Topics in Shear Flow

Chapter 10 – The Wall Jet

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Chapter 10

THE WALL JET

A plane jet flowing parallel to an adjacent wall—a wall jet—is a configuration often encountered in ejector design, in film-cooling applications, and in boundary-layer control. The radial wall jet is a variation that is important in problems of heat and mass transfer, as in heating by a torch or drying by an impinging jet. The situations of interest are almost always turbulent. The latter flows are sensitive to residual effects of transition, and the approach to experimental similarity is awkward because a simple displacement of the origin is not compatible with the radial geometry.

In CHAPTER 4, the model for the turbulent boundary layer is a continuously evolving turbulent wake, modified in a definite way by the insertion of a wall along the plane of symmetry. The no-slip condition reduces the velocity to zero at the wall and strongly affects the flow near the original plane of symmetry. In particular, the presence of the wall radically changes the normal or v' fluctuations, which are now reduced to zero at $y = 0$. The no-slip condition also changes the other two fluctuations u' and w' in a more complicated way, as discussed in various places in this monograph. As far as I am aware, the corresponding model has never been considered seriously for the wall jet. This model might be expected to lead to something called the law of the jet and to further development of the concept of equilibrium flow, and it will be addressed in SECTION 10.3.2.

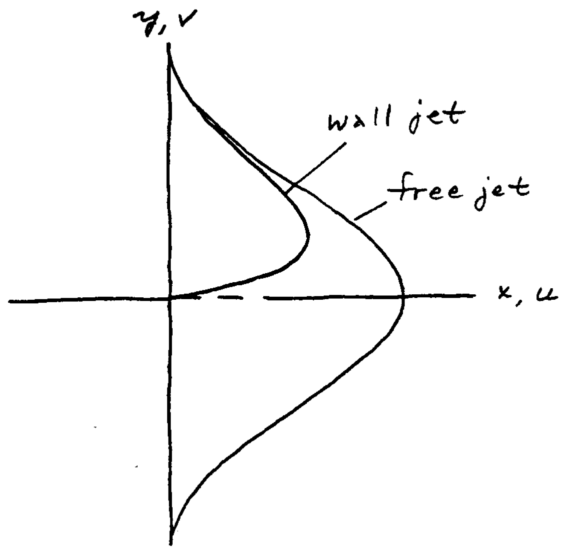

Figure 10.1: Schematic connection between the laminar plane free jet and wall jet.

The laminar wall jet can be visualized as a laminar plane jet with a thin plate inserted on the plane of symmetry, as shown in FIGURE 10.1. The main mathematical consequence of the loss of symmetry for the wall-jet flow is a qualitative change in the similarity argument, which now leads to an eigenvalue problem. It is therefore more important than usual to practice technique with the problem of laminar flow. Part of this technique is an application in SECTION 10.1.6 of the Mangler transformation, which relates a plane flow and a radial flow in the manner shown earlier for the free jet. Many of the other operations carried out in this chapter have already been encountered in CHAPTER 9 on the free jet, where they are described in somewhat greater detail and supported by more extensive arguments.

10.1 Laminar plane wall jet into fluid at rest

10.1.1 The eigenvalue problem

Priority in solving the problem of the laminar plane wall jet with similarity is generally assigned to GLAUERT (1956), although an essentially complete account was published earlier by TETERVIN (1948). The problem is more subtle than the problem of the plane free jet, and the subtleties were fully appreciated by Glauert. The momentum equation in the boundary-layer approximation is the same as for the free jet;

$$\rho\left(\frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y}\right) = \mu\,\frac{\partial^2 u}{\partial y^2} = \frac{\partial \tau}{\partial y} \ . \tag{10.1}$$

The boundary conditions are suitably chosen from

$$\psi = u = v = 0 \ \text{ at } \ y = 0\,, \qquad u = \tau = 0 \ \text{ at } \ y = \infty \ . \tag{10.2}$$

The momentum-integral equation is easily written down by inspection of equation (10.1);

$$\rho\,\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty u\;u\;\mathrm{d}y = \frac{\mathrm{d}J}{\mathrm{d}x} = -\tau_w \ , \tag{10.3}$$

where $\tau_w = \mu(\partial u/\partial y)_w$ and J is the momentum integral previously defined for the plane free jet by equation (9.8). The fact that J is no longer a constant, as it was for the free jet, prevents the introduction at the outset of intrinsic scales for mass, length, and time. The friction at the wall continuously removes momentum from the wall jet, beginning at the origin of the flow at $x = 0$, at a rate that is slow but significant. It will be found for the case of laminar flow that similarity requires the terms in equation (10.3) to behave like $x^{-5/4}$ near the origin (see equations (10.60) and (10.61) below). Thus the singularity in τ_w at $x = 0$ is not integrable. Moreover, since the integral in equation (10.3) behaves like $x^{-1/4}$ near $x = 0$, the initial momentum flux J in the similarity formulation is infinite. I emphasize these points because some experimenters have assumed that the momentum flux J measured at the jet exit or elsewhere in a laminar laboratory flow has some important role to play in similarity formulations of their data. Similar problems with turbulent flow are taken up in SECTION 10.3.1.

One primitive but popular version of dimensional analysis is to assume a power-law behavior and to determine the exponents for two local scales $U(x) \sim x^p$ and $L(x) \sim x^q$ by substitution of a suitable ansatz in the momentum equation (10.1) and its integral (10.3). This approach is demonstrated for several different flows by BIRKHOFF and ZARANTONELLO (1957), for example. These authors did not anticipate the problem of the wall jet, but did comment on an eigenvalue problem for a different flow, the momentumless wake. The new feature in the case of the momentumless wake is that the global constant defined by the momentum integral, the drag, vanishes identically. The new feature in the case of the wall jet is that the two equations (10.1) and (10.3) have essentially the same dimensional structure. In either case, only one condition can be found for the two exponents p and q unless the problem is attacked at a deeper level.

Glauert. To arrive at a dimensionless ansatz, Glauert assumed power-law behavior. He took the free jet as a model and a point of departure. The analysis that follows is faithful in spirit to Glauert's presentation, but the notation and certain details have been changed to suit the style of this monograph. I have also chosen to begin with plane flow rather than radial flow. Glauert postulated the existence of a local velocity scale U and a corresponding local length scale ν/U, and assumed a solution of the form

$$\frac{\psi/\nu}{(Ux/\nu)^a} = f\left[\frac{(Uy/\nu)}{(Ux/\nu)^b}\right] = f(\eta) \ , \tag{10.4}$$

where

$$u = \frac{\partial\psi}{\partial y} \ , \qquad v = -\frac{\partial\psi}{\partial x} \ , \tag{10.5}$$

as usual. Substitution in equation (10.1) yields

$$(a-b)f'f' - aff'' = (Ux/\nu)^{1-a-b}f''' \ , \tag{10.6}$$

where primes indicate differentiation with respect to η. If f is required to depend only on η and not separately on x, this equation supplies one relation between the exponents a and b;

