method for producing an action on a UHF-algebra from an action on R. Moreover, no obvious modification of the construction in [4] seems to produce an action on a UHF-algebra. (They use the fact that the crossed product of R by a Bernoulli shift of an amenable torsion-free group is isomorphic to R, and the UHF-analog of this fact is far from true.) Even more, the actions we construct in Theorem B are shown to remain cocycle inequivalent in the weak closure of D. Hence, our results imply the result of Brothier-Vaes for groups with relative property (T).

The assertion that the relation of cocycle conjugacy of free actions of Λ on A is a complete analytic set can be interpreted as follows. There does not exist an explicit uniform procedure that, given two strongly outer actions of Λ on D, runs for countably many (but possibly transfinitely many) steps, at each step testing membership in some given open sets, and at the end decides whether the given actions are cocycle conjugate or not. In fact, the problem of deciding whether two such actions are cocycle conjugate is as hard as testing membership in any analytic set. Similar conclusions hold for conjugacy and weak cocycle conjugacy. For a more detailed discussion on this interpretation, see [11, Section 2.4].

The proof of our main theorem consists in assigning, to any second countable abelian pro-p group G, a strongly outer action of Λ on D whose weak cocycle conjugacy class completely "remembers" the group G. Using Popa's superrigidity results from [39], the group G is reconstructed from this action via its (weak, localized) 1-cohomology set, endowed with a canonical (2-sorted) group structure. The starting point of our construction is a canonical model action of G on the UHF-algebra M_{p^∞}, which we construct in Section 3. The rest of the construction can be seen as a C*-algebra analogue of the construction of factors of measure-preserving Bernoulli actions due to Popa [38] and Törnquist [46]; see also [11].

The methods used in this construction are not specific to our context, and can be used to compute (weak) 1-cohomology sets in other interesting cases. As an instance of this, the last section of this paper is devoted to constructing actions of infinite property (T) groups on R with prescribed (weak) 1-cohomology. In this context, these cohomology sets do not have a canonical group structure. The actions we construct are self-absorbing (in a strong sense), and there is a canonical 'pairing' function $m^\alpha \colon H^1_w(\alpha)\times H^1_w(\alpha) \to H^1_w(\alpha\otimes\alpha)$; see Theorem 2.10. Even this by itself does not guarantee the existence of a group structure, but this turns out to be the case for the actions we construct.

More specifically, for infinite groups with property (T), we prove the following analog of the main result of [38] for actions on R (the result we prove is somewhat more general):

Theorem C. *Let Λ be an infinite countable property (T) group, and let Γ be any countable abelian group. Then there exist an outer action $\alpha\colon \Lambda \to \mathrm{Aut}(R)$ and bijections $\eta\colon H^1_w(\alpha) \to \Gamma$ and $\eta^{(2)}\colon H^1_w(\alpha\otimes\alpha) \to \Gamma$ making the following diagram commute:*

$$\begin{tikzcd}
H^1_w(\alpha)\times H^1_w(\alpha) \arrow[r, "\eta\times\eta"] \arrow[d, "m^\alpha"'] & \Gamma\times\Gamma \arrow[d, "\text{multiplication}"] \\
H^1_w(\alpha\otimes\alpha) \arrow[r, "\eta^{(2)}"'] & \Gamma
\end{tikzcd}$$

This gives a different proof of Theorem B of [4] in the case that Λ has (a subgroup with the relative) property (T). For comparison, observe that Ocneanu's result implies that all outer actions of amenable groups on R have canonically isomorphic cohomology.

In the following, all topological groups are supposed to be *Hausdorff* and *second countable*. All tensor products of C*-algebras are supposed to be minimal (also called spatial); see [2, Section II.9]. If A is a C*-algebra and S is a finite set, then we let $A^{\otimes S}$ be the (minimal) tensor product of a family of copies of A indexed by S. Similarly, when A is unital and X is a countable set, then we let $A^{\otimes X}$ denote the limit of the direct system $\left(A^{\otimes S}\right)$, where S varies in the collection of finite subsets of X ordered by containment, and the connective maps are the canonical unital *-homomorphisms $\iota_{S,T}\colon A^{\otimes S} \to A^{\otimes T}$ for $S\subset T\subset X$. In the von Neumann-algebraic setting, we will only consider tensor products of tracial von Neumann algebras with respect to distinguished normal tracial states, which we denote by $\overline{\otimes}$; see [2, Section III.3.1].

Acknowledgments. We are grateful to Samuel Coskey, Łukasz Grabowski, Daniel Hoff, Alexander Kechris, André Nies, Stefaan Vaes, and Stuart White for many helpful conversations. Particularly, we would like to thank Alexander Kechris for suggesting a proof of Proposition 5.7, and Stuart White for suggesting the formulation of Lemma 2.23 below. Finally, we thank the referee for their careful reading of the manuscript, and for suggesting numerous improvements.

2. Preliminary notions on group actions

2.1. **Actions of groups on tracial von Neumann algebras.** We recall some terminology about group actions on von Neumann algebras. A *tracial von Neumann algebra* is a pair (M,τ), where M is a von Neumann algebra and τ is a normal tracial state on τ. We denote by $\mathrm{Aut}(M,\tau)$ the group of τ-preserving automorphisms of M. Let Λ be a discrete group. An *action* of Λ on (M,τ) is a group homomorphism $\alpha\colon \Lambda \to \mathrm{Aut}(M,\tau)$. An automorphism $\theta\in\mathrm{Aut}(M,\tau)$ is said to be *inner* if there exists a unitary $u\in M$ with $\theta(x) = uxu^*$ for all $x\in M$. It is said to be *outer* if it is not inner, and *properly outer* if for every θ-invariant projection $p\in M$, the restriction of θ to pMp is outer; see [44, Definition XVII.1.1].

Remark 2.1. As it is remarked in [30, Section 4], in the definition of properly outer autorphism one can equivalently only consider θ-invariant *central* projections; see also the comment after Theorem XVII.1.2 in [44]. In particular, an automorphism of a *factor* is properly outer if and only if it is outer.

Let $\theta_0 \in \mathrm{Aut}(M_0, \tau_0)$ and $\theta_1 \in \mathrm{Aut}(M_1, \tau_1)$ be automorphisms of tracial von Neumann algebras. It is shown in [27, Corollary 1.12] that, if either θ_0 or θ_1 is properly outer, then $\theta_0 \otimes \theta_1$ is a properly outer automorphism of $(M_0 \overline{\otimes} M_1, \tau_0 \otimes \tau_1)$.

Definition 2.2. Let (M, τ) be a tracial von Neumann algebra and let Λ be a discrete group. An action $\alpha \colon \Lambda \to \mathrm{Aut}(M, \tau)$ is called:

(1) *ergodic*, if the fixed point algebra $M^\alpha = \{x \in M \colon \alpha_\gamma(x) = x \text{ for all } \gamma \in \Lambda\}$ contains only the scalar multiples of the identity; see [43, Definition 7.3];
(2) *weakly mixing*, if for any finite subset $F \subseteq M$ and $\varepsilon > 0$, there exists $\gamma \in \Lambda$ such that
$$|\tau(x\alpha_\gamma(y)) - \tau(x)\tau(y)| < \varepsilon$$
for every $x, y \in F$; see [47, Definition D.1];
(3) *mixing*, if for every $a, b \in M$ one has $\tau(a\alpha_\gamma(b)) \to \tau(a)\tau(b)$ for $\gamma \to \infty$; see [47, Definition D.1];
(4) *outer*, if α_γ is not inner for every $\gamma \in \Lambda \setminus \{1\}$;
(5) *free*, if α_γ is properly outer for every $\gamma \in \Lambda \setminus \{1\}$; see [30, Subsection 4.1].

Observe that any free action is, in particular, outer. When M is a factor, the converse holds in view of Remark 2.1.

Remark 2.3. An action α is weakly mixing if and only if the only finite-dimensional vector subspace of $L^2(M, \tau)$ which is invariant under the representation associated with α is the space of scalar multiples of the identity; see [39, Proposition 2.4.2.] and [47, Proposition D.2].

Let α and β be actions of Λ on tracial von Neumann algebras (M_0, τ_0) and (M_1, τ_1), respectively. We let $(M_0 \overline{\otimes} M_1, \tau_0 \otimes \tau_1)$ be the tensor product of M_0 and M_1 with respect to the normal tracial states τ_0, τ_1 [2, Section III.3.1]. Define $\alpha \otimes \beta \colon \Lambda \to \mathrm{Aut}(M_0 \overline{\otimes} M_1, \tau_0 \otimes \tau_1)$ to be the action given by $(\alpha \otimes \beta)_\gamma = \alpha_\gamma \otimes \beta_\gamma$ for $\gamma \in \Lambda$. It is easy to check that $\alpha \otimes \beta$ is (weakly) mixing if both α and β are.

Definition 2.4. Let π a unitary representation of Λ on a Hilbert space H. Following [29], we say that π has *almost invariant vectors*, and write $1_\Lambda \prec \pi$, if for every $\varepsilon > 0$ and finite subset $F \subseteq \Lambda$, there exists a vector $\xi \in H$ such that $\|\pi(\gamma)\xi - \xi\| \leq \varepsilon$ for every $\gamma \in F$. A unitary representation $\pi \colon \Lambda \to U(H)$ is said to be a c_0-*representation* if for every $\xi, \eta \in H$, the function $\gamma \mapsto \langle \pi(\gamma)\xi, \eta \rangle$ belongs to $c_0(\Lambda)$.

Let X be a countable set endowed with an action of Λ. We say that the action is *amenable* if it satisfies the following *Følner condition*: for any finite subset $Q \subseteq \Lambda$ and $\varepsilon > 0$, there exists a finite subset $F \subseteq X$ such that $|\gamma F \triangle F| \leq \varepsilon |F|$ for every $\gamma \in Q$. For an action $\Lambda \curvearrowright X$, we consider the corresponding *left regular representation* $\lambda_X \colon \Lambda \to U(\ell^2(X))$ determined by $\lambda_X(\gamma)(\delta_x) = \delta_{\gamma^{-1}x}$ for $\gamma \in \Lambda$ and $x \in X$. Theorem 1.1 in [29] asserts that $\Lambda \curvearrowright X$ is amenable if and only if $1_\Lambda \prec \lambda_X$.

For a tracial von Neumann algebra (M, τ), we denote by $(M, \tau)^{\overline{\otimes} X}$ the tensor product of copies of M indexed by X with respect to the normal tracial state τ; see [2, III.3.1]. Then $(M, \tau)^{\overline{\otimes} X}$ carries a canonical trace obtained from τ, which we still denote by τ. We denote by $M^{\odot X}$ the algebraic tensor product, which is dense in $(M, \tau)^{\overline{\otimes} X}$. If Y is a subset of X, then we canonically identify $M^{\overline{\otimes} Y}$ with a subalgebra of $(M, \tau)^{\overline{\otimes} X}$, and $M^{\odot Y}$ with a subalgebra of $M^{\odot X}$.

Notation 2.5. *Let X be a countable set endowed with an action $\Lambda \curvearrowright X$, and let (M, τ) be a tracial von Neumann algebra. We denote by $\beta_{\Lambda \curvearrowright X, M} \colon \Lambda \to \mathrm{Aut}((M, \tau)^{\overline{\otimes} X})$ the associated Bernoulli $(\Lambda \curvearrowright X)$-action with base (M, τ), defined by permuting the indices according to the action of Λ on X.*

Example 2.6. *In the context above, when $M = L^\infty(Z, \mu)$ for a probability space (Z, μ) and $\tau(f) = \int f d\mu$, one has $(M, \tau)^{\overline{\otimes} X} = L^\infty(Z^X, \mu^X)$ with trace $\tau(f) = \int f d\mu^X$. The action on $(M, \tau)^{\overline{\otimes} X}$ corresponds in this case to the Bernoulli action of Λ on (Z^X, μ^X) as considered in* [29].

We denote by κ the corresponding Koopman representation of Λ on $L^2(M^{\overline{\otimes} X}, \tau)$, and by κ_0 the restriction of κ to the orthogonal complement in $L^2(M^{\overline{\otimes} X}, \tau)$ of the space of scalar multiples of the identity.

The following characterization of mixing Bernoulli actions is well know; see [39, Lemma 2.4.3] and [29, Proposition 2.1 and Proposition 2.3] for the commutative case.

Proposition 2.7. *Let X be a countable set endowed with an action of Λ, and let (M,τ) be a tracial von Neumann algebra with a projection $p\in M$ such that $0<\tau(p)<1$. Then the Bernoulli action $\beta_{\Lambda\curvearrowright X,M}\colon\Lambda\to\mathrm{Aut}((M,\tau)^{\overline{\otimes}X})$ is mixing if and only if the stabilizers of the action $\Lambda\curvearrowright X$ are finite.*

2.2. **Actions of groups on C*-algebras.** Let Λ be a discrete group, and let A be a unital C*-algebra. Write $\mathrm{Aut}(A)$ for the automorphism group of A. An *action* of Λ on A is a group homomorphism $\alpha\colon\Lambda\to\mathrm{Aut}(A)$. In this case, we also say that the pair (A,α) is a Λ-C*-algebra. We denote by A^α the *fixed point algebra* $A^\alpha=\{a\in A\colon\alpha_\gamma(a)=a \text{ for all } \gamma\in\Lambda\}$. We say that elements x,y of A are *equivalent modulo scalars*, and write $x=y\mathrm{mod}\mathbb{C}$, if $x=\lambda y$ for some $\lambda\in\mathbb{C}\setminus\{0\}$. We denote by $U(A)$ the unitary group of A.

Definition 2.8. Let $\alpha\colon\Lambda\to\mathrm{Aut}(A)$ be an action of a discrete group Λ on a unital C*-algebra A, and let $u\colon\Lambda\to U(A)$ be a function.

1. We say that u is a 1-cocycle for α if $u_\gamma\alpha_\gamma(u_\rho)=u_{\gamma\rho}$ for every $\gamma,\rho\in\Lambda$.
2. We say that u is a *weak* 1-cocycle if $u_\gamma\alpha_\gamma(u_\rho)=u_{\gamma\rho}\ \mathrm{mod}\mathbb{C}$ for every $\gamma,\rho\in\Lambda$.

The notion of weak 1-cocycles allows one to define the weak 1-cohomology of actions. We will mostly use it for actions on tracial von Neumann algebras, but the definition can be given in general.

Definition 2.9. Let $\alpha\colon\Lambda\to\mathrm{Aut}(A)$ be an action of a discrete group Λ on a unital C*-algebra A. Following [39], we say that two weak 1-cocycles u and u' for α are *weakly cohomologous* (or cohomologous modulo scalars), if there exists a unitary $v\in U(A)$ such that $u'_\gamma=v^*u_\gamma\alpha_\gamma(v)\ \mathrm{mod}\mathbb{C}$ for every $\gamma\in\Lambda$. We say that u is a *weak coboundary* if it is weakly cohomologous to the weak 1-cocycle constantly equal to 1.

We denote by $Z^1_w(\alpha)$ the set of weak 1-cocycles for α. The relation of being weakly 1-cohomologous is an equivalence relation on $Z^1_w(\alpha)$, and we let $H^1_w(\alpha)$ be the corresponding quotient set, called the *weak cohomology set*. The class of the weak 1-cocycle u will be denoted by $[u]$.

Let us use the notation as in the definition above. If A is abelian, then the product of two weak 1-cocycles for α is again a weak 1-cocycle for α, and thus $H^1_w(\alpha)$ can be given a canonical group structure. In general, however, one can not define a group operation on $H^1_w(\alpha)$ in a similar fashion. To make up for the lack of multiplication in the 1-cohomology set $H^1_w(\alpha)$, we consider a natural "two-sorted group structure" on $H^1_w(\alpha)$, given by a pairing function $H^1_w(\alpha)\times H^1_w(\alpha)\to H^1_w(\alpha\otimes\alpha)$. Such a pairing function will be used to encode the group operation of a given countable group.

Definition 2.10. Let (M,τ) be a tracial von Neumann algebra, let $\alpha\colon\Lambda\to\mathrm{Aut}(M,\tau)$ be an action, and denote by $\alpha\otimes\alpha$ be the diagonal action of Λ on $M\overline{\otimes}M$. Then there is a canonical function

$$\widetilde{m}^\alpha\colon Z^1_w(\alpha)\times Z^1_w(\alpha)\to Z^1_w(\alpha\otimes\alpha)$$

given by $\widetilde{m}^\alpha(u,w)=u\otimes w$ for $u,w\in Z^1_w(\alpha)$. Observe that if u is weakly cohomologous to w and u' is weakly cohomologous to w', then $u\otimes w$ is weakly cohomologous to $u'\otimes w'$. Therefore, the map $\widetilde{m}^\alpha$ induces pairing function $m^\alpha\colon H^1_w(\alpha)\times H^1_w(\alpha)\to H^1_w(\alpha\otimes\alpha)$.

Following [46] one can also define the notion of weak cohomology set *localized* to a subgroup Δ of Λ, as follows. Say that two weak 1-cocycles u and u' for an action $\alpha\colon\Lambda\to\mathrm{Aut}(A)$ are Δ-*locally weakly cohomologous* if there exists a unitary $v\in U(A)$ such that $u'_\gamma=v^*u_\gamma\alpha_\gamma(v)\ \mathrm{mod}\mathbb{C}$ for every $\gamma\in\Delta$. Similarly, u is a Δ-*local weak coboundary* if there exists $v\in U(A)$ such that $u_\gamma=v^*\alpha_\gamma(v)\ \mathrm{mod}\mathbb{C}$ for every $\gamma\in\Delta$.

Definition 2.11. The Δ-localized weak cohomology set $H^1_{\Delta,w}(\alpha)$ is the quotient of $Z^1_w(\alpha)$ by the relation of being Δ-locally weakly cohomologous, endowed with the pairing function $m^\alpha_\Delta\colon H^1_{\Delta,w}(\alpha)\times H^1_{\Delta,w}(\alpha)\to H^1_{\Delta,w}(\alpha\otimes\alpha)$, given by $m^\alpha_\Delta([u],[u'])=[u\otimes u']$ for $u,v\in Z^1_w(\alpha)$.

