Topics in Shear Flow

Chapter 11 – The Plane Plume

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Chapter 11

THE PLANE PLUME

11.1 Generalities

Plumes are an important subject in civil and environmental engineering, because of the frequent need to dispose of waste heat and/or combustion products. In a uniform ambient fluid at rest, similarity solutions exist for round and plane plumes for both laminar and turbulent flow. The new property of such flows is that momentum is continuously added to the fluid, so that the velocity on the centerline increases, or at least does not decrease, with increasing distance from the source. If the initial momentum flux is not negligible, the flow may behave like a jet for a time, and then like a plume, as the acquired momentum begins to dominate the motion. The problem of a plume in a stably stratified ambient fluid in also important. Finally, the plume in a crossflow is an even more difficult problem than the jet in a crossflow, a problem already discussed in section x.

11.1.1 Dimensional preamble

Plumes are usually treated within the Boussinesq approximation, which assumes that variations in density can be ignored everywhere except in the energy equation and in the driving buoyancy term in the momentum equation. In the engineering literature, the approximation is often stated as if it were self-evident. In what follows, a more rigorous argument will be attempted.

Recall the result of taking the limit $M \to 0$ in the equations of motion for a perfect gas in SECTION 1.2.3. The equations of continuity, momentum, energy, and state have the form

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \vec{u} = 0 \tag{11.1}$$

$$\rho \frac{D\,\vec{u}}{Dt} = -\operatorname{grad} p + \rho \vec{\mathrm{F}} + \operatorname{div} \underline{\tau} \tag{11.2}$$

$$\rho c_p \frac{DT}{Dt} = -\operatorname{div} \vec{q} + \rho Q \tag{11.3}$$

$$\rho \mathrm{T} = \text{constant} \tag{11.4}$$

where the last two equations assume a calorically and thermally perfect fluid ($h = c_p T$ and $p = \rho R T$, respectively, with c_p and R constant). To these are added Newton's hypothesis for viscous stress and Fourier's hypothesis for heat flow,

$$\underline{\tau} = \mu(\underline{\text{grad } \vec{u}} + \underline{\text{grad } \vec{u}^*}) = \mu\, \underline{\text{def } \vec{u}} \tag{11.5}$$

$$\vec{q} = -k \operatorname{grad} \mathrm{T} \tag{11.6}$$

where the two constants μ and k are assumed to depend only on the state of the fluid.

The steady laminar plane plume is defined in FIGURE 11.1. The flow is driven by a line heat source at the origin, with E the energy input per unit time per unit length. The fluid is characterized by the temperature and density in the uniform ambient region, and by four secondary state variables, the viscosity, the heat conductivity, the specific heat at constant pressure, and the volume coefficient of expansion. The coordinates are labelled for consistency with other flows already considered, with x increasing upward. The acceleration of gravity g is directed downward, in the negative x-direction.

There are altogether eight parameters, listed below in the form of dimensional statements. For the present, the argument will proceed

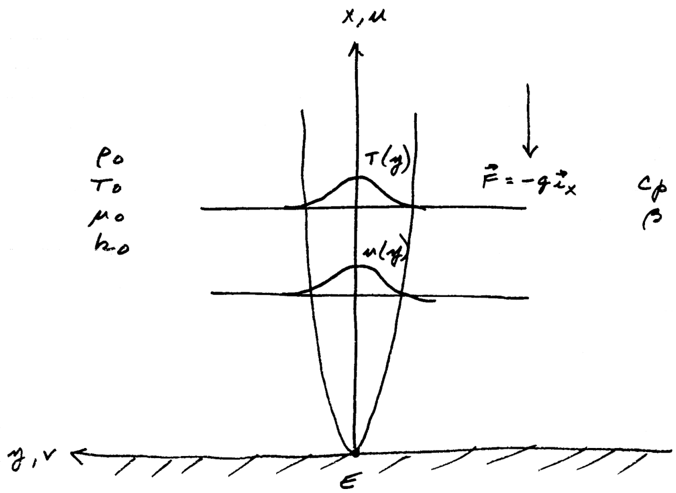

Figure 11.1: Steady laminar plane plume. (Caption provided by B. Coles)