$$a + b = 1 \ , \tag{10.7}$$

together with an ordinary differential equation for f,

$$f''' + (1-b)ff'' - (1-2b)f'f' = 0 \ , \tag{10.8}$$

whose boundary conditions, from equations (10.2), are

$$f(0) = f'(0) = f'(\infty) = 0 \ . \tag{10.9}$$

Glauert's first major contribution was to establish that there exists at least one non-trivial similarity solution of equation (10.8), satisfying the null boundary conditions (10.9), provided that the exponent b has the eigenvalue 3/4. The analysis begins with an integration whose purpose is to examine the shearing stress f'' and to deal with the absence of symmetry. Replace ff'' by $(ff')' - f'f'$ and integrate equation (10.8) formally from some arbitrary positive value of η to $\eta = \infty$ to obtain

$$f'' + (1-b)ff' + (2-3b)\,g = 0 \ , \tag{10.10}$$

where

$$g(\eta) = \int_{\eta}^{\infty} f'f' \mathrm{d}\eta \ . \tag{10.11}$$

The range of integration is evidently chosen to exploit the fact that f' and f'' vanish at infinity for both the free jet and the wall jet. In particular,

$$f''(0) = (3b-2)g(0) \ , \tag{10.12}$$

where

$$g(0) = \int_{0}^{\infty} f'f' \,\mathrm{d}\eta \ . \tag{10.13}$$

A brief digression disposes of the symmetric problem (the free jet). The boundary conditions in the plane of symmetry are then $f(0) = 0$ and $f''(0) = 0$, corresponding to $\psi(x,\,0) = 0$ and $\tau(x,\,0) =$

0, with $f'(0) \sim u_c(x)$ left unspecified. Since $g(0)$ is a positive constant, it follows from equation (10.12) that the boundary condition $f''(0) = 0$ can be satisfied only if $b = 2/3$, $a = 1/3$, in agreement with the result obtained more directly in SECTION 9.1.2 above.

Now return to the unsymmetric problem, the wall jet. The boundary conditions at the wall are $f(0) = 0$ and $f'(0) = 0$, corresponding to $\psi(x,\, 0) = 0$ and $u(x,\, 0) = 0$, with $f''(0) \sim \tau_w(x)$ left unspecified. Nothing can be learned from equation (10.12), and something more is required. Glauert eliminated f'' by multiplying equation (10.10) by f' and integrating through the thickness of the wall jet. After some integration by parts and use of the identity $g' = -f'f'$ and the boundary condition $g(\infty) = 0$, the result is

$$(3 - 4b) \int_0^\infty f'g \,\mathrm{d}\eta = 0 \ . \tag{10.14}$$

The integral in equation (10.14) is a positive constant, provided that the velocity f' is non-negative everywhere, and the equation can therefore be satisfied only for the exponents

$$b = \frac{3}{4} \ , \qquad a = \frac{1}{4} \ . \tag{10.15}$$

This value for b requires, from equation (10.12),

$$f''(0) = \frac{1}{4}\, g(0) \ . \tag{10.16}$$

It reduces the differential equation (10.8) to

$$4f''' + ff'' + 2f'f' = 0 \tag{10.17}$$

and also provides the necessary invariant, which can have different forms;

$$\int_0^\infty f'g \,\mathrm{d}\eta = \int_0^\infty f' \int_\eta^\infty f'f' \mathrm{d}\eta \,\mathrm{d}\eta = -\int_0^\infty fg' \,\mathrm{d}\eta = \\ = \int_0^\infty ff'f' \,\mathrm{d}\eta = \text{constant} \ . \tag{10.18}$$

Of these, the two dominant forms in physical variables are the second and the fourth;

$$\rho \int_0^\infty u \int_y^\infty uu \, \mathrm{d}y \, \mathrm{d}y = \rho \int_0^\infty \psi uu \, \mathrm{d}y = F = \text{constant} \ . \tag{10.19}$$

Like Glauert, I have some difficulty in assigning a physical meaning to the quantity F. His best effort produced the phrase "flux of exterior momentum flux."

Having established the structure of his problem, Glauert repeated his derivation from the beginning in physical variables for readers who do not object to a strong element of *deus ex machina*. Note, as did Glauert, that this second derivation does not require the assumption of similarity or of power-law behavior. First, write an incomplete integral corresponding to equation (10.3) in the form

$$\frac{\partial}{\partial x} \rho \int_y^\infty uu \, \mathrm{d}y - \rho \, u \, v + \tau = 0 \ . \tag{10.20}$$

Denote the integral by W, say;

$$W = \rho \int_y^\infty u \, u \, \mathrm{d}y \ , \tag{10.21}$$

and observe that good things happen if the equation

$$\frac{\partial W}{\partial x} - \rho \, u \, v + \tau = 0 \tag{10.22}$$

is multiplied by the streamwise velocity u and if it is noticed that $-\rho uu = \partial W / \partial y$ from equation (10.21). Thus

$$u \, \frac{\partial W}{\partial x} + v \, \frac{\partial W}{\partial y} + \tau u = 0 \ . \tag{10.23}$$

Add to this the continuity equation multiplied by W to obtain

$$\frac{\partial uW}{\partial x} + \frac{\partial vW}{\partial y} + \tau u = 0 \ . \tag{10.24}$$

Finally, integrate over the thickness of the layer and use the boundary conditions $v(0) = 0$, $W(\infty) = 0$. The result is

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty uW\,\mathrm{d}y + \int_0^\infty \tau u\,\mathrm{d}y = 0 \ . \tag{10.25}$$

A last crucial step can be carried out provided that the flow is laminar, with $\tau = \mu\,\partial u/\partial y$. Then the second term in equation (10.25) drops out;

$$\int_0^\infty \tau u\,\mathrm{d}y = \mu\int_0^\infty \frac{\partial u^2/2}{\partial y}\,\mathrm{d}y = 0 \ , \tag{10.26}$$

since u is zero at both limits. For laminar flow, this procedure has reproduced the conserved quantity (10.19);

$$\int_0^\infty uW\,\mathrm{d}y = \rho\int_0^\infty u\int_y^\infty uu\,\mathrm{d}y\,\mathrm{d}y = F = \text{constant} \ . \tag{10.27}$$

For turbulent flow, neither equation (10.26) nor equation (10.27) is valid.

(Interpret this process in terms of work done on fluid? Minimize the integral of τu? W is the momentum flux outboard of a particular point in the flow. Equation (10.23), written as

$$\frac{DW}{Dt} + \tau u = 0 \ , \tag{10.28}$$

suggests that the rate of change of this quantity following a streamline is given by the rate that work is done by the shearing stress (this needs work). Look at the difference between F and the conserved quantity J for the free jet. Interpret as divergence. Look at energy. Comment on vorticity as variable, with no symmetry and zero integral. See the Rayleigh problem in the introduction.)

