Topics in Shear Flow

Chapter 12 – Flow Control

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Chapter 12

FLOW CONTROL

Flow through a plane gauze, or screen, is accompanied by a pressure drop and, if the flow is not normal to the screen, by a flow deflection toward the normal, much like the refraction of light when moving from an optically less dense into an optically more dense medium.

Screens are usually woven wire, but may be cloth or may be perforated plates or have other geometries. There are two effects to be considered. One is attenuation of turbulence existing upstream and the other is generation of new turbulence to be studied for its own sake or for its effect on other phenomena, such as transition, surface friction, or heat transfer. That is, make a non-uniform flow uniform or vice versa. I propose not to become involved with turbulence for its own sake, as this subject is very difficult and is covered in monographs by Hinze, Batchelor, Townsend, Monin and Yaglom, and elsewhere. (*Comment on curious identity of sizes of woven-wire screens available from different manufacturers, as if they bought from each other.*)

(*Look up the reasons why each author was interested in screens to show wide applicability. Note that early wind tunnels had no contractions; see paper by Prandtl. Mention Wright brothers, van der Hegge Zijnen.*)

The earliest competent study of the behavior of screens is by

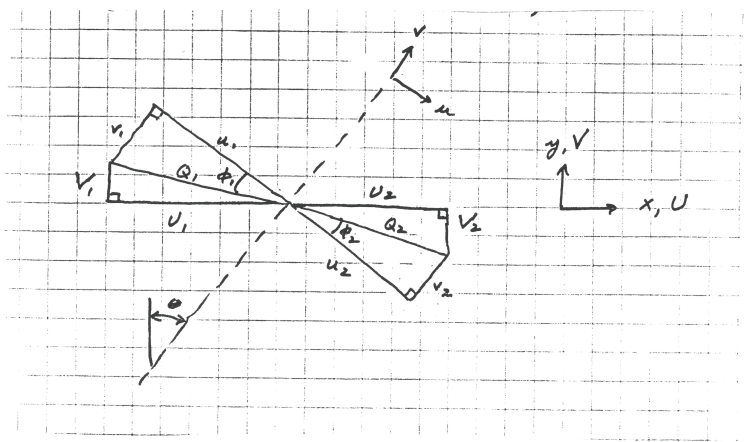

Figure 12.1: Flow through a screen. (Figure and caption added by K. Coles.)

TAYLOR and BATCHELOR (1949).

Assume that the resistance of the screen depends only on the component of velocity normal to its plane. If $p_2 - p_1$ is the pressure drop, the loss coefficient for flow normal to the screen ($\theta = 0$) will be defined as **(consider including solidity; what happens? Note that overall velocity decreases if there is a deflection; compare to shock wave.) (Put solidity s in denominator.)** (Need FIGURE X.) [1]

$$C_n = \frac{p_1 - p_2}{\frac{1}{2}\rho Q_1^2} \tag{12.1}$$

where Q, u, v are velocities in screen coordinates; u is the component normal to the screen and is necessarily conserved; i.e., $u_2 = u_1$. The component parallel to the screen is not conserved, being reduced by the drag of the screen elements. Here a second loss coefficient can be defined as

$$C_t = \frac{F}{\frac{1}{2}\rho u_1 v_1} \tag{12.2}$$

[1] A sketch found in ms that may be the one cited is included here as Figure 12.1.

where T [2] is the force per unit area in the plane of the screen and the subscript t refers to the tangential component. **(Notation is a problem. Can the Taylor and Batchelor argument be put in vector form?)** Note that both coefficients are designed to be of order unity, although they can also be expected to depend on solidity as well as on Reynolds number, Mach number, and geometrical details. The relationships

$$\begin{aligned} u_1 &= Q_1 \cos\,\phi_1 \\ u_2 &= Q_2 \cos\,\phi_2 \\ v_1 &= Q_1 \sin\,\phi_1 \\ v_2 &= Q_2 \sin\,\phi_2 \end{aligned} \tag{12.3}$$

and the tangential momentum equation

$$T = \rho u_1 (v_1 - v_2) \tag{12.4}$$

allow equation (12.2) to be put in the form

$$C_t = 2\left(1 - \frac{\cos\,\phi_1}{\sin\phi_1}\frac{\sin\,\phi_2}{\cos\,\phi_2}\right)\ . \tag{12.5}$$

