

# Topics in Shear Flow

## Front Matter

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Assembled and Edited by

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The current maintainers of this work are Kenneth S. Coles (kcoles@iup.edu) and Betsy Coles (betsycoles@gmail.com)

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# Preface

## AUTHOR'S DRAFT PREFACE

*This section was found in an early draft of this monograph. While it was not revised after the author added much of the content of this work, we include it as it conveys his motivation and approach.*

– K. Coles

Even limited exposure to industrial problems involving fluid mechanics shows that there is a pervasive need for a comprehensive, critical compilation of empirical knowledge in the field of turbulent shear flow. The enormous experimental literature published during the last fifty years, literally thousands of papers, is still mostly undigested. This literature is generated in governmental and industrial research laboratories and in university departments of aeronautical, chemical, civil, environmental, and mechanical engineering at institutions all over the world. There are also important applications in ocean engineering, vehicle and building aerodynamics, dynamic meteorology and physical oceanography, and even in planetary physics and astrophysics.

With industrial problems in mind, Anatol Roshko and I have developed over the past 20 years a graduate course in aeronautics at Caltech called “Technical Fluid Mechanics.” The emphasis is strongly on turbulent shear flow. The course is normally given every year and is well attended. It draws graduate students from several engineering options other than aeronautics and is also open to se-

lected undergraduates who have had at least a first course in fluid mechanics.

For various reasons, none of the existing monographs that might be consulted by engineers or used as textbooks in advanced engineering courses on turbulent flow is a satisfactory reference for such a course. The most common defect is a lack of adequate experimental content. When I took the decision in 1986 to write this monograph, much of the lecture material for the course had already been laboriously collected from the experimental literature, although without much critical compilation or comparison of data from different sources. Nevertheless, a large part of the necessary bibliographical preparation and interpretation of data had been carried out, and a solid foundation for a monograph on technical fluid mechanics did exist. To produce this book has involved a substantial commitment for several years, once adequate support was obtained for assembling and collating experimental data.

The fundamental premise for the book is that the only reliable information about turbulent flow is experimental information. This varies greatly in quality and completeness, and needs to be carefully screened. Some additional premises will be self-evident in the text. First, it is advisable to understand thoroughly the laminar version of a particular flow, because some conceptual problems are not peculiar to turbulent flow; e.g., the third boundary condition for the mixing layer, or the integral invariant for the wall jet. Second, the most powerful organizing principle so far available for both laminar and turbulent flow is the principle of similarity. Third, the most important phenomenological concept for many turbulent flows is the concept of entrainment. The need of the user is often likely to be for hard numbers and practical insights, rather than for elegance. I have therefore made some use of mixing-length and eddy-viscosity ideas, and even power-law methods, as primitive links between fundamental and technical problems.

Each chapter of the book deals with one of the classical shear flows (mixing layers, jets, plumes, wakes, boundary layers, pipe flow, and so on) and with its ramifications, or with an important technical problem such as flow management. Wherever possible, the presen-

tation is intended to suggest how various flow problems might be connected analytically and experimentally one to another, using as far as possible a consistent notation and a consistent level of rigor and detail.

In practically all cases, I have organized and presented the data in the language of the Reynolds-averaged equations of motion, since there is general agreement that these equations, although incomplete, are at least correct. The important areas of turbulence modeling and numerical simulation are served indirectly, by extensive documentation of the experimental facts that modeling and simulation attempt to reproduce. I have made every effort to ensure that the material of the monograph will not quickly become dated. The rapidly evolving subject of coherent structure is therefore discussed only in cases where the Reynolds-averaged equations clearly do not suffice for describing the phenomenology of particular turbulent flows. An example is the sublayer of a turbulent flow near a smooth wall.

Finally, I found it essential to limit the objectives of the book. Combustion is not discussed. Neither is the very large subject of compressibility, including aerodynamic noise. Transition is viewed primarily from the turbulent side, with the elements of randomness and three-dimensionality already present. Effects of body forces associated with buoyancy or curvature are discussed, but not in the context of the classical Benard or Couette flows. Grid turbulence is mentioned mainly in connection with flow management. Instrumentation and experimental techniques are discussed mainly in connection with questions of good experimental practice.