Intrinsic scales. Given the existence of the integral invariant F, it is now a simple matter to work out intrinsic scales for the laminar wall jet. The dimensional statements

$$[F] = \frac{\mathbf{ML}^2}{\mathbf{T}^3} \ , \qquad [\rho] = \frac{\mathbf{M}}{\mathbf{L}^3} \ , \qquad [\mu] = \frac{\mathbf{M}}{\mathbf{LT}} \tag{10.29}$$

imply, in their alternative role as definitions,

$$\mathbf{M} = \frac{\rho^4 \nu^9}{F^3} \ , \qquad \mathbf{L} = \frac{\rho \nu^3}{F} \ , \qquad \mathbf{T} = \frac{\rho^2 \nu^5}{F^2} \ , \tag{10.30}$$

with

$$\mathbf{U} = \frac{\mathbf{L}}{\mathbf{T}} = \frac{F}{\rho \nu^2} \tag{10.31}$$

and, as for the free jet,

$$\frac{\mathbf{UL}}{\nu} = 1 \ . \tag{10.32}$$

The relation (10.31) provides *a posteriori* justification for Glauert's original ansatz (10.4), because **U** is now precisely defined. In fact, substitution for **U** yields immediately

$$\left(\frac{\rho}{F\nu x}\right)^{1/4} \psi = f\left[\left(\frac{F}{\rho \nu^3 x^3}\right)^{1/4} y\right] \ . \tag{10.33}$$

Another brief calculation shows that this expression is equivalent to

$$\frac{\psi}{\mathbf{UL}^{3/4} x^{1/4}} = f\left(\frac{y}{\mathbf{L}^{1/4} x^{3/4}}\right) \ . \tag{10.34}$$

Tetervin. Tetervin's earlier approach to the same problem was handicapped by a dreadful notation and by failure to introduce a stream function until the last possible moment. What follows is a radical paraphrase of his argument. In effect, he assumed similarity in terms of local scales for velocity U and layer thickness L;

$$\frac{\psi}{UL} = f\left(\frac{y}{L}\right) = f(\eta) \ , \tag{10.35}$$

where $U(x)$ and $L(x)$ have to be determined. Substitution in the momentum equation (10.1) gives, just as in the case of the laminar free jet (see SECTION 9.1.2),

$$f''' + \frac{L}{\nu} \frac{\mathrm{d}UL}{\mathrm{d}x} f f'' - \frac{L^2}{\nu} \frac{\mathrm{d}U}{\mathrm{d}x} f' f' = 0 \ . \tag{10.36}$$

Substitution in the momentum-integral equation (10.3) gives

$$\frac{L}{\nu U}\,\frac{\mathrm{d}U^2L}{\mathrm{d}x}\int_0^\infty f'f'\,\mathrm{d}\eta = -f''(0)\ . \tag{10.37}$$

Tetervin eventually normalized the integral to unity;

$$\int_0^\infty f'f'\,\mathrm{d}\eta = g(0) = 1\ , \tag{10.38}$$

so that

$$\frac{L}{\nu U}\,\frac{\mathrm{d}U^2L}{\mathrm{d}x} = -f''(0)\ . \tag{10.39}$$

Only two of the three constant coefficients involving U and L in equations (10.36) and (10.39) are independent, and these two are not sufficient to determine $U(x)$ and $L(x)$ explicitly. Neither is the device of the moving observer useful for resolving the question of exponents. Tetervin, like Glauert, found another way.

When ν is eliminated between equations (10.36) and (10.39), and the variables depending on x and on η are separated, the result is

$$\frac{L\,\mathrm{d}U/\mathrm{d}x}{U\,\mathrm{d}L/\mathrm{d}x} = \frac{-f''' + f''(0)ff''}{2f''' - f''(0)(ff'' - f'f')} = -k\ , \tag{10.40}$$

where k must be a positive constant because x and η are arbitrary and $\mathrm{d}U/\mathrm{d}x < 0$, $\mathrm{d}L/\mathrm{d}x > 0$. This expression strongly suggests that power laws are appropriate for $U(x)$ and $L(x)$, and guarantees in any case that

$$UL^k = \text{constant}\ . \tag{10.41}$$

Tetervin noted in passing that the boundary condition $f''(0) = 0$ in equation (10.40) implies $k = 1/2$ and thus U^2L = constant, so that the case of the plane free jet is accounted for. The present interest is in the case of lost symmetry with its eigenvalue k. This eigenvalue appears along with $f''(0)$ in the differential equation obtained from the second part of equation (10.40);

$$f''' + \left(\frac{1-k}{2k-1}\right)f''(0)ff'' + \left(\frac{k}{2k-1}\right)f''(0)f'f' = 0\ . \tag{10.42}$$

At this point, Tetervin's argument becomes opaque. The essence of his procedure, suitably revised to leave open the question of normalization, is to multiply equation (10.42) by f and integrate over the thickness of the wall jet. After the usual integration by parts and use of the boundary conditions, the result is

$$\left(\frac{3k-2}{2k-1}\right) f''(0) \int_0^\infty f\, f' f' \mathrm{d}\eta = 0 \ . \tag{10.43}$$

Both the integral and the factor $f''(0)$ are necessarily positive, so that the desired invariant emerges from this equation together with the eigenvalue

$$k = \frac{2}{3} \ . \tag{10.44}$$

Equation (10.42) becomes

$$\frac{f'''}{f''(0)} + f f'' + 2\, f' f' = 0 \ . \tag{10.45}$$

Equation (10.41) becomes

$$U^3 L^2 = \text{constant} \ , \tag{10.46}$$

and it follows from this result and equation (10.39) that

$$U \sim x^{-1/2} \ , \qquad L \sim x^{3/4} \ . \tag{10.47}$$

Tetervin integrated equation (10.45) numerically for the particular initial conditions $f(0) = f'(0) = 0$ and $f''(0) = 1/4$. His conversion of a two-point boundary-value problem to an initial-value problem was successful, although he may not have been aware of the reason, which involves a property first pointed out for the Blasius equation by TÖPFER (1912). The argument is easily extended by inspection to equation (10.45), which also has no pressure-gradient term. If $f(\eta)$ is a solution, so is $\phi(\eta) = \alpha f(\alpha\eta)$, where α is any constant. It follows that $f''(0)$ can be chosen arbitrarily, with $\phi(\infty)$ adjusted later to any desired value by a proper choice of α (see SECTION X).

10.1.2 Similarity

The affine transformation. Discovery of the integral invariant F allows the problem of the laminar plane wall jet to be treated by the method of the affine transformation. Let a stream function ψ be introduced in the usual way to satisfy the continuity equation. Rewrite equation (10.1) as

$$\rho\left(\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}\right)=\mu\frac{\partial^3\psi}{\partial y^3} \tag{10.48}$$

and apply the affine transformation

$$\begin{aligned} x &= a\widehat{x} \ , \\ y &= b\widehat{y} \ , \\ \psi &= c\widehat{\psi} \ , \\ \rho &= d\widehat{\rho} \ , \\ \mu &= e\widehat{\mu} \ , \\ F &= f\widehat{F} \ . \end{aligned} \tag{10.49}$$

This is the same group as equations (9.28) for the plane free jet, except that F replaces J. The result is

$$\frac{c^2d}{ab^2}\,\widehat{\rho}\left(\frac{\partial\widehat{\psi}}{\partial\widehat{y}}\frac{\partial^2\widehat{\psi}}{\partial\widehat{x}\partial\widehat{y}}-\frac{\partial\widehat{\psi}}{\partial\widehat{x}}\frac{\partial^2\widehat{\psi}}{\partial\widehat{y}^2}\right)=\frac{ce}{b^3}\widehat{\mu}\frac{\partial^3\widehat{\psi}}{\partial\widehat{y}^3} \ . \tag{10.50}$$