This coefficient C_t is called F_θ/θ by Taylor and Batchelor. The analogy with optics can be made explicit by writing

$$n = \frac{\sin\,\phi_1}{\sin\,\phi_2} \tag{12.6}$$

in which case, to first order in θ (**give also exact expression**),

$$C_t = 2\left(\frac{n-1}{n}\right) \tag{12.7}$$

(*Cite experiments on ϕ_2 vs ϕ_1, especially Schubauer, Spangenberg, and Klebanoff. Mention experimental setup. Mention fit $F_\theta/\theta = 2 - 2.2/(1+k_\theta)^{\frac{1}{2}}$ and T & B's figure 5. See JAS.*)

[2] T may be same as F in 12.2, i.e., a transcription error from ms.

The proper geometric parameters for analyzing screen preformance are the solidity s and the index of refraction n. For a square wire-mesh screen, the solidity is defined by the sketch;

$$s = \frac{\text{blocked area}}{\text{total area}} = \frac{2dD - d^2}{D^2} = 2\frac{d}{D} - \frac{d^2}{D^2} \tag{12.8}$$

The resistance coefficient C_n can be expected to increase with increasing solidity, as in sketch A.

There is a weak Reynolds-number effect, as indicated in sketch B, which can be expected to look like the drag coefficient of a cylinder. Finally, there is a Mach number effect, as shown in sketch C *(if density changes are appreciable, take them into account).*

(Combine to reproduce figure 5 of Taylor and Batchelor. Can solidity effect in A be estimated by adding up cylinders? See Wieghardt.)

(Structural strength is a factor; work out some details. Different companies sell the same screens. Better to put screen in low-velocity region, for sake of lower loads and lower losses. Keep Re based on stream velocity and wire diameter below shedding frequency.)

(In handout, note better scheme used by Dryden and Schubauer for determining angle; Simmons and Cowdrey were clumsy.)

To determine the effect of the screen on a small disturbance in the oncoming flow, linearize the problem. Suppose that the upstream flow is two-dimensional, with a perturbation that depends only on y; say **(now U, not Q; comment)**

$$U_1 = U_0 + u_1 \cos \kappa y \ \ . \tag{12.9}$$

This flow is assumed to be normal to the screen, and u_1 is now the amplitude of the perturbation. If viscosity is neglected, except perhaps in the close vicinity of the screen, the vorticity is constant on streamlines;

$$\frac{D\zeta}{Dy} = 0 \ \ . \tag{12.10}$$

In the two-dimensional flow, define a stream function ψ, with

$$\zeta = -\nabla^2 \psi \ \ . \tag{12.11}$$

To first order, therefore, $\partial\nabla^2\psi/\partial x = 0$, and

$$\nabla^2\psi = f(y) \ . \tag{12.12}$$

Note that this analysis is essentially for a two-dimensional screen; in three dimensions the vortex-stretching terms would appear and also there would be no stream function. But see pp. 11–12 of Taylor and Batchelor for a counter-argument. Far upstream,

$$\zeta = -\frac{\partial U_1}{\partial y} = \kappa u_1 \sin \kappa\, y \tag{12.13}$$

and in general, for the perturbation,

$$\nabla^2\psi_1 = -\kappa u_1 \sin \kappa\, y \ . \tag{12.14}$$

The solution, easily obtained by separation of variables, is of the form

$$\psi(x,\ y) = Ce^{\pm\kappa x} \begin{matrix}\sin \\ \cos\end{matrix}\ \kappa\, y \ . \tag{12.15}$$