Given a weak 1-cocycle u for an action $\alpha\colon\Lambda\to\mathrm{Aut}(A)$, one can define the *cocycle perturbation* $\alpha^u\colon\Lambda\to\mathrm{Aut}(A)$ of α by setting $\alpha^u_\gamma=\mathrm{Ad}(u_\gamma)\circ\alpha_\gamma$ for every $\gamma\in\Lambda$. (The weak cocycle condition implies that α^u is also an action.)

Definition 2.12. Let Λ be a countable discrete group, and let α and β be actions of Λ on unital C*-algebras A and B, respectively.

1. We say that α and β are *conjugate* if there exists an isomorphism $\psi\colon A\to B$ such that $\psi\circ\alpha_\gamma=\beta_\gamma\circ\psi$ for every $\gamma\in\Lambda$,
2. We say that α and β are *cocycle conjugate* if β is conjugate to α^u for some 1-cocycle u for α,
3. We say that α and β are *weakly cocycle conjugate* if β is conjugate to α^u for some weak 1-cocycle u for α.

Remark 2.13. Let $\alpha, \beta \colon \Lambda \to \mathrm{Aut}(A)$ be actions of a discrete group Λ on a C*-algebra A. It is easy to check that if α and β are (weakly) cocycle conjugate, then there is a canonical bijection between the Δ-localized (weak) 1-cohomology sets of α and β, for any subgroup Δ of Λ.

When the actions α and β are conjugate, we also say that the Λ-C*-algebras (A, α) and (B, β) are *equivariantly isomorphic*. An *equivariant unital embedding* from (A, α) to (B, β) is an injective unital *-homomorphism $\phi \colon A \to B$ satisfying $\phi \circ \alpha_\gamma = \beta_\gamma \circ \phi$ for every $\gamma \in \Lambda$.

Suppose that A is a unital C*-algebra. A linear functional τ on A is said to be a *trace* if $\tau(1) = \|\tau\| = 1$ and $\tau(ab) = \tau(ba)$ for every $a, b \in A$. We let $\mathrm{T}(A)$ be the simplex of traces on A. Suppose that τ is a trace on A, θ is an automorphism of A, and α is an action of Λ on A. We say that τ is θ-invariant if $\tau \circ \theta = \tau$, and that it is α-invariant if it is α_γ-invariant for every $\gamma \in \Lambda$. If τ is α-invariant, then we also say that α is τ-preserving. We let $\mathrm{T}(A)^\alpha \subseteq \mathrm{T}(A)$ be the closed convex subset of α-invariant traces. Observe that, if Λ is amenable, then $\mathrm{T}(A)^\alpha$ is nonempty whenever $\mathrm{T}(A)$ is nonempty.

For a trace τ on A, consider the corresponding *left regular representation* $\pi_\tau \colon A \to B(L^2(A, \tau))$ obtained via the GNS construction. We let $\overline{A}^\tau$ be the closure of $\pi_\tau(A)$ inside $B(L^2(A, \tau))$ with respect to the weak operator topology. We regard $\overline{A}^\tau$ as a tracial von Neumann algebra, endowed with the unique extension of τ to $\overline{A}^\tau$. The unit ball of A is dense in the unit ball of $\overline{A}^\tau$ with respect to the 2-norm $\|a\|_\tau = \tau(a^*a)^{1/2}$ defined by τ. If α is a τ-preserving action of Λ on A, then it induces a canonical action $\overline{\alpha}^\tau \colon \Lambda \to \mathrm{Aut}(\overline{A}^\tau, \tau)$.

Notation 2.14. *As in the case of actions on tracial von Neumann algebras, given a unital C*-algebra A, we denote by $\beta_{\Lambda \curvearrowright X, A} \colon \Lambda \to \mathrm{Aut}(A^{\otimes X})$ the Bernoulli $(\Lambda \curvearrowright X)$-action with base A induced by an action $\Lambda \curvearrowright X$ of a countable discrete group Λ on a countable set X.*

The following lemma is well known, and we will use it without further reference.

Lemma 2.15. Let A be a C*-algebra, let Λ be a countable discrete group, let $\Lambda \curvearrowright X$ be an action of Λ on a countable set X, and let τ_0 be a trace on A. Denote by τ the trace $\tau_0^{\otimes X}$ on $A^{\otimes X}$. Then the action $\overline{\beta}^\tau_{\Lambda \curvearrowright X, A}$ is conjugate to the von Neumann-algebraic Bernoulli action $\beta_{\Lambda \curvearrowright X, \overline{A}^{\tau_0}}$.

Definition 2.16. Let Λ be a discrete group, let A be a unital C*-algebra, and let $\tau \in \mathrm{T}(A)$.

(1) We say that α is *strongly outer* if for every $\gamma \in \Lambda \setminus \{1\}$ and for every α_γ-invariant trace σ on A, the weak extension $\overline{\alpha}^\sigma_\gamma$ is outer [34, Definition 2.7];
(2) If α is τ-preserving, then we say that α is (*weakly*) *τ-mixing* if $\overline{\alpha}^\tau \colon \Lambda \to \mathrm{Aut}(\overline{A}^\tau, \tau)$ is (weakly) mixing in the sense of Definition 2.2.
(3) We say that two τ-preserving actions $\alpha, \beta \colon \Lambda \to \mathrm{Aut}(A)$ are *τ-conjugate* (respectively, *cocycle τ-conjugate*, or *weakly cocycle τ-conjugate*), if $\overline{\alpha}^\tau$ and $\overline{\beta}^\tau$ are conjugate (respectively, *cocycle conjugate*, or *weakly cocycle conjugate*), in the sense of Definition 2.12.

Remark 2.17. Suppose that α is an automorphism of a C*-algebra A, σ is an α-invariant trace, $\overline{\alpha}$ is the canonical extension of α to $\overline{A}^\sigma$, and $p \in \overline{A}^\sigma$ is a $\overline{\alpha}$-invariant central projection. Then defining $\tau(x) = \sigma(px)/\sigma(p)$ gives an α-invariant (normalized) trace on A, such that the canonical extension of α to $\overline{A}^\tau$ can be identified with the restriction of $\overline{\alpha}$ to $p\overline{A}^\sigma$. In view of this and Remark 2.1, one can equivalently replace "outer" with "properly outer" in the definition of strongly outer action. We will tacitly use this fact in the rest of the paper.

The notion of strongly action from Definition 2.16 recovers the notion of free action on a locally compact Hausdorff space when one considers actions on commutative C*-algebras, as the next proposition shows.

Proposition 2.18. *Let $\Lambda \curvearrowright X$ be a topological action of a discrete group Λ on a locally compact Hausdorff space X, and denote by $\alpha \colon \Lambda \to \mathrm{Aut}(C_0(X))$ the induced action. Then α is strongly outer if and only if $\Lambda \curvearrowright X$ is free.*

Proof. Suppose that α is strongly outer, and let $\gamma \in \Lambda \setminus \{1\}$. To reach a contradiction, assume that there exists $x \in X$ such that $\gamma \cdot x = x$. Then the Dirac probability measure concentrated on $\{x\}$ is Borel and γ-invariant. This measure induces, via integration, an α_γ-invariant trace τ_x on $C_0(X)$. Since $\overline{C_0(X)}^{\tau_x}$ is isomorphic to $\mathbb{C}$, the weak extension of α_γ cannot be outer. This contradiction implies that $\Lambda \curvearrowright X$ is free.

Conversely, assume that $\Lambda \curvearrowright X$ is free and let $\gamma \in \Lambda \setminus \{1\}$. Let τ be an α_γ-invariant trace on $C_0(X)$. Then τ is given by integration with respect to a Borel probability measure μ on X which satisfies $\mu(\gamma \cdot U) = \mu(U)$ for every open subset $U \subseteq X$. Moreover, $\overline{C_0(X)}^\tau$ is isomorphic to $L^\infty(X, \mu)$. Suppose, to reach a contradiction, that $\overline{\alpha}_\gamma$ is inner (and hence trivial). It follows that the set of fixed point of α_γ has μ-measure 1 and, in particular, it is nonempty. This is a contradiction. $\square$

Let A be a C*-algebra. Then $\mathrm{Aut}(A)$ is a topological group when endowed with the topology of pointwise convergence. An action of a topological group G on A is said to be *continuous* if it is continuous as a group homomorphism $G \to \mathrm{Aut}(A)$. *In the following, all the actions of topological groups are supposed to be continuous.* The following is a natural example of a continuous action:

Notation 2.19. *For a compact group G, let $C(G)$ be the commutative C*-algebra of continuous complex-valued functions on G. We denote by $\mathtt{Lt}^G\colon G \to \mathrm{Aut}(C(G))$ the canonical action by left translation, given by $\mathtt{Lt}^G_g(f)(h) = f(g^{-1}h)$ for $g, h \in G$ and $f \in C(G)$. When the group G is clear from the context, we write $\mathtt{Lt}$ instead of $\mathtt{Lt}^G$ to lighten the notation.*

We recall the definition of the Rokhlin property for compact group actions on unital C*-algebras from [20, Definition 3.2]. The formulation given here is taken from [17, Lemma 3.7].

Definition 2.20. Let G be a compact group, let A be a unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. We say that α has the *Rokhlin property* if for every $\varepsilon > 0$, for every finite subset $S \subseteq C(G)$, and every finite subset $F \subseteq A$, there exists a unital completely positive linear map $\psi\colon C(G) \to A$ satisfying

- $\|(\psi \circ \mathtt{Lt}_g)(f) - (\alpha_g \circ \psi)(f)\| < \varepsilon$ for all $f \in S$ and all $g \in G$;
- $\|\psi(f)a - a\psi(f)\| < \varepsilon$ for all $f \in S$ and all $a \in F$;
- $\|\psi(f_0 f_1) - \psi(f_0)\psi(f_1)\| < \varepsilon$ for any $f, f_0, f_1 \in S$.

2.3. **Direct and inverse limit constructions.**

Definition 2.21. An *inverse system of topological groups* is a family $(G_i, \pi_{i,j})_{i,j\in I}$, where I is an ordered set, G_i are topological groups, and $\pi_{i,j}\colon G_j \to G_i$, for $i \leq j$, is a surjective continuous group homomorphism. Given such a countable inverse system, we denote by $G = \varprojlim(G_i, \pi_{i,j})$ the inverse limit, together with the canonical continuous surjective group homomorphisms $\pi_{i,\infty}\colon G \to G_i$ for $i \in I$.

Similarly, a *direct system of unital C*-algebras* is a family $(A_i, \iota_{i,j})_{i,j\in I}$, where I is an ordered set, A_i is a unital C*-algebra, and $\iota_{i,j}\colon A_i \to A_j$, for $i \leq j$, is an injective unital *-homomorphism. We denote by $A = \varinjlim(A_i, \iota_{i,j})$ the corresponding direct limit, together with the canonical injective unital *-homomorphisms $\iota_{i,\infty}\colon A_i \to A$ for $i \in I$.

Next, we will see that one can construct actions of inverse limits of groups on direct limits of C*-algebras in a natural way.

Lemma 2.22. Let I be an ordered set, let $(A_i, \iota_{i,j})_{i,j\in I}$ be a direct system of unital C*-algebras with limit A, and let $(G_i, \pi_{i,j})_{i,j\in I}$ be an inverse system of topological groups with limit G. For every $i \in I$, let $\alpha^{(i)}\colon G_i \to \mathrm{Aut}(A_i)$ be an action satisfying

$$\alpha^{(j)}_g \circ \iota_{i,j} = \iota_{i,j} \circ \alpha^{(i)}_{\pi_{i,j}(g)} \tag{1}$$

for every $i, j \in I$ with $i \leq j$ and every $g \in G_j$. Then there exists a unique action $\alpha\colon G \to \mathrm{Aut}(A)$ such that

$$\alpha_g \circ \iota_{i,\infty} = \iota_{i,\infty} \circ \alpha^{(i)}_{\pi_{i,\infty}(g)} \tag{2}$$

for every $i \in I$ and $g \in G$.

Proof. It is clear that Equation (2) defines a unique action of G on A in view of Equation (1). We check that such an action is continuous. For every $i \in I$, we identify A_i with its image under $\iota_{i,\infty}$.

Fix $\varepsilon > 0$, let $i \in I$ and let $F \subseteq A_i$ be a finite subset. Since $\alpha^{(i)}$ is continuous, there exists a neighborhood U of the identity of G_i such that $\|\alpha^{(i)}_g(x) - x\| < \varepsilon$ for every $x \in F$ and $g \in U$. Set $V = \pi^{-1}_{i,\infty}(U)$, which is a neighborhood of the identity of G. For every $g \in V$ and $x \in F$, we have

$$\left\|(\alpha^{(i)}_g \circ \iota_{i,\infty})(x) - \iota_{i,\infty}(x)\right\| \leq \left\|\alpha^{(i)}_{\pi_{i,\infty}(g)}(x) - x\right\| \leq \varepsilon.$$

Since $i \in I$ is arbitrary and $\bigcup_{i\in I} \iota_{i,\infty}(A_i)$ is dense in A, this concludes the proof. □

The definition of an amenable trace on a unital C*-algebra A can be found in [6, Definition 6.2.1]. Observe that the set $\mathrm{T}_{\mathrm{am}}(A)$ of amenable traces on A is a face of the simplex $\mathrm{T}(A)$ of traces on A. Particularly, any extreme point of $\mathrm{T}_{\mathrm{am}}(A)$ is also an extreme point of $\mathrm{T}(A)$. Recall that every trace on a nuclear C*-algebra is amenable [5, Theorem 4.2.1]. The notion of locally reflexive C*-algebra can be found in [5, Definition 4.3.1]. Every exact C*-algebra is locally reflexive [6, Corollary 9.4.1]. The following result is folklore, and we thank Stuart White for suggesting this formulation.

Lemma 2.23. Let A be a separable, locally reflexive C*-algebra, and let τ be a nonzero trace on A. Then $\overline{A}^{\tau}$ is isomorphic to the hyperfinite II_1-factor with separable predual R if and only if τ is amenable and extreme, and A is infinite dimensional.

Proof. It is well known that a trace is extreme if and only if $\overline{A}^{\tau}$ is a factor (in which case it will be of type II_1 or I_n for some $n \in \mathbb{N}$). If τ is amenable, then $\overline{A}^{\tau}$ is hyperfinite by [5, Corollary 4.3.4], because A is locally reflexive. Finally, since A is infinite dimensional, $\overline{A}^{\tau}$ must be isomorphic to R by its uniqueness. Conversely, assume that $\overline{A}^{\tau} \cong R$. Since the trace on R is amenable, its restriction to A, which agrees with τ, must also be amenable. Infinite dimensionality of A is clear, so the proof is complete. □

2.4. **Subgroups with relative property (T).** In this subsection, we recall the definition of relative property (T) for a subgroup Δ of a discrete group Λ.

Definition 2.24. Let Λ be a discrete group and let Δ be a subgroup. We say that Δ has *relative property* (T) (of Kazhdan–Margulis), if there exist a finite subset $F \subseteq \Lambda$ and $\varepsilon > 0$ such that whenever $u\colon \Lambda \to U(H)$ is a unitary representation of Λ on a Hilbert space H, and $\xi \in H$ is a unit vector satisfying $\|u_\gamma(\xi) - \xi\| < \varepsilon$ for all $\gamma \in F$, then H has a nonzero vector which is fixed by the restriction of u to Δ.

For $\Lambda = \Delta$, the definition above recovers the notion of property (T) group. More generally, it is clear that if either Λ *or* Δ has property (T), then $\Delta \subseteq \Lambda$ has relative property (T). There also exist inclusions of groups with relative property (T), for which neither the subgroup nor the containing group have property (T). One such example is $\mathbb{Z}^2 \subseteq \mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$. (One can also replace $\mathrm{SL}_2(\mathbb{Z})$ with any of its nonamenable subgroups, by a result of Burger.) Subgroups with relative property (T) have been studied, among others, by Margulis [32], Burger [7], and Jolissaint [24].

3. Model action for profinite abelian groups

3.1. **Profinite groups.** Let $\mathcal{C}$ be a class of groups closed under quotients, finite products, and subgroups. A *pro-*$\mathcal{C}$ group is a topological group G that can be realized as the inverse limit of groups from $\mathcal{C}$. Particularly, a group G is said to be

- *profinite* if it is pro-$\mathcal{C}$ for the class $\mathcal{C}$ of finite groups;
- *pro-p* if it is a pro-$\mathcal{C}$ for the class $\mathcal{C}$ of finite p-groups.

It is clear that a profinite group is abelian if and only if it is pro-$\mathcal{C}$ for the class $\mathcal{C}$ of finite abelian groups. Similarly, a pro-p group is abelian if and only if it is pro-$\mathcal{C}$ for the class of finite abelian p-groups. Equivalent characterizations of pro-$\mathcal{C}$ groups can be found in [40, Theorem 2.1.3]. In particular, these characterizations show that a topological group is profinite if and only if it is totally disconnected, if and only if the identity of G has a basis of neighborhoods made of open subgroups [40, Theorem 2.1.3]. Recall that, by [40, Lemma 2.1.2], a subgroup of a profinite abelian group is open if and only if it is closed and has finite index.

Denote by $\mathcal{P}$ the set of prime numbers. A *supernatural number* is a function $\boldsymbol{n}\colon \mathcal{P} \to \{0, 1, 2, \dots, \infty\}$. Recall also that a separable UHF-algebra is a unital C*-algebra that is obtained as the direct limit of a countable direct system of full matrix algebras. By a fundamental result of Glimm, any separable UHF-algebra has the form $\otimes_{p\in\mathcal{P}} M_p^{\otimes \boldsymbol{n}(p)}$ for some supernatural number $\boldsymbol{n}$. The supernatural number can be obtained intrinsically from the given separable UHF-algebra, and it is a complete invariant for separable UHF-algebras up to *-isomorphism. Next, we associate to each second countable profinite group, a canonical supernatural number.

Definition 3.1. Let G be a profinite group. The *supernatural number associated with* G is defined by

$$\boldsymbol{n}_G(p) = \begin{cases} \infty & \text{if } p \text{ divides the index of an open subgroup of } G, \\ 0 & \text{otherwise.} \end{cases}$$

We let D_G be the UHF-algebra $\bigotimes_{p\in\mathcal{P}} M_p^{\otimes \boldsymbol{n}_G(p)}$ corresponding to $\boldsymbol{n}_G$.