Invariance of equation (10.48) thus requires

$$\frac{bcd}{ae}=1 \ , \tag{10.51}$$

just as in the case of the plane free jet. Transformation of equation (10.19),

$$\rho\int_0^\infty \psi\,\frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial y}\mathrm{d}y=F \ , \tag{10.52}$$

yields

$$\frac{c^3 d}{b}\,\widehat{\rho}\int_0^\infty \widehat{\psi}\,\frac{\partial\widehat{\psi}}{\partial\widehat{y}}\,\frac{\partial\widehat{\psi}}{\partial\widehat{y}}\,\mathrm{d}\widehat{y} = f\widehat{F} \ , \tag{10.53}$$

and requires for invariance

$$\frac{c^3 d}{bf} = 1 \ . \tag{10.54}$$

As usual, I take the primary variables to be ψ and y. When equations (10.51) and (10.54) are revised to isolate for c and b, the result is

$$\frac{c^4 d^2}{aef} = 1 \ , \qquad \frac{b^4 d^2 f}{a^3 e^3} = 1 \ . \tag{10.55}$$

Hence the proper ansatz, including constants A and B for later normalization, is again equation (10.33),

$$A\left(\frac{\rho}{F\nu x}\right)^{1/4}\psi = f\left[B\left(\frac{F}{\rho\nu^3 x^3}\right)^{1/4} y\right] = f(\eta) \ . \tag{10.56}$$

Substitution of this ansatz in the momentum equation (10.48) yields

$$4ABf''' + ff'' + 2f'f' = 0 \ , \tag{10.57}$$

with boundary conditions

$$f(0) = f'(0) = 0, \qquad f'(\infty) = 0 \ , \tag{10.58}$$

corresponding to $\psi = u = 0$ at $y = 0$ and $u = 0$ at $y = \infty$. If $AB = 1$, equation (10.57) is identical with my version of Glauert's result, equation (10.17). Substitution of equation (10.56) in equation (10.52) for F gives

$$\int_0^\infty ff'f'\mathrm{d}\eta = \frac{A^3}{B} \ . \tag{10.59}$$

The singular behavior of the flow at the origin, mentioned earlier, is demonstrated by the relations

$$J = \rho\int_0^\infty uu\,\mathrm{d}y = \frac{B}{A^2}\left(\frac{F^3\rho}{\nu x}\right)^{1/4}\int_0^\infty f'f'\mathrm{d}\eta \tag{10.60}$$

and

$$\frac{\tau_w}{\rho} = \frac{B^2}{A}\left(\frac{F^3}{\rho^3 \nu x^5}\right)^{1/4} f''(0) \ . \tag{10.61}$$

10.1.3 The boundary-layer solution

Glauert's second major contribution was to obtain the eigenfunction $f(\eta)$ in closed form. First, multiply equation (10.57) by f and integrate to obtain

$$4AB\left(ff'' - \frac{f'f'}{2}\right) + fff' = 0 \ , \tag{10.62}$$

where the constant of integration vanishes by virtue of the first two boundary conditions (10.58). Multiply this result by $f^{-3/2}$ and integrate again, to obtain

$$4AB\frac{f'}{f^{1/2}} + \frac{2}{3}\left(f^{3/2} - C^{3/2}\right) = 0 \ , \tag{10.63}$$

where $C > 0$ is a constant of integration. The boundary condition (10.58) at infinity requires

$$C = f(\infty) \ . \tag{10.64}$$

Finally, integrate equation (10.63) with the aid of the change of variable

$$f = Ch^2 = CH \tag{10.65}$$

and the method of partial fractions. An intermediate result is

$$\frac{C}{4AB}\mathrm{d}\eta = \frac{\mathrm{d}h}{(1-h)} + \frac{2\mathrm{d}h}{(1+h+h^2)} + \frac{h\mathrm{d}h}{(1+h+h^2)} \ . \tag{10.66}$$

The final result in terms of h, after use of the boundary condition $f(0) = h(0) = 0$ to evaluate the constant of integration, can be written

$$\frac{C}{2AB}\eta = \ln\frac{(1-h^3)}{(1-h)^3} + 2\sqrt{3}\tan^{-1}\left(\frac{\sqrt{3}h}{2+h}\right) \ . \tag{10.67}$$

Equations (10.65) and (10.67) are a parametric system for $f(\eta)$, with h as parametric variable. Note that h depends not directly on η but on $C\eta/2AB$.

Pause here to look at experimental data for the laminar profile; see

BAJURA and SZEWCZYK (1970)
BAJURA and CATALANO (1975)
TSUJI et al. (1977)
TSUJI and MORIKAWA (1977)
HORNE and KARAMCHETI (1979)
SCIBILIA and DUROX (1980)
PAIGE (1988)
ZHOU et al. (1992)

It remains to consider the streamlines of the boundary-layer flow in compact outer variables (x, y) having equal scales. A unique representation of the flow can again be found, without regard for the values of the three constants A, B, and C. Rewrite equation (10.56) in terms of H as

$$\frac{A}{C}\left(\frac{\rho}{F\nu x}\right)^{1/4}\psi = \frac{f(\eta)}{C} = H\left(\frac{C}{2AB}\eta\right) = H\left[\frac{C}{2A}\left(\frac{F}{\rho\nu^3 x^3}\right)^{1/4} y\right] . \tag{10.68}$$

In the combinations containing ψ and y, use the second of equations (10.30) to eliminate the quantity F in favor of $\mathbf{L} = \rho\nu^3/F$. Thus write

$$\frac{A}{C}\left(\frac{\mathbf{L}}{x}\right)^{1/4}\frac{\psi}{\nu} = H\left[\frac{C}{2A}\left(\frac{y^4}{\mathbf{L}x^3}\right)^{1/4}\right] . \tag{10.69}$$

Compact outer variables define themselves immediately as

$$\Psi = \frac{\psi}{2\nu} , \qquad X = \left(\frac{C}{2A}\right)^4 \frac{x}{\mathbf{L}} , \qquad Y = \left(\frac{C}{2A}\right)^4 \frac{y}{\mathbf{L}} , \tag{10.70}$$

and equation (10.68) takes the form

$$\Psi = X^{1/4} H\left(\frac{Y}{X^{3/4}}\right) . \tag{10.71}$$

Note that $H = h^2 = f(\eta)/C$, but that the argument of H is the quantity $C\eta/2AB$ on the left in equation (10.67). The example of the free jet suggests that a useful relation involving the constants A and C should emerge when the integral invariant (10.59) is evaluated for Glauert's closed-form solution. Use equation (10.65) and its derivative, together with equation (10.63), to replace the variable f by h. The result is

$$\int_0^\infty f f' f' \mathrm{d}\eta = \frac{C^4}{3AB}\int_0^1 h^4(1-h^3)\mathrm{d}h = \frac{C^4}{40AB} = \frac{A^3}{B} \ , \qquad (10.72)$$

from which

$$\left(\frac{C}{2A}\right)^4 = \frac{5}{2} \ . \qquad (10.73)$$

The variables in equation (10.71) can therefore be written

$$\Psi = \frac{\psi}{2\nu} \ , \qquad X = \frac{5x}{2\mathbf{L}} \ , \qquad Y = \frac{5y}{2\mathbf{L}} \ . \qquad (10.74)$$

Streamlines for the boundary-layer approximation (10.71) are shown in FIGURE 10.2[1] for the case of a laminar wall jet flowing from the origin along a plane wall that extends to infinity in the positive x-direction. Rather than calculate Ψ on a large rectangular array $(X,\ Y)$ and find level curves on which Ψ is constant, it is simpler here to define each streamline separately. The algorithm is: fix Ψ, vary X. Calculate $H = \Psi/X^{1/4} = h^2$. Calculate h. Calculate $C\eta/2\,AB = Y/X^{3/4}$ from equation (10.67). Calculate Y.

