Topics in Shear Flow

Appendix A

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Appendix A

NOTE: This section appears to have been part of the chapter on the boundary layer (Chapter 4) but was pulled out as a separate file and labeled "Appendix A." It has no title of its own. The file name is "highre.tex," presumably for "high Reynolds number." –B. Coles

In 1962, I undertook an extensive survey of experimental data in low-speed turbulent boundary layers at constant pressure, in an attempt to identify a fully developed (standard, normal, ideal, equilibrium, asymptotic) state and determine its properties. My objective at that time was to establish a point of departure for a study of compressibility. My survey appeared as Appendix A of a RAND report (COLES 1962), but was never published outside of the subliterature. Because the work is not readily accessible, I will summarize here my methods and conclusions. I set out to test a large number of mean-velocity profiles for their consistency with the momentum-integral equation and the momentum-defect law, which is to say the departure of the outer part of the profile from the logarithmic law of the wall. The next four figures are copied from Appendix A to show the test method. Given a profile and a value for ν, I first determined a value for u_τ in equation (xxx)[1] that would put one point belonging to the hypothetical log region on the straight line defined by the particular constants of the time, $\kappa = 0.41$ and $c = 5.0$, as shown in FIGURE A.1. I then drew a parallel straight line through the point of maximum departure of the profile from the log line. Great precision was not needed for this oper-

[1] Given as equation (A-1) on p. 54 of COLES 1962 Rand Report, available at http://www.rand.org/content/dam/rand/pubs/reports/2006/R403.pdf

PLACEHOLDER

Figure A.1: Caption for Figure with label 7.40 (figure on p. 54 of COLES 1962).

ation, and there was no formal curve-fitting. (Both the non-linear regression scheme of Levenberg and Marquardt (SECTION X) and the corner modification proposed by Sandham (SECTION Y) were still in the future.) The vertical distance between the two lines, labeled $\Delta u/u_\tau$, is a measure of the strength of the wake component of the profile. I found that this quantity, although is represents only about ten percent of u_∞ for flow at constant pressure, was distinguished by an almost exquisite sensitivity to the history and environment of each particular flow. This property in turn made possible not only a close classification of boundary-layer flows at constant pressure, but a refinement and rationalization of the similarity laws for the profile.

The upper part of FIGURE A.2 shows $\Delta u/u_\tau$ as a function of the local Reynolds number R_θ for flows that I classified as normal. The lower part of the figure compares two estimates of the surface friction τ_w for the same data; first, the momentum-integral result $\rho u_\infty^2\, d\theta/dx$, where θ is the momentum thickness, and second, the result from the graphical procedure for u_τ in the form ρu_τ^2. The agreement is generally within ten percent, and usually better. I take this agreement as strong evidence in favor of identifying u_τ with $(\tau_w/\rho)^{1/2}$.

By way of contrast, FIGURE A.3 shows the corresponding

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Figure A.2: Caption for Figure with label 7.41 (figures 10 and 11 on p. 56 of COLES 1962).

PLACEHOLDER

Figure A.3: Caption for Figure with label 7.43 (figures 12 and 13 on p. 58 of COLES 1962).

quantities for flows that I classified as abnormal. I noted as a significant point of technique that the normal data were obtained for the most part in closed wind tunnels, either on plates having blunt leading edges or some equally effective tripping means, or on tunnel walls having a long approach length. Some of the anomalous data were obtained in open-jet tunnels using models not equipped with adequate side plates. In view of the generally poor momentum balance in FIGURE A.3, I blamed the anomalies in these flows for the most part on three-dimensionality of the mean flow.

One major finding of the study, which perhaps should have been anticipated, was that $\Delta(u/u_\tau)$ decreases, and hence that the traditional defect law fails, as the Reynolds number R_θ decreases below a value of about 5000. In fact, the wake component in FIGURE A.2 disappears entirely and rather abruptly by the time R_θ has decreased to a value near 500. This behavior was present in all of the data, I therefore do not view it normal or not. The behavior might be viewed as a residual effect of transition, because I would not expect such a high degree of commonality in such a diverse population of data. A better hypothesis is that the flows are fully turbulent and in equilibrium, in the special sense that two characteristic scales δ and ν/u_τ are emerging and separating from each other as discrete parameters for turbulent flow near a wall. It is possible, although difficult to prove experimentally, that the constants in the log law are also evolving. If so, I doubt that the graphical classification scheme of FIGURE A.1 is seriously compromised.

