Topics in Shear Flow

Chapter 1 - Introduction

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Chapter 1

INTRODUCTION

1.1 Generalities

The subject of turbulent shear flow is not simply connected. Some organization is essential, and I have tried to arrange that material required in a particular discussion appears earlier in the text. Thus pipe flow is discussed first because it provides the best evidence for the existence and value of Karman's two constants in Prandtl's law of the wall. It might seem easier to begin with the simpler topic of free shear flows, such as the plane jet. However, there is then a difficulty with the natural progression to wall jets, impinging jets, and other topics that require experience with Karman's constants. The main advantage of pipe flow is that the magnitude of the wall friction in fully developed flow can be obtained unambiguously from the pressure gradient, although a preliminary study is needed to determine what conditions guarantee that a given pipe flow is axially symmetric and fully developed.

It is an accepted axiom in basic research on the classical turbulent shear flows, elegantly expressed by NARASIMHA (1984), that there exists in each case a unique equilibrium state that can be realized in different experiments and thus made the basis of a general synthesis of empirical knowledge. The equilibrium may be stationary (as in pipe or channel flow) or dynamic or developing (as in the plane jet, the boundary layer, and most other flows). This axiom will be tested repeatedly in various parts of this monograph, usually by an emphasis on similarity laws, an emphasis that generates its own problems. There are no widely accepted similarity laws for several important turbulent flows, including boundary-layer flows with compressibility, mass transfer, heat transfer, or lateral or longitudinal curvature. Other difficult cases are the wall jet and the three-dimensional boundary layer. Because most of these problems involve walls, some attention is paid to the issue of the behavior of various mean quantities in turbulent flow near a wall. For example, there was at one time some ambiguity in the literature about the leading term in the expansion for the Reynolds shearing stress $-\rho\overline{u'v'}$. This leading term has sometimes been identified as a term in y^3, and sometimes as a term in y^4 (see HINZE 19XX, p. 621; SPALDING 1981). In fact, a rigorous argument is analytically straightforward, and generalizations that take into account changes in coordinates and boundary conditions may eventually shed some light on the nature of the proper similarity variables for some of the flows just mentioned. Experimental evidence for the magnitude of the first few coefficients is unreliable, but these coefficients can sometimes be estimated from numerical work on solutions of the full Navier-Stokes equations for flow in channels and boundary layers. There is a clear and present need for detail here when choosing boundary conditions for large-eddy simulations.

It is remarkable that one of the ostensibly most difficult problems in fluid mechanics, the problem of surface roughness, should be in a relatively comfortable state. Missing are sound methods for characterizing roughness. It is also remarkable that an apparently unrelated problem, the flow of a dilute solution of a high-molecular-weight polymer, exhibits properties that might be more easily understood if there were such a thing as negative roughness. Although expectations are not high, the prospect of finding possible connections in the transport mechanisms for these two problems is certainly worth some effort.

A topic that involves the mechanisms of turbulence in an essential way is the problem of relaxation, especially from one classical flow to another. One example is the strong plane jet into a moving fluid ($\delta \sim x$), with a final state describable by a linearized analysis ($\delta \sim x^{1/2}$). The round jet into a moving fluid has an equivalent behavior. Another example is the turbulent boundary layer on a finite flat plate ($\delta \sim x^{4/5}$) which relaxes downstream from the trailing edge to a plane wake ($\delta \sim x^{1/2}$). Several experimental studies exist of pipe, channel, or boundary-layer flows during a smooth-rough or rough-smooth transition of the wall boundary condition. Another instructive case is relaxation of a rectangular wake or jet to a round one. Such flows often overshoot the final state at least once. Finally, there are several flows with initially variable density that relax toward constant density as mixing proceeds. These include jets into a different medium, as well as plumes with finite initial momentum. A global view of these problems may lead to useful inferences about characteristic scales in time or space. The list of issues mentioned here is not intended to be comprehensive, but only to suggest various approaches that may or may not be productive in the future, given the fact that insight cannot be programmed.

It is also relevant that papers on models and mechanisms of turbulence tend to cite a limited standard set of experimental papers (these papers are sometimes different from one discipline to another). Endorsement by repetition often fails to present the best evidence. A positive development is that survey papers on various topics in turbulent shear flow are an increasingly important component of the contemporary literature. In a smaller setting, such surveys are also a common ingredient in Ph.D. theses although most of these latter surveys are not as critical as they could be. In any event, all of this material is a valuable resource for this monograph.

Sources of information about turbulent flow exist on several levels. In decreasing order of authority, I distinguish

1. the laws of mechanics
2. expert measurements (or numerical simulations)
3. insight
4. peripheral vision
5. brute force.

In what follows I will emphasize the first three levels of information. By "the laws of mechanics" I mean the Navier-Stokes equations and their boundary conditions. The question of "expert measurements" is more subjective. In any survey of experimental data, it is necessary somehow to assign a degree of confidence to each particular measurement. If the measurement has been made many times by different observers, like the pressure drop in a pipe, this is fairly easy to do. But if the measurement has been made only a few times, or even only once, judgment has to be backed by experience. "Insight" tends to be rare. Instances occur at intervals typically measured in years. The problem is then to pick signal out of noise in the literature, and the turbulence community is reasonably effective at doing so. An example of "peripheral vision" is the use of power laws that are not intrinsic, such as in Bradshaw's treatment of equilibrium turbulent boundary layers. What I hope to establish in this book is a set of ideas, chosen according to criteria defined by evidence rather than by faith or tradition. I will try to avoid the last level, "brute force," except when I am obliged to live up to the claim that part of my purpose is to deal with technical problems.[1]

Two constraints dominate the whole subject of turbulent shear flow at the contemporary stage of development. One is the boundary-layer approximation and the other is the idea of Reynolds stresses. I will almost always be considering incompressible fluids in the sense that variations in density are caused by variations in temperature, not by high velocity, and are important mainly because of associated body forces in a gravitational field.

[1] *The following two paragraphs appeared at this point in the 1997 draft of this work:*

The notation of this monograph tends toward usage in aeronautical engineering. I have tried to use mnemonic notation where this is possible, and I have therefore avoided arbitrary use of Greek symbols except where these are firmly established in the literature.

The literature in Russian is not well represented, primarily because the text is usually terse, the figures small, and tables nonexistent. The Russian subliterature — institute reports and theses — is not accessible at all.

1.2 Analytical prologue

1.2.1 Definitions and identities

Effects of compressibility are deliberately not emphasized in this monograph. However, effects of buoyant body forces and of heat transfer at walls are considered, so that the density of the fluid cannot always be taken as constant. In particular, I want to comment on something called the BOUSSINESQ approximation (1903), which I believe is not always well presented in the engineering literature. I will therefore outline briefly the structure of the Navier-Stokes equations for a compressible fluid and consider the limiting form of these equations, first as the Mach number approaches zero, and then as the Froude number approaches zero.