without benefit of equations. **(Change sub 0 to sub α?)**

$$
\begin{aligned}
[E] &= \frac{\mathbf{ML}}{\mathbf{T}^3} \\
[\rho_0] &= \frac{\mathbf{M}}{\mathbf{L}^3} \\
[\mu_0] &= \frac{\mathbf{M}}{\mathbf{LT}} \\
[T_0] &= \mathbf{\Theta} \\
[k_0] &= \frac{\mathbf{ML}}{\mathbf{T}^3\mathbf{\Theta}} \\
[g] &= \frac{\mathbf{L}}{\mathbf{T}^2} \\
[c_p] &= \frac{\mathbf{L}^2}{\mathbf{T}^2\mathbf{\Theta}} \\
[\beta] &= \frac{1}{\mathbf{\Theta}} \ .
\end{aligned}
\qquad (11.7)
$$

Let the first four statements be interpreted as defining equations for $\mathbf{M}$, $\mathbf{L}$, $\mathbf{T}$, $\mathbf{\Theta}$ and solved, to obtain

$$
\mathbf{M} = \left(\frac{\rho_0^5\nu_0^9}{E^3}\right)^{1/2}, \quad \mathbf{L} = \left(\frac{\rho_0\nu_0^3}{E}\right)^{1/2}, \quad \mathbf{T} = \frac{\rho_0\nu_0^2}{E}, \quad \mathbf{\Theta} = T_0 \ . \qquad (11.8)
$$

In passing, define

$$
\mathbf{U} = \frac{\mathbf{L}}{\mathbf{T}} = \left(\frac{E}{\rho_0\nu_0}\right)^{1/2} \ . \qquad (11.9)
$$

and note that, as usual,

$$
\frac{\mathbf{UL}}{\nu_0} = 1 \ . \qquad (11.10)
$$

The relationships (11.8) can now be inserted in the last four of equations (11.7) to obtain four dimensionless groups,

$$P_1 = \frac{k_0 T_0}{E}\ , \quad P_2 = \frac{\rho_0^3 \nu_0^5 g^2}{E^3}\ , \quad P_3 = \frac{\rho_0 \nu_0 c_p T_0}{E}\ , \quad P_4 = \beta\, T_0 \quad (11.11)$$

in which the four physical parameters k_0, g, c_p, and β are isolated. Within limits, a different group of four statements might have been selected initially, with the same result, since the four groups (11.11) can be multiplied or divided arbitrarily by one another. This operation is evidently an ad hoc illustration of Buckingham's Π theorem (**ref**), which states that p dimensional parameters and $q < p$ dimensional units imply $p-q$ dimensionless groups. One of these, obtained by dividing the third group in (11.11) by the first, is familiar;

$$\Pi_3 = \frac{P_3}{P_1} = \frac{\rho_0 \nu_0 c_p}{k_0} = \frac{\nu_0}{\kappa_0} = Pr\ . \quad (11.12)$$

Another, a global Froude number, follows on rearrangement of P_2;

$$\Pi_2 = P_2^{-1/2} = \frac{1}{g}\left(\frac{E^3}{\rho_0^3 \nu_0^5}\right)^{1/2} = \frac{1}{g}\left(\frac{E}{\rho_0 \nu_0}\right)\left(\frac{E}{\rho_0 \nu_0^3}\right)^{1/2} = \frac{\mathbf{U^2}}{\mathbf{gL}} = Fr^2\ . \quad (11.13)$$