In the RAND report I also looked at the strength of the wake component in the presence of high stream turbulence and in the flow downstream from very strong tripping devices. These effects will not be discussed here. (**Elsewhere?**) I turn instead to my second major finding, which was that the quantity $\Delta u/u_\tau$ seemed to decrease substantially at Reynolds numbers larger than the upper limit in FIGURES A.2–A.3. This second finding, if correct, signals a serious and perhaps fatal defect in the defect law.

In 1962, almost the only reliable data in low-speed boundary-layer flow at high Reynolds numbers were the careful and extensive measurements by SMITH and WALKER (1958), which offered themselves by default as definitive for Reynolds numbers R_θ from 15000 to 50000. I found no evidence that these measurements might be affected by pressure gradient, stream turbulence, or three-dimensionality. My conclusion at the time was that these data could only be questioned on some other ground. Failing this, the defect law is not valid at the high level of precision attempted in my survey.

Fortunately, there is other ground. Because the freestream velocities in the experiments by Smith and Walker reached 110 meters per second, I propose here to make one more test of these and certain other data, a test based on the premise that the apparent problem with the defect law may be solved by considering the effect of compressibility. I cannot recall why I did not test this hypothesis in my 1962 report, except that suitable descriptions of mean-velocity profiles, including my own failed description of 1962, were not part of the machinery of the time. In particular, in 1962 the proposal by VAN DRIEST (1951) was still a decade or more away from being generally accepted as the best available means for organizing the effects of compressibility. This proposal will be examined to what follows.

Standard methods exist for data processing in studies of turbulent boundary layers in compressible fluids. The fluid is invariably assumed to be a perfect gas, with equation of state

$$p = \rho R T \ . \tag{A.1}$$

The two specific heats c_p and c_v are taken as constants, as are their combinations $R = c_p - c_v$ and $\gamma = c_p/c_v$. The instrument of choice is the impact or total-pressure tube. Almost without exception, each measurement of velocity begins with the local Mach number M, which is inferred from the ratio of impact pressure to static pressure. If the flow is supersonic, the operational equation (LIEPMANN and ROSHKO 1957) is

$$\frac{p_0'}{p} = \frac{\left(\dfrac{\gamma+1}{2}\, M^2\right)^{\frac{\gamma}{\gamma-1}}}{\left(\dfrac{2\gamma}{\gamma+1}\, M^2 - \dfrac{\gamma-1}{\gamma+1}\right)^{\frac{1}{\gamma-1}}} \ , \tag{A.2}$$

where the prime denotes probe impact pressure behind a normal shock wave, and p is the static pressure at the probe entrance in the absence of the probe. The latter pressure is usually measured at an adjacent wall or is computed by assuming isentropic expansion to a Mach number M_∞ in the free stream. In either case the static pressure is taken as constant through the boundary layer. If the flow is subsonic and thus free of shock waves, equation (A.2) is replaced by

$$\frac{p_0}{p} = \left(1 + \frac{\gamma - 1}{2}\, M^2\right)^{\frac{\gamma}{\gamma-1}} \quad . \tag{A.3}$$

Finally, if the Mach number is much less than unity, the last equation reduces to Bernoulli's integral,

$$p_0 = p + \frac{1}{2}\rho u^2 \quad . \tag{A.4}$$

The last two equations are the ones plotted in FIGURE 1.1 of the introduction. I will assume in what follows that accuracy in measurement of p and p_0' and hence of the local Mach number M have been accurately measured. I will also ignore for the moment any corrections for effects of turbulence or mean-flow gradients on the probe readings.