The first part of the discussion, and the notation, are taken from the classical article by LAGERSTROM (1964, 1996) in Volume IV of the Princeton handbook series. Many details are omitted here that can be found in Lagerstrom's article. The main reason for this choice of model is that the compact notation of vector calculus, with an appropriate generalization to operations on tensors, allows easy manipulations whose results are independent of any particular coordinate system.

A number of definitions and identities will be used in this and later sections of this monograph. In what follows an arrow over a symbol indicates a vector, and an underline indicates a tensor.

Some definitions from vector geometry and vector calculus, with generalizations, include the divergence of a vector,

$$\iiint \mathrm{div}\ \vec{a}\,\mathrm{d}V = \iint \vec{a}\cdot\vec{n}\,\mathrm{d}S\ \ , \tag{1.1}$$

where V is a stationary control volume bounded by a surface S, and $\vec{n}$ is the unit outward normal.

The divergence of a tensor is defined similarly,

$$\iiint \mathrm{div}\ \underline{A}\,\mathrm{d}V = \iint \underline{A}\vec{n}\,\mathrm{d}S\ \ , \tag{1.2}$$

where $\underline{A}\vec{n}$ is a multiplication defined by $(\underline{A}\vec{n})_i = A_{ij}n_j$.

The gradient of a scalar is

$$\text{grad } \alpha \cdot \mathrm{d}\vec{x} = \mathrm{d}\alpha \tag{1.3}$$

and of a vector is

$$(\underline{\text{grad } \vec{a}}\,)\, \mathrm{d}\vec{x} = \mathrm{d}\vec{a} \ \ . \tag{1.4}$$

The dyadic product of two vectors is

$$(\underline{\vec{a} \circ \vec{b}}\,)\vec{c} = \vec{a}(\vec{b} \cdot \vec{c}\,) \qquad \vec{c} \text{ arbitrary} \ \ . \tag{1.5}$$

The deformation tensor is symmetric;

$$\underline{\text{def } \vec{a}} = \underline{\text{grad } \vec{a}} + (\underline{\text{grad } \vec{a}}\,)^* \ \ , \tag{1.6}$$

where $*$ means the transpose. The corresponding antisymmetric tensor defines the curl operator;

$$(\text{curl } \vec{a}) \times \vec{b} = \left[\,\underline{\text{grad } \vec{a}} - (\underline{\text{grad } \vec{a}})^*\,\right]\vec{b}\,, \qquad \vec{b} \text{ arbitrary} \ \ . \tag{1.7}$$

The substantial derivative of a scalar is

$$\frac{D\alpha}{Dt} = \frac{\partial \alpha}{\partial t} + \text{grad } \alpha \cdot \vec{u} \tag{1.8}$$

and of a vector is

$$\frac{D\vec{a}}{Dt} = \frac{\partial \vec{a}}{\partial t} + (\underline{\text{grad } \vec{a}}\,)\,\vec{u} \ \ . \tag{1.9}$$

The identity tensor $\underline{I}$ is defined by

$$\underline{I}\vec{a} = \vec{a} \tag{1.10}$$

and the scalar product of two tensors by

$$\underline{A} \cdot \underline{B} = \sum_{i,j} A_{ij}\, B_{ij} \ \ . \tag{1.11}$$

Various identities are also useful. These are written here in a form independent of the choice of coordinates, and are easily verified in any convenient orthogonal coordinate system, say rectangular;

$$\text{div } (\alpha\vec{a}\,) = \alpha \text{ div } \vec{a} + \vec{a} \cdot \text{grad } \alpha \ \ ; \tag{1.12}$$

$$\text{div } (\alpha \underline{I}) = \text{grad } \alpha \;\; ; \tag{1.13}$$

$$\text{div curl } \vec{a} = 0 \;\; ; \tag{1.14}$$

$$\text{div } (\underline{\vec{a} \circ \vec{b}}\,) = (\underline{\text{grad } \vec{a}}\,)\, \vec{b} + (\text{div } \vec{b})\, \vec{a} \;\; ; \tag{1.15}$$

$$\text{div } (\underline{A}\vec{a}\,) = (\text{div}\underline{A}^*) \cdot \vec{a} \; + \underline{A}^* \cdot \underline{\text{grad } \vec{a}} \;\; ; \tag{1.16}$$

$$\text{div } (\vec{a} \times \vec{b}\,) = \vec{a} \cdot \text{curl } \vec{b} - \vec{b} \cdot \text{curl } \vec{a} \;\; ; \tag{1.17}$$

$$\text{div } (\underline{\text{grad } \vec{a}}\,)^* = \text{grad } (\text{div } \vec{a}) \;\; ; \tag{1.18}$$

$$\text{curl } (\text{grad } \alpha) = 0 \;\; ; \tag{1.19}$$

$$\text{curl } (\alpha \vec{a}) = \alpha \text{ curl } \vec{a} + (\text{grad } \alpha) \times \vec{a} \;\; ; \tag{1.20}$$

$$\text{curl } (\vec{a} \times \vec{b}\,) = \text{div } \left[(\underline{\vec{a} \circ \vec{b}}\,) - (\,\underline{\vec{b} \circ \vec{a}}) \right] \;\; ; \tag{1.21}$$

$$(\underline{A}\vec{a}) \cdot \vec{b} = (\underline{A}\vec{b}) \cdot \vec{a} \;\; , \qquad \underline{A} \text{ symmetric } \;\; ; \tag{1.22}$$

$$\vec{a} \times (\,\vec{b} \times \vec{c}\,) = \vec{b}\,(\vec{a} \cdot \vec{c}\,) - \vec{c}\,(\vec{a} \cdot \vec{b}\,) \;\; ; \tag{1.23}$$

$$(\,\underline{\text{grad } \vec{a}}\,)\, \vec{a} = \text{grad } \frac{a^2}{2} + (\text{curl } \vec{a}\,) \times \vec{a} \;\; , \quad \alpha^2 = \vec{a} \cdot \vec{a} \;\; . \tag{1.24}$$

The three relations (1.19), (1.14), and (1.1) might be considered the basis of a *carpe diem* school of mechanics. See a gradient, take the curl. See a curl, take the divergence. See a divergence, integrate over a control volume.