A local Reynolds number can be expressed in compact outer variables by beginning with dimensionless versions of equations (10.47);

$$U(x) = \mathbf{U}\mathbf{L}^{1/2}x^{-1/2} \ , \qquad L(x) = \mathbf{L}^{1/4}x^{3/4} \ . \qquad (10.75)$$

Use of equations (10.30) and (10.31) leads to

$$U = \left(\frac{F^2}{\rho^2\nu^2x^2}\right)^{1/4} \ , \qquad L = \left(\frac{\rho\nu^3x^3}{F}\right)^{1/4} \ , \qquad (10.76)$$

[1] A longer handwritten version of the caption for this figure in the 1996 ms. reads "Streamlines $\Psi = \psi/2\nu =$ *constant* of the boundary-layer approximation for the laminar plane wall jet according to equation (10.71). The range of Ψ is 0(1)10 (**check**).

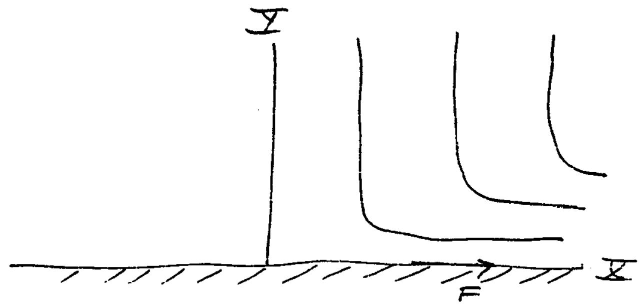

Figure 10.2: Streamlines of the boundary-layer model for the laminar plane wall jet according to equation (10.71). The range of........

and thus to

$$Re(x) = \frac{UL}{\nu} = \left(\frac{Fx}{\rho\nu^3}\right)^{1/4} = \left(\frac{x}{\mathbf{L}}\right)^{1/4} = \left(\frac{2}{5}\,X\right)^{1/4} \quad . \tag{10.77}$$

10.1.4 Normalization

The three constants A, B, and C for the plane wall jet can be assigned sensible values by operations that run in parallel with similar operations for the plane free jet in SECTION 9.1.4. The condition

$$4AB = 1 \tag{10.78}$$

establishes the standard operator $f''' + ff''$ in equation (10.57). A second and mandatory condition, just derived, is

$$\frac{C}{A} = (40)^{1/4} \quad . \tag{10.79}$$

(See end of Part A of this chapter. This is too messy. No tidy normalization seems to be in view. Sort through this material to find

something simple, elegant, and redundant. Consider equation (10.67) for h near 1 and for h large and negative. Note that for $h = 1$ the angle is 30 degrees. Should square root be $\pm$? Need to match to plane jet at infinity. Try osculating parabola.)

The third condition determining the constants A, B, C requires definition of a length or velocity scale. The simplest choice, suggested by the example of the free jet (see SECTION 9.1.4), is to set the argument of h; i.e., the left-hand side of equation (10.67), equal to η itself, so that $C/2AB = 1$ or $C = 1/2$.

The maximum velocity $U = u_m$ (m for maximum) is easily worked out;

$$Re = \frac{UL}{\nu} = \frac{1}{\nu}\int_0^\infty u dy = \frac{\psi(x,\,\infty)}{\nu} = \frac{1}{A}\left(\frac{Fx}{\rho\nu^3}\right)^{1/4} f(\infty) = 2X^{1/4} \ . \tag{10.80}$$

This Reynolds number is small compared with the corresponding value $Re = 12\,X^{1/3}$ for the plane free jet. In FIGURE 10.2, which extends **(ten)** times farther than FIGURE 9.4, the Reynolds number at the right boundary is **(six)** times smaller. Comparison of equation (10.80) with equation (9.57) for the free jet suggests that the two measures just cited are associated with the exponent and the coefficient, respectively.

The maximum streamwise velocity η_m occurs when $\partial u/\partial y \sim f'' = 0$. With this condition, equations (10.62) and (10.63) can be restated in terms of h and h' and solved algebraically to produce **(see Tetervin)**

$$f(\eta_m) = \left(\frac{1}{4}\right)^{2/3} C \ ; \tag{10.81}$$

$$f'(\eta_m) = \left(\frac{1}{4}\right)^{1/3} \frac{C^2}{8AB} \ . \tag{10.82}$$

Thus if $f'(\eta_m) = 1$, then $C^2/4AB$ or $C = (32)^{1/6}$. Conversely, if $C = 1/2$, then $f'(\eta_m) = (2)^{1/3}/16$.

Several other choices suggest themselves, chief among them the integral scale L for the profile. Define this integral scale in terms of the maximum velocity u_m (m for maximum) by

$$u_m L = \int_0^\infty u \mathrm{d}y = \psi(x,\, \infty) \ . \qquad (10.83)$$

After use of the ansatz (10.56) and the second of conditions (10.82), this turns into

$$\widetilde{\eta} = B\left(\frac{F}{\rho\nu^3 x^3}\right)^{1/4} L = \int_0^\infty \frac{f'(\eta)}{f'(\eta_m)}\, \mathrm{d}\eta = (4)^{1/3}\, \frac{8AB}{C} \ . \qquad (10.84)$$

Hence if $\widetilde{\eta} = 1$, $C/4AB = 2(4)^{1/3}$.

A similar calculation, with h replacing f, leads from the definition (10.13) to

$$g(0) = \int_0^\infty f' f' \mathrm{d}\eta = \frac{1}{18}\, \frac{C^3}{AB} \ . \qquad (10.85)$$

Finally, integration of the primary differential equation (10.57) between the limits zero and infinity leads to

$$f''(0) = \frac{g(0)}{4AB} = \frac{1}{72}\, \frac{C^3}{A^2 B^2} \ . \qquad (10.86)$$

Either of these relationships, as well as

$$f(\infty) = \int_0^\infty f' \mathrm{d}\eta = C \ , \qquad (10.87)$$

could provide a third condition if its right-hand side is arbitrarily set equal to unity, say. The results, respectively, are $C^3 = 9/2$ if $g(0) = 1$, $C^3 = 9/2$ if $f''(0) = 1$, and $C = 1$ if $f(\infty) = 1$, where $4AB$ is read as unity.

The inflection point in the profile at $\eta = \eta_i$, say, is found by putting $f''' = 0$. This point also marks the maximum velocity along a streamline, since $Du/Dt = \nu\partial^2 u/\partial y^2 = 0$ for laminar flow. Then equation (10.57) becomes $ff'' + 2f'f' = 0$, and other derivatives at the inflection point can be calculated from this truncated form together with equations (10.62) and (10.63). The results are

$$f(\eta_i) = \left(\frac{5}{8}\right)^{2/3} C \ ; \tag{10.88}$$

$$f'(\eta_i) = \frac{1}{16}\left(\frac{5}{8}\right)^{1/3} \frac{C^2}{AB} \ ; \tag{10.89}$$

$$f''(\eta_i) = -\frac{1}{128}\,\frac{C^3}{A^2B^2} \ . \tag{10.90}$$

Hence if $f'(\eta_i) = 1$, then $C^2/4AB = 8/5^{1/3}$.