The state equation (A.1) and the condition $p = p_w = p_\infty$ reduce the number of independent thermodynamic variables from three to one. One thermodynamic quantity must therefore be measured or assumed. The usual choice is the local stagnation temperature T_0 or the local static temperature T. The definition of T_0 for a perfect gas with $M = u/a = u/(\gamma RT)^{1/2}$ is

$$T_0 = T + \frac{u^2}{2c_p} = T\left[1 + \left(\frac{\gamma - 1}{2}\right) M^2\right] \quad . \tag{A.5}$$

If the flow is laminar, there exist under certain conditions an energy integral, which is to say a relation between temperature and velocity, or more accurately between enthalpy and kinetic energy, that satisfies the equations of motion and is valid independent of position in the flow. Suppose that the wall temperature T_w is constant and the Prandtl number

$$Pr = \frac{c_p \mu}{k} \tag{A.6}$$

is equal to unity. With the further restriction that there is no heat transfer, a primitive energy integral, first found by BUSEMANN (1931), is

$$T_0 = T + \frac{u^2}{2c_p} = \text{ constant } = T_{0_\infty} = T_w \ . \tag{A.7}$$

There is no restriction on pressure gradient. A different integral obtains if there is heat transfer, still with $T_w =$ constant and $Pr = 1$, but now with the restriction of constant pressure. For these conditions a generalization of Busemann's energy integral was found by CROCCO (1932), and independently by BUSEMANN (1935) (**check**);

$$T + bu + \frac{u^2}{2c_p} = \text{ constant } = T_w \ , \tag{A.8}$$

where the parameter b will be shown shortly to be a measure of heat transfer at the wall.

The relations just given are often summarily adopted as a model for turbulent flow. FERNHOLZ and FINLEY (1980) recommend a relationship between temperature and velocity originally proposed by WALZ (1966) (**check**),

$$\frac{T}{T_\infty} = A + B\frac{u}{u_\infty} + C\frac{u^2}{u_\infty^2} \ . \tag{A.9}$$

This expression is formally identical with the Crocco-Busemann integral (A.8) but has no physical basis in turbulent flow except that it is capable of satisfying the two boundary conditions

$$T = T_w \text{ at } u = 0 \ , \quad T = T_\infty \text{ at } u = u_\infty \ . \tag{A.10}$$

It follows that

$$A = \frac{T_w}{T_\infty} \ , \quad B + C = 1 - \frac{T_w}{T_\infty} \ . \tag{A.11}$$

The derivative of equation (A.9) at the wall is

$$\frac{1}{T_\infty}\left(\frac{\partial T}{\partial y}\right)_w = \frac{B}{u_\infty}\left(\frac{\partial u}{\partial y}\right)_w \quad . \tag{A.12}$$

Hence $B = 0$ corresponds to adiabatic flow at constant wall temperature. The discussion hereafter will be limited to this special case of zero heat transfer. (**A reference**?)

The energy integral (A.9) with $B = 0$ becomes

$$\frac{T}{T_\infty} = \frac{T_w}{T_\infty} + \left(1 - \frac{T_w}{T_\infty}\right)\frac{u^2}{u_\infty^2} \quad . \tag{A.13}$$

In most experiments in compressible fluids, as already pointed out, the measured quantity is the Mach number, and it is necessary to prepare equation (A.13) for this situation. For a perfect gas, the temperature, velocity, and Mach number are related by the definition of M;

$$\rho u^2 = \gamma p M^2 \tag{A.14}$$

Whether or not the pressure p depends only on x, the ratio p/ρ is always equal to RT. Hence a suitable normalized form of equation (A.14) is

$$\frac{u^2}{u_\infty^2} = \frac{T}{T_\infty}\frac{M^2}{M_\infty^2} \quad . \tag{A.15}$$

When this equation is used to eliminate the velocity in equation (A.13), the result is

$$\frac{T_w}{T} = 1 - \left(1 - \frac{T_w}{T_\infty}\right)\frac{M^2}{M_\infty^2} \quad . \tag{A.16}$$

For the sake of symmetry, the energy integral (A.13) can be rewritten as

$$\frac{T}{T_w} = 1 + \left(\frac{T_\infty}{T_w} - 1\right)\frac{u^2}{u_\infty^2} \quad . \tag{A.17}$$

Within this formulation, the Mach number determines the temperature, and the temperature determines the velocity (and the density). In dimensional form, the argument assumes a knowledge of T_w and one of the three parameters T_{0_∞}, T_∞, u_∞, together with use of the rigorous definitions (A.5) and (A.14).