1.2.2 Equations of motion

Mass. I have grown up with a derivation of the Navier-Stokes equations of motion using the device of a stationary control volume. Conservation of mass is expressed with complete clarity by the relation

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint \rho \mathrm{d}V = - \iint \rho\, \vec{u} \cdot \vec{n}\, \mathrm{d}S \;\; , \tag{1.25}$$

where ρ is density, $\vec{u}$ is velocity, and $\mathrm{d}V$ and $\mathrm{d}S$ are elements of volume and surface, respectively. The negative sign on the right-hand side is required by the convention that $\vec{n}$ is the unit outward normal to the surface of the control volume. No provision is ordinarily made for sources or sinks within the control volume. If these are needed, they can be added at an appropriate later stage. With the aid of the definition (1.1) for divergence, equation (1.25) can be rewritten in terms of volume integrals only;

$$\iiint \left(\frac{\partial \rho}{\partial t} + \text{div } \rho \vec{u} \right) \mathrm{d}V = 0 \ . \tag{1.26}$$

Because the control volume is arbitrary, the integrand must be zero everywhere. Conservation of mass therefore requires

$$\frac{\partial \rho}{\partial t} + \text{div } \rho \vec{u} = 0 \ . \tag{1.27}$$

A different form is obtained by use of the vector identity (1.12);

$$\frac{\partial \rho}{\partial t} + \vec{u} \cdot \text{grad } \rho \ + \rho \text{ div } \vec{u} = 0 \ . \tag{1.28}$$

The first two terms are now a rate of change of density following an element of the fluid, already defined by equation (1.8). A final form for the continuity equation is therefore

$$\frac{D\rho}{Dt} + \rho \text{ div } \vec{u} = 0 \ . \tag{1.29}$$

Momentum. Conservation of momentum is expressed in the same simple terms by

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint \rho \, \vec{u} \, \mathrm{d}V = - \iint \rho \, \vec{u} \, (\vec{u} \cdot \vec{n}\,) \, \mathrm{d}S \ + \\ + \iiint \rho \, \vec{F} \, \mathrm{d}V + \iint \underline{\sigma} \, \vec{n} \, \mathrm{d}S \ . \tag{1.30}$$

The term on the left is the time rate of change of the vector momentum within the control volume. The first term on the right is flux of momentum through the boundary, with a negative sign for the same reason given earlier. In this term the velocity $\vec{u}$ appears twice in different roles. The first is as vector momentum per unit mass. The second is as volume flux per unit area per unit time. I consider the distinction to be important and will maintain it throughout this monograph. The quantity $\vec{F}$ is an internal body force per unit mass, usually due to gravity. The surface stress $\underline{\sigma}$ is a tensor, or linear vector operator, with $\underline{\sigma}\,\vec{n}$ the vector force per unit area on the boundary of the control volume.

Two steps are required to obtain a differential equation. The first step introduces the dyadic product of two vectors, $(\underline{\vec{a} \circ \vec{b}})$, defined by equation (1.5) with $\vec{a} = \rho\vec{u}$, $\vec{b} = \vec{u}$, and $\vec{c} = \vec{n}$. The second step introduces the generalized divergence of a tensor by use of equation (1.2). With these relationships, equation (1.30) can be written in terms of volume integrals only[2];

$$\iiint \left(\frac{\partial \rho\,\vec{u}}{\partial t} + \text{div}\ \rho\,(\underline{\tilde{\text{u}} \circ \tilde{\text{u}}}\,) - \rho\vec{F} - \text{div}\ \underline{\sigma} \right) \text{d}V = 0\ . \qquad (1.31)$$

Since the control volume is arbitrary, it follows that

$$\frac{\partial \rho\,\vec{u}}{\partial t} + \text{div}\ \rho\,(\underline{\tilde{\text{u}} \circ \tilde{\text{u}}}\,) = \rho\vec{F} + \text{div}\ \underline{\sigma} \qquad (1.32)$$

everywhere.

This form for the transport terms is well suited for the introduction of what are called Reynolds stresses in turbulent flow. It is also the most useful form when the objective is to derive integral laws (such as Karman's momentum integral) from the differential equations, because the volume integral of a divergence usually begins life as the surface integral of a flux. It is therefore often convenient to return to the control volume for this operation.

Another form for the transport terms follows from the identity (1.15) with $\vec{a} = \vec{u}$, $\vec{b} = \rho\,\vec{u}$, so that

$$\text{div}\ \rho\,(\underline{\tilde{\text{u}} \circ \tilde{\text{u}}}\,) = \rho\,(\underline{\text{grad}\ \vec{u}}\,)\,\vec{u} + \vec{u}\ \text{div}\ \rho\,\vec{u}\ . \qquad (1.33)$$

Now the momentum equation (1.32) takes the form

$$\rho\,\frac{\partial \vec{u}}{\partial t} + \vec{u}\,\frac{\partial \rho}{\partial t} + \vec{u}\ \text{div}\ \rho\,\vec{u} + \rho\,(\underline{\text{grad}\ \vec{u}}\,)\,\vec{u} = \rho\vec{F} + \text{div}\ \underline{\sigma}\ . \qquad (1.34)$$

[2] *The tilde notation is not defined here but section 2.1.2 states*, "[T]he tilde, here and elsewhere, is intended as a mnemonic for an integral mean value."

The second and third terms drop out, by virtue of the continuity equation (1.27), leaving

$$\rho\left[\frac{\partial\vec{u}}{\partial t}+(\underline{\text{grad }\vec{u}})\,\vec{u}\right]=\rho\vec{F}+\text{div }\underline{\sigma}\ . \tag{1.35}$$

The quantity in brackets on the left is the substantial derivative (1.9) (the derivative following a fluid element) of the vector $\vec{u}$, so that finally

$$\rho\,\frac{D\vec{u}}{Dt}=\rho\vec{F}+\text{div }\underline{\sigma}\ . \tag{1.36}$$

This equation may also have explicit source or sink terms for momentum, although these are usually left in implicit form. A notation sometimes used in equation (1.36) is

$$\frac{D\vec{u}}{Dt}=\frac{\partial\vec{u}}{\partial t}+(\vec{u}\cdot\text{grad})\,\vec{u}\ . \tag{1.37}$$

I think that this notation might be misleading in any coordinate system except a rectangular one.

The body force $\vec{F}$ will normally be a gravity force, $\vec{F}=-g\vec{i}_y$, where $\vec{i}_y$ is a unit vector directed vertically upward. It might also be a local force expressed as a Dirac δ-function to represent a concentrated source or sink of momentum; e.g., thrust or drag. For a Newtonian fluid, the tensor $\underline{\sigma}$ is symmetric, isotropic, and linear in the spatial first derivatives of the velocity. The most general form meeting these conditions is (cite Jeffries, Stefan)

$$\underline{\sigma}=-p\,\underline{I}+\lambda\text{ div }\vec{u}\,\underline{I}+\mu\,\underline{\text{def }\vec{u}}=-p\ \underline{I}+\underline{\tau}\ , \tag{1.38}$$

where $\underline{I}$ is the identity tensor, defined by $\underline{I}\vec{a}=\vec{a}$, and where

$$\underline{\tau}=\lambda\text{ div }\vec{u}\,\underline{I}+\mu\,\underline{\text{def }\vec{u}}\ . \tag{1.39}$$

The stress tensor $\underline{\tau}$ thus involves two viscosities, λ and μ, of which the first is immaterial if div $\vec{u}=0$. In this formulation the three state variables p, μ, λ are introduced in a single operation and are not conceptually different.