It is clear that G is a pro-p group if and only if $\boldsymbol{n}_G = p^\infty$ or, equivalently, $D_G = M_{p^\infty}$.

3.2. **Model action.** The goal of this subsection is to construct a model action δ^G of G on D_G with the Rokhlin property; see Theorem 3.5. This action will be crucial in the next section, where for certain nonamenable groups, we construct many non weakly cocycle conjugate strongly outer actions on UHF-algebras.

Remark 3.2. Suppose that G is finite. Then $D_G = M_{|G|^\infty}$, and the model action in this case is rather easy to describe. If $\lambda^G\colon G \to U(\ell^2(G))$ denotes the left regular representation, then the model action $\delta^G\colon G \to \mathrm{Aut}(D_G)$ is given by $\delta^G_g = \mathrm{Ad}(\lambda^G_g)^{\otimes\mathbb{N}}$.

The following is folklore; see, for example, [21, Subsection 2.4] (but note that the reference given there only proves the statement about fixed point algebras for $G = \mathbb{Z}_p$). Since we have not been able to find a reference, we include a short proof for the convenience of the reader. (The proof given below uses the classification results of [22], but a direct and elementary, although longer, proof can also be given.)

Lemma 3.3. Let G be a finite group. Then the action $\delta^G \colon G \to \mathrm{Aut}(D_G)$ described in the remark above has the Rokhlin property. When G is abelian, then $D_G \rtimes_{\delta^G} G$ is naturally isomorphic to $D_{\widehat{G}}$, in such a way that the dual action $\widehat{\delta^G} \colon \widehat{G} \to \mathrm{Aut}(D_G \rtimes_{\delta^G} G)$ is conjugate to $\delta^{\widehat{G}}$.

Proof. It is easy to see that δ^G has the Rokhlin property, since there is a unital and equivariant embedding $C(G) \to B(\ell^2(G)) \cong M_{|G|}$ as multiplication operators. The crossed product $D_G \rtimes_{\delta^G} G$ is a UHF-algebra by [16, Corollary 3.11]. Since there are unital inclusions

$$D_G \subseteq D_G \rtimes_{\delta^G} G \subseteq \mathcal{K}(\ell^2(G)) \otimes D_G \cong D_G,$$

it follows that D_G and $D_G \rtimes_{\delta^G} G$ have the same corresponding supernatural number, and hence they are isomorphic. In particular, $D_G \rtimes_{\delta^G} G$ is isomorphic to $D_{\widehat{G}}$.

It remains to identify the dual action of δ^G. Observe that δ^G is approximately representable in the sense of [22, Definition 3.6], as one may take the unitaries $u(g)$ appearing in said definition to be $u(g) = \lambda_g^{\otimes n}$ for a large enough $n \in \mathbb{N}$. By [22, part (2) of Lemma 3.8], it follows that the dual action $\widehat{\delta^G}$ has the Rokhlin property as an action of $\widehat{G}$ on $D_G \rtimes_{\delta^G} G \cong D_{\widehat{G}}$. Since for $\chi \in \widehat{G}$, the automorphisms $\delta^{\widehat{G}}_\chi$ and $\widehat{\delta^G_\chi}$ are both approximately inner, it follows from [22, Theorem 3.5] that $\widehat{\delta^G}$ is conjugate to $\delta^{\widehat{G}}$, and the proof is finished. $\square$

The model action δ^G for an arbitrary profinite abelian group G will be constructed from its finite quotients using Lemma 3.3. The following proposition is the inductive step in the construction.

Proposition 3.4. *Let H be a finite abelian group, let N be a subgroup of H, and set $Q = H/N$ with quotient map $\pi \colon H \to Q$. Denote by $\delta^H \colon H \to \mathrm{Aut}(D_H)$ and $\delta^Q \colon Q \to \mathrm{Aut}(D_Q)$ the actions described in Remark 3.2. Then there is an injective unital homomorphism $\iota \colon D_Q \to D_H$ satisfying*

$$\delta^H_h \circ \iota = \iota \circ \delta^Q_{\pi(h)}$$

for all $h \in H$.

Proof. For a finite group K, we denote by $\{\xi^K_k\}_{k \in K}$ the canonical basis of $\ell^2(K)$. Also, when K is abelian, we write $\widehat{K}$ for its dual group, and an element of $\widehat{K}$ will be denoted, with a slight abuse of notation, by $\widehat{k}$. Observe that $\widehat{Q}$ is a subgroup of $\widehat{H}$, and that $\widehat{H}/\widehat{Q} \cong \widehat{N}$. Fix a section $s \colon \widehat{N} \to \widehat{H}$. Then s induces a unitary $U \colon \ell^2(\widehat{Q}) \otimes \ell^2(\widehat{N}) \to \ell^2(\widehat{H})$ given by

$$U(\xi^{\widehat{Q}}_{\widehat{q}} \otimes \xi^{\widehat{N}}_{\widehat{n}}) = \xi^{\widehat{H}}_{\widehat{q}s(\widehat{n})}$$

for every $\widehat{q} \in \widehat{Q}$ and every $\widehat{n} \in \widehat{N}$. Define a unital embedding $\varphi \colon B(\ell^2(\widehat{Q})) \to B(\ell^2(\widehat{H}))$ by

$$\varphi(a)(\xi^{\widehat{H}}_{\widehat{q}s(\widehat{n})}) = U(a(\xi^{\widehat{Q}}_{\widehat{q}}) \otimes \xi^{\widehat{N}}_{\widehat{n}})$$

for every $a \in B(\ell^2(\widehat{Q}))$, for every $\widehat{q} \in \widehat{Q}$ and every $\widehat{n} \in \widehat{N}$. Let $\widehat{q} \in \widehat{Q}$. We claim that $\varphi(\lambda^{\widehat{Q}}_{\widehat{q}}) = \lambda^{\widehat{H}}_{\widehat{q}}$. To see this, let $\widehat{p} \in \widehat{Q}$ and let $\widehat{n} \in \widehat{N}$. Then

$$\varphi(\lambda^{\widehat{Q}}_{\widehat{q}})(\xi^{\widehat{H}}_{\widehat{p}s(\widehat{n})}) = U(\lambda^{\widehat{Q}}_{\widehat{q}}(\xi^{\widehat{Q}}_{\widehat{p}}) \otimes \xi^{\widehat{N}}_{\widehat{n}}) = U(\xi^{\widehat{Q}}_{\widehat{q}\widehat{p}} \otimes \xi^{\widehat{N}}_{\widehat{n}}) = \xi^{\widehat{H}}_{\widehat{q}\widehat{p}s(\widehat{n})} = \lambda^{\widehat{H}}_{\widehat{q}}(\xi^{\widehat{H}}_{\widehat{p}s(\widehat{n})}).$$

This proves the claim. It follows that φ induces, upon taking its infinite tensor product, a unital injective homomorphism $\psi \colon D_{\widehat{Q}} \to D_{\widehat{H}}$, which moreover satisfies

$$\psi \circ \delta^{\widehat{Q}}_{\widehat{q}} = \delta^{\widehat{H}}_{\widehat{q}} \circ \psi$$

for all $\widehat{q} \in \widehat{Q}$, by the claim above. After taking crossed products by $\widehat{Q}$ and $\widehat{H}$, and using Lemma 3.3, we obtain a unital embedding $\iota \colon D_Q \to D_H$, which satisfies $\delta^H_h \circ \iota = \iota \circ \delta^Q_{\pi(h)}$ for all $h \in H$. This completes the proof. $\square$

Here is the main result of this section.

Theorem 3.5. *Let G be a second countable, abelian, profinite group. Let D_G denote the UHF-algebra associated with G as in Definition 3.1. Then there exists a canonical action $\delta^G \colon G \to \mathrm{Aut}(D_G)$ with the following properties.*

(1) *There exists an equivariant unital embedding* $(C(G), \mathtt{Lt}^G) \to (D_G, \delta^G)$.
(2) $(D_G^{\otimes\mathbb{N}}, (\delta^G)^{\otimes\mathbb{N}})$ *is equivariantly isomorphic to* (D_G, δ^G).
(3) δ^G *has the Rokhlin property.*
(4) *The fixed point algebra* $D_G^{\delta^G}$ *is isomorphic to* D_G.

Moreover, δ^G *is—up to conjugacy—the* unique *action of* G *on* D_G *with the Rokhlin property. Furthermore, if* E *is a unital C*-algebra with* $D_G \otimes E \cong E$ *and* $\beta\colon G \to \mathrm{Aut}(E)$ *is an action with the Rokhlin property, then* β *is conjugate to* $\delta^G \otimes \beta$.

Proof. Since the group G is fixed, we drop the subscript G from all algebras and actions, in order to lighten the notation. We first construct the action, and then show that it has the desired properties. Let $\mathcal{V}$ be the collection of open subgroups of G, and observe that $\mathcal{V}$ is countable. Define an inverse system $(G_i, \pi_{i,j})_{i,j\in I}$ of finite groups as follows. Set $I = \mathcal{V}$ ordered by *reverse inclusion*. For $i \in I$, let $G_i = G/i$, and for $i, j \in I$ with $i \leq j$, let $\pi_{i,j}\colon G_j \to G_i$ be the canonical quotient map. Then $G = \varinjlim(G_i, \pi_{i,j})$. By Proposition 3.4, for every $i, j \in I$ with $i \leq j$, there exists a unital embedding $\iota_{i,j}\colon D_{G_i} \to D_{G_j}$ satisfying $\delta_g^{G_j} \circ \iota_{i,j} = \iota_{i,j} \circ \delta^{G_i}_{\pi_{i,j}(g)}$ for all $g \in G_j$. Observe that D can be identified with the direct limit of the UHF-algebras D_{G_i}, for $i \in I$, with connective maps $\iota_{i,j}$ for $i, j \in I$ with $i \leq j$. By Lemma 2.22, there exists an induced action $\delta\colon G \to \mathrm{Aut}(D)$ given by

$$\delta_g(\iota_{i,\infty}(a)) = \iota_{i,\infty}(\delta^{G_i}_{\pi_{i,\infty}(g)}(a))$$

for all $g \in G$, for all $i \in I$, and all $a \in D_{G_i}$.

(1): Let $i \in I$. Observe that the restriction of δ^{G_i} to $\mathbb{C} \rtimes \widehat{G_i} \cong C(G_i)$ is naturally conjugate to the left translation action $\mathtt{Lt}^{G_i}$. In particular, there is a unital equivariant embedding $\phi_i\colon (C(G_i), \mathtt{Lt}^{G_i}) \to (D_{G_i}, \delta^{G_i})$. For $i, j \in I$ with $i \leq j$, denote by $\pi^*_{i,j}\colon C(G_i) \to C(G_j)$ the injective unital *-homomorphism given by $\pi^*_{i,j}(f) = f \circ \pi_{i,j}$ for all $f \in C(G_i)$. Then the maps ϕ_i are easily seen to satisfy $\iota_{i,j} \circ \phi_i = \phi_j \circ \pi^*_{i,j}$ for all $i, j \in I$ with $i \leq j$. By the universal property of direct limits, it follows that there exists a unital equivariant embedding $(C(G), \mathtt{Lt}) \to (D, \delta)$, as desired.

(2): This is an easy consequence of the fact that δ^{G_i} is conjugate to $(\delta^{G_i})^{\otimes\mathbb{N}}$ for every $i \in I$.

(3): This is an immediate consequence of (1) and (2).

(4): By part (1) of Corollary 3.11 in [16], the fixed point algebra D^δ is a UHF-algebra, and it absorbs D by [16, Theorem 4.3]. Since it is obviously unitally embedded in D, it follows from [45, Proposition 5.12] that D^δ is isomorphic to D. The last part of the theorem is a consequence of [18, Theorem X.4.5]. □

4. Uncountably many actions

In this section, given a countable group Λ containing an infinite subgroup Δ with relative property (T)—which we fix once and for all—and given a UHF-algebra A of infinite type, we construct uncountably many strongly outer actions of Λ on A, which are not weakly cocycle conjugate; see Theorem 4.7. In fact, we perform the construction for an arbitrary separable unital C*-algebra A satisfying the following properties: A is locally reflexive, M_{p^∞}-absorbing for some prime p, has an amenable trace, and is isomorphic to its infinite tensor product $A^{\otimes\mathbb{N}}$.

Let G be a second countable abelian pro-p group, and let $\delta^G\colon G \to \mathrm{Aut}(D_G)$ be the action constructed in Theorem 3.5. We denote in the same fashion the extension of δ^G the weak closure of D_G with respect to its unique trace, which can be regarded as an action on the hyperfinite II_1 factor R. In the following lemma, we will use the pairing function from Definition 2.11. We write Γ for the Pontryagin dual of G, and we let $m^\Gamma\colon \Gamma \times \Gamma \to \Gamma$ be the multiplication operation. Recall also that $\mathtt{Lt}\colon G \to \mathrm{Aut}(C(G))$ denotes the action by left translation.

Lemma 4.1. Let N be the algebra $(R \otimes R)^{\overline{\otimes}\Lambda}$, and let ρ be the action $\left(\delta^G \otimes \mathrm{id}_R\right)^{\otimes\Lambda}$ of G on N. Define B to be the fixed point algebra N^ρ of ρ, which is isomorphic to the hyperfinite II_1 factor by Theorem 3.5. Consider the Bernoulli $(\Lambda \curvearrowright \Lambda)$-action β with base $R \otimes R$, which is an action on N, and its restriction α to B. Then there exist bijections $\eta\colon H^1_{\Delta,w}(\alpha) \to \Gamma$ and $\eta^{(2)}\colon H^1_{\Delta,w}(\alpha \otimes \alpha) \to \Gamma$ satisfying

$$\eta^{(2)} \circ m^\alpha_\Delta = m^\Gamma \circ (\eta \times \eta).$$

Proof. Let ζ be the restriction of β to Δ. Let $u\colon \Lambda \to U(B)$ be a weak 1-cocycle for α. Since α is the restriction of β to B, we deduce that u is also a 1-cocycle for β. It is shown in [39, Section 3] that the von Neumann-algebraic Bernoulli $(\Lambda \curvearrowright \Lambda)$-action β satisfies the assumptions of [39, Theorem 4.1]. Applying [39, Theorem 4.1] in the case of weak 1-cocycles, with $S = S_1 = \{1\}$ in the notation of [39], we conclude that u is a

Δ-local weak coboundary for β. Therefore, there exist a unitary $v \in U(N)$ and a function $\mu\colon \Delta \to \mathbb{T}$ satisfying $u_\gamma = \mu_\gamma v^* \beta_\gamma(v)$ for every $\gamma \in \Delta$. Fix $g \in G$. Applying ρ_g to the previous identity, and using that $u_\gamma \in N^\rho = B$, yields

$$\mu_\gamma v^* \beta_\gamma(v) = u_\gamma = \rho_g(u_\gamma) = \mu_\gamma \rho_g(v)^* \beta_\gamma(\rho_g(v))$$

for every $\gamma \in \Delta$. Hence $\rho_g(v)v^*$ is fixed by ζ. By Proposition 2.7, ζ is mixing. Therefore by Remark 2.3 we conclude that $\rho_g(v)v^*$ is a scalar. Define a function $\chi_u\colon G \to \mathbb{T}$ by $\chi_u(g) = \rho_g(v)v^*$ for $g \in G$.

Claim. *χ_u is well-defined (that is, independent of the choice of μ and v).*

Proof of claim. Fix $v, v' \in U(N)$ and $\mu, \mu'\colon \Delta \to \mathbb{T}$ satisfying

$$u_\gamma = \mu_\gamma v^* \beta_\gamma(v) = \mu'_\gamma (v')^* \beta_\gamma(v')$$

for every $\gamma \in \Delta$. We want to show that $\rho_g(v)v^* = \rho_g(v')(v')^*$ for all $g \in G$. The above identity implies that

$$\beta_\gamma(v'v^*) = \mu_\gamma \overline{\mu}'_\gamma v'v^*$$

for all $\gamma \in \Delta$. In particular, the 1-dimensional subspace of $L^2(N)$ spanned by $v'v^*$ is invariant by ζ. Hence, it follows from Remark 2.3 and the fact that ζ is mixing that $v'v^*$ is a scalar, which we abbreviate to $z \in \mathbb{T}$. Thus,

$$\rho_g(v')(v')^* = \rho_g(zv)(zv)^* = z\overline{z}\rho_g(v)v^* = \rho_g(v)v^*$$

for all $g \in G$, as desired. □

Claim. *χ_u is a character on G*

Proof of claim. First, observe that χ_u is a continuous function, since ρ_g is a continuous action. To check the character condition, let $g, h \in G$. Then

$$\chi_u(gh) = \rho_{gh}(v)v^* = \rho_g(\rho_h(v)v^*)\rho_g(v)v^* = \chi_u(g)\chi_u(h),$$

so the claim is proved. □

Claim. *For $u \in Z^1_w(\alpha)$, the character χ_u only depends on the Δ-local weak cohomology class of u.*

Proof of claim. Let $u' \in Z^1_w(\alpha)$ be Δ-locally weakly cohomologous to u, and let $w \in U(N)$ satisfy $u'_\gamma = w^* u_\gamma \alpha_\gamma(w) \mathrm{mod}\mathbb{C}$ for every $\gamma \in \Delta$. Let $v \in U(N)$ be an eigenvector for ρ with eigenvalue χ_u, such that $u_\gamma = v^* \overline{\beta}_\gamma(v) \,\mathrm{mod}\mathbb{C}$ for every $\gamma \in \Delta$. Then

$$u'_\gamma = \mu_\gamma (vw)^* \beta_\gamma(vw) \mathrm{mod}\mathbb{C}$$

for every $\gamma \in \Delta$, and hence $vw \in U(N)$ is an eigenvector for ρ with eigenvalue χ_u. Therefore $\chi_{u'} = \chi_u$. □

In view of the previous claims, we can define a function $\eta\colon H^1_{\Delta,w}(\alpha) \to \Gamma$ by $\eta([u]) = \chi_u$ for all $[u] \in H^1_{\Delta,w}(\alpha)$.