To this must be added the particular solution. For an anti-symmetric flow, with $\psi(x, 0) = 0$ and with the upstream boundary condition (12.9),

$$\psi_1 = \left(\frac{u_1}{\kappa} + A\, e^{\kappa\, x}\right) \sin \kappa\, y \ . \tag{12.16}$$

A similar argument for the downstream region gives

$$\psi_2 = \left(\frac{u_2}{\kappa} + B\ e^{-\kappa x}\right) \sin \kappa\, y \tag{12.17}$$

where

$$U_2 = U_0 + u_2 \cos \kappa\, y \ . \tag{12.18}$$

Three conditions are needed to determine A, B, and u_2/u_1. First, the component $u = \partial\psi/\partial y$ must be continous. At the screen $x = 0$, therefore,

$$u_1 + \kappa\, A = u_2 + \kappa\, B = u_s,\ \text{say.} \tag{12.19}$$

The meaning of the quantity u_s (for screen) is indicated in the sketch. Where the stream velocity is higher, more resistance will be encountered, and the stream will diverge and reach the screen at an angle **(how does linearization prevent appearance of $\sin 2\ \kappa y$, etc.?)**. The amplitude u_1 will decrease to u_3; see below. Since $n = \sin\phi_1/\sin\phi_2 = v_1/v_2$, the v-components across the screen are related by

$$v_2 = -\frac{\partial \psi_2}{\partial x} = \frac{v_1}{n} = -\frac{1}{n}\frac{\partial \psi_1}{\partial x} \tag{12.20}$$

from which

$$A + n\,B = 0 \ . \tag{12.21}$$

The final condition is obtained from Bernoulli's equation. To first order, with $U_1 = U_2 = U$, the total pressures differ by the loss at the screen;

$$p_1 + \frac{\rho}{2}\big(U^2 + 2U\ u_1\ \cos\kappa y\big) - p_2 - \frac{\rho}{2}\big(U^2 + 2U\ u_2\ \cos\kappa y\big) = \\ = C_n\frac{\rho}{2}\big(U^2 + 2U\ u_s \cos\kappa\ y\big) \ . \tag{12.22}$$

This becomes, after cancelling terms of order unity,

$$u_1 - u_2 = C_n\, u_s \ . \tag{12.23}$$

When A, B, and u_s are eliminated from equations (12.19), (12.21), and (12.23), the result is a formula for attenuation;

$$\frac{u_2}{u_1} = \frac{1 + n - C_n}{1 + n + nC_n} \ . \tag{12.24}$$

(Note that this result is independent of κ and that the numerator may vanish. There is no practical upper limit on C_n. If $n = 3/2$, then $C_n = 5/2$ removes the upstream perturbation completely. Note that v-perturbations are attenuated by a factor $1/n$. This might be neater in a vector notation.)

The velocity u_s can be expressed in two ways:

$$\frac{u_s}{u_1} = \frac{1+n}{1+n+n\,C_n} \tag{12.25}$$

$$\frac{u_s}{u_2} = \frac{1+n}{1+n-C_n} \tag{12.26}$$

from which it is obvious that

$$u_1 > u_s > u_2 \ . \tag{12.27}$$

In all of this, the screen is assumed to have no structure, so that the effect on the turbulence spectrum is not treated. For a screen with high resistance, there is a high price in power required. The method has been used in diffusers (**comment on effect on boundary layer or on separated region**) and is worth the cost if a good downstream flow is essential. The same kind of analysis can be used for turning vanes. Compare S-duct in Lockheed 1011, Boeing 727.

Reprise of Taylor and Batchelor; see 20 April, 22 April.

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A gauze, or screen, or grid, is a high-drag device that is normally used to redistribute the flow in a channel. Other uses include prevention of separation in diffusers, generation of turbulence, and reduction of turbulence, depending on the properties of the screen. (See Taylor and Batchelor for turbulence reduction.)

The basic problem considered by ELDER (**ref**) is modification of flow in a straight channel by a single shaped screen located in the vicinity of $x = 0$.