With the aid of the identity (1.13) with $\alpha = p$, the momentum equation can be written finally as

$$\rho \, \frac{D\vec{u}}{Dt} = -\text{grad } p + \rho\vec{F} + \text{div } \underline{\tau} \ . \tag{1.40}$$

Energy. A law for conservation of energy can be derived from first principles by visualizing a molecular structure for the fluid, although the result is often classified as part of continuum mechanics. Hard spherical molecules in a state of agitation have two kinds of energy: internal kinetic energy e, associated with random motion and represented as temperature, and directed or organized motion $\vec{u} \cdot \vec{u}/2$ associated with bulk velocity. Energy can be added or subtracted in the interior of a control volume by at least two processes. One is heat release Q per unit mass by chemical reactions, including phase changes such as evaporation and condensation. The other is work done by body forces $\vec{u} \cdot \vec{F}$. Energy can also be transferred at the boundary of a control volume by heat conduction $\vec{q}$ and by work done by surface forces $\underline{\sigma}$. The latter two processes will appear as divergence terms and are neutral in the interior of the control volume. Thus write

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \iiint \rho\left(e + \frac{u^2}{2}\right) \mathrm{d}V = &- \iint \ \rho\left(e + \frac{u^2}{2}\right) \vec{u} \cdot \vec{n} \, \mathrm{d}S \ + \\ &+ \iiint \rho \, Q \mathrm{d}V + \iiint \vec{u} \cdot \vec{F} \mathrm{d}V \ - \\ &- \iint \vec{q} \cdot \vec{n} \, \mathrm{d}S + \iint (\underline{\sigma} \, \vec{n}\,) \cdot \vec{u} \, \mathrm{d}S \end{aligned} \tag{1.41}$$

where $u^2 = \vec{u} \cdot \vec{u}$. The three surface integrals can be converted to volume integrals using equations (1.1) and (1.2), with the result, after use of the identity (1.22) in the last term,

$$\frac{\partial}{\partial t}\rho\left(e + \frac{u^2}{2}\right) + \text{div}\,\rho\left(e + \frac{u^2}{2}\right)\vec{u} = \rho Q + \vec{u} \cdot \vec{F} - \text{div } \vec{q} + \text{div}(\underline{\sigma}\,\vec{u}\,) \ . \tag{1.42}$$

The first two terms can be modified by differentiating the second term as a product and using the continuity equation (1.27). The last term can be modified using equation (1.38) for $\underline{\sigma}$ and the identity (1.10). Finally, with the definition (1.8) for the derivative of a scalar following a fluid element, there is obtained

$$\rho \frac{D}{Dt}\left(e + \frac{u^2}{2}\right) = \rho Q - \text{div } \vec{q} + \vec{u} \cdot \vec{F} - \text{div } p\vec{u} + \text{div } \underline{\tau}\, \vec{u} \ . \quad (1.43)$$

For a Fourierian fluid, the heat conduction vector is linear in the spatial first derivatives of the temperature,

$$\vec{q} = -k \text{ grad } T \ , \quad (1.44)$$

with a scalar heat conductivity k. The negative sign indicates that energy is transferred down the temperature gradient.

There is also available a mechanical energy equation, derived independently of the thermodynamic equation (1.43) by taking the scalar product of the momentum equation (1.40) with the vector velocity $\vec{u}$ to obtain

$$\rho \frac{Du^2/2}{Dt} = -\vec{u} \cdot \text{grad } p + \vec{u} \cdot \vec{F} + \vec{u} \cdot \text{div } \underline{\tau} \ . \quad (1.45)$$

Equations (1.43) and (1.45) and an obvious formula for $D(p/\rho)/Dt$ lead, with the aid of the identity (1.16), to the array of energy equations displayed in TABLE 1.1. The scalar product of two tensors is defined by the identity $\underline{A} \cdot \underline{B} = A_{ij}B_{ij}$ (summed over i and j).

The middle three equations, the last three, and the first three (the equation for h is listed twice) form natural groups that describe the evolution of the quantities e and $u^2/2$ and their sum in equation (1.43), and the evolution of the terms appearing in the definitions for static and stagnation enthalpy,

$$h = e + \frac{p}{\rho} \quad (1.46)$$

and

$$h_0 = h + \frac{u^2}{2} \ . \quad (1.47)$$

Table 1.1

Energy equations

$$
\begin{array}{rlllllll}
\rho \dfrac{D}{Dt} h = & \dfrac{\partial p}{\partial t} & + \rho Q & & - \mathrm{div}\ \vec{q} & + \underline{\tau} \cdot \underline{\mathrm{grad}\ \vec{u}} & + \vec{u} \cdot \mathrm{grad}\ p \\
 & & & & & \Uparrow & \\
\rho \dfrac{D}{Dt} h_0 = & \dfrac{\partial p}{\partial t} & + \rho Q & + \rho \vec{F} \cdot \vec{u} & - \mathrm{div}\ \vec{q} & + \mathrm{div}\ (\underline{\tau}\vec{u}) & \\
 & & & & & \Downarrow & \\
\rho \dfrac{D}{Dt} \dfrac{u^2}{2} = & & & \rho \vec{F} \cdot \vec{u} & & + \vec{u} \cdot \mathrm{div}\ \underline{\tau} & - \vec{u} \cdot \mathrm{grad}\ p \\
 & & & & & \Uparrow & \Uparrow \\
\rho \dfrac{D}{Dt} \left(e + \dfrac{u^2}{2} \right) = & & \rho Q & + \rho \vec{F} \cdot \vec{u} & - \mathrm{div}\ \vec{q} & + \mathrm{div}\ (\underline{\tau}\vec{u}) & - \mathrm{div}\ (p\vec{u}) \\
 & & & & & \Downarrow & \Downarrow \\
\rho \dfrac{D}{Dt} e = & & \rho Q & & - \mathrm{div}\ \vec{q} & + \underline{\tau} \cdot \underline{\mathrm{grad}\ \vec{u}} & - p\ \mathrm{div}\ \vec{u} \\
 & & & & & & \Uparrow \\
\rho \dfrac{D}{Dt} h = & \dfrac{\partial p}{\partial t} & + \rho Q & & - \mathrm{div}\ \vec{q} & + \underline{\tau} \cdot \underline{\mathrm{grad}\ \vec{u}} & + \vec{u} \cdot \mathrm{grad}\ p \\
 & & & & & & \Downarrow \\
\rho \dfrac{D}{Dt} \dfrac{p}{\rho} = & \dfrac{\partial p}{\partial t} & & & & & + \mathrm{div}\ (p\vec{u})
\end{array}
$$

So far the nature of the fluid is left open. In this monograph the fluid will be either an ordinary liquid or a thermally perfect gas with an equation of state

$$p = \rho RT \ . \tag{1.48}$$

The gas will also be assumed to be calorically perfect; that is, the specific heats c_p and c_v will be taken as constant in the definitions

$$e = c_v T \ ; \tag{1.49}$$

$$h = c_p T \ ; \tag{1.50}$$

and in the combinations

$$R = c_p - c_v \, , \qquad \gamma = c_p / c_v \ . \tag{1.51}$$

Finally, the scalar quantities k, λ, and μ are state variables that can be taken to depend on temperature only.