The list of references cited in this monograph is extensive but not exhaustive. The list is most complete when my objective is to assemble and compare the experimental evidence on some special topic. In the face of a large volume of material, my task has been made easier by the evolution of the scientific literature from archival journals to abstract journals to survey and review volumes. I have made heavy use of Science Citation Index and of the surveys that are a common component of Ph.D. theses. I have also used the series of unpublished reports prepared by various groups of experts for the second (1980-1981) Stanford conference on computation of turbulent

flow. Finally, I have taken the time to study most of the original papers that have laid the foundations of fluid mechanics since the middle of the 19th century. This study changed my ideas about the way that classical contributions to mechanics have been introduced and developed. It also influenced this monograph in a way that I hope will be seen as respect for the uses of the past.

In a real sense, the part of fluid mechanics treated in this monograph is a mature subject. A histogram in time constructed for the references cited here shows that paper production is level or decreasing. A reasonable inference is that the classical turbulent shear flows are thought, rightly or wrongly, to be under good control. Activity is shifting to study of coherent structure and to exploitation of the power of large computing machines. Another area developing slowly but promising important contributions is the relationship of turbulence to dynamical systems theory. I hope that this monograph will be useful in support of these efforts as well as in solution of engineering problems.

D.C.  
(July 1995)

## EDITOR'S NOTE

The pages that follow represent an attempt to reconstruct the unfinished book left by Donald Coles at his death in May of 2013. Combining a large number of computer files, drafts and printouts of figures, and various scraps of manuscript and lecture notes has proven challenging. In places internal clues indicate the intent of the author the last time he reviewed or revised a section. The editing was done between late 2013 and the present by myself and Betsy Coles, who also assembled and updated the computer files.

Don Coles often said he wanted to write this work, and over twenty years ago he began writing to colleagues requesting original experimental data so he could replot them in a uniform way. The task of creating the plots for figures, with the help of assistants he

hired, occupied the great majority of time and effort in the author's later years. His intention to complete the figures before turning his attention to revising the respective chapters accounts for the fragmentary state of parts of the text.

The working title was "Topics in turbulent shear flow." We shortened this title at the suggestion of Prof. Anatol Roshko, who pointed out that the author treats laminar flow, typically at length, before considering corresponding turbulent flows. We did not track down many of the references to unspecified sections or figures elsewhere in the work; some were never created or have not been found.

The manuscript used arbitrary chapter numbers. Some references (e.g., to missing figures) use these old numbers rather than the consecutive chapter numbers of this edition. The author's typographic conventions, which we have followed where possible, include:

**CAPITALS:** Names of cited or referenced authors; numbered figures and tables; cited sections of this work.

**Boldface:** Notes to self, such as items to check or add.

*Italic:* Early draft or tentative material; longer notes about topics to be covered or material to be included.

Clearly the work is incomplete. Some figures and text may yet turn up in paper files, amounting to 50 to 75 cubic feet, that survive. While it is my intent to survey these, it does not make sense to delay publication of this work in the name of what will be a lengthy process that may add little. Similarly, we have not attempted to reconstruct a bibliography of the numerous literature references. We would be happy to hear from anyone who cares to compile any of these or who can suggest errors, omissions, or possible alternate readings of the text to include in a future edition. In the words of Anatol Roshko, "It pained [Don] to see anything not done absolutely as well as it could possibly be done." He would be frustrated that this work appears in less than complete and correct form. Nevertheless, we suspect it still contains much that may be useful and chose to share what we have. It is time for others to pursue the ideas that lie herein. We simply ask that those who make use of this work credit it by citation in the usual and customary manner of scholars.

We owe thanks to many. Over the years Don Coles was assisted by a number of people. We do not have all their names, but they included Dr. Paul Schatzle, Dr. Misha Pesenson, Dr. Gregory Cardell, Jim Edberg, and Evan Coles-Harris. Prof. Emeritus Anatol Roshko (Caltech) and Prof. Hassan Nagib (Illinois Tech) gave helpful input and suggestions. While no dedication survives, we do not need that evidence to know to whom Don Coles would have dedicated this book: Ellen Coles, the lifelong companion who assisted him with everything he wrote.

Ken Coles  
Indiana, Pennsylvania  
November 2017