The coordinates are $(x,\ y)$, and the corresponding velocities are $(U,\ V)$. The flow is uniform but rotational far upstream and far downstream, and the effect of the screen is to introduce a discontinuity in vorticity at $x = 0$. The screen has no structure and is treated

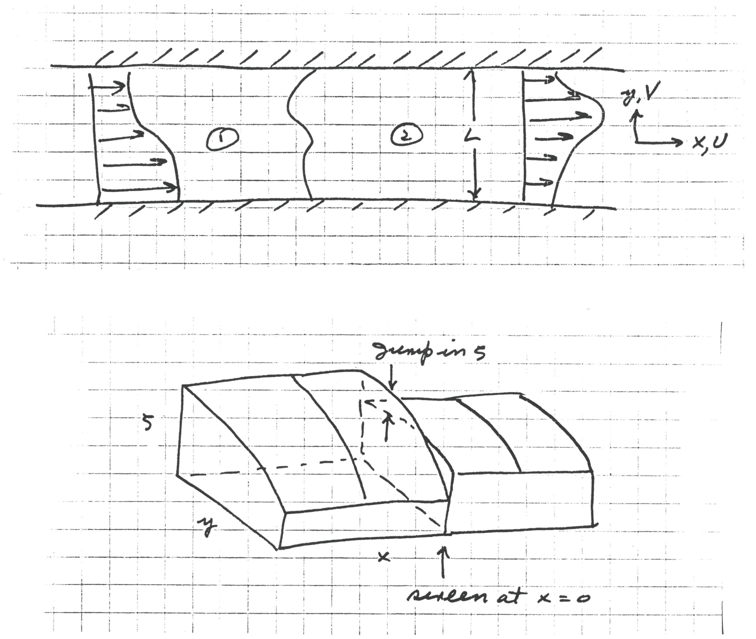

Figure 12.2: Flow in a channel modified by a single shaped screen. (Figure and caption added by K. Coles.)

like an actuator sheet.

Suppose that the flow is rotational but steady, incompressible, inviscid, and two-dimensional. Start with the flow in the sketch.[3] A subscript 1 or 2 denotes upstream conditions or downstream conditions, respectively. Denote by a superscript 0 the one-dimensional flow that coincides with the initial or final state far from the screen; typical variables are ψ^0, $U^0 = \partial\psi^0/\partial y$, $\zeta^0 = -\partial U^0/\partial y$.

In the two-dimensional flow, the continuity equation is satisfied if

$$\vec{U} = \nabla\psi \times \nabla z \tag{12.28}$$

[3]Two sketches found in ms that appear related to this discussion, the first of which may be the one cited, are included here as Figure 12.2.

where ψ is a stream function. Taking the curl yields

$$\zeta = -\nabla^2 \psi \quad . \tag{12.29}$$

The conditions already stated also imply

$$\frac{D\zeta^0}{Dt} = \vec{U}^0 \cdot \ \text{grad}\ \zeta^0 \quad . \tag{12.30}$$

For the flow in the sketch,

$$\frac{D\zeta^0}{Dt} = U^0 \frac{\mathrm{d}\zeta^0}{\mathrm{d}x} = (\zeta_2^0 - \zeta_1^0)\delta(x) \quad . \tag{12.31}$$

However, the velocity cannot be continuous, because $\partial U^0/\partial y$ is different for the upstream and downstream regions. It is necessary to add another flow near the screen. If the basic flow carries the vorticity, the proper composition is

$$\nabla^2 \psi = \nabla^2 \psi^0 + \nabla^2 \psi' \tag{12.32}$$

where the perturbation ψ' is irrotational;

$$\nabla^2 \psi^0 = -\zeta_0 \quad , \tag{12.33}$$

$$\nabla^2 \psi' = 0 \quad . \tag{12.34}$$

The assumptions are:

1. The jump in vorticity is carried by the basic flow ψ^0.
2. The condition that U is constant through the screen is carried by the combined flow.
3. A jump in V, to implement the jump in ζ, is carried by the perturbation flow. In particular, both ψ^0 and ψ' are discontinuous at the screen, but the sum is continuous.