To recapitulate, the preceding discussion refers to impact-probe measurements of the Mach-number profile in adiabatic boundary layers. Nothing in the discussion requires that the flow be specified as laminar or turbulent. If the flow is laminar and adiabatic and the Prandtl number is unity, the relationships are rigorous within the boundary-layer approximation ($\partial p/\partial y = 0$) and the usual considerations of experimental accuracy. Moreover, the wall temperature is equal to the free-stream stagnation temperature, according to equation (A.7). It is a difficulty, readily overcome by numerical means if the flow is laminar, that the Prandtl number for common gases is nearly constant but usually at a value near 0.7 rather than unity. This difficulty is usually expressed by introduction of a recovery factor r, defined as

$$r = \frac{T_r - T_\infty}{T_{0_\infty} - T_\infty} \ , \tag{A.18}$$

where T_r, the recovery temperature, is the wall temperature T_w when there is no heat transfer. In practice, the recovery factor is close to unity, although it is different for laminar and turbulent flows.

(Paragraph on high thermal price, case $T_w \to T_{0_\infty}$ if $Pr > 1$, etc.)

Experimenters may sometimes have direct access to the velocity in high-speed flow through laser Doppler velocimetry or particle-image velocimetry or the like; all these techniques face formidable difficulties in supersonic flow. In such cases, the temperature can be estimated from equation (A.17) in order to determine the density, which is required for any test of momentum balance.

(Section on experimental T or T_0 profile.)

There is not much evidence that the energy ansatz (A.9) is a real improvement over other possible forms, such as the form $T_0 = T_{0_\infty}$, commonly used, or the form $T_0 = T_w$, both of which have no less claim to validity than the form (A.9). To see the effect of errors in T_0, suppose first that p, M, and T_0 are known exactly. Hence so is T, from $T_0/T = 1 + (\gamma - 1)M^2/2$. So is u, from $u/(\gamma RT)^{1/2} = M$. So is ρ, from $\rho u^2 = \gamma p M^2$. If M is known but T_0 is estimated rather than measured, with a local relative error of ϵ in T_0, then the local relative errors in T, u, ρ, and ρu are ϵ, $\epsilon/2$, $-\epsilon$, and $-\epsilon/2$, respectively.

Van Driest. The most widely accepted scheme for comparing data for $M \neq 0$ with data for $M = 0$ is the scheme now called Van Driest II. It has been endorsed by Fernholz and Finley in their massive survey of data for $M \neq 0$, and by numerous other authors (**Name some**.). These authors sometimes refer to the scheme as a transformation, but I prefer to reserve this term for a relationship based on the equations of motion, and to use the term "mapping" for a relationship based on variables only. The compressibility mapping proposed by VAN DRIEST (1951) begins with the mixing-length expression

$$\tau = \tau_w = \rho \ell^2 \left(\frac{\mathrm{d}u}{\mathrm{d}y} \right)^2 \quad , \tag{A.19}$$

together with Prandtl's hypothesis

$$\ell = \kappa y \ , \tag{A.20}$$

and arrives at the ansatz

$$\rho^{1/2} \frac{\mathrm{d}u}{\mathrm{d}y} = \frac{{\tau_w}^{1/2}}{\kappa y} \quad . \tag{A.21}$$

The appearance of the combination $(\rho^{1/2}\mathrm{d}u)$ suggests that the physical velocity u can be replaced by an effective velocity u^*, say, defined by $\mathrm{d}u^* \sim \rho^{1/2}\mathrm{d}u$, or by its definite integral,

$$u^* = \int_0^u \left(\frac{\rho}{\rho^*} \right)^{1/2} \mathrm{d}u \ , \tag{A.22}$$

where ρ^* is a constant reference density included for dimensional reasons. It is apparent that the Van Driest mapping (A.22) has the effect of rotating the profile in the counter-clockwise direction in the usual semi-logarithmic coordinates, since ρ is small where u is small, near the wall, and ρ is large where u is large, near the free stream. This is the property that makes the scheme an attractive device in any attempt to restore the defect law to respectability, as noted at the beginning of this section.

An alternative expression for the mixing length is Karman's similarity hypothesis,

$$l = x \frac{\mathrm{d}u/\mathrm{d}y}{\mathrm{d}^2u/\mathrm{d}y^2} \quad . \tag{A.23}$$

This form was proposed simultaneously and independently by Wilson (1950), and both Wilson and Van Driest developed their hypothesis into formulas for skin friction as a function of Mach number and Reynolds number (**check**). These two authors were not alone; several other authors, working independently, used the same or similar methods and approximations to complete their analyses. The situation in 1953 was surveyed by COLES (1953) and by CHAPMAN and KESTER (1953), and the various proposals to that time were collected by Chapman and Kester in a celebrated figure that is reproduced here as FIGURE 4.xx.[2] There were few competent measurements in 1951, and it should not be surprising that the various predictions filled a plot of C_f against M almost uniformly densely.