Claim. *The map $\eta\colon H^1_{\Delta,w}(\alpha) \to \Gamma$ is surjective.*

Proof of claim. Fix $\omega \in \Gamma$. Since ω is a continuous function $\omega\colon G \to \mathbb{C}$, we can regard ω as a (unitary) element in $C(G)$. Observe that ω is an eigenvector for $\mathtt{Lt}$ with eigenvalue ω. By part (1) of Theorem 3.5, there exists an equivariant unital embedding $(C(G), \mathtt{Lt}) \to (D, \delta)$. Furthermore, there exists an equivariant unital embedding

$$(D, \delta) \to \left((D \otimes A)^{\otimes \Lambda}, (\delta \otimes \mathrm{id}_A)^{\otimes \Lambda}\right).$$

Composing these maps, one can conclude that there exists an equivariant unital embedding

$$(C(G), \mathtt{Lt}) \to \left((D \otimes A)^{\otimes \Lambda}, (\delta \otimes \mathrm{id}_A)^{\otimes \Lambda}\right).$$

Identifying $C(G)$ with its image inside $(D \otimes A)^{\otimes \Lambda}$, we can regard ω as an element of $(D \otimes A)^{\otimes \Lambda}$, which is an eigenvector for $(\delta \otimes \mathrm{id}_A)^{\otimes \Lambda}$ with eigenvalue ω. In turn, this gives an element v of the weak closure N of $(D \otimes A)^{\otimes \Lambda}$ which is an eigenvector for ρ with eigenvalue ω. Define a function $u\colon \Lambda \to N$ by $u_\gamma = v^* \beta_\gamma(v)$ for all $\gamma \in \Lambda$. For every $g \in G$, we have

$$\rho_g(u_\gamma) = \rho_g(v)^* \beta_\gamma(\rho_g(v)) = v^* \beta_\gamma(v) = u_\gamma$$

for all $\gamma \in \Lambda$. It follows that u takes values in $B = N^\rho$. On the other hand, given $\gamma, \lambda \in \Lambda$, we have

$$u_\gamma \alpha_\gamma(u_\lambda) = v^* \beta_\gamma(v) \alpha_\gamma(v^* \beta_\lambda(v)) = v^* \beta_{\gamma\lambda}(v) = u_{\gamma\lambda} \mathrm{mod}\mathbb{C}.$$

Therefore u is a weak 1-cocycle for α, and $\chi_u = \omega$. It follows that η is surjective, as desired. □

Claim. *The map $\eta\colon H^1_{\Delta,w}(\alpha) \to \Gamma$ is injective (and hence a bijection).*

Proof of claim. Let $u_0, u_1 \in Z^1_w(\alpha)$ satisfy $\chi_{u_0} = \chi_{u_1}$. Denote by ω this character. Find eigenvectors $v_0, v_1 \in U(B)$ for ρ with eigenvalue ω such that $u_{j,\gamma} = v_j^* \beta_\gamma(v_j) \mathrm{mod}\mathbb{C}$ for all $\gamma \in \Delta$ and $j = 0, 1$. Set $w = v_0^* v_1$, which is a unitary in B. For every $\gamma \in \Delta$, we have

$$w^* u_{0,\gamma} \alpha_\gamma(w) = (v_0^* v_1)^* u_{0,\gamma} \alpha_\gamma(v_0^* v_1) = v_1^* v_0 (v_0^* \beta_\gamma(v_0)) \beta_\gamma(v_0^* v_1) = v_1^* \beta_\gamma(v_1) = u_{1,\gamma} \ \mathrm{mod}\mathbb{C}.$$

Therefore w witnesses the fact that u_0 and u_1 are Δ-locally weakly cohomologous. Thus $[u_0] = [u_1] \in H^1_{\Delta,w}(\overline{\alpha})$, and η is injective. $\square$

We now turn to the construction of the map $\eta^{(2)} \colon H^1_{\Delta,w}(\alpha \otimes \alpha) \to \Gamma$. Observe that $\alpha \otimes \alpha$ is conjugate to α. Let $u \in Z^1_w(\alpha \otimes \alpha)$, and choose a unitary $v \in U(N \overline{\otimes} N)$ satisfying $u_\gamma = v^*(\beta_\gamma \otimes \beta_\gamma)(v) \mathrm{mod}\mathbb{C}$ for all $\gamma \in \Delta$. As before, one checks that v is an eigenvector for $\rho \otimes \rho$, and that its eigenvalue κ_u is a character in Γ, which is independent of v. Similarly to what was done above, one defines the map $\eta^{(2)} \colon H^1_{\Delta,w}(\alpha \otimes \alpha) \to \Gamma$ by $\eta^{(2)}([u]) = \kappa_u$ for all $[u] \in H^1_{\Delta,w}(\alpha \otimes \alpha)$.

It remains to prove the identity $\eta^{(2)} \circ m^\alpha_\Delta = m^\Gamma \circ (\eta \times \eta)$. Let $[u], [u'] \in H^1_{\Delta,w}(\alpha)$, and set $\omega = \eta([u])$ and $\omega' = \eta([u'])$. Find eigenvectors $v, v' \in U(B)$ for ρ with eigenvalues ω and ω', respectively, satisfying

$$u_\gamma = v^* \beta_\gamma(v) \mathrm{mod}\mathbb{C} \quad \text{and} \quad u'_\gamma = v'^* \beta_\gamma(v') \mathrm{mod}\mathbb{C}$$

for all $\gamma \in \Delta$. Hence $(u \otimes u')_\gamma = (v \otimes v')^* (\overline{\beta}_\gamma \otimes \overline{\beta}_\gamma)(v \otimes v') \mathrm{mod}\mathbb{C}$ for every $\gamma \in \Delta$. Since $v \otimes v'$ is an eigenvector for $\rho \otimes \rho$ with eigenvalue $\omega\omega'$, this shows that

$$(\eta^{(2)} \circ m^\alpha_\Delta)([u], [u']) = \omega\omega' = \eta([u])\eta([u']) = (m^\Gamma \circ (\eta \times \eta))([u], [u']).$$

This concludes the proof of the lemma. $\square$

We fix now a C*-algebra A which is locally reflexive, M_{p^∞}-absorbing for some prime p, has an amenable trace, and is isomorphic to its infinite tensor product $A^{\otimes \mathbb{N}}$. We also fix a prime p such that $A \cong A \otimes M_{p^\infty}$. We will frequently use the notation for Bernoulli actions from Notation 2.14. We write D for M_{p^∞}. We also fix an isomorphism $\phi \colon A \to A^{\otimes \Lambda}$. Using this isomorphism, we let $\sigma \colon \Lambda \to \mathrm{Aut}(A)$ denote the action given by

$$\sigma_\gamma = \phi^{-1} \circ (\beta_{\Lambda \curvearrowright \Lambda, A})_\gamma \circ \phi$$

for all $\gamma \in \Lambda$.

Consider the diagonal action $(\delta^G)^{\otimes \Lambda} \colon G \to \mathrm{Aut}(D^{\otimes \Lambda})$, and denote by E_G its fixed point algebra, which, by parts (2) and (4) of Theorem 3.5, is isomorphic to D. Since $(\delta^G)^{\otimes \Lambda}$ and $\beta_{\Lambda \curvearrowright \Lambda, D}$ commute, $\beta_{\Lambda \curvearrowright \Lambda, D}$ restricts to an action $\beta_{\Lambda \curvearrowright \Lambda, D}|_{E_G} \colon \Lambda \to \mathrm{Aut}(E_G)$.

Definition 4.2. For each pro-p group G, we choose an isomorphism $\xi_G \colon A \to E_G \otimes A$. Now, we define an action $\alpha^G \colon \Lambda \to \mathrm{Aut}(A)$ by

$$\alpha^G_\gamma = \xi_G^{-1} \circ (\beta_{\Lambda \curvearrowright \Lambda, D}|_{E_G} \otimes \sigma)_\gamma \circ \xi_G$$

for all $\gamma \in \Lambda$.

It will be shown in Theorem 4.6 that for non-isomorphic pro-p groups G_0 and G_1, the actions α^{G_0} and α^{G_1} are not weakly cocycle conjugate. In order to do this, we will need to study the weak extensions of these actions with respect to certain invariant traces. Our next result provides us with a canonical subset of $T(A)$ consisting of traces that are α^G-invariant for every pro-p group G. Later, in Proposition 4.4, we will show that for any of these traces and for any G, the weak extension of any of α^G is mixing. For a pro-p group G, we denote by τ_{E_G} the (unique) trace on E_G.

Proposition 4.3. *Adopt the notation from the previous discussion, and define a continuous, affine map $\iota \colon T(A) \to T(A)$ by $\iota(\tau) = \tau^{\otimes \Lambda} \circ \phi$ for all $\tau \in T(A)$. If τ is extreme and amenable, then so is $\iota(\tau)$.*

Moreover, if G is any pro-p group, then $\iota(\tau) = (\tau_{E_G} \otimes \iota(\tau)) \circ \xi_G$ for all $\tau \in T(A)$. In particular, $\iota(\tau)$ is α^G-invariant for all $\tau \in T(A)$.

Proof. The first assertion is standard (and in our case, it follows from Lemma 2.23, since the weak closure of $A^{\otimes \Lambda}$ with respect to $\tau^{\otimes \Lambda}$ is canonically isomorphic to $(\overline{A}^\tau)^{\otimes \Lambda} \cong R^{\otimes \Lambda} \cong R$.)

Let G be a pro-p group and let $\tau \in T(A)$. Observe that $(\tau_{E_G} \otimes \iota(\tau)) \circ \xi_G$ is an α^G-invariant trace on A. Hence, it suffices to show that this trace equals $\iota(\tau)$.

Observe that since E_G is a UHF-algebra of infinite type, the isomorphism $\xi_G \colon A \to E_G \otimes A$ is approximately unitarily equivalent to the second tensor factor embedding $\kappa \colon A \to E_G \otimes A$ given by $\kappa(a) = 1_{E_G} \otimes a$ for all

$a \in A$; see [45, Corollary 1.12]. It follows that ξ_G and κ induce the same map at the level of traces. Using this at the first step in the following computation, we conclude that

$$(\tau_{E_G} \otimes \iota(\tau)) \circ \xi_G = (\tau_{E_G} \otimes \iota(\tau)) \circ \kappa = \iota(\tau)$$

for all traces $\tau \in T(A)$, as desired. □

Our next goal is to establish a number of properties for α^G; this will be done in Proposition 4.4. In order to do this, we need an alternative description of α^G. Since the group G will be fixed from now on and until Theorem 4.6, we will drop it from the notation for the actions δ^G and α^G, as well as from the notation for the algebra E_G. In Theorem 4.6, we will show that for nonisomorphic G_0 and G_1, the actions constructed above are not weakly cocycle conjugate. Until then, we will work with a fixed pro-p group G.

Observe that the Bernoulli action $\beta_{\Lambda\curvearrowright\Lambda,D\otimes A}$ commutes with the diagonal action

$$(\delta \otimes \mathrm{id}_A)^{\otimes\Lambda} \colon G \to \mathrm{Aut}((D \otimes A)^{\otimes\Lambda}).$$

Thus, with B denoting the fixed point algebra of $(\delta \otimes \mathrm{id}_A)^{\otimes\Lambda}$, the action $\beta_{\Lambda\curvearrowright\Lambda,D\otimes A}$ restricts to an action $\widetilde{\alpha} \colon \Lambda \to \mathrm{Aut}(B)$.

Fix an amenable extreme trace τ_0 on $D \otimes A$ and let $\widetilde{\tau}$ be the trace $\tau_0^{\otimes\Lambda}$ on $(D \otimes A)^{\otimes\Lambda}$. Then $\overline{D \otimes A}^{\tau_0}$ is isomorphic to R by Lemma 2.23, and the extension of τ_0 to $\overline{D \otimes A}^{\tau_0}$ is the unique trace on R. We identify $\overline{\beta}^{\widetilde{\tau}}_{\Lambda\curvearrowright\Lambda,D\otimes A}$ with the von Neumann-algebraic Bernoulli action $\beta_{\Lambda\curvearrowright\Lambda,R} \colon \Lambda \to \mathrm{Aut}(R^{\otimes\Lambda})$, and $\overline{\beta}^{\widetilde{\tau}}_{\Delta\curvearrowright\Lambda,D\otimes A}$ with $\beta_{\Delta\curvearrowright\Lambda,R} \colon \Delta \to \mathrm{Aut}(R^{\otimes\Lambda})$. Similarly, the extension of $(\delta \otimes \mathrm{id}_A)^{\otimes\Lambda}$ to the weak closure with respect to $\widetilde{\tau}$ can be identified with $(\overline{\delta}^{\tau_D} \otimes \mathrm{id}_R)^{\overline{\otimes}\Lambda}$, where τ_D is the unique trace on D. Furthermore, since G is compact, $\overline{B}^{\widetilde{\tau}}$ can be identified with the fixed point algebra of $(\overline{\delta}^{\tau_D} \otimes \mathrm{id}_R)^{\overline{\otimes}\Lambda}$, and the weak extension of $\widetilde{\alpha}$ can be identified with the restriction of $\overline{\beta}^{\widetilde{\tau}}$ to $\overline{B}^{\widetilde{\tau}}$.

In the next proposition, we first show that α is conjugate to $\widetilde{\alpha}$. Then we use this alternative descriptions to verify some properties of α.

Proposition 4.4. *Adopt the notation of the discussion above. Let τ_0 be a trace on A, and τ be the image of τ_0 under the map ι from Proposition 4.3. Define $\widetilde{\tau}$ to be the trace $(\tau_D \otimes \tau_0)^{\otimes\Lambda}$ on $(D \otimes A)^{\otimes\Lambda}$. Then:*

1. *There is a Λ-equivariant trace-preserving isomorphism $(A, \tau, \alpha) \cong (B, \widetilde{\tau}, \widetilde{\alpha})$.*
2. *There is a Λ-equivariant isomorphism $(A, \alpha) \cong (A \otimes M_p^{\otimes\Lambda}, \alpha \otimes \beta_{\Lambda\curvearrowright\Lambda,M_p})$.*
3. *There is a Λ-equivariant trace-preserving isomorphism $(A, \tau, \alpha) \cong (A \otimes A, \tau \otimes \tau, \alpha \otimes \alpha)$.*
4. *The action α is strongly outer;*
5. *The action $\overline{\alpha}^\tau$ is mixing.*

Proof. (1): By rearranging the tensor factors, it is clear that there exists a Λ-equivariant trace-preserving isomorphism

$$((D \otimes A)^{\otimes\Lambda}, \widetilde{\tau}, \beta_{\Lambda\curvearrowright\Lambda,D\otimes A}) \cong (D^{\otimes\Lambda} \otimes A^{\otimes\Lambda}, \tau_D^{\otimes\Lambda} \otimes \tau_0^{\otimes\Lambda}, \beta_{\Lambda\curvearrowright\Lambda,D} \otimes \beta_{\Lambda\curvearrowright\Lambda,A}).$$

Upon identifying $A^{\otimes\Lambda}$ with A via the isomorphism ϕ, we obtain a Λ-equivariant isomorphism

$$((D \otimes A)^{\otimes\Lambda}, \widetilde{\tau}, \beta_{\Lambda\curvearrowright\Lambda,D\otimes A}) \cong (D^{\otimes\Lambda} \otimes A, \tau_D^{\otimes\Lambda} \otimes \tau, \beta_{\Lambda\curvearrowright\Lambda,D} \otimes \sigma).$$

This isomorphism can be regarded as a G-equivariant isomorphism

$$((D \otimes A)^{\otimes\Lambda}, \widetilde{\tau}, (\delta \otimes \mathrm{id}_A)^{\otimes\Lambda}) \cong (D^{\otimes\Lambda} \otimes A, \tau_D^{\otimes\Lambda} \otimes \tau, \delta^{\otimes\Lambda} \otimes \mathrm{id}_A).$$

Upon taking G-fixed point algebras, and recalling that E denotes the fixed point algebra of $\delta^{\otimes\Lambda}$, we obtain a trace-preserving isomorphism $\psi \colon (B, \widetilde{\tau}) \to (E \otimes A, \tau_E \otimes \tau)$. Moreover, ψ can be regarded as a Λ-equivariant trace-preserving isomorphism

$$\psi \colon (B, \widetilde{\tau}, \widetilde{\alpha}) \to (E \otimes A, \tau_E \otimes \tau, \beta_{\Lambda\curvearrowright\Lambda,D}|_E \otimes \sigma).$$

Since $\xi \colon (A, \alpha) \to (E \otimes A, \beta_{\Lambda\curvearrowright\Lambda,D}|_E \otimes \sigma)$ is an equivariant isomorphism by definition, and $(\tau_E \otimes \tau) \circ \xi = \tau$ by Proposition 4.3, it follows that $\xi^{-1} \circ \psi$ is a Λ-equivariant trace-preserving isomorphism $(B, \widetilde{\tau}, \widetilde{\alpha}) \to (A, \tau, \alpha)$.

(2): By (1), it is enough to prove that there is a Λ-equivariant isomorphism

$$(B, \widetilde{\alpha}) \cong (B \otimes M_p^{\otimes\Lambda}, \widetilde{\alpha} \otimes \beta_{\Lambda\curvearrowright\Lambda,M_p}).$$

Using that A is isomorphic to $A \otimes M_p$, it is clear that there is an equivariant isomorphism

$$\left((D \otimes A)^{\otimes\Lambda}, \beta_{\Lambda\curvearrowright\Lambda,D\otimes A}\right) \cong \left((D \otimes A)^{\otimes\Lambda} \otimes M_p^{\otimes\Lambda}, \beta_{\Lambda\curvearrowright\Lambda,D\otimes A} \otimes \beta_{\Lambda\curvearrowright\Lambda,M_p}\right).$$

This isomorphism intertwines the actions

$$(\delta \otimes \mathrm{id}_A)^{\otimes\Lambda} \colon G \to \mathrm{Aut}\left((D \otimes A)^{\otimes\Lambda}\right) \quad \text{and} \quad (\delta \otimes \mathrm{id}_A)^{\otimes\Lambda} \otimes \mathrm{id}_{M_p^{\otimes\Lambda}} \colon G \to \mathrm{Aut}\left((D \otimes A)^{\otimes\Lambda} \otimes M_p^{\otimes\Lambda}\right).$$

The fixed point algebra of the second action is isomorphic to $B \otimes M_p^{\otimes\Lambda}$, in such a way that the restriction of $\beta_{\Lambda \curvearrowright \Lambda, D\otimes A} \otimes \beta_{\Lambda\curvearrowright\Lambda, M_p}$ to this algebra is conjugate to $\widetilde{\alpha} \otimes \beta_{\Lambda\curvearrowright\Lambda,M_p}$. Thus $(B, \widetilde{\alpha})$ is equivariantly isomorphic to $(B \otimes M_p^{\otimes\Lambda}, \widetilde{\alpha} \otimes \beta_{\Lambda\curvearrowright\Lambda,M_p})$, as desired.