The vorticity thickness η_ζ is defined graphically in FIGURE X and is defined algebraically by

$$\eta_\zeta = -\frac{f'(\eta_m)}{f''(\eta_i)} = 16\left(\frac{1}{4}\right)^{1/3} \frac{AB}{C} \ , \tag{10.91}$$

where $f'(\eta_m)$ is given by equation (10.82). Hence if $\eta_\zeta = 1$, $C/4AB = (4)^{2/3}$.

The normalizations used by Glauert and Tetervin can be inferred by using my notation in the ansatz (10.56) and the relations that follow. Glauert put $C = 1$ and also $4AB = 1$, according to his equation (4.1). It follows that $A = (1/40)^{1/4}$, $B = (5/32)^{1/4}$; and these are the numbers that appear in Glauert's equations (4.9) for the plane case. Tetervin's final normalization can be deduced from his equation (20), in which his $G(\xi)$ is the same as my $f(\eta)$. He put $4AB = 4/3$ and $f''(0) = 1/4$, and these together with equation (10.86) above imply $C = f(\infty) = (2)^{1/3} = 1.259921$. His numerical result for large η, namely $f(14.95) = 1.259916$, is evidence that his integration was carried out with remarkable accuracy.

(Give normalized f, f', etc.)

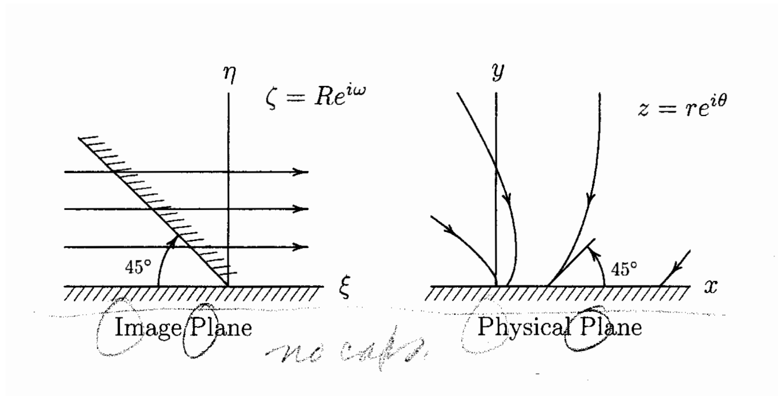

Figure 10.3: Mapping of the outer entrained flow for the laminar plane wall jet.

10.1.5 Entrainment and composite flow

The outer or entrained flow associated with the boundary-layer solution in FIGURE 10.2 can again be obtained by the method of conformal mapping, as indicated in FIGURE 10.3. The complex potential for uniform flow in the ζ-plane is

$$F(\zeta) = \phi + i\psi = U_0 \zeta \ . \tag{10.92}$$

The mapping can be assumed to be of the form

$$\zeta = L_0^{3/4} z^{1/4} e^{i\alpha} \ , \tag{10.93}$$

where $\zeta = Re^{i\omega}$ and $z = re^{i\theta}$, with U_0, L_0, and α to be determined. Angles are related by

$$\omega = \frac{\theta}{4} + \alpha \ . \tag{10.94}$$

Consequently, if $\theta = \pi$ when $\omega = \pi$ on OA,

$$\alpha = \frac{3\pi}{4} \ . \tag{10.95}$$

Since now $\theta = 0$ when $\omega = 3\pi/4$ on OB, streamlines of the outer flow will intersect the x-axis at an angle of 45 degrees. The complex potential in the z-plane becomes

$$F(z) = \phi_o + i\psi_o = U_0 \zeta(z) = U_0 L_0^{3/4}\, r^{1/4}\, e^{i\left(\frac{\theta+3\pi}{4}\right)} \quad , \tag{10.96}$$

and the outer stream function is

$$\psi_o(r,\, \theta) = U_0 L_0^{3/4}\, r^{1/4}\, \sin\left(\frac{\theta + 3\pi}{4}\right) \quad . \tag{10.97}$$

At the wall, where $\theta = 0$ and $r = x$,

$$\psi_o(x,\, 0) = \frac{1}{\sqrt{2}}\, U_0 L_0^{3/4}\, x^{1/4} \quad . \tag{10.98}$$

This outer flow on the positive x-axis in the physical plane is to be matched to the inner stream function at infinity, from equation (10.56) with $f(\infty) = C$;

$$\psi_i(x,\, \infty) = \frac{C}{A}\left(\frac{F\nu x}{\rho}\right)^{1/4} \quad . \tag{10.99}$$

Matching therefore requires

$$\frac{1}{\sqrt{2}}\, U_0 L_0^{3/4} = \frac{C}{A}\left(\frac{F\nu}{\rho}\right)^{1/4} = \frac{C}{A}\, \mathbf{UL}^{3/4} \quad , \tag{10.100}$$

where the last equality makes use of equations (10.30). Finally, therefore,

$$\psi_o\,(r,\, \theta) = \sqrt{2}\, \frac{C}{A}\left(\frac{F\nu r}{\rho}\right)^{1/4} \sin\left(\frac{\theta + 3\pi}{4}\right) \quad . \tag{10.101}$$

(Work out pressure here.)

The composite stream function ψ_c is the sum of the inner component (10.56) and the outer component (10.101) with the common part (10.99) subtracted,

$$\psi_c = \frac{C}{A}\left(\frac{F\nu r}{\rho}\right)^{1/4}\left\{\left(\frac{x}{r}\right)^{1/4}\left[\frac{f(\eta)}{C} - 1\right] + \sqrt{2}\sin\left(\frac{\theta + 3\pi}{4}\right)\right\} \quad . \tag{10.102}$$

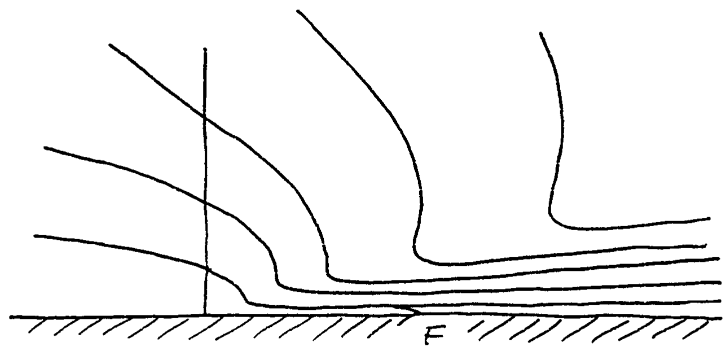

Figure 10.4: Streamlines Ψ = constant of the composite model for the laminar plane wall jet into a stagnant fluid according to equation (10.103).