The solution of the equation $\nabla^2 \psi' = 0$ in rectangular coordinates is easily obtained by separation of variables. The solution can be written in dimensionless form for the upstream region

$$\frac{\psi_1'}{L\overline{\overline{U}}} = \sum \frac{1}{m\pi} P_m e^{m\pi\, x/L} \sin m\pi \frac{y}{L} \qquad (x < 0) \tag{12.35}$$

and for the downstream region,

$$\frac{\psi_2'}{L\overline{U}} = \sum \frac{1}{m\pi} Q_m e^{-m\pi\, x/L} \sin m\pi \frac{y}{L} \qquad (x > 0) \tag{12.36}$$

where L is the channel width and $\overline{U}$ is the mean velocity over the cross section,

$$\overline{U} = \frac{1}{L} \int U \; \mathrm{d}y \ . \tag{12.37}$$

It follows from the geometry that this velocity is the same far from the screen in both directions.

The flow represented by ψ' vanishes at $x = \pm\infty$. The form automatically satisfies the requirement that the flow follow the two walls, since $\psi' = 0$ at $y = 0$ and at $y = L$. Note that there is no net flow, so that the condition $\overline{U}_1 = \overline{U}_2$ must be satisfied by the primary flow.

The velocity components associated with ψ_1' are

$$\frac{U_1'}{\overline{U}} = \sum P_m \, e^{m\pi x/L} \cos \, m\pi \frac{y}{L} \ , \tag{12.38}$$

$$\frac{V_1'}{\overline{U}} = -\sum P_m \, e^{m\pi x/L} \sin \, m\pi \frac{y}{L} \ , \tag{12.39}$$

and for ψ_2' are

$$\frac{U_2'}{\overline{U}} = \sum Q_m \, e^{m\pi x/L} \cos \, m\pi \frac{y}{L} \ , \tag{12.40}$$

$$\frac{V_2'}{\overline{U}} = \sum Q_m \, e^{-m\pi x/L} \sin \, m\pi \frac{y}{L} \ . \tag{12.41}$$

The screen properties are referred to screen coordinates, as shown in the sketch;

The velocity normal to the screen is

$$u_1 = U_1 \cos \theta - V_1 \sin \theta \ , \tag{12.42}$$

$$u_2 = U_2 \cos \theta - V_2 \sin \theta \ . \tag{12.43}$$

Since $u_1 = u_2$, it follows that

$$(U_1 - U_2) = (V_1 - V_2) \tan \theta \ . \tag{12.44}$$

The velocity parallel to the screen is

$$v_1 = U_1 \sin\theta + V_1 \cos\theta \ , \tag{12.45}$$
$$v_2 = U_2 \sin\theta + V_2 \cos\theta \ , \tag{12.46}$$

from which

$$v_1 - v_2 = (U_1 - U_2)\sin\theta + (V_1 - V_2)\cos\theta \tag{12.47}$$

or, in view of equation (12.44),

$$v_1 - v_2 = \frac{V_1 - V_2}{\cos\theta} \ . \tag{12.48}$$

Elder notices that the combination (his BUT)

$$G = \left(\frac{v_1 - v_2}{v_1}\right) U_1 \frac{\sin\theta}{\cos\theta} \tag{12.49}$$

can be developed by using equation (12.45) to eliminate $\sin\theta$;

$$\begin{aligned} G &= \frac{(v_1 - v_2)}{v_1} \frac{(v_1 - V_1 \cos\theta)}{\cos\theta} \\ &= \frac{(v_1 - v_2)}{\cos\theta} - \frac{v_1 - v_2)}{v_1} V_1 \\ &= \frac{(V_1 - V_2)}{\cos^2\theta} - \frac{(v_1 - v_2)}{v_1} V_1 \end{aligned} \tag{12.50}$$

where the last step requires equation (12.48). When terms in V_1 and V_2 are collected and the identity $1/\cos^2\theta = 1 + \tan^2\theta$ is used, this becomes

$$G = V_1 \left[1 - \frac{(v_1 - v_2)}{v_1} + \tan^2\theta\right] - V_2\big(1 + \tan^2\theta\big) \tag{12.51}$$

or finally, if $\tan^2\theta$ is neglected on the right-hand side, U_1 is replaced by $\overline{U}$, and the original form (12.49) is restored,

$$\frac{(v_1 - v_2)}{v_1} \overline{U} \tan\theta = \left[1 - \frac{(v_1 - v_2)}{v_1}\right] V_1 \ - \ V_2 \ . \tag{12.52}$$