(3): This is similar to (2), using the fact that $(A^{\otimes\Lambda}, \tau_0^{\otimes\Lambda}) \cong (A, \tau)$ via ϕ.

(4): By (1) and (2), it suffices to show that $\widetilde{\alpha} \otimes \beta_{\Lambda\curvearrowright\Lambda,M_p}$ is strongly outer. Let $\gamma \in \Lambda \setminus \{1\}$ and let $\hat{\tau}$ be an $(\widetilde{\alpha} \otimes \beta_{\Lambda\curvearrowright\Lambda,M_p})_\gamma$-invariant trace on $B \otimes M_p^{\otimes\Lambda}$. Since $M_p^{\otimes\Lambda} \cong D$ has a unique trace τ_D, we deduce that $\hat{\tau}$ has the form $\tau_B \otimes \tau_D$ for some α_γ-invariant trace τ_B on B. The weak extension of $(\widetilde{\alpha} \otimes \beta_{\Lambda\curvearrowright\Lambda,M_p})_\gamma$ with respect to $\hat{\tau}$ is conjugate to $(\overline{\widetilde{\alpha}}^{\tau_B} \otimes \overline{\beta}_{\Lambda\curvearrowright\Lambda,M_p})_\gamma$, where $\overline{\beta}_{\Lambda\curvearrowright\Lambda,M_p}$ is the von Neumann-algebraic Bernoulli $(\Lambda \curvearrowright \Lambda)$-action with base M_p. Such an action is easily seen to be outer.

(5): By (1), it suffices to check that $\overline{\widetilde{\alpha}}^{\widetilde{\tau}}$ is mixing. Observe that $\overline{\widetilde{\alpha}}^{\widetilde{\tau}}$ is the restriction to $\overline{B}^{\widetilde{\tau}}$ of $\overline{\beta}^{\widetilde{\tau}}_{\Lambda\curvearrowright\Lambda,D\otimes A}$. The latter action is conjugate to the von Neumann-algebraic Bernoulli action $\beta_{\Lambda\curvearrowright\Lambda,\overline{D\otimes A}^{\tau_0}}$, which is mixing by Proposition 2.7. Therefore $\overline{\widetilde{\alpha}}^{\widetilde{\tau}}$ is mixing. □

We retain the notation from before Proposition 4.4. Given a trace τ in the image of the map ι from Proposition 4.3, we will use the pairing function $m_\Delta^{\overline{\alpha}^\tau} \colon H^1_{\Delta,w}(\overline{\alpha}^\tau) \times H^1_{\Delta,w}(\overline{\alpha}^\tau) \to H^1_{\Delta,w}(\overline{\alpha}^\tau \otimes \overline{\alpha}^\tau)$ from Definition 2.11. As above, we write Γ for the Pontryagin dual of G, and we let $m^\Gamma \colon \Gamma \times \Gamma \to \Gamma$ be the multiplication operation. Recall also that $\mathtt{Lt}\colon G \to \mathrm{Aut}(C(G))$ denotes the action by left translation.

In what follows, and since all weak extensions will be taken with respect to $\widetilde{\tau}$, we will omit this trace from the notation. Also, we will regard $\widetilde{\alpha}$ as an alternative description of α, and, with a slight abuse of notation, we will write α to mean $\widetilde{\alpha}$. In particular, the symbol $\overline{\alpha}$ will always represent the weak extension of $\widetilde{\alpha}$ with respect to the trace $\widetilde{\tau}$.

Theorem 4.5. *Adopt the notation from Definition 4.2. Let $\tau_0 \in T(A)$, and set $\tau = \iota(\tau_0)$. Then there exist bijections $\eta \colon H^1_{\Delta,w}(\overline{\alpha}^\tau) \to \Gamma$ and $\eta^{(2)} \colon H^1_{\Delta,w}(\overline{\alpha}^\tau \otimes \overline{\alpha}^\tau) \to \Gamma$ satisfying*

$$\eta^{(2)} \circ m_\Delta^{\overline{\alpha}^\tau} = m^\Gamma \circ (\eta \times \eta).$$

Proof. Let τ_D denote the unique trace on D. For the trace τ_0 in the statement, set $\widetilde{\tau} = (\tau_D \otimes \tau_0)^{\otimes\Lambda}$, which is naturally a trace on $(D \otimes A)^{\otimes\Lambda}$. Recall the definition of $\widetilde{\alpha}$ from before Proposition 4.4. By part (1) of Proposition 4.4 the action α is conjugate to $\widetilde{\alpha}$ via an isomorphism that maps τ to $\widetilde{\tau}$. In particular, the weak extension of α with respect to τ is conjugate to the weak extension of $\widetilde{\alpha}$ with respect to $\widetilde{\tau}$. Therefore it is enough to prove the corresponding statement for $\widetilde{\alpha}$ and $\widetilde{\tau}$.

Let β be the Bernoulli $(\Lambda \curvearrowright \Lambda)$-action with base $D \otimes A$, and ζ its restriction to Δ which is the Bernoulli $(\Delta \curvearrowright \Lambda)$-action with base $D \otimes A$. Observe that $\widetilde{\alpha}$ is the restriction of β to $B \subseteq (D \otimes A)^{\otimes\Lambda}$, and the unique extension $\overline{\widetilde{\alpha}}$ of $\widetilde{\alpha}$ to the weak closure $\overline{B}$ with respect to the trace $\widetilde{\tau}$. Therefore the conclusion follows from Lemma 4.1. □

Using the previous result, we will show below that if one starts with two non-isomorphic abelian pro-p groups G_0 and G_1, then the actions α^{G_0} and α^{G_1} of Λ on A, as in Definition 4.2, are not weakly cocycle conjugate. Since the pro-p group is no longer fixed, we again use superscripts (for the actions) and subscripts (for the algebras) to keep track of which pro-p group they come from.

Theorem 4.6. *Let the notation be as in Definition 4.2 and Proposition 4.3. Fix $\tau_0 \in T(A)$ and set $\tau = \iota(\tau)$. Let G_0 and G_1 be second countable abelian pro-p groups. The following assertions are equivalent:*

1. *The groups G_0 and G_1 are topologically isomorphic;*
2. *The Λ-actions α^{G_0} and α^{G_1} are conjugate;*
3. *The Λ-actions α^{G_0} and α^{G_1} are cocycle conjugate;*
4. *The Λ-actions α^{G_0} and α^{G_1} are weakly cocycle τ-conjugate.*

Proof. Since G_0 and G_1 are pro-p groups, there are isomorphisms $D_{G_0} \cong D_{G_1} \cong M_{p^\infty}$, and we denote this algebra by D. Any isomorphism $G_0 \cong G_1$ is immediately seen to induce an equivariant *-isomorphism $(D, \delta^{G_0}) \cong (D, \delta^{G_1})$. From this, it is easy to construct an equivariant *-isomorphism $(B_{G_0}, \alpha^{G_0}) \cong (B_{G_1}, \alpha^{G_1})$. This proves the implication (1)⇒(2). It is clear that (2) implies (3), and that (3) implies (4), because τ is α^{G_0}- and α^{G_1}-invariant by Proposition 4.3. Therefore, it only remains to prove that (4) implies (1).

Suppose that α^{G_0} and α^{G_1} are weakly cocycle τ-conjugate. Let τ_D denote the unique trace on D, and set $\widetilde{\tau} = (\tau_D \otimes \tau_0)^{\otimes\Lambda}$, which is a trace on $(D \otimes A)^{\otimes\Lambda}$. By part (1) of Proposition 4.4, for $i = 0, 1$, the weak extension of α^{G_i} with respect to τ can be identified with the weak extension of $\widetilde{\alpha}^{G_i}$ with respect to $\widetilde{\tau}$. (The action $\widetilde{\alpha}^{G_i}$ is described before Proposition 4.4.) Hence, it suffices to show that $\widetilde{\alpha}^{G_0}$ and $\widetilde{\alpha}^{G_1}$ are not weakly cocycle $\widetilde{\tau}$-conjugate. Since the trace $\widetilde{\tau}$ is fixed, we will omit it from the notation for weak closures and weak extensions. With a slight abuse of notation, we will write α^{G_i} to mean $\widetilde{\alpha}^{G_i}$. This way, the symbol $\overline{\alpha}^{G_i}$ will always represent the weak extension of $\widetilde{\alpha}^{G_i}$ with respect to the trace $\widetilde{\tau}$. Let β be the Bernoulli $(\Lambda \curvearrowright \Lambda)$-action with base $D \otimes A$. We also denote the algebra $\overline{(D \otimes A)}^{\otimes\Lambda}$ by N (omitting the trace $\tau_D \otimes \tau_0$), and abbreviate the G_i-action $(\overline{\delta}^{G_i} \otimes \mathrm{id}_R)^{\overline{\otimes}\Lambda}$ on N to $\rho^{(i)} \colon G_i \to \mathrm{Aut}(N)$. (In particular, $\overline{B}_{G_i} = N^{\rho_i}$.) We let Γ_i be the dual group of G_i.

Let $\psi\colon \overline{B}_{G_0} \to \overline{B}_{G_1}$ be an isomorphism and let $w\colon \Lambda \to U(\overline{B}_{G_1})$ be a weak 1-cocycle for α^{G_1} satisfying

$$\mathrm{Ad}(w_\gamma) \circ \overline{\alpha}^{G_1}_\gamma = \psi \circ \overline{\alpha}^{G_0}_\gamma \circ \psi^{-1} \tag{3}$$

for every $\gamma \in \Lambda$. Using Popa's superrigidity theorem [39, Theorem 4.1] in the case of weak 1-cocycles as in the proof of Theorem 4.5, one can find unitaries $z \in U(\overline{B}_{G_1})$ and $v \in U(N)$, and a character $\chi \in \Gamma_1$ such that

$$w_\gamma = z^* v^* \overline{\beta}_\gamma(v) \overline{\alpha}_\gamma(z) \bmod \mathbb{C} \quad \text{and} \quad \rho^{(1)}_g(v) = \chi(g) v$$

for every $\gamma \in \Delta$ and for every $g \in G_1$. Therefore, upon replacing ψ with $\psi \circ \mathrm{Ad}(z^*)$, we can assume that $z = 1$ and $w_\gamma = v^* \overline{\beta}_\gamma(v)$ for every $\gamma \in \Delta$.

Next, we want to define a bijection $\varphi\colon H^1_{\Delta,w}(\overline{\alpha}^{G_0}) \to H^1_{\Delta,w}(\overline{\alpha}^{G_1})$. Given a function $u\colon \Lambda \to U(\overline{B}_{G_1})$, define $\psi(u)w\colon \Lambda \to U(\overline{B}_{G_1})$ to be the function given by $(\psi(u)w)_\gamma = \psi(u_\gamma)w_\gamma$ for all $\gamma \in \Lambda$.

Claim. *If $u \in Z^1_w(\overline{\alpha}^{G_0})$, then $\psi(u)w \in Z^1_w(\overline{\alpha}^{G_1})$.*

Proof of claim. Let $\gamma, \sigma \in \Lambda$. In the following computation (where all equalities are up to scalars), we use the fact that w is a weak 1-cocycle for $\overline{\alpha}^{G_1}$ at the first step, and equation (3) at the second step, to get

$$\psi(u_{\gamma\sigma}) = \psi(u_\gamma)(\psi \circ \overline{\alpha}^{G_0}_\gamma \circ \psi^{-1})(\psi(u_\sigma)) = \psi(u_\gamma)(\mathrm{Ad}(w_\gamma) \circ \overline{\alpha}^{G_1}_\gamma)(\psi(u_\sigma)) = \psi(u_\gamma) w_\gamma \overline{\alpha}^{G_1}_\gamma(\psi(u_\sigma)) w_\gamma^* \bmod \mathbb{C}$$

Therefore, using the above identity and the fact that u is a weak 1-cocycle for $\overline{\alpha}^{G_0}$, we deduce that

$$\psi(u_{\gamma\sigma}) w_{\gamma\sigma} = \psi(u_\gamma) w_\gamma \overline{\alpha}^{G_1}_\gamma(\psi(u_\sigma)) w_\gamma^* w_\gamma \overline{\alpha}^{G_1}_\gamma(w_\sigma) = \psi(u_\gamma) w_\gamma \overline{\alpha}^{G_1}_\gamma(\psi(u_\sigma) w_\sigma) \bmod \mathbb{C}.$$

This shows that $\psi(u)w$ is a weak 1-cocycle for $\overline{\alpha}^{G_1}$, proving the claim. □

It follows that there is a well-defined map $\widehat{\psi}\colon Z^1_w(\overline{\alpha}^{G_0}) \to Z^1_w(\overline{\alpha}^{G_1})$ given by $\widehat{\psi}(u) = \psi(u)w$ for $u \in Z^1_w(\overline{\alpha}^{G_0})$.

Claim. *If $u, u' \in Z^1_w(\overline{\alpha}^{G_0})$ are Δ-locally weakly cohomologous, then so are $\widehat{\psi}(u)$ and $\widehat{\psi}(u')$.*

Proof of claim. Find a unitary $z \in U(\overline{B}_{G_0})$ satisfying $u'_\gamma = z^* u_\gamma \overline{\alpha}^{G_0}_\gamma(z) \bmod \mathbb{C}$ for $\gamma \in \Delta$. Then

$$\psi(u'_\gamma) w_\gamma = \psi(z)^* \psi(u_\gamma)(\psi \circ \overline{\alpha}^{G_0}_\gamma \circ \psi^{-1})(z) w_\gamma = \psi(z)^* \psi(u_\gamma)(\mathrm{Ad}(w_\gamma) \circ \overline{\alpha}^{G_1}_\gamma)(z) w_\gamma = \psi(z)^* \psi(u_\gamma) w_\gamma \overline{\alpha}^{G_1}_\gamma(z) \bmod \mathbb{C}$$

for all $\gamma \in \Delta$. This shows that $\psi(u')w$ and $\psi(u)w$ are Δ-locally weakly cohomologous. □

It follows that $\widehat{\psi}$ induces a well-defined map $\varphi\colon H^1_{\Delta,w}(\overline{\alpha}^{G_0}) \to H^1_{\Delta,w}(\overline{\alpha}^{G_1})$.

Claim. *The map φ is invertible.*

Proof of claim. It follows from equation (3) that

$$\overline{\alpha}^{G_0}_\gamma = \psi^{-1} \circ \mathrm{Ad}(w_\gamma) \circ \overline{\alpha}^{G_1}_\gamma \circ \psi = \mathrm{Ad}(\psi^{-1}(w_\gamma)) \circ \psi^{-1} \circ \overline{\alpha}^{G_0}_\gamma \circ \psi$$

for all $\gamma \in \Lambda$. Therefore, the same argument as before shows that the function that assigns to the cocycle u for $\overline{\alpha}^{G_1}$ the cocycle $\gamma \mapsto \psi^{-1}(u_\gamma w_\gamma)$ for $\overline{\alpha}^{G_0}$ induces a well-defined function $H^1_{\Delta,w}(\overline{\alpha}^{G_1}) \to H^1_{\Delta,w}(\overline{\alpha}^{G_0})$, which is easily seen to be the inverse of φ. This proves the claim. □

Similarly as above, we define a bijection $\varphi^{(2)}\colon H^1_{\Delta,w}(\overline{\alpha}^{G_0} \otimes \overline{\alpha}^{G_0}) \to H^1_{\Delta,w}(\overline{\alpha}^{G_1} \otimes \overline{\alpha}^{G_1})$, by

$$\varphi^{(2)}([u]) = [(\psi \otimes \psi)(u)(w \otimes w)]$$

for all $u \in Z^1_w(\overline{\alpha} \otimes \overline{\alpha})$, where $(\psi \otimes \psi)(u)(w \otimes w)\colon \Lambda \to U(\overline{B}_{G_1})$ is the weak 1-cocycle for $\overline{\alpha}^{G_1} \otimes \overline{\alpha}^{G_1}$ given by $\gamma \mapsto (\psi \otimes \psi)(u_\gamma)(w_\gamma \otimes w_\gamma)$ for all $\gamma \in \Lambda$. Moreover, a routine calculation shows that $\varphi^{(2)} \circ m^{\overline{\alpha}^{G_0}}_\Delta = m^{\overline{\alpha}^{G_1}}_\Delta \circ \varphi$.