A third amounts to a global Rayleigh number,

$$\begin{aligned}\Pi_1 = \frac{P_2^{1/2} P_3 P_4}{P_1} &= g\left(\frac{\rho_0^3 \nu_0^5}{E^3}\right)^{1/2} \frac{\nu_0}{\kappa_0} \beta T_0 \\ &= g\left(\frac{\rho_0^3 \nu_0^9}{E^3}\right)^{1/2} \frac{\beta T_0}{\kappa_0} = \frac{g \beta T_0 \mathbf{L^3}}{\kappa_0 \nu_0} = Ra\ . \end{aligned} \quad (11.14)$$

The ratio Π_1/Π_3 also has a name (see Eckert for comment),

$$\frac{Ra}{Pr} = \frac{g \beta T_0 \mathbf{L^3}}{\nu_0^2} = Gr\ . \quad (11.15)$$

Finally, a fourth parameter is P_4 itself,

$$\Pi_4 = P_4 = \beta T_0\ . \quad (11.16)$$

Note from equation (x) of section x that $\Pi_4 = 1$ for a perfect gas. For water at 20 $^\circ C$, $\Pi_4 = xxxx$. **(Do the moving observer argument.)**

The similarity argument for the equations of motion is essentially independent of the dimensional argument just completed, which did not use equations except to establish the relevance of the eight quantities in equations (11.7). The two most common sources of buoyancy forces are heating **(why is cooling not allowed?)** as in the sketch, or programmed density differences, as for flow of fresh or hot water into salt or cold water. The first case will be considered here and the second, which is a jet-plume relaxation, in section x. (**Should also do thermal.**)

The equations of motion for the laminar plane plume, with the boundary-layer and Boussinesq approximations, are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \ , \tag{11.17}$$

$$\rho_0 \left(\frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} \right) = \beta \rho_0 (T - T_0) g + \mu_0 \frac{\partial^2 u}{\partial y^2} \ , \tag{11.18}$$

$$\rho_0 c_p \left(\frac{\partial u(T - T_0)}{\partial x} + \frac{\partial v(T - T_0)}{\partial y} \right) = k_0 \frac{\partial^2 (T - T_0)}{\partial y^2} \ . \tag{11.19}$$

Suitable boundary conditions for $u(x,\ y)$ and $T(x,\ y)$ are

$$u(x,\ \pm\infty) = 0 \tag{11.20}$$

$$T(x,\ \pm\infty) = 0 \tag{11.21}$$

and the symmetry condition

$$\psi(x,\ 0) = v(x,\ 0) = 0 \tag{11.22}$$

with null conditions on higher derivatives as needed.

Two integrals can be found immediately by integrating (11.18) and (11.19) from $-\infty$ to ∞ in y;

$$\rho \frac{\mathrm{d}}{\mathrm{d}x} \int_{-\infty}^{\infty} uu \, \mathrm{d}y = \beta \ \rho \ g \int_{-\infty}^{\infty} (T - T_0) \, \mathrm{d}y \tag{11.23}$$

$$\rho \ c_p \frac{\mathrm{d}}{\mathrm{d}x} \int_{-\infty}^{\infty} u(T - T_0) \, \mathrm{d}y = 0 \tag{11.24}$$

from which, in the second case,

$$\rho\ c_p \int_{-\infty}^{\infty} u(T - T_0)\,\mathrm{d}y = E = \text{constant} \ . \qquad (11.25)$$

Equation (11.23) describes the budget for the changing momentum, which increases with increasing x if $T > T_0$. The integral in equation (11.24), when expressed in terms of $(\rho - \rho_0)$, is often referred to as "buoyancy flux", for reasons that are not apparent to me; it is obviously an energy flux. The constant E has the units of energy per unit time per unit length passing any station x = constant and is here identified with the power per unit length at the heat source at the origin.