In terms of the index of refraction, with the approximation $\sin\phi = \tan\phi$ and the condition $u_1 = u_2$,

$$\tan\,\phi_1 = \frac{v_1}{u_1} = \sin\,\phi_1 \ , \qquad \tan\,\phi_2 = \frac{v_2}{u_1} = \sin\,\phi_2 \tag{12.53}$$

and therefore $v_1 = nv_2$. Equation (12.52) becomes

$$\left(\frac{n-1}{n}\right)\overline{U}\tan\,\theta = \frac{V_1}{n} - V_2 \ . \tag{12.54}$$

This expression defines the jump in V at the screen when the screen properties n and θ are specified. The approximations include neglecting $\tan^2\,\theta$ compared to unity, replacing $\tan\,\phi$ by $\sin\,\phi$, and replacing U_1 by $\overline{U}$.

When the right-hand side of equation (12.54) is rewritten using equations (12.39) and (12.41) with $x = 0$, the result is

$$\left(\frac{n-1}{n}\right)\tan\,\theta = \sum\left(-\frac{P_m}{n} - Q_m\right)\sin\,m\pi\frac{y}{L} \ . \tag{12.55}$$

This relation provides the coefficients in a Fourier series for $\tan\,\theta$ if the coefficients P_m and Q_m are known. (**Define n for a honeycomb.**)

It remains to express the jump in pressure in the same way. The momentum equation can be written

$$-\frac{1}{\rho}\nabla\,p = \nabla\frac{\vec{u}\cdot\vec{u}}{2} + (curl\ \vec{u})\times\vec{u} \ . \tag{12.56}$$

The y-component of this equation is

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{\partial}{\partial y}\left(\frac{U^2+V^2}{2}\right) + \zeta U \ . \tag{12.57}$$

Far upstream and downstream of the screen (**close to screen?**)

$$-\frac{1}{\rho}\frac{\partial p_1}{\partial y} = \frac{\partial}{\partial y}\frac{U_1^2}{2} + \zeta_1 U_1 \tag{12.58}$$

$$-\frac{1}{\rho}\frac{\partial p_2}{\partial y} = \frac{\partial}{\partial y}\frac{U_2^2}{2} + \zeta_2 U_2 \tag{12.59}$$

and therefore

$$\frac{1}{\rho}\frac{\partial}{\partial y}(p_1 - p_2) = \frac{\partial}{\partial y}\frac{\left(U_2^2 - U_1^2\right)}{2} + \zeta_2 U_2 - \zeta_1 U_1 \ . \tag{12.60}$$

This condition should be applied at the screen, and the approximation is made that streamline displacements are small. If the second term is discarded, on the ground that $U_1 = U_2 = \overline{U}$ approximately, then

$$\frac{1}{\rho}\frac{\partial}{\partial y}(p_1 - p_2) = \zeta_2 U_2 - \zeta_1 U_1 \ . \tag{12.61}$$