These full equations of motion for a compressible fluid are so complex as to be intractable. Analytical progress toward their solution therefore tends to occur in small increments, in which the equations are truncated in various ways and solved for special classes of problems. The simplest method of truncation is brute force. For example, it can be stipulated that the density of a fluid is constant, or that the viscosity and heat conductivity are zero, or that the flow depends on only one space coordinate and time. More systematic truncations can often be based on dimensional considerations. For example, suppose that the ostensible data for a class of problems, including boundary conditions and characteristic fluid properties, are sufficient to define a complete set of global scales for length, velocity, temperature, and so on. Then the equations of motion can immediately be put in non-dimensional form. The relative magnitude of various terms can be estimated, and certain terms can be discarded as negligible, along with the physical processes that they model. A limit process is often involved. If the limit is regular, suitable expansions define themselves. However, the expansion procedure is usually neither transparent nor trivial, so that it is best illustrated by a few examples.

1.2.3 Incompressible fluids

A large part of fluid mechanics deals either with liquids or with gases moving at low speeds, so that effects of compressibility are not important. The limit process that allows a gas to be treated as incompressible was first accurately described by LAGERSTROM (1964) in his handbook article on laminar flow. This limit process preserves the mechanical role of the pressure in the momentum equation (1.40) while suppressing the thermodynamic role of the pressure in the energy equation (1.43) and the state equation (1.48). The reasoning here proceeds from the general to the particular, on the premise that it is logically easier (and safer) to derive the correct equations for a compressible fluid from first principles, and then to apply the correct limit, than it is to go in the opposite direction. However, it is important to keep in mind that the reasoning is also *ad hoc*, being strictly valid in each instance only for a particular class of flows specified in advance.

Consider the class of flows of a viscous perfect gas past a finite body. Begin by converting the equations of motion to dimensionless form. Assume that there is a constant reference length **L** in the problem, together with a constant reference velocity **U** and a reference fluid state in which p, ρ, T have the values p_a, ρ_a, T_a (a for ambient, usually at infinity), and similarly for μ and k. The body force per unit mass, $\vec{F}$, is made dimensionless with g, the acceleration of gravity. Nothing essential is lost if the flow is taken to be steady, with $Q = 0$, and if λ and μ are both represented by μ. With an overbar to indicate dimensionless variables like $\overline{\vec{u}} = \vec{u}/\mathbf{U}$ and $\overline{\rho} = \rho/\rho_a$ (and, implicitly, dimensionless operators div, grad, and D/Dt) in coordinates $\overline{\vec{x}} = \vec{x}/\mathbf{L}$, the equations of motion can be written

$$\frac{D\overline{\rho}}{Dt} + \overline{\rho}\ \mathrm{div}\ \overline{\vec{u}} = 0\ ; \tag{1.52}$$

$$\overline{\rho}\ \frac{D\overline{\vec{u}}}{Dt} = -\frac{1}{\gamma M^2}\ \mathrm{grad}\ \overline{p} + \frac{1}{Fr^2}\ \overline{\rho}\,\overline{\vec{F}} + \frac{1}{Re}\ \mathrm{div}\ \underline{\overline{\tau}}\ ; \tag{1.53}$$

$$\overline{\rho}\ \frac{D\overline{h}}{Dt} = \left(\frac{\gamma-1}{\gamma}\right)\overline{\vec{u}}\cdot\mathrm{grad}\ \overline{p} - \frac{1}{P\acute{e}}\ \mathrm{div}\ \overline{\vec{q}} + (\gamma-1)\frac{M^2}{Re}\underline{\overline{\tau}}\cdot\underline{\mathrm{grad}\ \overline{\vec{u}}}\ ; \tag{1.54}$$

$$\overline{p} = \overline{\rho}\,\overline{T} \ . \tag{1.55}$$

Several dimensionless parameters materialize as coefficients of terms on the right-hand sides. They are called, respectively, the Mach number M, the Froude number Fr, the Reynolds number Re, the Péclet number $Pé$, and the ratio of specific heats γ;

$$M^2 = \frac{\mathbf{U}^2}{\gamma\, p_a/\rho_a} \ ; \tag{1.56}$$

$$Fr^2 = \frac{\mathbf{U}^2}{g\mathbf{L}} \ ; \tag{1.57}$$

$$Re = \frac{\rho_a\, \mathbf{UL}}{\mu_a} \ ; \tag{1.58}$$

$$Pé = \frac{\rho_a c_p \mathbf{UL}}{k_a} \ ; \tag{1.59}$$

$$\gamma = \frac{c_p}{c_v} \ . \tag{1.60}$$

The ratio $Pé/Re$, which compares the relative rates of diffusion for heat and vorticity, is a derived parameter called the Prandtl number Pr;

$$Pr = \frac{Pé}{Re} = \frac{c_p \mu_a}{k_a} \ . \tag{1.61}$$

I was not aware, until some of my students pointed it out to me, that the definition of Froude number in current textbooks and monographs on fluid mechanics is not uniform. When I made an informal survey of books within easy reach in my office, I found seven authors, some of them very distinguished, who use the definition $Fr = \mathbf{U}^2/g\mathbf{L}$. Eleven other authors, equally distinguished, use the definition $Fr^2 = \mathbf{U}^2/g\mathbf{L}$. I will adopt the second definition, as in equation (1.57) above; first, because it is the form originally proposed by FROUDE (ref); second, because it is the form commonly used by writers on topics that directly involve surface waves or buoyancy forces; and third, and most important, because the symbol Fr then runs in parallel with the symbol M as the ratio of a fluid velocity to a characteristic wave velocity.

It is not necessary to use or even to know the equations of motion in order to discover these five dimensionless parameters. The Buckingham Π theorem (see SECTION X)[3] is sufficient, given the presence of nine independent physical quantities in the problem, together with four independent physical units (mass, length, time, and temperature). For each of the dimensionless parameters just listed, and others to come, experience shows that much of fluid mechanics and the associated applied mathematics is concentrated near the three special values 0, 1, ∞. Lagerstrom discusses several of these special values at length, especially the difficult cases $Re \to 0$ and $Re \to \infty$. The case at hand, the case $M \to 0$, is relatively straightforward because the perturbation is entirely regular.

For each class of problems, the global reference values used to make the variables and operators dimensionless should be chosen in such a way that the essential terms in the equations, weighted by their dimensionless coefficients, are of order unity in the limit. In the present instance of flow past a body, the transport terms on the left in the momentum equation (1.53) and in the energy equation (1.54) are essential by assumption. So is the pressure gradient term in equation (1.53). All other terms can be left to follow these leaders. This format is not universal. For example, the transport terms in the momentum equation are not important in lubrication theory, and different arguments are needed. Fortunately, in such cases the equations are usually capable of indicating the direction the argument should take as well as the nature of higher approximations.