The momentum equation (11.18) and energy equation (11.19), together with the integral conservation law (11.25),

$$\rho_0 c_p \int_{-\infty}^{\infty} u(T - T_0)\,\mathrm{d}y = E \ , \qquad (11.26)$$

are the substance of the similarity formulation. The affine transformation operates on a large number of quantities; there are eight parameters (11.7) and five variables x, y, u, v, T. **(Recall the rule "transform everything in sight." Note that T transforms like**

T_0. **Discuss** ψ**, boundary conditions.)** Put

$$\begin{aligned}
x &= a\widehat{x} \\
y &= b\widehat{y} \\
\psi &= c\widehat{\psi} \\
\mu_0 &= d\widehat{\mu}_0 \\
\rho_0 &= e\widehat{\rho}_0 \\
\kappa_0 &= f\widehat{\kappa}_0 \\
c_p &= p\widehat{c}_p \\
\beta &= q\widehat{\beta} \\
T &= r\widehat{T} \\
T_0 &= r\widehat{T}_0 \\
E &= s\widehat{E} \\
g &= t\widehat{g}
\end{aligned} \tag{11.27}$$

Four alphabetic invariants are obtained, constituting a complicated but faithful image of the equations of the problem **(where is** T

replaced by $T - T_0$?);

$$\frac{bce}{ad} = 1 \ , \tag{11.28}$$

$$\frac{b^3eqrt}{cd} = 1 \ , \tag{11.29}$$

$$\frac{bcep}{af} = 1 \ , \tag{11.30}$$

$$\frac{cepr}{s} = 1 \ . \tag{11.31}$$

If equation (11.30) is divided by equation (11.28), the result is

$$\frac{dp}{f} = 1 \tag{11.32}$$

which requires invariance for one of the dimensionless parameters;

$$\frac{c_p\mu_0}{k_0} = \frac{\nu_0}{\kappa_0} = Pr = \widehat{Pr} \ . \tag{11.33}$$

The remaining three equations, say (11.28), (11.29), and (11.31), can be solved for b, c, r to establish dimensionless forms for y, ψ, $(T-T_0)$;

$$\frac{b^5e^2qst}{a^2d^3p} = 1 \ , \tag{11.34}$$

$$\frac{c^5e^3p}{a^3d^2qst} = 1 \ , \tag{11.35}$$

$$\frac{r^5a^3d^2e^2p^4qt}{s^4} = 1 \ . \tag{11.36}$$

Hence choose an ansatz of the form

$$A\left(\frac{c_p\rho_0}{\beta gE\nu_0^2x^3}\right)^{1/5}\psi = f\left[B\left(\frac{\beta gE}{\rho_0c_p\nu_0^3x^2}\right)^{1/5}y\right] = f(\eta) \ , \tag{11.37}$$

$$D\left(\frac{\beta g c_p^4 \rho_0^4 \nu_0^2 x^3}{E^4}\right)^{1/5}(T - T_0) = \theta\,(\eta) \tag{11.38}$$

with boundary conditions

$$f(0) = f'(\pm\infty) = f''(\pm\infty) = \theta(\pm\infty) = \theta'(\pm\infty) = 0\ \ . \tag{11.39}$$

(Note from the form of η that δ varies like $\nu^{3/5}$ and like $x^{2/5}$. What about u_c and Fr?)

Substitution in (11.34)-(11.36)[1] yields

$$\frac{5}{3}ABf''' + ff'' - \frac{1}{3}f'f' - \frac{5}{3}\frac{A^2}{B^2 D}\ \theta = 0\ \ , \tag{11.40}$$

$$\frac{5}{3}\frac{AB}{Pr}\ \theta'' + f\theta' + f'\theta = 0\ \ , \tag{11.41}$$

$$\int_{-\infty}^{\infty} f'\ \theta\,\mathrm{d}\,\eta = AD \tag{11.42}$$

with one parameter, the Prandtl number. **(Check the other integral** (11.23).)