In terms of the reduced similarity variables X and Y defined by equations (10.70), with $R = (X^2 + Y^2)^{1/2}$ and $\tan \Theta = Y/X$, this is

$$\Psi_c = R^{1/4} \left\{ I(\Theta) \left[H\left(\frac{Y}{X^{3/4}} \right) - 1 \right] + \sqrt{2} \sin \left(\frac{\Theta + 3\pi}{4} \right) \right\} , \tag{10.103}$$

where

$$\begin{aligned} I(\Theta) &= (\cos \Theta)^{1/4} && \text{for } x > 0 , \\ &= 0 && \text{for } x < 0 . \end{aligned} \tag{10.104}$$

Streamlines for the composite flow are shown in FIGURE 10.4. The calculation here requires an iteration for $h(\eta)$ and a contour subroutine. The figure can be viewed as a conceptual model for flow near the nozzle of a plane wall ejector with small induced flow (see SECTION X).

(Want S, U, V, T; see free jet. Plot corrected profile, etc.)

10.1.6 The laminar radial wall jet

10.1.7 Stability and transition

10.2 Laminar plane wall jet into moving fluid

10.2.1 Similarity

10.3 Turbulent plane wall jet into fluid at rest

10.3.1 Similarity

A preliminary step is to determine if the turbulent plane wall jet is also an eigenvalue problem. Physically, it can never be established whether or not there is a finite initial momentum flux J, because turbulent flow cannot be observed at sufficiently low Reynolds numbers. The argument below refers only to the boundary-layer approximation, with the momentum equation written as

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial \tau}{\partial y} \quad . \tag{10.105}$$

The boundary conditions are $u = 0$ and $\psi = 0$ at $y = 0$ and $u = 0$ and $\tau = 0$ at $y = \infty$. (**Use the best and highest version of the invariant.**)

With the laminar problem of SECTION 10.1.1 in mind as both a model and a special case, multiply equation (10.105) by the product ψu, where ψ is the usual stream function, and add to the result the two identities

$$\psi\, u\,\frac{\partial u}{\partial x} + \psi\, u\,\frac{\partial v}{\partial y} = 0 \tag{10.106}$$

and

$$uu\frac{\partial \psi}{\partial x} + uv\frac{\partial \psi}{\partial y} = -uuv + uvu = 0 \tag{10.107}$$

to obtain

$$\frac{\partial \psi uu}{\partial x} + \frac{\partial \psi uv}{\partial y} = \frac{\psi}{\rho}\frac{\partial \tau}{\partial y} \quad . \tag{10.108}$$

Integration, with $\psi = 0$ at the wall and $\tau = 0$ at infinity, gives

$$\int_0^\infty \frac{\partial \psi uu}{\partial x}\,\mathrm{d}y = \frac{1}{\rho}\int_0^\infty \psi\,\frac{\partial \tau}{\partial y}\,\mathrm{d}y = -\frac{1}{\rho}\int_0^\infty \tau\,\frac{\partial \psi}{\partial y}\,\mathrm{d}y\ . \tag{10.109}$$

Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty \psi uu\,\mathrm{d}y = -\frac{1}{\rho}\int_0^\infty \tau u \mathrm{d}y\ . \tag{10.110}$$

If the flow is laminar, $\tau = \mu\,\partial u/\partial y$, and

$$\int_0^\infty \tau u\,\mathrm{d}y = \mu\int_0^\infty \frac{\partial u^2/2}{\partial y}\,\mathrm{d}y = 0 \tag{10.111}$$

because u^2 vanishes at both limits. Thus

$$\int_0^\infty \psi uu\,\mathrm{d}y = F = \text{constant}\ . \tag{10.112}$$

If the flow is turbulent, the integral (10.111) does not necessarily vanish. The velocity is presumably always positive, and the stress τ changes sign, being positive near the wall and negative farther out, so there is the possibility. **Calculate τ from $\int Du/Dt\,dy$ and see what happens.**

Forthmann did not look at wall law because he did not know the wall friction. He also introduced the half-velocity scheme for defining δ.

Schwartz and Cosart separated the profile into two parts at the point of maximum velocity. The wall jet should be treated by the same methods that are successful for the boundary layer.

The growth rate for a wall jet is about 3/4 of the growth rate for a free jet. This is because the vertical fluctuations are inhibited and the effect is felt in all of the Reynolds stresses and in entrainment. Comment on cases where τ and du/dy do not go to zero at the same point. This is a blow to the idea of eddy viscosity.

Need a handout on methods for measuring surface friction.

10.3.2 The law of the jet

Measured mean-velocity profiles in plane wall jets can be found in

FORTHMANN (1934)
BAKKE (1957)
SIGALLA (1958)
SCHWARZ and COSART (1961)
PATEL (1962)
RAJARATNAM (1965)
SRIDHAR and TU (1966)
TAILLAND and MATHIEU (1967)
GUITTON (1968)
KOHAN (1968)
HUBBARTT and NEALE (1972)
SPETTEL et al. (1972)
KIND and SUTHANTHIRAN (1973)
HO and HSIAO (1983)
SCHNEIDER (1987)
ABRAHAMSSON et al. (1991)
KATZ et al. (1992)
WYGNANSKI et al. (1992)

Measured Reynolds-stress profiles can be found in

FORTHMANN (1934)
TAILLAND and MATHIEU (1967)
GUITTON (1968)
SPETTEL et al. (1972)
SCHNEIDER (1987)
ABRAHAMSSON et al. (1991)
WYGNANSKI et al. (1992)

10.3.3 Entrainment and composite flow

10.3.4 Coanda effects

10.4 The turbulent radial wall jet

10.4.1 Similarity

The mean-velocity profile in turbulent radial wall jets is reported in

POREH (1959)
POREH and CERMAK (1959)
SCHRODER (1961)
TSUEI (1962)
LUDWIEG (1964)
CHAO and SANDBORN (1966)
DONALDSON (1966)
JOHNSON (1967) 19D
POREH et al. (1967)
HRYCAK (1970)
SCHOLTZ and TRASS (1970)
DONALDSON et al. (1971)
GOVINDAN and RAJU (1974)
ERA and SAIMA (1976)
BOLDMAN and BRINICH (1977)
LEISTER (1977)
MITACHI and ISHIGURO (1977)
TANAKA and TANAKA (1977, 1978)
TANI and KOMATSU (1977)
ARAUJO et al. (1981)
DESHPANDE and VAISHNAV (1982)
KATAOKA et al. (1983)
CODAZZI et al. (1983)

Tanaka. Figure 13 is useful.

Tanaka. Combined jet far downstream behaves like a single jet. Includes single jet for reference.

Bradshaw. Clumping.

Knystautas. Study this again.

Hegge Zijnen (two papers). Note no side plates. Figure 6 implies turbulent Prandtl number.

Foss and Jones. Effect of low aspect ratio with side walls. Mechanism? Mean velocity profile is insensitive.

Curtet. Not ejector; secondary stream is controlled. Figure V.4 has fitting constant. Figure V.7 shows positive and negative flow ratios. Figure VII.1 shows separation bubble.