After these preliminaries, consider the limit $M \to 0$ in equations (1.52)–(1.55). There is an obvious difficulty in the momentum equation, where the pressure term blows up. This problem can be avoided by making the pressure dimensionless with the dynamic pressure rather than the static pressure. Instead of

$$\overline{p} = \frac{p}{p_a} \ , \tag{1.62}$$

define

$$\widetilde{p} = \frac{\overline{p}}{\gamma\, M^2} = \frac{p}{\rho_a \mathbf{U}^2} \ . \tag{1.63}$$

[3] *Sections that discuss Buckingham* Π *include 2.4.1, 9.1.2, and 11.1.1*

The difficulty in the momentum equation clears up, but a new problem appears in the state equation (1.55), which becomes

$$\overline{\rho}\,\overline{T} = \gamma\, M^2 \widetilde{p} \ , \tag{1.64}$$

with a right-hand side that vanishes in the limit. This new problem is caused by the fact that p is defined only within an additive constant in grad p, but is defined in an absolute sense in the state equation. Both cases are accounted for if the dimensionless pressure is expressed in the form usually called a pressure coefficient by aeronautical and mechanical engineers. Thus define

$$\widehat{p} = \widetilde{p} - \frac{1}{\gamma\, M^2} = \frac{\overline{p} - 1}{\gamma\, M^2} = \frac{p - p_a}{\rho_a \mathbf{U}^2} \ . \tag{1.65}$$

With this change, the dimensionless equations become

$$\frac{D\overline{\rho}}{Dt} + \overline{\rho}\ \mathrm{div}\ \overline{\vec{u}} = 0 \ ; \tag{1.66}$$

$$\overline{\rho}\ \frac{D\overline{\vec{u}}}{Dt} = -\mathrm{grad}\ \widehat{p} + \frac{1}{Fr^2}\,\overline{\rho}\,\overline{\vec{F}} + \frac{1}{Re}\ \mathrm{div}\ \underline{\overline{\tau}} \ ; \tag{1.67}$$

$$\overline{\rho}\ \frac{D\overline{h}}{Dt} = (\gamma - 1)\, M^2\ \overline{\vec{u}} \cdot \mathrm{grad}\ \widehat{p} - \frac{1}{P\acute{e}}\ \mathrm{div}\ \overline{\vec{q}} + (\gamma - 1)\,\frac{M^2}{Re}\ \underline{\overline{\tau}} \cdot \underline{\mathrm{grad}\ \overline{\vec{u}}} \ ; \tag{1.68}$$

$$\overline{\rho}\,\overline{T} = 1 + \gamma\, M^2\, \widehat{p} \ . \tag{1.69}$$

Now in the limit $M \to 0$ the flow of a viscous perfect gas about a finite body is described by the equations

$$\frac{D\overline{\rho}}{Dt} + \overline{\rho}\ \mathrm{div}\ \overline{\vec{u}} = 0 \ ; \tag{1.70}$$

$$\overline{\rho}\ \frac{D\overline{\vec{u}}}{Dt} = -\mathrm{grad}\ \widehat{p} + \frac{1}{Fr^2}\overline{\rho}\,\overline{\vec{F}} + \frac{1}{Re}\ \mathrm{div}\ \underline{\overline{\tau}} \ ; \tag{1.71}$$

$$\overline{\rho}\ \frac{D\overline{h}}{Dt} = -\frac{1}{P\acute{e}}\ \mathrm{div}\ \overline{\vec{q}} \ ; \tag{1.72}$$

$$\overline{\rho}\,\overline{T} = 1 \ . \tag{1.73}$$

In particular, the pressure-work term and the dissipation term drop out of the energy equation (1.68), leaving only conduction to balance transport of heat. Restored to dimensional form, the equations for $M \to 0$ are

$$\frac{D\rho}{Dt} + \rho \,\mathrm{div}\, \vec{u} = 0 \ ; \tag{1.74}$$

$$\rho \,\frac{D\vec{u}}{Dt} = -\mathrm{grad}\ p + \rho \vec{F} + \mathrm{div}\ \underline{\tau} \ ; \tag{1.75}$$

$$\rho \,\frac{Dh}{Dt} = -\mathrm{div}\ \vec{q} \ ; \tag{1.76}$$

$$\rho T = \rho_a T_a \ . \tag{1.77}$$

The limit $M \to 0$ can also be approached more directly. For a perfect gas, the definition of stagnation enthalpy,

$$c_p T_0 = c_p T + \frac{u^2}{2} = \frac{c_p}{R}\frac{p}{\rho} + \frac{u^2}{2} \ , \tag{1.78}$$

and the definition of local Mach number,

$$M^2 = \frac{\rho u^2}{\gamma p} \ , \tag{1.79}$$

imply

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2 \ . \tag{1.80}$$

Now suppose that the flow is steady and isentropic (inviscid, adiabatic), so that

$$\frac{T}{T_0} = \left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}} \ . \tag{1.81}$$

Then

$$\frac{\rho u^2}{2p_0} = \frac{\gamma}{2}\frac{pM^2}{p_0} = \left(\frac{\gamma}{\gamma-1}\right)\frac{p}{p_0}\left[\left(\frac{p_0}{p}\right)^{\frac{\gamma-1}{\gamma}} - 1\right] \ . \tag{1.82}$$

Finally, suppose that p/p_0 is nearly unity. Put $p_0/p = 1 + \epsilon$ and take the first term in an expansion in powers of ϵ of the quantity in brackets in equation (1.82);

$$\frac{\rho u^2}{2} = \left(\frac{\gamma}{\gamma-1}\right)\left[\frac{\gamma-1}{\gamma}\epsilon\right] p = \epsilon\, p \ , \tag{1.83}$$

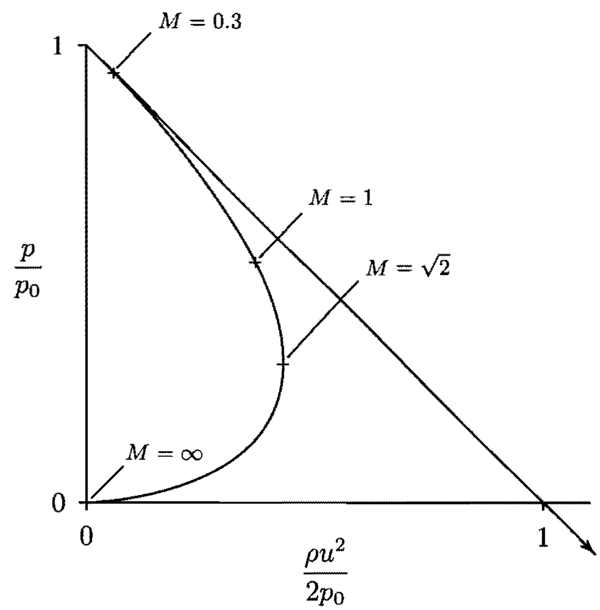

Figure 1.1: The Bernoulli integral for an incompressible fluid and for a perfect gas with $\gamma = 1.4$.

or

$$\frac{\rho u^2}{2} = p_0 - p \ . \tag{1.84}$$

Thus the low-speed Bernoulli integral is recovered in the limit $\epsilon \to 0$. Comparison of equations (1.79) and (1.83) shows that $\epsilon = \gamma M^2/2$, so that this is also the limit $M \to 0$. It is equation (1.82) and not equation (1.78) that should be referred to as the Bernoulli integral for a compressible perfect gas. To illustrate this limit graphically, the relationship between p/p_0 and $\rho u^2/2p_0$ from equations (1.82) and (1.84) is shown in FIGURE 1.1 for $\gamma = 7/5$. The absence of a lower bound for p in an incompressible fluid is evident, as is the basis of an expansion for p/p_0 near unity.