Equation (11.41) can be integrated once to obtain

$$\frac{5}{3}\frac{AB}{Pr}\theta' + f\theta = 0\ \ . \tag{11.43}$$

The velocities u and v are (note $\beta gE/c_p\rho_0$ recurs)

$$u = \frac{B}{A}\left(\frac{\beta^2 g^2 E^2 x}{c_p^2 \rho_0^2 \nu_0}\right)^{1/5} f'(\eta) \tag{11.44}$$

and

$$v = \frac{1}{A}\left(\frac{\beta g \nu_0^2 E}{c_p \rho_0 x^2}\right)^{1/5}\left(\frac{2}{5}\eta\, f' - \frac{3}{5}f\right)\ \ . \tag{11.45}$$

[1] Original ms is unclear about these equation numbers.

Provided that $\eta f' \to 0$ as $\eta \to \infty$, the entrainment velocity at the edge of the plume is

$$v(x,\ \infty) = -\frac{3}{5}\frac{1}{A}\left(\frac{\beta g \nu_0^2 E}{c_p \rho_0 x^2}\right)^{1/5} f(\infty)\ . \tag{11.46}$$

The notation will eventually put $f(\infty) = C$. A constant local Froude number can be formed from the centerline velocity, which varies like $x^{1/5}$, and the plume thickness, which varies like $x^{2/5}$. A maximum-slope thickness for the plume can be defined in the same way as for the plane jet in SECTION X, namely

$$\triangle = 2\frac{f(\infty)}{f'(0)}\ . \tag{11.47}$$

A second relationship is obtained by putting $\eta = \triangle/2$ when $y = \delta/2$, in the argument of f in equation (11.37);

$$\triangle = B\left(\frac{\beta g E}{\rho_0 c_p \nu_0^3 x^2}\right)^{1/5} \delta\ . \tag{11.48}$$

Elimination of δ gives

$$\delta = \frac{2}{B}\frac{f(\infty)}{f'(0)}\left(\frac{c_p \rho_0 \nu_0^3 x^2}{\beta g E}\right)^{1/5}\ . \tag{11.49}$$

Finally, from equation (11.44) at $y = 0$,

$$u_c = \frac{B}{A}\left(\frac{\beta^2 g^2 E^2 x}{c_p \rho_0^2 \nu_0}\right)^{1/5} f'(0)\ . \tag{11.50}$$

It follows that **(must be small)**

$$Fr^2 = \frac{u_c^2}{g\delta} = \frac{B^3}{2A^2}\frac{[f'(0)]^3}{f(\infty)}\left(\frac{\beta E}{c_p \rho_0 \nu_0}\right) \tag{11.51}$$

which is constant **(of what order?)** and formally independent of T_0, k_0, and g. The dimensionless combination in parentheses can be recognized as the ratio of the fourth to the third of the dimensionless groups (11.11). (*How does $T - T_0$ vary with x? Who solved the problem first? Proceed to entrained flow, with and without wall. If source is cold and there is a wall, is the outcome a wall plume? Do thermal. Discuss Grashof number, Richardson number. See Schmidt and Beckmann.*)

11.1.2 The plane turbulent plume

In the turbulent case, the boundary-layer approximation for the equations of motion for a plane plume, with the usual stream function $(u = \partial\psi/\partial y,\ v = -\partial\psi/\partial x)$ is

$$\rho_0(\psi_y\psi_{xy} - \psi_x\psi_{yy}) = \rho_0\,\beta\,g(T - T_0) + \frac{\partial\tau}{\partial y}\ , \tag{11.52}$$

$$\rho_0 c_p\Big[\psi_y(T - T_0)_x - \psi_x(T - T_0)_y\Big] = \frac{\partial q}{\partial y}\ . \tag{11.53}$$

The integral conservation law is again **(mention buoyancy flux)**

$$\rho_0 c_p \int\limits_{-\infty}^{\infty} u\,(T - T_0)\mathrm{d}y = E\ . \tag{11.54}$$

In these equations,

$$\tau = -\rho_0\overline{u'v'}\ , \tag{11.55}$$

$$q = -\rho_0\overline{T'v'}\ . \tag{11.56}$$

The second of these is usually defined without the factor c_p.