For $i \in \{0,1\}$, let $\eta_{G_i} \colon H^1_{\Delta,w}(\overline{\alpha}^{G_i}) \to \Gamma_i$ and $\eta^{(2)}_{G_i} \colon H^1_{\Delta,w}(\overline{\alpha}^{G_i} \otimes \overline{\alpha}^{G_i}) \to \Gamma_i$ be the maps from Theorem 4.5, and set

$$\pi = \eta_{G_1} \circ \varphi \circ \eta_{G_0}^{-1} \colon \Gamma_0 \to \Gamma_1 \quad \text{and} \quad \pi^{(2)} = \eta^{(2)}_{G_1} \circ \varphi^{(2)} \circ (\eta^{(2)}_{G_0})^{-1} \colon \Gamma_0 \to \Gamma_1.$$

By Theorem 4.5, the following diagram is commutative:

$$\begin{tikzcd}
\Gamma_0 \times \Gamma_0 \arrow[d, "\pi\times\pi"'] & H^1_{\Delta,w}(\overline{\alpha}^{G_0}) \times H^1_{\Delta,w}(\overline{\alpha}^{G_0}) \arrow[l, "\eta_{G_0}\times\eta_{G_0}"'] \arrow[r, "m^{\overline{\alpha}^{G_0}}"] \arrow[d, "\varphi\times\varphi"'] & H^1_{\Delta,w}(\overline{\alpha}^{G_0} \otimes \overline{\alpha}^{G_0}) \arrow[r, "\eta^{(2)}_{G_0}"] \arrow[d, "\varphi^{(2)}"] & \Gamma_0 \arrow[d, "\pi^{(2)}"] \\
\Gamma_1 \times \Gamma_1 & H^1_{\Delta,w}(\overline{\alpha}^{G_1}) \times H^1_{\Delta,w}(\overline{\alpha}^{G_1}) \arrow[l, "\eta_{G_1}\times\eta_{G_1}"] \arrow[r, "m^{\overline{\alpha}^{G_1}}"'] & H^1_{\Delta,w}(\overline{\alpha}^{G_1} \otimes \overline{\alpha}^{G_1}) \arrow[r, "\eta^{(2)}_{G_1}"] & \Gamma_1.
\end{tikzcd}$$

Recall that χ denotes the character of G_1 associated with the weak 1-cocycle w for $\overline{\alpha}^{G_1}$. Then $\pi^{(2)}(\omega\omega') = \pi(\omega)\pi(\omega')$ and $\pi(1_{\Gamma_0}) = \chi$. It follows that $\pi^{(2)}(\omega) = \pi(\omega)\chi$ for every $\omega \in \Gamma_0$. Therefore the map $\widetilde{\pi} \colon \Gamma_0 \to \Gamma_1$ given by $\widetilde{\pi}(\omega) = \pi(\omega)\chi^{-1}$ for all $\omega \in \Gamma_0$, is a group isomorphism. Indeed, we have

$$\pi(\omega)\chi^{-1}\pi(\omega')\chi^{-1} = \pi^{(2)}(\omega\omega')\chi^{-2} = \pi(\omega\omega')\chi^{-1}$$

for $\omega, \omega' \in \Gamma_0$. Since clearly $\widetilde{\pi}$ is a bijection, we conclude that $\widetilde{\pi}$ is a group isomorphism, and hence $\Gamma_0 \cong \Gamma_1$. By Pontryagin duality, we conclude that $G_0 \cong G_1$, and the proof is finished. □

We now arrive at the main result of this section. Its conclusion will be significantly strengthened in Corollary 5.10.

Theorem 4.7. *Let Λ be a countable discrete group with an infinite relative property* (T) *subgroup, let p be a prime number, and let A be separable, locally reflexive, M_{p^∞}-absorbing, unital C*-algebra admitting an amenable trace, and such that $A \cong A^{\otimes \mathbb{N}}$. Then there exists a continuum $(\alpha^{(t)})_{t\in\mathbb{R}}$ of pairwise not weakly cocycle conjugate, strongly outer actions of Λ on A. In fact, there exists an amenable, extreme trace τ that is invariant under $\alpha^{(t)}$ for every $t \in \mathbb{R}$, and such that the actions $\alpha^{(t)}$ are all τ-mixing and pairwise not weakly cocycle τ-conjugate.*

Proof. Let $(G_t)_{t\in\mathbb{R}}$ be a continuum family of pairwise nonisomorphic abelian pro-p groups. For $t \in \mathbb{R}$, set $\alpha^{(t)} := \alpha^{G_t}$, where $\alpha^{G_t} : \Lambda \to \mathrm{Aut}(A)$ is the action of Λ on A given by Definition 4.2. By part (4) of Proposition 4.4, $\alpha^{(t)}$ is strongly outer. Since A has an amenable trace and $T_{\mathrm{am}}(A)$ is a face in the simplex $T(A)$, there exists an extreme, amenable trace τ_0 on A. Let $\iota \colon T(A) \to T(A)$ be the map from Proposition 4.3. Then $\tau = \iota(\tau_0)$ is extreme and amenable, and it is $\alpha^{(t)}$-invariant for every $t \in \mathbb{R}$ by Proposition 4.3. By part (5) of Proposition 4.4, $\alpha^{(t)}$ is τ-mixing for every $t \in \mathbb{R}$. Finally, Theorem 4.6 implies that the weak extensions of the $\alpha^{(t)}$ to $\overline{A}^{\tau}$ are pairwise not weakly cocycle conjugate. This concludes the proof. □

We make some comments on the assumptions of the theorem above. First, subgroups with relative property (T) are abundant: if either Λ or Δ has property (T), then the inclusion $\Delta \subseteq \Lambda$ has relative property (T). On the other hand, it is easy to find many C*-algebras satisfying the assumptions of Theorem 4.7. Indeed, if A_0 is any separable, unital, exact C*-algebra with an amenable trace, then $A = M_p^\infty \otimes A_0^{\otimes\mathbb{N}}$ satisfies the assumptions of said theorem. In particular, A_0 and A need not be simple. We also remark that every trace on a nuclear C*-algebra is necessarily amenable.

To end this section, we explicitly state our result for UHF-algebras, to highlight the contrast with the results in [31, 33, 42].

Corollary 4.8. *Let D be a UHF-algebra of infinite type, and let Λ be a countable group with an infinite subgroup with relative property (T). Then there exists a continuum of pairwise non (weakly) cocycle-conjugate, strongly outer actions of Λ on D.*

5. Conjugacy, cocycle conjugacy, and weak cocycle conjugacy are not Borel

In this section, we discuss how the construction from Section 4 can be used to prove that, under the assumptions of Theorem 4.7, conjugacy, cocycle conjugacy, and weak cocycle conjugacy of strongly outer actions of Λ on A are complete analytic sets.

5.1. **Borel complexity of equivalence relations.** We recall here some notions from Borel complexity theory. In this setting, a *classification problem* is identified with an equivalence relation E on a Polish space X. Virtually any concrete classification problem in mathematics is of this form, perhaps after a suitable parameterization. For example, a countable discrete group can be identified with a set of triples of natural numbers, coding a group operation on $\mathbb{N}$. The space of such sets of triples is a G_δ subset of the compact metrizable space $\{0,1\}^{\mathbb{N}^3}$ endowed with the product topology. (A G_δ subspace of a Polish space is Polish by [28, Theorem 3.11].)

Definition 5.1. (See [15, Definitions 5.1.1 and 5.1.2]). A *Borel reduction* from an equivalence relation E on a Polish space X to an equivalence relation F on a Polish space Y is a Borel function $f\colon X \to Y$ such that $[x]_E \mapsto [f(x)]_F$ is a well-defined injective function from the space X/E of E-classes to the space Y/F of F-classes. The equivalence relation E is said to be *Borel reducible* to F, in formulas $E \leq_B F$, if there exists a Borel reduction from E to F.

Remark 5.2. When E is Borel reducible to F, the objects of X up to E can be *explicitly* classified using F-classes as complete invariants. In other words, the classification problem represented by F is *at least as complex* as the classification problem represented by E. (Observe that this notion does not depend on the topologies of X and Y, but only on the standard Borel structures that they induce.)

The notion of Borel reducibility can be used to measure the complexity of a given classification problem. The first natural measure of complexity is simply the *number of classes* of the corresponding equivalence relation. Theorem 4.7 addresses this problem in the case of conjugacy, cocycle conjugacy, and weak cocycle conjugacy of strongly outer actions of Λ on A: they have a continuum of equivalence classes.

The natural next step in the study of the complexity of a classification problem consists in determining whether the classes can be *explicitly* parameterized as the points of a Polish space. This is equivalent to the corresponding equivalence relation being *smooth*, that is, Borel reducible to the relation of *equality* in some Polish space. As an example, Glimm's classification of separable UHF-algebras implies that the relation of *-isomorphism for these algebras is smooth (even though there exists a continuum of isomorphism classes). Similarly, the orbit equivalence relation of a continuous action of a compact group on a Polish space is smooth. However, isomorphism of countable rank-one torsion-free abelian groups, for instance, is not smooth. Another canonical example of a nonsmooth equivalence relation is the relation of *tail equivalence* for binary sequences.

A more generous notion of being well-behaved for an equivalence relation E on X is being *Borel* as a subset of $X \times X$. For instance, isomorphism of countable rank-one torsion-free abelian groups is Borel. Similarly, tail equivalence of binary sequences is Borel and, more generally, the orbit equivalence relation of a *free* continuous action of a Polish group on a Polish space is Borel. (The orbit equivalence relation of a continuous action of a Polish group G on a Polish pace X is Borel if and only if the map that assigns to each point x of X the corresponding *stabilizer subgroup* G_x of G is Borel; see [1, 7.1.2].) Since the relation of equality on any Polish space is clearly Borel, any smooth equivalence relation is, in particular, Borel.

One can also define a similar notion of comparison among *sets*, rather than equivalence relations.

Definition 5.3. (See [28, Section 14.A and Definition 26.7].) A subset A of a Polish space X is said to be *analytic*, or $\mathbf{\Sigma}^1_1$, if there exist a Polish space Z and a Borel function $f\colon Z \to X$ such that A is the image under f of a Borel subset of Z. A *complete analytic set* (also called $\mathbf{\Sigma}^1_1$-complete set) is an analytic subset A of a Polish space X such that, for any other analytic subset B of a Polish space Y, there exists a Borel function $f\colon Y \to X$ such that $f^{-1}(A) = B$.

We recall here the fundamental fact that a complete analytic set is not Borel. The canonical example of a complete analytic set is the set of ill-founded trees on $\mathbb{N}$; see [28, Section 27.A].

As above, we regard an equivalence relation E on a Polish space X as a subset of the product space $X \times X$ endowed with the product topology. Consistently, we say that E is a complete analytic set if it is complete analytic as a subset of $X \times X$. It is clear that if E is Borel reducible to an equivalence relation F, and E is a complete analytic set, then F is a complete analytic set as well.

In Theorem 5.9, we will prove that the construction of actions from profinite groups described in Section 4 can be used to show that, under the assumption of Theorem 4.7, the relations of conjugacy, cocycle conjugacy, and weak cocycle conjugacy of strongly outer actions of Λ on A are complete analytic sets. This is a significant strengthening of the conclusions of Theorem 4.7.

5.2. **Parametrizing actions.** For the rest of this section, we fix a countable discrete group Λ and a separable unital C*-algebra A. We proceed to explain how the classification problem for strongly outer actions of Λ on A can be naturally regarded as equivalence relations on a Polish space. We regard $\mathrm{T}(A)$ as a compact metrizable space endowed with the w*-topology. The set $\mathrm{Act}_\Lambda(A)$ of actions of Λ on A is a closed subset of the product space $\mathrm{Aut}(A)^\Lambda$ endowed with the product topology, giving it the structure of a Polish space.

Notation 5.4. *Let τ be a trace on A.*

- *We denote by $\mathrm{Act}_\Lambda(A,\tau)$ the set of τ-preserving actions of Λ on A;*
- *We denote by $\mathrm{WM}_\Lambda(A,\tau)$ the set of τ-preserving weakly τ-mixing actions of Λ on A;*
- *We denote by $\mathrm{SO}_\Lambda(A)$ the set of strongly outer actions of Λ on A;*
- *We denote by $\mathrm{SOWM}_\Lambda(A,\tau)$ the set of τ-preserving weakly τ-mixing strongly outer actions of Λ on A.*

It is easy to see that $\mathrm{Act}_\Lambda(A,\tau)$ and $\mathrm{WM}_\Lambda(A,\tau)$ are G_δ subsets of $\mathrm{Act}_\Lambda(A)$. We will show below that $\mathrm{SO}_\Lambda(A)$ and $\mathrm{SOWM}_\Lambda(A,\tau)$ are also G_δ subsets of $\mathrm{Act}_\Lambda(A)$.

Given a C*-algebra A, we let A_{sa} be the set of selfadjoint elements of A. An element a of A_{sa} is a contraction if $\|a\| \leq 1$. Given a trace τ on A, we let $\|a\|_\tau = \sqrt{\tau(a^*a)}$ be the 2-norm induced by τ on A and $\overline{A}^\tau$. Using Borel functional calculus [2, Section I.4.3], we fix a continuous function $\varpi\colon [0,+\infty) \to [0,+\infty)$ satisfying the following properties:

- $\varpi(0) = 0$ and $\varpi(t) \geq t$ for all $t \in [0,\infty)$;
- Let (M,τ) be a tracial von Neumann algebra, let $a \in M_{\mathrm{sa}}$ be a contraction, and let $\varepsilon > 0$. If $\|a^2 - a\|_\tau < \varepsilon$, then there exists a projection $p \in M$ such that $\|p - a\|_\tau < \varpi(\varepsilon)$.

The following lemma will be used to show that the space of strongly outer (weak mixing) actions of a fixed countable group on a unital, separable C*-algebra is a Polish space; see Proposition 5.7. Its proof follows from Lemma 4.2 using an easy approximation argument, which is presented for the sake of completeness.

Lemma 5.5. Let A be a unital, separable C*-algebra, let $\theta \in \mathrm{Aut}(A)$ be an automorphism of A, and let τ be a θ-invariant trace on A. Fix a countable dense subset A_0 of the unit ball of A_{sa}. Then $\overline{\theta}^\tau$ is properly outer if and only if for every $a \in A_0$, and every $\varepsilon \in (0,+\infty) \cap \mathbb{Q}$ satisfying $\|a^2 - a\|_\tau \leq \varepsilon$, there exists a contraction $b \in A_{\mathrm{sa}}$ satisfying

$$\|b^2 - b\|_\tau < \varepsilon, \quad \|ab - b\|_\tau < \varpi(\varepsilon), \quad \|b\theta(b)\|_\tau < \varepsilon, \quad \text{and} \quad \tau(b) > \frac{1}{3}\tau(a) - \varepsilon.$$

Proof. Let $\pi_\tau : A \to B\left(L^2(A,\tau)\right)$ be the GNS representation associated with τ. Suppose that $\overline{\theta}^\tau$ is properly outer. Let $\varepsilon \in (0,\infty) \cap \mathbb{Q}$, and let $a \in A_0$ satisfy $\|a^2 - a\|_\tau < \varepsilon$. By the choice of ϖ, there exists a projection $p \in \overline{A}^\tau$ such that $\|\pi_\tau(a) - p\|_\tau < \varpi(\varepsilon)$. By (1)⇒(4) in [30, Lemma 4.2], there exists a projection $q \in \overline{A}^\tau$ such that

$$q \leq p, \quad \|q\overline{\theta}^\tau(q)\|_\tau < \varepsilon, \quad \text{and} \quad \tau(q) \geq \frac{1}{3}\tau(p) > \frac{1}{3}\tau(a) - \varepsilon.$$

Therefore, $\|q\pi_\tau(a) - q\|_\tau \leq \|q - qp\|_\tau + \|qp - q\pi_\tau(a)\|_\tau < \varpi(\varepsilon)$, and similarly $\|\pi_\tau(a)q - q\|_\tau < \varpi(\varepsilon)$. Since the norm-unit ball of A is $\|\cdot\|_\tau$-dense in the unit ball of $\overline{A}^\tau$, there exists a contraction $b \in A_{\mathrm{sa}}$ satisfying the conditions in the statement.

We prove the converse. We want to prove that $\overline{\theta}^\tau$ is properly outer. Fix a nonzero projection $p \in \overline{A}^\tau$ and $\varepsilon, \varepsilon_0 > 0$. By (4)⇒(1) in [30, Lemma 4.2], it is enough to prove that there exists a projection $q \in \overline{A}^\tau$ such that

$$q \leq p, \quad \|q\overline{\theta}^\tau(q)\|_\tau < \varepsilon, \quad \text{and} \quad \tau(q) \geq \frac{1}{3}\tau(p) > \frac{1}{3}\tau(a) - \varepsilon.$$

Let $a \in A_0$ satisfy $\|\pi_\tau(a) - p\|_\tau < \varepsilon$ and $\left\|a^2 - a\right\|_\tau < \varepsilon$. By assumption, there exists a contraction $b \in A_{\mathrm{sa}}$ with

$$\|b^2 - b\|_\tau < \varepsilon, \quad \|ab - b\|_\tau < \varpi(\varepsilon) \quad \|b\theta(b)\|_\tau < \varepsilon, \quad \text{and} \quad \tau(b) > \frac{1}{3}\tau(a) - \varepsilon.$$

By the choice of ϖ, there exists a projection $r \in \overline{A}^\tau$ such that $\|\pi_\tau(r) - b\|_\tau < \varpi(\varepsilon)$. By choosing ε small enough, one can ensure that

$$\|pr - r\|_\tau < \varepsilon_0, \quad \|rp - p\|_\tau < \varepsilon_0, \quad \left\|r\overline{\theta}^\tau(r)\right\|_\tau < \varepsilon_0, \quad \text{and} \quad \tau(r) > \frac{1}{3}\tau(a) - \varepsilon_0.$$

By choosing ε_0 small enough, one can then find a projection $q \in \overline{A}^\tau$ satisfying the conditions in item (4) of [30, Lemma 4.2] mentioned above. This concludes the proof. □

For convenience, we record the following easy lemma. For a relation $\mathcal{R} \subset X \times Y$, its *projection onto* X is

$$\mathrm{proj}_X(\mathcal{R}) = \{x \in X\colon \text{ there is } y \in Y \text{ with } (x,y) \in \mathcal{R}\}.$$

Lemma 5.6. Let X be a Polish space, let Y be a compact metrizable space, and let $\mathcal{R} \subset X \times Y$ be a subset. If $\mathcal{R}$ is closed, then $\mathrm{proj}_X(\mathcal{R})$ is closed. If $\mathcal{R}$ is F_σ, then $\mathrm{proj}_X(\mathcal{R})$ is F_σ.

Proof. It is enough to prove the first assertion, so assume that $\mathcal{R}$ is closed. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{proj}_X(\mathcal{R})$ converging to $x \in X$. Our goal is to prove that $x \in \mathrm{proj}_X(\mathcal{R})$. For every $n \in \mathbb{N}$, let $y_n \in Y$ satisfy $(x_n, y_n) \in \mathcal{R}$. Since Y is compact, after passing to a subsequence, we can assume that the sequence $(y_n)_{n\in\mathbb{N}}$ converges to some $y \in Y$. Since $\mathcal{R}$ is closed, we have $(x, y) \in \mathcal{R}$ and hence $x \in \mathrm{proj}_X(\mathcal{R})$, as desired. □

Recall the definitions of the sets $\mathrm{SO}_\Lambda(A)$ and $\mathrm{SOWM}_\Lambda(A)$ from Notation 5.4.