Throughout this discussion of the limit $M \to 0$, the main issue is the behavior of the variable called the pressure. In its thermodynamic role, the pressure must be non-negative. In its dynamic role, there is no lower limit for the pressure. For an incompressible fluid it can go to negative infinity, according to the Bernoulli equation (1.84), when the velocity $\vec{u}$ goes to positive infinity, as for potential flow at a sharp external corner or for flow at a source or sink. The issue is resolved formally by measuring the magnitude of the pressure in both of its roles from a local reference value, here called p_0, and requiring changes in p to be small when compared with p_0 but of order unity when compared with $\rho_a \mathbf{U}^2$. It should not be surprising, with the pressure almost constant, that the only thermodynamic process that leaves any residue in the limit is the process at constant pressure, as represented by the enthalpy $h = c_p T$. The specific heat at constant volume c_v has dropped out, along with the internal energy $e = c_v T$ and the state constants $\gamma = c_p/c_v$ and $R = c_p - c_v$.

The argument just presented does not require the density ρ to be constant. It does require changes in density to be associated with changes in temperature, not with compressibility. If the temperature T is constant, the energy equation (1.76) with $\vec{q} = -k \text{grad}\, T$ is moot, and the state equation (1.77) requires the density ρ to be constant also. The continuity equation (1.74) is then reduced to div $\vec{u} = 0$, and the fluid is effectively incompressible. No distinction need be made between a gas and a liquid.

1.2.4 Low-speed heat transfer

For many low-speed flows, variations in temperature are forced by the boundary conditions. If these variations are small, both gases and liquids can be accommodated as working fluids through linearization of the state equation. Consider again the low-speed equations (1.74)–(1.77) in dimensional form. Omit the body-force term temporarily, and take $h = c_p T$. Use Newton's hypothesis $\underline{\tau} = \mu\ \underline{\text{def}\ \vec{u}}$ for the viscous terms, and Fourier's hypothesis $\vec{q} = -k$ grad T for the heat-conduction terms. For low-speed thermal problems, replace the state equation (1.77) by a tangent approximation,

$$\rho - \rho_a = -\beta\rho_a(T - T_a) \tag{1.85}$$

where the constant $\beta = -(\partial\rho/\partial T)_a/\rho_a$ is called the volume coefficient of expansion and depends on the reference temperature T_a. If the inventors of the linearized state equation (1.85) had chosen to include a factor T_a in the denominator of the right-hand side, then the parameter β would represent a dimensionless slope in logarithmic coordinates and would have the value unity for a perfect gas. The reference parameter T_a would still have to be specified for any other fluid, as it does now. There are also problems that may require more complex measures; for example, $\beta \to 0$ for water near 4 °C, or $\beta \to \infty$ for a liquid near its critical point.

The dimensional equations of motion become

$$\frac{D\rho}{Dt} + \rho \text{ div } \vec{u} = 0 \;; \tag{1.86}$$

$$\rho \frac{D\vec{u}}{Dt} = -\text{grad } p + \text{div } (\mu \underline{\text{def } \vec{u}}) \;; \tag{1.87}$$

$$\rho\, c_p \frac{DT}{Dt} = \text{div } (k \text{ grad } T) \;; \tag{1.88}$$

together with the state equation (1.85). It will be trivial in the sequel that μ and k can be taken as constant.

An important issue involves the limiting form of the continuity equation (1.86). Because the energy equation (1.88) with constant k is linear and homogeneous in T, it can be written with the aid of equation (1.85) as an equation for ρ;

$$\frac{D\rho}{Dt} = \frac{k}{\rho c_p} \text{ div grad } \rho \;. \tag{1.89}$$

Note that the commonly accepted form of the continuity equation (1.86) for low-speed flow with heat transfer is

$$\text{div } \vec{u} = 0 \;, \tag{1.90}$$

and that this seems to imply, according to equation (1.86),

$$\frac{D\rho}{Dt} = 0 \ . \tag{1.91}$$

Thus $D\rho/Dt$ is both zero and not zero. Some writers ignore the inconsistency. Others (see, for example, CHANDRASEKHAR (1961, pp. 16–17) explain it by estimates that depend on the smallness of the parameter β in equation (1.85). However, I find it more useful to think that it is the temperature difference $(T - T_a)$ in this equation that is small in some appropriate sense. The argument proceeds in the same spirit as that of the preceding section, and again contemplates flow around a body. Denote a constant reference temperature by T_w (w for wall) and an ambient temperature by T_a. Define two dimensionless parameters (the symbol P should be read as a Greek capital *rho*)

$$\mathrm{P} = \frac{\rho_w - \rho_a}{\rho_a} \ , \qquad \Theta = \frac{T_w - T_a}{T_a} \ , \tag{1.92}$$

and consider the limit $\mathrm{P} \to 0$, $\Theta \to 0$. Suitable non-dimensional variables of order unity suggest themselves as

$$\widehat{\rho} = \frac{\overline{\rho} - 1}{\mathrm{P}} = \frac{\rho - \rho_a}{\rho_w - \rho_a} \ , \qquad \widehat{T} = \frac{\overline{T} - 1}{\Theta} = \frac{T - T_a}{T_w - T_a} \ . \tag{1.93}$$

Now write the dimensionless equations of motion (1.70)–(1.72) in terms of $\widehat{\rho}$,

$$\mathrm{P}\frac{D\widehat{\rho}}{Dt} + (1 + \mathrm{P}\widehat{\rho}) \ \mathrm{div} \ \overline{\vec{u}} = 0 \ ; \tag{1.94}$$

$$(1 + \mathrm{P}\widehat{\rho})\frac{D\overline{\vec{u}}}{Dt} = -\mathrm{grad} \ \widehat{p} + \frac{1}{Re}\mathrm{div} \ \overline{\mu} \ \underline{\mathrm{def} \ \overline{\vec{u}}} \ ; \tag{1.95}$$

$$(1 + \mathrm{P}\widehat{\rho})\frac{D\widehat{\rho}}{Dt} = \frac{1}{P\acute{e}} \ \mathrm{div} \ \overline{k} \ \mathrm{grad} \ \widehat{\rho} \ ; \tag{1.96}$$

$$P\widehat{\rho} = -\beta \, T_a \, \Theta \, \widehat{T} \ ; \tag{1.97}$$

and take the limit $\mathrm{P} \to 0$, $\Theta \to 0$. The incompressible form $\mathrm{div} \ \overline{\vec{u}} = 0$ is evidently the correct limit of the continuity equation. The derivative $D\widehat{\rho}/Dt$ is not zero in the limit; only its coefficient P is zero.