To implement an affine transformation, list the variables and parameters in a table, together with their dimensions;

$$
\begin{array}{rclll}
x &=& a\widehat{x} & \mathbf{L} & \\
y &=& b\widehat{y} & \mathbf{L} & \\
\psi &=& c\widehat{\psi} & \mathbf{L^2/T} & \\
\rho_0 &=& d\widehat{\rho}_0 & \mathbf{M/L^3} & \\
c_p &=& p\widehat{c}_p & \mathbf{L^2/T^2\Theta} & \\
\beta &=& q\widehat{\beta} & \mathbf{1/\Theta} & \quad (11.57) \\
T &=& r\widehat{T} & \mathbf{\Theta} & \\
T_0 &=& r\widehat{T}_0 & \mathbf{\Theta} & \\
E &=& s\widehat{E} & \mathbf{ML/T^3} & \\
g &=& t\widehat{g} & \mathbf{L/T^2} & \\
\tau &=& m\widehat{\tau} & \mathbf{M/LT^2} & \\
q &=& n\widehat{q} & \mathbf{M/T^3}\ . &
\end{array}
$$

From the six parameters $\rho_0,\ c_p,\ \beta,\ T_0,\ E,$ and g, characteristic scales appear as

$$
\mathbf{M} = \left(\frac{E^6}{g^9\rho_0}\right)^{1/5},\ \mathbf{L} = \left(\frac{E^2}{g^3\rho_0^2}\right)^{1/5},\ \mathbf{T} = \left(\frac{E}{g^4\rho_0}\right)^{1/5},\ \mathbf{\Theta} = T_0 \tag{11.58}
$$

together with two dimensionless groups,

$$
\beta\, T_0\,, \quad \frac{c_p^5\rho_0^2T_0^5}{g^2E^2}\ . \tag{11.59}
$$

The second of these is also a group for the laminar problem **(therefore basic?)**.

Affine transformation of equations (11.52) - (11.54) yields the following alphabetical relations:

$$\frac{c^2 d}{abm} = 1 \ , \tag{11.60}$$

$$\frac{bdqrt}{m} = 1 \ , \tag{11.61}$$

$$\frac{cdr}{an} = 1 \ , \tag{11.62}$$

$$\frac{cdpr}{s} = 1 \ . \tag{11.63}$$

Equations (11.62) and (11.63) can first be combined in the form **(why?)**

$$\frac{anp}{s} = 1 \tag{11.64}$$

and (11.62) discarded. An equivalent condition to (11.62) is that q transforms like $\rho\,\psi_x(T - T_0)$. If it is also assumed that τ transforms like $\rho\,\psi_y^2$, the relation implied is

$$\frac{mb^2}{c^2 d} = 1 \ . \tag{11.65}$$

When this is compared with (11.60), it appears that

$$\frac{a}{b} = 1 \ . \tag{11.66}$$

(**Question by Hall: is this above the line or below the line? Answer: below the line.**) Hence the combination y/x is invariant under the transformation, and is evidently the proper independent variable. When b is replaced by a throughout, three relations remain, in addition to (11.64);

$$\frac{c^2 d}{a^2 m} = 1, \qquad \frac{adqrt}{m} = 1, \qquad \frac{cdpr}{s} = 1 \ . \tag{11.67}$$

These can be solved for c, m and r. The result is

$$\frac{c^3 dp}{a^3 qst} = 1, \qquad \frac{r^3 a^3 d^2 p^2 qt}{s^2} = 1, \qquad \frac{m^3 p^2}{d^2 q^2 s^2 t^2} = 1 \ . \tag{11.68}$$