It is probably time to invent the law of the jet. The sketch[2] shows the decomposition of the profile. The corresponding formula is

$$\frac{u}{u_\tau} = f\left(\frac{yu\tau}{\nu}\right) - \frac{uc}{u_\tau} j\left(\frac{y}{\delta}\right) \tag{10.113}$$

where a tentative form for $j(y/\delta)$ is $\sin^2(\pi y/2\delta)$. The defect form is obtained by subtracting the local friction law

$$0 = f\left(\frac{\delta u\tau}{\nu}\right) - \frac{uc}{u_\tau} \tag{10.114}$$

to obtain

$$\frac{u}{u_\tau} = \frac{1}{\kappa}\frac{y}{\delta} + \frac{uc}{u_\tau}\cos^2\left(\frac{\pi}{2}\frac{y}{\delta}\right) \ . \tag{10.115}$$

In wall-law variables, equation (1)[3] has the form shown in the sketch. If the defect law is equivalent to equilibrium, then u_c/u_τ must be constant, and so must $\delta u_\tau/\nu$. Each equilibrium flow has an invariant profile, which changes with (**??**) The data do not seem to have this property, perhaps because none of the flows are fully developed.

The boundary-layer problem has been wrapped up for 30 years. The wall jet has a number of properties in common, but is still being treated by 19th-century methods.

---

[2]No such sketch has been found.

[3]Unclear reference.

Tanaka and Tanaka also studied the free jet and wall jet. They divide the profile at the maximum velocity. Note that r is measured from the outside of the pipe, not from the axis of symmetry. The estimates of C_f from the momentum equation and the fit to the wall law do not agree.

The stability paper seems to show a vena contracta, but this may be the cylindrical geometry. The paper allows the observed vortex pairs to be treated along with the eigenfunctions. This is a good paper on stability.

Describe profile formula for wall jet. Invent the law of the jet to supply rigor and detail. (Need sketch). Since the jet function is not known, the process is iterative. (Need sketch in wall-law coordinates.) If u_c/u_τ is constant, so is $\delta u_\tau/\nu$, and there is only one profile for a given flow. Different profiles may apply for different Reynolds numbers.

The colliding round jets (Witze and Dwyer) form a radial jet that grows at an abnormally large rate.

Several authors have compared the flow near the wall in a wall jet to the log law, with considerable variations on both sides of the Prandtl law.

Irwin's data have low scatter.

These wall-jet data should reinforce a conviction that the law of the wall, which was originally recovered from pipe data, is really universal. The law of the wall is the largest handle striking out of the problem of shear flow near a wall. It should be central in any global study.

It remains to consider the usual momentum-integral equation. The development is at first completely general. Rewrite equation (**xxx**), with the aid of the continuity equation $\partial u/\partial x + \partial v/\partial y = 0$, as

$$\rho\left(\frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y}\right) = \rho u_\infty \frac{du_\infty}{dx} + \frac{\partial \tau}{\partial y} \tag{10.116}$$

and integrate from $y = 0$ to some fixed value of y, with the boundary condition $v = 0$ at $y = 0$ (either because of symmetry or because of the presence of a wall). The result can be written

$$\rho \int_0^y \frac{\partial uu}{\partial x} \mathrm{d}y + \rho uv = \rho \frac{\mathrm{d}u_\infty}{\mathrm{d}x} \int_0^y u_\infty \mathrm{d}y + \tau - \tau_w \tag{10.117}$$

where τ_w is the value of τ at $y = 0$; this value may be zero.

The continuity equation also yields an integral,

$$\int_0^y \frac{\partial u}{\partial x} \mathrm{d}y + v = 0 \tag{10.118}$$

so that equation (10.117) can be written

$$\rho \int_0^y \frac{\partial uu}{\partial x} \mathrm{d}y - \rho u \int_0^y \frac{\partial u}{\partial x} \mathrm{d}y - \rho \frac{\mathrm{d}u_\infty}{\mathrm{d}x} \int_0^y u_\infty \mathrm{d}y - \tau + \tau_w = 0 \ . \tag{10.119}$$

To eliminate the divergent integrals when the upper limit goes to infinity, add to this the identity

$$-\rho \int_0^y \frac{\partial u u_\infty}{\partial x} \mathrm{d}y + \rho u_\infty \int_0^y \frac{\partial u}{\partial x} \mathrm{d}y + \rho \frac{\mathrm{d}u_\infty}{\mathrm{d}x} \int_0^y u \mathrm{d}y = 0 \tag{10.120}$$

to obtain

$$\rho \frac{\partial}{\partial x} \int_0^y (uu - uu_\infty) \mathrm{d}y + \rho (u_\infty - u) \frac{\partial}{\partial x} \int_0^y u \mathrm{d}y \ + $$

$$+\rho \frac{\mathrm{d}u_\infty}{\mathrm{d}x} \int_0^y (u - u_\infty) \mathrm{d}y - \tau + \tau_w = 0 \ . \tag{10.121}$$

Finally, let the upper limit go to infinity and change the signs;

$$\rho \frac{\mathrm{d}}{\mathrm{d}x} \int_0^\infty u(u_\infty - u) \mathrm{d}y + \rho \frac{\mathrm{d}u_\infty}{\mathrm{d}x} \int_0^\infty (u_\infty - u) \mathrm{d}y - \tau_w = 0 \ . \tag{10.122}$$

Equation (10.122) is the momentum-integral equation of Karman. It is customary to define a displacement thickness δ^* or δ_1 by

$$u_\infty \delta^* = \int_0^\infty (u_\infty - u)\mathrm{d}y \qquad (10.123)$$

and a momentum thickness θ or δ_2 by

$$u_\infty^2 \theta = \int_0^\infty u(u_\infty - u)\mathrm{d}y \ . \qquad (10.124)$$

(**Use a sketch to define these graphically, particularly concept of displacement**.) Equation (10.122) can then be succinctly written

$$\tau_w = \frac{\rho \mathrm{d}}{\mathrm{d}x} u_\infty^2 \theta + \rho u_\infty \delta^* \frac{\mathrm{d}u_\infty}{\mathrm{d}x} \ . \qquad (10.125)$$

If $u_\infty =$ constant, this becomes

$$\tau_w = \rho u_\infty^2 \frac{\mathrm{d}\theta}{\mathrm{d}x} \ . \qquad (10.126)$$

This expression is often used to determine the surface friction τ_w (as is 10.125) in the more general case). If a turbulent boundary layer is visualized as a wake, with a wall continuously removing momentum, this equation exposes the process of local momentum removal.

The invariant J for the radial jet should probably include a factor 2π to represent integration over the azimuthal angle.

Practice source-sink method for plane laminar jet, where outer flow is known (obtained by conformal mapping). (Will separation of variables work? Variables are separated in known answer.)

Handout on boundary layer. Ideas are rare, one every few years or occasionally much longer. Millikan looked at the departure h from the log law, but had no good data except for pipe and possible channel in 1938. Millikan's paper was absolutely ignored. Are there any citations before my thesis? Ludwieg and Tillmann were interested in the failure of the momentum-integral equation to give the correct friction in flows going to separation. Ludwieg invented the heated element as an *alternative. They stumbled on the result that the log law is independent of pressure gradient as well as Reynolds number. (Put $\partial w/\partial z$ in derivation.) Clauser also has a plot of h but did not comment on the shape or meaning.*

10.5 Turbulent plane wall jet into moving fluid

10.5.1 Similarity

10.5.2 Relaxation

10.5.3 Effectiveness

10.5.4 Boundary-layer control

10.6 Three-dimensional wall jets

10.6.1 Single jets

10.6.2 Film cooling through holes