Chapman and Kester used the designations Van Driest I and II and for the Prandtl and Karman forms, although they misplaced the second one in their figure. A later survey by SPALDING and CHI (1964) used the same designations, but reversed taking Karman as I and Van Driest as II. In fact, Van Driest and Wilson were fully informed very early about their respective contributions, according to the proceedings of a Navy conference in 1951. Van Driest in 1956 described both models, Prandtl first and Karman second, without attributing either. The confusion was made permanent by Spalding and Chi, and made permanent by FERNHOLZ and FINLEY (1978?) (**Check all this.**) It is only necessary to know that Van Driest's first and only analysis was based on the Prandtl model and is now universally referred to as Van Driest II. Details follow.

The definition (A.22) is readily integrated in closed form for the energy integral (**??**) [3] for adiabatic flow, putting T_∞/T for ρ/ρ_∞.

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[2]Unclear reference.

[3]Unclear equation reference.

The result is the Van Driest mapping for velocity;

$$m \left(\frac{\rho^*}{\rho_w}\right)^{1/2} \frac{u^*}{u_\infty} = \sin^{-1}\left(m \frac{u}{u_\infty}\right) , \tag{A.24}$$

where m, defined by (**check**)

$$m^2 = \frac{T_w - T_\infty}{T_w} = \frac{r\left(\dfrac{\gamma-1}{2}\right) M_\infty^2}{1 + r\left(\dfrac{\gamma-1}{2}\right) M_\infty^2} , \tag{A.25}$$

lies between 0 and 1 and in any case requires $T_w > T_\infty$. Given the specified energy integral, the dimensionless mean-velocity profile u/u_∞ can be replaced at each value of y by

$$\frac{u^*}{u^*_\infty} = \frac{\sin^{-1}\left(m \dfrac{u}{u_\infty}\right)}{\sin^{-1}(m)} , \tag{A.26}$$

without regard to the definition of ρ^*. Incidentally, the $\sin^{-1}$ operator is the unique signature of the Van Driest mapping for any energy law that has T quadratic in u.

The mapping defined by equation (A.26) is not based on any observable process or mechanism. Whether or not the accepted similarity laws remain valid for the mean velocity profile after the Van Driest mapping is a question to be settled experimentally. Before the evidence can be tested, the friction velocity u_τ and some other constants have to be redefined in a plausible way for the case of variable density.

For this purpose, integrate the mixing-length equation (A.21) formally, with $\mathrm{d}u = (\rho^*/\rho)^{1/2}\, \mathrm{d}u^*$, to obtain the modified law of the wall,

$$u^* = \frac{1}{\kappa}\left(\frac{\tau_w}{\rho^*}\right)^{1/2} \ln\left(\frac{y}{y^*}\right) + \text{ constant} , \tag{A.27}$$

where y^* is a constant reference length also included for dimensional reasons. Equation (A.27) is typical of mixing-length formulas in that it is at best an unclear description of a fragment of the mean-velocity profile. The choices for ρ^* and y^* and the value of the constant in equation (A.27) are customarily resolved by an extension of the mapping to the wall, taking care to be consistent with the earlier treatment of an incompressible fluid in SECTION X. First, write the functional dependence in the profile formula (A.27) in dimensionless form as

$$\left(\frac{\rho^*}{\tau_w}\right)^{1/2} u^* = f\left(\frac{y}{y^*}\right) \quad . \tag{A.28}$$

Near the wall, this becomes approximately

$$\left(\frac{\rho^*}{\tau_w}\right)^{1/2} u^* = \frac{y}{y^*} \quad . \tag{A.29}$$