The proper similarity variables, with constants for future use, are therefore

$$A\frac{\psi}{x}\left(\frac{\rho_0\, c_p}{\beta\, g\, E}\right)^{1/3} = f\left(B\ \frac{y}{x}\right) = f(\eta)\ , \tag{11.69}$$

$$C\ (T - T_0)x\left(\frac{\rho_0^2\, \beta\, g\, {c_p}^2}{E^2}\right)^{1/3} = \theta\,(\eta)\ , \tag{11.70}$$

$$\tau\left(\frac{{c_p}^2}{\rho_0\, \beta^2\, g^2\, E^2}\right)^{1/3} = g\,(\eta)\ , \tag{11.71}$$

$$q\frac{c_p x}{E} = h(\eta)\ . \tag{11.72}$$

When these are substituted in equations (11.52) - (11.54), the result is

$$f\ f'' + \frac{A^2}{B^2 C}\ \theta + \frac{A^2}{B}\ g' = 0\ , \tag{11.73}$$

$$f'\theta + f\theta' + ACh' = 0\ , \tag{11.74}$$

$$\int\limits_{-\infty}^{\infty} f'\theta\, d\,\eta = AC\ . \tag{11.75}$$

Note from equation (11.74) that there is a relation between f, θ and h independent of position in the dimensionless flow;

$$f\theta + ACh = \text{constant} = 0 \tag{11.76}$$

since f and h are zero on the plane of symmetry. In physical variables, this is

$$\psi(T - T_0) - x\overline{T'v'} = 0\ . \tag{11.77}$$

The mean-velocity components in the turbulent plane plume are

$$u = \frac{B}{A}\left(\frac{\beta g E}{c_p \rho_0}\right)^{1/3} f'\ , \tag{11.78}$$

$$v = \frac{1}{A}\left(\frac{\beta g E}{c_p \rho_0}\right)^{1/3} (\eta\, f' - f) \ . \tag{11.79}$$

Hence the velocity in the plane of symmetry, u_c, is constant. The entrainment velocity is

$$v(x,\ \infty) = -\frac{1}{A}\left(\frac{\beta g E}{c_p \rho_0}\right)^{1/3} f(\infty) \tag{11.80}$$

and is also constant. In the plane of symmetry the temperature ($T_c - T_0$) and the turbulent heat transfer vary like $1/x$, while the Reynolds shearing stress is constant. (**Why not $\tau \sim u_c^2$, $T - T_0 \sim T_c - T_0$? Should do without boundary-layer approximation.**)

Note from (11.78) that u_c is independent of x;

$$u_c = \frac{B}{A}\left(\frac{\beta g E}{C_p \rho_o}\right)^{1/3} f'(0) \tag{11.81}$$

and from (11.70) that

$$T_c - T_0 = \frac{1}{Cx}\left(\frac{E^2}{\rho_0^2 \beta g c_p^2}\right)^{1/3} \theta(0) \tag{11.82}$$

so that the Reynolds shearing stress (11.71) and the Reynolds heat transfer (11.72) can be written

$$\tau = \frac{A^2}{B^2[f'(0)]^2} \rho_0 u_c^2 g(\eta) \tag{11.83}$$

and

$$q = \frac{AC}{B f'(0)\, \theta\,(0)} \rho_0 u_c (T_c - T_0) h(\eta) \ . \tag{11.84}$$

(**Why not do this in the beginning?**) *From handout:*

Laminar plume: nice work. No evidence of instability. Velocity in plane of symmetry is increasing with x. Thickness goes like $\nu^{2/3}$.

Sparrow. If (similarity) theory and experiment do not agree, I choose to believe the theory and look for sources of experimental error. This *policy might be useful for the turbulent problem, where there is no theory.*

Sparrow. Note turbulent case; independent variable is y/x, and uses up 0.6*: this flow is not a boundary layer. The entrained flow for the round turbulent plume uses Legendre polynomials.*

Kotsovinos. Good on relaxation from jet to plume, given some initial momentum. K's plume not very wide; tank is small. See also picture of jet.

Anwar. Does not comment on development into pair of counter-rotating vortices.