By definition,

$$p_1 - p_2 = \frac{1}{2}\rho u^2 C_n \tag{12.62}$$

where $u = u_1 = u_2$. Moreover, from (12.42), $u = U\ \cos\theta$ approximately. Substitution gives

$$\frac{1}{2}\frac{\partial}{\partial y}\, U^2 \cos^2\,\theta\, C_n = \overline{U}\left(-\frac{\mathrm{d}U_2^0}{\mathrm{d}y} + \frac{\mathrm{d}U_1^0}{\mathrm{d}y}\right) \tag{12.63}$$

where it is also assumed on the right that $U_1 = U_2 = U$. One integration gives

$$\frac{1}{2}U^2\cos^2\theta\, C_n = \overline{U}\left(U_1^0 - U_2^0\right) + C \tag{12.64}$$

where C is a constant of integration. Put

$$U = \overline{U}(1+\epsilon) \tag{12.65}$$

to obtain

$$\frac{1}{2}\overline{U}(1+\epsilon)\cos^2\,\theta\, C_n = \overline{U}\ \left(U_1^0 - U_2^0\right) + \frac{C}{\overline{U}}(1-\epsilon)\ . \tag{12.66}$$

A second integration from $y = 0$ to $y = L$, with $\cos^2\theta$ treated as constant, and with

$$\int_0^L \epsilon\, \mathrm{d}y = 0 \tag{12.67}$$

by virtue of equation (?)[4], gives

$$C = \frac{\overline{U}^2}{2} C_n \quad . \tag{12.68}$$

Equation (12.64) can therefore be written, to leading order, and with $\cos^2\theta$ taken as unity,

$$(U - \overline{U})\frac{C_n}{2} = U_1^0 - U_2^0 \quad . \tag{12.69}$$

The stage is now set for the solution of equations (?)[5] and (12.69) above. Continuity of the streamwise velocity at the screen requires (Elder's Eq. 2.5)

$$U_1^0 + \overline{U}\sum P_m \cos m\pi\frac{y}{L} = U_2^0 + \overline{U}\sum P_m \cos m\pi\frac{y}{L} \quad . \tag{12.70}$$

The difference in the V-component across the screen is

$$V_1' - V_2' = -\overline{U}\sum(P_m + Q_m)\sin m\pi\frac{y}{L} \quad . \tag{12.71}$$

(**There is some missing algebra here.**) After some algebra, there is obtained

$$\left(\frac{U_2^0}{\overline{U}} - 1\right)\frac{\left(n + 1 + n\frac{C_n}{2}\right)}{n\frac{C_n}{2}} - \left(\frac{U_1^0}{\overline{U}} - 1\right)\frac{\left(n + 1 - \frac{C_n}{2}\right)}{n\frac{C_n}{2}} \tag{12.72}$$

$$= -\sum \alpha_m \cos m\pi\frac{y}{L} = 0 \tag{12.73}$$

where

$$\alpha_m = \frac{P_m}{n} + Q_m \quad . \tag{12.74}$$

Equation (12.73) has to be compared to equation 12.55, which can be written

$$\left(\frac{n-1}{n}\right)\overline{U}\tan\theta = -\sum \alpha_m \sin m\pi\frac{y}{L} \quad . \tag{12.75}$$

[4]Equation number not recorded

[5]Equation number not recorded

Equations (12.73) and (12.75) define two functions expressed as Fourier series. The series have the same coefficients, but one is in terms of $\sin n\pi\, y/L$ and the other is in terms of $\cos n\pi\, y/L$. There is a theorem due to Hardy that applies to this situation. Given

$$g(\theta) = \sum h_m \sin m\,\theta \tag{12.76}$$

$$g^*(\theta) = \sum h_m \cos m\,\theta \tag{12.77}$$

valid in the interval $0 < \theta < \pi$, it follows that

$$g^* = H(g),\ \ g = H^*(g^*) \tag{12.78}$$

where

$$H(g) = \frac{1}{2\pi}\int_0^{\pi} [g(\theta+t) - g(\theta-t)] \cot\frac{t}{2}\mathrm{d}t\ \ . \tag{12.79}$$

(*Look up the theorem and describe the symmetry. Did this theorem drive Elder's analysis, or was it discovered in time to save the analysis? Ask him?*) The theorem connects the screen angle to the velocities upstream and downstream. The analysis should reduce for $\theta = 0$ and $m = 1$ to the Taylor-Batchelor formula.

(There is an error in Elder's application of his analysis; see Livesey and Laws and others.)

Appendices