As u and y approach zero, the definition (A.22) and its integral (A.24) both lead to

$$u^* = \left(\frac{\rho_w}{\rho^*}\right)^{1/2} u \quad . \tag{A.30}$$

Finally, the requirement of Newtonian friction at the wall implies, to the same approximation,

$$\tau_w = \mu_w \, \frac{u}{y} \quad . \tag{A.31}$$

When u^* is eliminated between equations (A.29) and (A.30), and u/y is eliminated between this result and equation (A.31), the constant of integration y^* emerges in terms of well-defined quantities,

$$y^* = \nu_w \left(\frac{\rho_w}{\tau_w}\right)^{1/2} \quad . \tag{A.32}$$

At the same time, a generalized friction velocity u_τ emerges as

$$u_\tau = \left(\frac{\tau_w}{\rho_w}\right)^{1/2} \quad , \tag{A.33}$$

and equation (A.27) becomes

$$\left(\frac{\rho^*}{\rho_w}\right)^{1/2} \frac{u^*}{u_\tau} = \frac{1}{\kappa} \, \ln \left(\frac{y u_\tau}{\nu_w}\right) + \text{constant} \quad . \tag{A.34}$$

This form and equation (A.24) strongly suggest, although they do not require, defining ρ^* by

$$\rho^* = \rho_w \ , \tag{A.35}$$

whereupon

$$u^* = u \tag{A.36}$$

very near the wall.

As usual, this reasoning for $y \to 0$ is not part of the mixing-length argument, which applies only for fully turbulent flow outside the sublayer. Given the choices (A.33) and (A.35), then in a usual notation equation (A.27) becomes

$$u^+ = \frac{1}{\kappa} \ln y^+ + c \ , \tag{A.37}$$

where now

$$u^+ = \frac{u^*}{u_\tau} \ , \qquad y^+ = \frac{y u_\tau}{\nu_w} \ , \qquad \rho_w u_\tau^2 = \tau_w \ , \tag{A.38}$$

and

$$m \frac{u^*}{u_\infty} = \sin^{-1}\left(m \frac{u}{u_\infty} \right) \ . \tag{A.39}$$

The fragile derivation just given, with Prandtl's equation (A.20) for ℓ, is commonly referred to as Van Driest II. The choice for ρ^*, u_τ, and y^* is important because it controls the dependence of the generalized κ and c on M_∞ and γ. What is wanted is the particular choice that minimizes this dependence. There is substantial evidence, for example, in papers by FENTER and STALMACH (1957), ROTTA (1960), MOORE and HARKNESS (1964), MAISE and MCDONALD (1968), MICHEL, QUEMART, and ELENA (1969), DANBERG (1971), SQUIRE (1971), and FERNHOLZ (1976) that use of wall quantities as in equations (A.37)–(A.39) is very nearly optimum from this point of view, at least for adiabatic flow at constant pressure at Mach numbers up to perhaps 5.

Most of these authors have also gone beyond the mixing-length argument to consider a more general fit to a defect law or to a combined wall-wake formulation of the mean profile, in the manner adopted by COLES (1968) for low-speed flow; i.e., a fit to

$$u^+ = \frac{1}{\kappa} \ln y^+ + c + 2 \frac{\Pi}{\kappa} \sin^2 \eta \quad , \tag{A.40}$$

where

$$\eta = \frac{\pi}{2} \frac{y}{\delta} \quad . \tag{A.41}$$

The present method for determining the strength of the wake component is the third in an evolving series. In 1962 the fit of the mean-velocity profile used only one point. In 1968 I tried to involve a fit of the entire profile to equation (A.40) but had to finesse the problem of a misfit near $y = \delta$ by omitting data for y/δ greater than some threshold value noted in the tabulation in the "Young person's guide." In this monograph, I have made room for the omitted data by using the Sandham scheme (SECTION 4.9.3) for rounding the profile near $y = \delta$. In addition, the constants κ and c now have new values based on Zagarola's pipe measurements (SECTION 2.5.7). The constants κ and c are here given their new incompressible values, $\kappa = 0.435$ and $c = 6.10$. The parameters u_τ, δ, and n (need equation) are then determined by a three-parameter least-squares fit of the experimental data to equation (A.40), after eliminating Π temporarily with the aid of the constraint imposed by the local friction law,

$$u^+_\infty = \frac{1}{\kappa} \ln \delta^+ + c + 2 \frac{\Pi}{\kappa} \quad . \tag{A.42}$$

The quality of Van Driest scaling, when universal constant values are assumed for κ and c, can be tested in different ways. One test is to compare values inferred for the local friction coefficient (**does this make sense**?)

$$C_f = 2 \frac{\rho_w}{\rho_\infty} \left(\frac{u_\tau}{u_\infty} \right)^2 \quad , \tag{A.43}$$

with values obtained by other means. A second test is to compare values obtained for the profile parameter Π with corresponding values for low-speed flow. This second comparison will be made first and will lead to the conclusion is that there is very little effect of compressibility on the shape of the mean-velocity profile in Van Driest II coordinates, at least for Mach numbers up to about 3.