Proposition 5.7. *Let A be a unital, separable C*-algebra, let Λ be a countable group, and let τ be a trace on A. Then the sets $\mathrm{SO}_\Lambda(A)$ and $\mathrm{SOWM}_\Lambda(A, \tau)$ are G_δ subsets of $\mathrm{Act}_\Lambda(A)$.*

Proof. Fix $\gamma \in \Lambda$. Let $\mathcal{R}_\gamma$ be the set of pairs $(\alpha, \tau) \in \mathrm{Act}_\Lambda(A) \times \mathrm{T}(A)$ such that τ is α_γ-invariant and $\overline{\alpha}^\tau_\gamma$ is *not* properly outer. By Lemma 5.5, $\mathcal{R}_\gamma$ is an F_σ subset of $\mathrm{Aut}_\Lambda(A)$. By Lemma 5.6, its projection $\mathcal{P}_\gamma$ onto $\mathrm{Act}_\Lambda(A)$ is F_σ as well. Let $\mathcal{C}_\gamma$ be the complement of $\mathcal{P}_\gamma$ in $\mathrm{Act}_\Lambda(A)$, which is G_δ. We have that $\mathrm{SO}_\Lambda(A)$ is the intersection of $\mathcal{C}_\gamma$ for $\gamma \in \Lambda$, and hence G_δ. We have already observed that $\mathrm{WM}_\Lambda(A, \tau)$ is G_δ. Therefore $\mathrm{FWM}_\Lambda(A, \tau) = \mathrm{WM}_\Lambda(A, \tau) \cap \mathrm{SO}_\Lambda(A)$ is G_δ as well. □

Adopt the notation of the lemma above. We regard $\mathrm{SO}_\Lambda(A)$ as the Polish space of strongly outer actions of Λ on A. Consistently, we regard the classification problems for strongly outer actions of Λ on A up to conjugacy, cocycle conjugacy, or weak cocycle conjugacy, as equivalence relations on $\mathrm{SO}_\Lambda(A)$. Similarly, if τ is a trace on A, we regard $\mathrm{SOWM}_\Lambda(A, \tau)$ as the space of τ-preserving weakly τ-mixing strongly outer actions of Λ on A. On the latter space we can also consider the relations of τ-conjugacy, cocycle τ-conjugacy, and outer τ-conjugacy.

5.3. **Parametrizing abelian pro-p groups.** Fix a prime number p. In this subsection, we define a compact metrizable space parametrizing in a canonical way all second countable abelian pro-p groups. The construction is analogous to the one from [36, Section 2.2].

Let $\mathbb{Z}^\infty$ be the free abelian group on a countably infinite set $\{x_k : k \in \mathbb{N}\}$ of generators. Let $\mathcal{N}$ be the (*countable*) collection of finite index subgroups of $\mathbb{Z}^\infty$ whose index is a multiple of p, and which contain all but finitely many of the generators of $\mathbb{Z}^\infty$. We consider $\mathbb{Z}^\infty$ as a topological group having the elements of $\mathcal{N}$ as basis of neighborhoods of the identity. Define $\hat{\mathbb{Z}}^\infty_p$ to be the completion of $\mathbb{Z}^\infty$ with respect to such a topology, which is a second countable abelian pro-p group. In the terminology of [40, Section 3.3], $\hat{\mathbb{Z}}^\infty_p$ is the free abelian pro-p group on a sequence of generators $(x_k)_{k\in\mathbb{N}}$ converging to 1.

Suppose that G is a second countable abelian pro-p group. By [40, Proposition 2.4.4 and Proposition 2.6.1] G has a generating sequence converging to the identity. It therefore follows from [40, Section 3.3.16] that there exists a surjective continuous group homomorphism $\pi \colon \hat{\mathbb{Z}}^\infty_p \to G$. In other words, G is isomorphic to the quotient of $\hat{\mathbb{Z}}^\infty_p$ by a closed subgroup. Conversely, any quotient of $\hat{\mathbb{Z}}^\infty_p$ by a closed subgroup is a second-countable abelian pro-p group. Thus the closed subgroups of $\hat{\mathbb{Z}}^\infty_p$ naturally parametrize all second-countable abelian pro-p groups.

We let $\mathcal{K}(\hat{\mathbb{Z}}^\infty_p)$ be the space of closed subsets of $\hat{\mathbb{Z}}^\infty_p$ endowed with the Vietoris topology [28, Section 4.F], which turns it into a compact metrizable space. Let also $\mathcal{S}(\hat{\mathbb{Z}}^\infty_p) \subseteq \mathcal{K}(\hat{\mathbb{Z}}^\infty_p)$ be the (closed) subset of closed subgroups of $\hat{\mathbb{Z}}^\infty_p$. Then $\mathcal{S}(\hat{\mathbb{Z}}^\infty_p)$ is a compact metrizable space with the relative topology. We regard isomorphism of second-countable abelian pro-p groups as an equivalence relation on $\mathcal{S}(\hat{\mathbb{Z}}^\infty_p)$.

Proposition 5.8. *Let p be a prime number. The relation of topological isomorphism of second-countable abelian pro-p groups is a complete analytic set.*

Proof. As it is observed in [36, Section 4], Pontryagin's duality theorem in the special case of profinite *abelian* groups is witnessed by a Borel map, and for p-groups, the parametrization given in [36, Section 4] is compatible with the one discussed above, as we now show.

In [36, Section 4], abelian profinite groups are parametrized as follows. Let F_ω be the free group on a countably infinite set of generators, let $\hat{F}_\omega$ be the completion of F_ω with respect to the collection of finite index subgroups of F_ω which contain all but finitely many of the generators. and let $\pi : \hat{F}_\omega \to \hat{\mathbb{Z}}^\infty_p$ be the canonical quotient mapping that sends generators to generators. One denotes by $\mathcal{N}_{\mathrm{ab}}(\hat{F}_\omega)$ the space of closed normal subgroup of $\hat{F}_\omega$ that contain the commutator subgroup of $\hat{F}_\omega$. This is a closed subspace of the space of closed subsets of $\hat{F}_\omega$ endowed with the Vietoris topology. Since any second countable profinite abelian group is a quotient of $\hat{F}_\omega$ by an element of $\mathcal{N}_{\mathrm{ab}}(\hat{F}_\omega)$, the space $\mathcal{N}_{\mathrm{ab}}(\hat{F}_\omega)$ can be seen as a parametrization of (presentations of) abelian profinite groups. In this parametrization, the class of abelian pro-p groups correspond to the closed subspace $\mathcal{N}_{\mathrm{ab}}(\hat{F}_\omega)_p$ of elements of $\mathcal{N}_{\mathrm{ab}}(\hat{F}_\omega)$ that contain $\mathrm{Ker}(\pi)$. Furthermore, the assignments $\mathcal{N}_{\mathrm{ab}}(\hat{F}_\omega)_p \to \mathcal{S}(\hat{\mathbb{Z}}^\infty_p)$, $A \mapsto \pi(A)$ and $\mathcal{S}(\hat{\mathbb{Z}}^\infty_p) \to \mathcal{N}_{\mathrm{ab}}(\hat{F}_\omega)_p$, $A \mapsto \pi^{-1}(A)$ are Borel functions that map presentations for a given abelian pro-p in one parametrization to presentations for the same abelian pro-p group in the other parametrization. This shows that the parametrization for abelian pro-p groups introduced above is compatible with the parametrization of arbitrary abelian profinite groups considered in [36, Section 4].

The duals of abelian pro-p groups are precisely the countable abelian p-groups; see [40, Theorem 2.9.6 and Lemma 2.9.3]. Therefore, the relation of isomorphism of countable abelian p-groups is Borel reducible (in fact, Borel isomorphic) to the relation of isomorphism of second-countable abelian pro-p groups. Since the relation of isomorphism of countable abelian p-groups is a complete analytic set [14, Theorem 6], the result follows. $\square$

5.4. **Reducing groups to actions.** In this last subsection, we obtain the main results of this work. Recall that for a discrete group Λ, a separable C*-algebra A, and a trace τ on A, we denote by $\mathrm{SOWM}_\Lambda(A,\tau)$ the Polish space of τ-preserving strongly outer weakly τ-mixing actions of Λ on A. Below, we will assume all the C*-algebras to be *separable*. In the proof of the following theorem we will tacitly use the fact—proved in [12, 13, 19]—that tensor products, direct limits, and crossed products of C*-algebras and C*-dynamical systems are given by Borel functions with respect to the parameterizations of C*-algebras and C*-dynamical systems considered in [12, 13, 19]. It is also not difficult to see that fixed point algebras of actions of *compact* groups on C*-algebras can be computed in a Borel way.

Theorem 5.9. *Let Λ be a countable group containing an infinite relative property* (T) *subgroup. Fix a prime number p. Let A be a separable, locally reflexive, M_{p^∞}-absorbing, unital C*-algebra with an amenable trace, satisfying $A \cong A^{\otimes \mathbb{N}}$. Then there exists an extreme, amenable trace τ on A such that the relation of isomorphism of second-countable abelian pro-p groups is Borel reducible to the following equivalence relations on* $\mathrm{SOWM}_\Lambda(A,\tau)$*:*

(1) conjugacy;
(2) cocycle conjugacy;
(3) weak cocycle conjugacy;
(4) τ-conjugacy;
(5) cocycle τ-conjugacy;
(6) weak cocycle τ-conjugacy

Proof. In view of Theorem 4.6, and parts (4) and (5) of Proposition 4.4, it is enough to prove that the function $G \mapsto \alpha^G$ that assigns to a second-countable abelian pro-p group G the action $\alpha^G\colon \Lambda \to \mathrm{Aut}(A)$ from Definition 4.2, is given by a Borel function with respect to the parametrization of second-countable abelian pro-p groups and strongly outer actions of Λ on A described in Subsection 5.2 and Subsection 5.3.

Recall that the action α^G is defined by
$$\alpha^G_\gamma = \xi_G^{-1} \circ (\beta_{\Lambda \curvearrowright \Lambda, M_p^\infty}|_{E_G} \otimes \sigma)_\gamma \circ \xi_G$$
for $\gamma \in \Lambda$, for some choice of isomorphism $\xi_G\colon E_G \otimes A \to A$. It is therefore enough to show that

(1) the assignment $G \mapsto E_G$ is given by a Borel function, and
(2) the isomorphism $\xi_G\colon E_G \otimes A \to A$ can be chosen in a Borel fashion from E_G.

We address the second assertion first. Several Borel parameterizations of separable unital C*-algebras are considered in [13]. Therein, these parameterizations are shown to be equivalent, in the sense that one can find Borel functions between any two of them, that map a code for a C*-algebra in one parametrization to a code for the same C*-algebra in the other parametrization. Furthermore, it is shown in [13, 12, 19] that the standard constructions of C*-algebra theory, including tensor products and direct limits, are given by Borel functions with respect to these parametrizations.

For simplicity, we consider here the parametrization Ξ from [13], which is defined as follows. Let $\mathbb{Q}(i)$ be the field of Gaussian rationals, and let $\mathcal{U}$ be the collection of noncommutative *-polynomials with constant term and with coefficients from $\mathbb{Q}(i)$ in the variables $(x_n)_{n\in\mathbb{N}}$. Let Ξ be the set of functions $f\colon \mathcal{U} \to \mathbb{R}$ such that f defines a seminorm on $\mathcal{U}$ with the property that, if $C^*(f)$ is the Hausdorff completion of $\mathcal{U}$ with respect to the metric defined by f, then the unital $\mathbb{Q}(i)$-$*$-algebra structure of $\mathcal{U}$ induces a unital C*-algebra structure on $C^*(f)$. For $\mathfrak{p} \in \mathcal{U}$, we let $\mathfrak{p}_f$ be the corresponding element of $C^*(f)$. It is shown in [13] that Ξ is a G_δ subset of $\mathbb{R}^{\mathcal{U}}$ endowed with the product topology. Furthermore, it follows from stability of the relations defining the matrix units for a unital copy of M_{p^n} that the set UHF_{p^∞} of codes $f \in \Xi$ such that $C^*(f)$ is isomorphic to M_{p^∞} is a G_δ subset of Ξ, and hence a Polish space with the induced topology. Given $f \in \mathrm{UHF}_{p^\infty}$, a *-isomorphism $\psi\colon C^*(f) \to M_{p^\infty}$ can be regarded as an element of $(M_{p^\infty})^{\mathcal{U}}$. Indeed, given $\psi\colon C^*(f) \to M_{p^\infty}$ one can consider the element $\bar{a} = (a_\mathfrak{p})_{\mathfrak{p}\in\mathcal{U}}$ of $(M_{p^\infty})^{\mathcal{U}}$ defined by setting $a_\mathfrak{p} = \psi(\mathfrak{p}_f)$ for $\mathfrak{p} \in \mathcal{U}$. Thus, it suffices to show that there exists a Borel assignment $\mathrm{UHF}_{p^\infty} \to (M_{p^\infty})^{\mathcal{U}}$, $f \mapsto \bar{a}^{(f)} = (a^{(f)}_\mathfrak{p})_{\mathfrak{p}\in\mathcal{U}}$ such that the assignment $\mathfrak{p}_f \mapsto a^{(f)}_\mathfrak{p}$ extends to a *-isomorphism from $C^*(f)$ to M_{p^∞}. To this purpose, we consider the set $\mathcal{A}$ of pairs

$(f,(a_{\mathfrak{p}})_{\mathfrak{p}\in\mathcal{U}}) \in \mathrm{UHF}_{p^\infty} \times (M_{p^\infty})^{\mathcal{U}}$ such that the assignment $\mathfrak{p}_f \mapsto a_{\mathfrak{p}}$ extends to a *-isomorphism from $C^*(f)$ to M_{p^∞}. It is easy to see that $\mathcal{A}$ is a G_δ subset of $\mathrm{UHF}_{p^\infty} \times (M_{p^\infty})^{\mathcal{U}}$. Furthermore, the automorphism group $\mathrm{Aut}(M_{p^\infty})$ of M_{p^∞} naturally acts on $(M_{p^\infty})^{\mathcal{U}}$, in such a way that, for every $f \in \mathrm{UHF}_{p^\infty}$, the corresponding fiber

$$\mathcal{A}_f = \left\{\bar{a} \in (M_{p^\infty})^{\mathcal{U}} : (f,\bar{a}) \in \mathcal{A}\right\}$$

forms a single orbit under the $\mathrm{Aut}(M_{p^\infty})$-action. It follows form these observations and [10, Theorem A] that $\mathcal{A}$ admits a *Borel uniformization*, i.e. there exists a Borel function $\mathrm{UHF}_{p^\infty} \to (M_{p^\infty})^{\mathcal{U}}$, $f \mapsto \bar{a}^{(f)}$ such that $(f,\bar{a}^{(f)}) \in \mathcal{A}$ for every $f \in \mathrm{UHF}_{p^\infty}$. This allows one to choose in a Borel fashion, given a C*-algebra E abstractly isomorphic to M_{p^∞}, a *-isomorphism $\psi_E : E \to M_{p^\infty}$. Since $M_{p^\infty} \otimes A$ is isomorphic to A, by fixing an isomorphism $M_{p^\infty} \otimes A \cong A$ beforehand, one can choose in a Borel fashion an isomorphism $\xi_E \colon E \otimes A \to A$. This justifies the second assertion.

We now justify the first assertion. Recall that E_G is the fixed point algebra of the action $(\delta^G)^{\otimes\Lambda} \colon G \to \mathrm{Aut}(D_G^{\otimes\Lambda})$. Furthermore, $(\delta^G)^{\otimes\Lambda}$ is conjugate to canonical model action $\delta^G : G \to \mathrm{Aut}\,(D_G)$ of G constructed in Theorem 3.5. Therefore, it is enough to show that δ^G can be constructed in a Borel fashion from G. This is clear when G is finite in view of Remark 3.2. In the general case, consider the following. In our parametrization, a second-countable abelian pro-p group G is given as the quotient $\hat{\mathbb{Z}}_p^\infty/N$ of $\hat{\mathbb{Z}}_p^\infty$ by some closed subgroup N of $\hat{\mathbb{Z}}_p^\infty$. The finite-index closed subgroups of G correspond to finite-index closed subgroups H of $\hat{\mathbb{Z}}_p^\infty$ that contain N. By the Kuratowski–Ryll-Nardzewski selection theorem [28, Theorem 12.13], the collection $\mathcal{V}$ of finite-index closed subgroups H of $\hat{\mathbb{Z}}_p^\infty$ that contain N can be chose in a Borel fashion starting from N. Since the relation of inclusion between closed subgroups is closed with respect to the Vietoris topology, the order on $\mathcal{V}$ given by containment is Borel. This shows that the canonical inverse system $(G_i,\pi_{i,j})_{i,j\in\mathcal{V}}$ of finite groups having G as inverse limit considered in the proof of Theorem 3.5 depends on G in a Borel way. The G-C*-algebra (D_G,δ_G) is obtained in the proof of Theorem 3.5 as the direct limit of the direct system $\left(\left(D_{G_i},\delta_{G_i}\right),\iota_{ij}\right)_{i,j\in\mathcal{V}}$, where (D_{G_i},δ_{G_i}) is the model action of the finite group G_i. It remains to observe now that the direct system $\left(\left(D_{G_i},\delta_{G_i}\right),\iota_{ij}\right)_{i,j\in\mathcal{V}}$ can be computed in a Borel fashion from $(G_i,\pi_{i,j})_{i,j\in\mathcal{V}}$. This concludes the proof. □

Corollary 5.10. *Under the hypotheses of Theorem 5.9, the relations of conjugacy, cocycle conjugacy, and weak cocycle conjugacy of strongly outer actions of Λ on A are complete analytic sets. Furthermore, there exists an amenable, extreme trace τ on A such that the relations of conjugacy, cocycle conjugacy, weak cocycle conjugacy, τ-conjugacy, cocycle τ-conjugacy, and weak cocycle τ-conjugacy of τ-preserving strongly outer weakly τ-mixing actions of Λ on A are complete analytic sets, and in particular not Borel.*

As mentioned after Theorem 4.7, it is easy to construct algebras A satisfied the hypotheses of Corollary 5.10. Indeed, if A_0 is any separable, unital, exact C*-algebra with an amenable trace, we may take $A = M_p^\infty \otimes A_0^{\otimes\mathbb{N}}$.

As we did after Theorem 4.7, we state the case of UHF-algebras separately, to highlight the contrast with the main results of [31, 33, 42].

Corollary 5.11. *Let D be a UHF-algebra of infinite type, and let Λ be a countable group with an infinite subgroup with relative property* (T). *Then the relations of conjugacy, cocycle conjugacy, and weak cocycle conjugacy of strongly outer actions of Λ on D are complete analytic sets, and in particular not Borel. The same applies to the relations of being conjugate, cocycle conjugate, and weakly cocycle conjugate in the weak closure with respect to the (necessarily Λ-invariant) unique trace on D.*

In fact, the same conclusions hold for any finite, strongly self-absorbing C*-algebra containing a nontrivial projection; see [45, Definition 1.3] and [45, Theorem 1.7].

6. Actions on R with prescribed cohomology

In this final section, we explain how the methods from Section 5 can be used to prove Theorem C in the introduction. In fact, we prove a somewhat more general statement; see Theorem 6.3.

We follow a strategy similar to the one used in Theorem 4.5, and for that we will need the following replacement of the action constructed in Theorem 3.5. Later on, we will be interested only in the weak extension of the action constructed below.

Recall that $\mathtt{Lt} \colon G \to \mathrm{Aut}(C(G))$ denotes the action of left translation. Similarly, we denote by $\mathtt{Rt}$ the action of G on $C(G)$ by right translation.

Proposition 6.1. *Let G be a second-countable, compact Hausdorff group. Then there exist a unital C*-algebra A_G and an action $\theta^G \colon G \to \mathrm{Aut}(A_G)$ with the following properties:*

(1) A_G is simple, separable, nuclear, and has a unique trace.
(2) There exists an equivariant unital embedding $(C(G), \mathtt{Lt}^G) \to (A_G, \theta^G)$.
(3) $(A_G^{\otimes \mathbb{N}}, (\theta^G)^{\otimes \mathbb{N}})$ is equivariantly isomorphic to (A_G, θ^G).
(4) θ^G has the Rokhlin property.
(5) The fixed point algebra $A_G^{\theta^G}$ is a simple, separable, nuclear, unital C-algebra with a unique trace.*

Proof. Fix a subset $\{x_n : n \in \mathbb{N}\}$ of G such that $\{x_n : n \geq m\}$ is dense in G for all $m \in \mathbb{N}$. For $n \in \mathbb{N}$, set $A_n = M_{2^n} \otimes C(G)$ with the G-action $\alpha^{(n)} = \mathrm{id}_{M_{2^n}} \otimes \mathtt{Lt}$. Let $\rho^{(n)}$ be the G-action on A_n given by $\rho^{(n)} = \mathrm{id}_{M_{2^n}} \otimes \mathtt{Rt}$, and define a unital equivariant homomorphism $\varphi_n \colon A_n \to A_{n+1}$ by $\varphi_n(a) = \begin{pmatrix} a & 0 \\ 0 & \rho_{x_n}(a) \end{pmatrix}$ for all $a \in A_n$. Define $\widetilde{A}_G$ and $\widetilde{\theta}^G$ to be the direct limits of $(A_n)_{n\in\mathbb{N}}$ and $(\alpha^{(n)})_{n\in\mathbb{N}}$ with respect to the connecting maps $(\varphi_n)_{n\in\mathbb{N}}$.

Observe that $\widetilde{A}_G$ is separable, unital and nuclear. We claim that it has a unique trace and that it is simple. Both fact are proved using similar arguments, so we only show uniqueness of the trace. Denote by τ the trace on $C(G)$ given by integration against the normalized Haar measure μ on G, and by tr_n the normalized trace on M_{2^n}. Then $\tau_n = \mathrm{tr}_n \otimes \tau$ is a normalized trace on A_n and $\tau_{n+1} \circ \varphi_n = \tau_n$ for all $n \in \mathbb{N}$. It follows that there is a direct limit trace on $\widetilde{A}_G$. Now let σ be another trace on $\widetilde{A}_G$. Then there exist $n_0 \in \mathbb{N}$ and traces $\sigma_n \in T(A_n)$, for $n \geq n_0$, satisfying $\sigma_{n+1} \circ \varphi_n = \sigma_n$ for all $n \geq n_0$ and $\tau(a) = \tau_n(a)$ for all $a \in A_n$, for $n \geq n_0$. For $n \geq n_0$, let ν_n be a probability measure on $C(G)$ such that, with $\widehat{\nu}_n$ denoting its associated functional on $C(G)$, we have $\sigma_n = \mathrm{tr}_n \otimes \widehat{\nu}_n$. The identity $\sigma_{n+1} \circ \varphi_n = \sigma_n$ amounts to $\nu_n(E) = \frac{1}{2}\left(\nu_{n+1}(x_n E) + \nu_{n+1}(E)\right)$ for every measurable subset $E \subseteq G$. Using the identity $\sigma_{n+k} \circ \varphi_{n+k-1} \circ \cdots \circ \varphi_n = \sigma_n$, valid for all $k \geq 1$, an using that $\{x_k : k \geq n\}$ is dense in G, one concludes that ν_n is translation invariant, and hence we must have $\nu_n = \mu$ for all $n \in \mathbb{N}$. In particular, it follows that $\sigma = \tau$, as desired.

The proof of simplicity is analogous, using open subsets of G which are translation invariant. We omit the details.

Now set $A_G = \otimes_{k\in\mathbb{N}} \widetilde{A}_G$ and $\alpha^G = \otimes_{k\in\mathbb{N}} \widetilde{\alpha}^G$. Then A_G is simple, separable, unital, nuclear, and has a unique trace, which proves (1). Observe that there are equivariant unital embeddings

$$C(G) \hookrightarrow A_1 \hookrightarrow \widetilde{A}_G \hookrightarrow A_G,$$

so part (2) is satisfied. Also, (3) holds by construction, while (4) follows from (2) and (3). Finally, the Rokhlin property for θ^G ensures that the properties for A_G listed in (1) are inherited by $A_G^{\theta^G}$, by the theorem in the introduction of [16]. This gives (5), and finishes the proof. □

Observe that A_G is never a UHF-algebra (unless G is the trivial group). In particular, even when G is a profinite group, the C*-dynamical system (A_G, θ^G) constructed in Proposition 6.1 is not the same as that constructed in Theorem 3.5.

Remark 6.2. Unlike in Theorem 3.5, the action constructed in the proposition above does not enjoy any reasonable uniqueness-type property among Rokhlin actions of G.

We now come to the main result of this section, which in particular implies Theorem C in the introduction.

Theorem 6.3. *Let Λ be a countable group containing an infinite subgroup Δ with relative property (T), and let Γ be any countable abelian group. Then there exist an outer action $\alpha^\Gamma \colon \Lambda \to \mathrm{Aut}(R)$ and bijections $\eta \colon H^1_{\Delta,w}(\alpha^\Gamma) \to \Gamma$ and $\eta^{(2)} \colon H^1_{\Delta,w}(\alpha^\Gamma \otimes \alpha^\Gamma) \to \Gamma$ making the following diagram commute:*

$$\begin{tikzcd}
H^1_{\Delta,w}(\alpha^\Gamma) \times H^1_{\Delta,w}(\alpha^\Gamma) \arrow[r, "\eta\times\eta"] \arrow[d, "m^{\alpha^\Gamma}"'] & \Gamma \times \Gamma \arrow[d, "m^\Gamma"] \\
H^1_{\Delta,w}(\alpha^\Gamma \otimes \alpha^\Gamma) \arrow[r, "\eta^{(2)}"'] & \Gamma
\end{tikzcd}$$

Proof. Let G denote the Pontryagin dual of Γ, which is a second-countable, compact Hausdorff group. Let $\theta^G \colon G \to \mathrm{Aut}(A_G)$ denote the action constructed in Proposition 6.1, and denote by $\overline{\theta}^G \colon G \to \mathrm{Aut}(R)$ its weak extension in the GNS representation associated to the unique (and hence θ^G-invariant) trace of A_G. (The fact that the weak closure of A_G is R follows from Lemma 2.23 and part (1) of Proposition 6.1.) Abbreviate $R^{\otimes\Lambda}$ to N, and abbreviate $(\overline{\theta}^G)^{\otimes\Lambda}$ to $\rho \colon G \to \mathrm{Aut}(N)$. Denote by $\beta \colon \Lambda \to \mathrm{Aut}(N)$ the Bernoulli shift $\beta_{\Lambda \curvearrowright \Lambda, R}$ of Λ

on $R^{\otimes\Lambda} = N$. Then β commutes with ρ, and hence induces an action α^Γ of Λ on the fixed point algebra N^ρ of ρ. Since $(\theta^G)^{\otimes\Lambda}$ is conjugate to θ^G by part (3) of Proposition 6.1, it follows that N, which is the weak closure of the fixed point algebra of $(\theta^G)^{\otimes\Lambda}$, is isomorphic to the weak closure of $A_G^{\theta^G}$. Hence N^ρ is isomorphic to R by part (5) of Proposition 6.1 and Lemma 2.23. Under this identification, we regard α^Γ as an action of Λ on R. Finally, the same proof as Lemma 4.1 gives the desired conclusion concerning the Δ-relative weak cohomology group of α^Γ. □

We close this work by pointing out that the argument used in Theorem 4.7 can be used in this context to give an alternative proof of Theorem B in [4] for the case of groups containing a subgroup with the relative property (T). Namely, it follows from Theorem 6.3 that for Λ as in its assumptions, there exist uncountably many weakly non-cocycle conjugate outer actions of Λ on R.

References

[1] Howard Becker and Alexander S. Kechris, *The descriptive set theory of Polish group actions*, London Mathematical Society Lecture Note Series, vol. 232, Cambridge University Press, Cambridge, 1996.

[2] Bruce Blackadar, *Operator algebras*, Encyclopaedia of Mathematical Sciences, vol. 122, Springer-Verlag, Berlin, 2006.

[3] O. Bratteli, D. E. Evans, and A. Kishimoto, *The Rohlin Property For Quasi-Free Automorphisms of the Fermion Algebra*, Proceedings of the London Mathematical Society **s3-71** (1995), no. 3, 675–694 (en).

[4] Arnaud Brothier and Stefaan Vaes, *Families of hyperfinite subfactors with the same standard invariant and prescribed fundamental group*, Journal of Noncommutative Geometry **9** (2015), no. 3, 775–796.

[5] Nathanial P. Brown, *Invariant means and finite representation theory of C*-algebras*, Mem. Amer. Math. Soc. **184** (2006), no. 865, viii+105.

[6] Nathanial P. Brown and Narutaka Ozawa, *C*-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics, vol. 88, American Mathematical Society, Providence, RI, 2008.

[7] Marc Burger, *Kazhdan constants for* $\mathrm{SL}(3,\mathbb{Z})$, Journal für die reine und angewandte Mathematik **413** (1991), 36–67.

[8] Alain Connes, *Outer conjugacy classes of automorphisms of factors*, Annales Scientifiques de l'École Normale Supérieure. Quatrième Série **8** (1975), no. 3, 383–419.

[9] ———, *Periodic automorphisms of the hyperfinite factor of type II_1*, Acta Universitatis Szegediensis. Acta Scientiarum Mathematicarum **39** (1977), no. 1-2, 39–66.

[10] George A. Elliott, Ilijas Farah, Vern I. Paulsen, Christian Rosendal, Andrew S. Toms, and Asger Törnquist, *The isomorphism relation for separable C*-algebras*, Mathematical Research Letters **20** (2013), no. 6, 1071–1080.

[11] Inessa Epstein and Asger Törnquist, *The Borel complexity of von Neumann equivalence*, arXiv:1109.2351 (2011).

[12] Ilijas Farah, Andrew S. Toms, and Asger Törnquist, *The descriptive set theory of C*-algebra invariants*, International Mathematics Research Notices (2012), 5196–5226.

[13] ———, *Turbulence, orbit equivalence, and the classification of nuclear C*-algebras*, Journal für die reine und angewandte Mathematik **688** (2014), 101–146.

[14] Harvey Friedman and Lee Stanley, *A Borel reducibility theory for classes of countable structures*, Journal of Symbolic Logic **54** (1989), no. 3, 894–914.

[15] Su Gao, *Invariant descriptive set theory*, Pure and Applied Mathematics, vol. 293, CRC Press, Boca Raton, FL, 2009.

[16] Eusebio Gardella, *Crossed products by compact group actions with the Rokhlin property*, Journal of Noncommutative Geometry, in press.

[17] ———, *Rokhlin dimension for compact group actions*, Indiana Journal of Mathematics, **66** (2017), 659–703.

[18] ———, *Compact group actions on C*-algebras: classification, non-classifiability, and crossed products; and rigidity results for L^p-operator algebras*, Ph.D. thesis, University of Oregon, 2015.

[19] Eusebio Gardella and Martino Lupini, *Conjugacy and cocycle conjugacy of automorphisms of $\mathcal{O}_2$ are not Borel*, Münster Journal of Mathematics **9** (2016), no. 1, 93–118.

[20] Ilan Hirshberg and Wilhelm Winter, *Rokhlin actions and self-absorbing C*-algebras*, Pacific Journal of Mathematics **233** (2007), no. 1, 125–143.

[21] Masaki Izumi, *Finite group actions on C*-algebras with the Rohlin property—II*, Advances in Mathematics **184** (2004), no. 1, 119–160.

[22] ———, *Finite group actions on C*-algebras with the Rohlin property, I*, Duke Mathematical Journal **122** (2004), no. 2, 233–280.

[23] Xinhui Jiang and Hongbing Su, *On a simple unital projectionless C*-algebra*, American Journal of Mathematics **121** (1999), no. 2, 359–413.

[24] Paul Jolissaint, *On property (T) for pairs of topological groups*, L'Enseignement Mathématique **51** (2005), no. 1-2, 31–45.

[25] Vaughan F. R. Jones, *Actions of finite groups on the hyperfinite type II_1 factor*, Memoirs of the American Mathematical Society **28** (1980), no. 237, v+70.

[26] ———, *A converse to Ocneanu's theorem*, Journal of Operator Theory **10** (1983), no. 1, 61–63.

[27] Robert R. Kallman, *A generalization of free action*, Duke Mathematical Journal **36** (1969), 781–789.

[28] Alexander Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995.

[29] Alexander Kechris and Todor Tsankov, *Amenable actions and almost invariant sets*, Proceedings of the American Mathematical Society **136** (2008), no. 2, 687–697.

[30] David Kerr, Hanfeng Li, and Mikaël Pichot, *Turbulence, representations, and trace-preserving actions*, Proceedings of the London Mathematical Society **100** (2010), no. 2, 459–484.

[31] Akitaka Kishimoto, *The Rohlin property for automorphisms of UHF algebras*, Journal für die reine und angewandte Mathematik **1995** (1995), no. 465, 183–196.

[32] Gregory A. Margulis, *Finitely-additive invariant measures on Euclidean spaces*, Ergodic Theory and Dynamical Systems **2** (1982), no. 3-4, 383–396 (1983).

[33] Hiroki Matui, $\mathbb{Z}^N$*-actions on UHF algebras of infinite type*, Journal für die reine und angewandte Mathematik **657** (2011), 225–244.

[34] Hiroki Matui and Yasuhiko Sato, $\mathcal{Z}$*-stability of crossed products by strongly outer actions*, Communications in Mathematical Physics **314** (2012), no. 1, 193–228.

[35] ———, $\mathcal{Z}$*-stability of crossed products by strongly outer actions II*, American Journal of Mathematics **136** (2014), no. 6, 1441–1496.

[36] André Nies, *The complexity of isomorphism between countably based profinite groups*, arXiv:1604.00609 (2016).

[37] Adrian Ocneanu, *Actions of discrete amenable groups on von Neumann algebras*, Lecture Notes in Mathematics, vol. 1138, Springer-Verlag, Berlin, 1985.

[38] Sorin Popa, *Some computations of 1-cohomology groups and construction of non-orbit-equivalent actions*, Journal of the Institute of Mathematics of Jussieu **5** (2006), no. 2, 309–332.

[39] ———, *Some rigidity results for non-commutative Bernoulli shifts*, Journal of Functional Analysis **230** (2006), no. 2, 273–328.

[40] Luis Ribes and Pavel Zalesskii, *Profinite groups*, second ed., Series of Modern Surveys in Mathematics, vol. 40, Springer-Verlag, Berlin, 2010.

[41] Yasuhiko Sato, *The Rohlin property for automorphisms of the Jiang-Su algebra*, Journal of Functional Analysis **259** (2010), no. 2, 453–476.

[42] Gábor Szabó, *Strongly self-absorbing C*-dynamical systems, III*, Advances in Mathematics **316** (2017), 356–380.

[43] M. Takesaki, *Theory of operator algebras. I*, Encyclopaedia of Mathematical Sciences, vol. 124, Springer-Verlag, Berlin, 2002.

[44] ———, *Theory of operator algebras. III*, Encyclopaedia of Mathematical Sciences, vol. 127, Springer-Verlag, Berlin, 2003.

[45] Andrew S. Toms and Wilhelm Winter, *Strongly self-absorbing C*-algebras*, Transactions of the American Mathematical Society **359** (2007), no. 8, 3999–4029.

[46] Asger Törnquist, *Localized cohomology and some applications of Popa's cocycle superrigidity theorem*, Israel Journal of Mathematics **181** (2011), 327–346.

[47] Stefaan Vaes, *Rigidity results for Bernoulli actions and their von Neumann algebras (after Sorin Popa)*, Astérisque (2007), no. 311, 237–294, Séminaire Bourbaki. Vol. 2005/2006.

[48] Wilhelm Winter, *Localizing the Elliott conjecture at strongly self-absorbing C*-algebras*, Journal für die Reine und Angewandte Mathematik **692** (2014), 193–231.

Eusebio Gardella, Westfälische Wilhelms-Universität Münster, Fachbereich Mathematik, Einsteinstrasse 62, 48149 Münster, Germany

E-mail address: `gardella@uni-muenster.de`

URL: `https://wwwmath.uni-muenster.de/u/gardella/`

Mathematics Department, California Institute of Technology, 1200 E. California Blvd, MC 253-37, Pasadena, CA 91125

E-mail address: `lupini@caltech.edu`

URL: `http://www.lupini.org/`