Such tests are not new. The first paper to compare various mappings of C_f was the extensive survey by SPALDING and CHI (1964). Tests were also carried out by JACKSON et al (1965), PETERSEN (19xx), MILES and KIM (19xx), DANBERG (1971), HOPKINS and INOUE (1971), and WINTER and GAUDET (1973). These efforts are not necessarily redundant, since they differed in their choice of data, viscosity law, and handling of temperature. The consensus is that the Van Driest scheme is at least as good as any other when taken as a high-level technical application.

Full profile fits and reports of wake strength have been carried out by (Winter, Gaudet, others). Among the most satisfactory studies to my mind is one by D. Collins at JPL, for which I was consultant and eventually co-author (COLLINS, COLES, and HICKS 1978). The invariance of the defect law under Van Driest mapping was strongly supported by these data for Mach numbers up to 2.2.

FERNHOLZ and FINLEY (1977) in their massive catalog of boundary-layer measurements involving compressibility, did not include the operation of curve fitting for the mean velocity profile. In a second volume (1980), they provided numerous plots in Van Driest coordinates, but still without a fitting operation. The survey has a large clerical component limited mainly to major issues such as effects of flow history and the validity of empirical energy integrals. I have relied very heavily on this survey in the new analysis that follows. (**Say how to get data**.)

I have used this scheme before in connection with work by D. Collins at JPL, in which I participated as consultant and coauthor. The objective was to document a set of flows for LDV measurements of $\overline{u'v'}$ in supersonic flow, a quantity that was then under a cloud, and perhaps still is. The next few paragraphs are borrowed from that report.

Values for viscosity are obtained from the Sutherland viscosity law,

$$\frac{\mu}{\mu_r} = \left(\frac{T_r + S}{T + S}\right)\left(\frac{T}{T_r}\right)^{3/2} \quad , \tag{A.44}$$

where $T_r = 291.75$ K, $S = 110$ K, and $\mu_r = 1.827 \times 10^{-4}$ gm/cm-sec.

Integral thicknesses for the boundary layer are computed from

$$\delta^* = \int_0^\delta \left(1 - \frac{\rho u}{\rho_\infty u_\infty}\right) dy \quad , \tag{A.45}$$

and

$$\theta = \int_0^\delta \frac{\rho u}{\rho_\infty u_\infty}\left(1 - \frac{u}{u_\infty}\right) dy \quad . \tag{A.46}$$

The boundary-layer form parameter H is defined as

$$H = \frac{\delta^*}{\theta} \quad . \tag{A.47}$$

For two-dimensional mean flow, the surface friction can be obtained from von Kármán's momentum-integral equation, (**What is cap** P?)

$$C_f = 2\frac{d\theta}{dx} - 2\big(2 + H - M_\infty^2\big)\,\frac{\theta}{\gamma M_\infty^2}\,\frac{1}{P}\,\frac{dP}{dx} \,. \tag{A.48}$$

The accuracy of equation (A.48) is expected to be low, primarily because of difficulty in differentiating experimental data for $\theta(x)$ and $u_\infty(x)$ (see Table A3 of the Appendix).[4] For the JPL measurements, the second term in equation (A.48) is at most 3 percent of the first term, and is uncertain by a comparable amount. Hence this term has been discarded. Values for $C_f = 2\,d\theta/dx$ are listed in Table 3, which compares values obtained for C_f by this and several other methods.

Note that these measurements at JPL were the last hurrah of the 20-inch tunnel before it was dismantled and moved to Langley in 1980. Hence there was ample time to do the work well. Note also that the boundary layer experiments reported in my thesis at Caltech were almost the first tests conducted in this tunnel in 1951, 29 years earlier.

[4]Table A3 or Table 3 mentioned later have not been found.