There is no effect on the momentum and energy equations. The state equation remains in the form (1.85).

In this development, the special case of a perfect gas turns out to be in no way special. In terms of the new variables (1.93), the state equation (1.73) for a perfect gas becomes

$$(1 + \mathrm{P}\widehat{\rho}\,)(1 + \Theta\widehat{T}\,) = 1 \ . \tag{1.98}$$

To first order in the small quantities P and Θ, this is

$$\mathrm{P}\widehat{\rho} + \Theta\widehat{T} = 0 \ , \tag{1.99}$$

or, in physical variables,

$$\rho - \rho_a = -\frac{\rho_a}{T_a}\ (T - T_a) \ . \tag{1.100}$$

This equation is the same as equation (1.85) if $\beta = 1/T_a$.

The equations (1.86)–(1.88) have now become, in physical variables,

$$\mathrm{div}\ \vec{u} = 0 \ ; \tag{1.101}$$

$$\rho\frac{D\vec{u}}{Dt} = -\mathrm{grad}\ p + \mu\ \mathrm{div}\ \underline{\mathrm{grad}\ \vec{u}} \ ; \tag{1.102}$$

$$\rho\, c_p\frac{DT}{Dt} = k\ \mathrm{div}\ \mathrm{grad}\ T \ ; \tag{1.103}$$

with ρ, μ, k all constant. These equations are the stuff of low-speed heat transfer. The form of the viscous terms in the momentum equation (1.102) now takes account of vector identity (1.18) (which states that div $(\underline{\mathrm{grad}\ \vec{u}})^* = 0$ if div $\vec{u} = 0$). The state equation has been discarded. The momentum and energy equations are uncoupled in the sense that the momentum equation does not involve the temperature. The energy equation is linear and homogeneous in T, so that solutions can be superposed as long as the variable coefficients u and v in the transport terms remain fixed. A large literature testifies to the importance of this property.

1.2.5 The Boussinesq approximation

It remains to consider the body-force term in the momentum equation. Suppose first that the gravitational force is normal to the general direction of flow, as in a free-surface water channel. In rectangular coordinates, the body-force term is then

$$\vec{\mathrm{F}} = -g\,\vec{\imath}_z \ , \tag{1.104}$$

where $\vec{\imath}_z$ is a unit vector directed vertically upward (the notation is that of the literature of meteorology). This force does not necessarily play a part in the dynamics of the fluid motion. Take the velocity to be zero in the vertical component of the momentum equation (1.75) (more accurately, take $Dv/DT \ll g$). In this hydrostatic limit, denoted by the subscript zero, vertical equilibrium implies

$$0 = -\frac{dp_0}{dz} - \rho_0 g \ . \tag{1.105}$$

If the density is constant, the integral is

$$p_0 = p_s - \rho_0 g z \ , \tag{1.106}$$

where p_s is the pressure at $z = 0$ (s for sea level, say). If the fluid is a perfect gas and is compressible but isothermal, with temperture T_s, equation (1.106) is replaced by

$$p_0 = p_s\, e^{-gz/RT_s} \ . \tag{1.107}$$

(Comment on Chapman's paper.) A more accurate model of the neutral atmosphere would be isentropic (well mixed) rather than isothermal, in which case

$$p_0 = p_s \left[1 - \left(\frac{\gamma - 1}{\gamma} \right) \frac{gz}{RT_s} \right]^{\frac{\gamma}{\gamma-1}} \ . \tag{1.108}$$

Equations (1.107) and (1.108) both reduce to equation (1.106) if z is small enough. For an isothermal atmosphere, the pressure and density decrease exponentially with increasing height, with an e-folding distance z_e given by $g\,z_e/RT_s = \gamma\,g\,z_e/a^2 = 1$, where a is the speed of sound. For the earth's atmosphere, z_e is several thousand meters. For any motion that occupies a sufficiently small fraction of this distance, the atmosphere can be treated as homogeneous.

Next, subtract equation (1.105) from equation (1.75) to obtain

$$\rho \frac{D\vec{u}}{Dt} = -\text{grad } (p - p_0) - (\rho - \rho_0)\, g\, \vec{\imath}_z + \text{div } \underline{\tau} \ . \tag{1.109}$$

In particular, if the density is constant the hydrostatic pressure is irrelevant, because pressure changes associated with fluid motion are measured from a variable hydrostatic datum p_0. An exception may occur if the vertical acceleration $w\partial w/\partial z$ is not negligibly small compared with the acceleration of gravity. This will certainly be the case if a liquid has a free surface that is not sensibly flat, so that the hydrostatic datum p_s is itself disturbed.

Now to the main point in connection with buoyancy effects, and the reason that an argument for the Boussinesq approximation is needed. The energy equation (11.3) can be written,[4] with the aid of equation (11.4), in the form

$$\frac{1}{\rho}\frac{D\rho}{Dt} = \frac{1}{\rho h} \text{ div } \vec{q} - \frac{Q}{h} \tag{1.110}$$

whereas the continuity equation (11.1) reads

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\text{div } \vec{u} \ . \tag{1.111}$$

The Boussinesq approximation sets the two sides of (1.111) separately to zero, but leaves (1.110) intact, in order to retain heat conduction as an essential process.

This apparent inconsistency can be resolved with the aid of another limiting process, this time involving the Froude number. Suppose that the motion is driven entirely by buoyancy forces, and that density changes are small enough to permit linearization of the relationship $\rho(T)$. The linearized state equation is usually taken in the form

$$\rho - \rho_0 = -\beta\rho_0(T - T_0) \tag{1.112}$$

[4] *The equations cited in this paragraph are discussed in Section 11.1.1.*

where β is the relative change in density, or with a sign change the relative change in specific volume, per degree change in temperature. The parameter β depends on temperature and is ordinarily positive, since for most fluids the density decreases as the temperature increases at constant pressure. There are occasional exceptions in nature. A familiar example is water between 0 °C and 4 °C. For a perfect gas, $\beta = 1/T$. For water at 20 °C, $\beta = xxxx$ **(check)**.

If heat addition Q is negligible, the energy equation (11.3) is linear in T and therefore in ρ, given the linearized state equation (1.112). The equations of motion for buoyancy-driven flows become

$$\frac{D\rho}{Dt} = -\rho \text{ div } \vec{u} \ , \tag{1.113}$$

$$\rho \frac{D\vec{u}}{Dt} = -\text{grad } (p - p_0) - (\rho - \rho_0)\, q\vec{\imath}_z + \text{div } \mu \ \underline{\text{grad } \vec{u}} \ , \tag{1.114}$$

$$\rho \frac{D\rho}{Dt} = \text{div } k \text{ grad } \rho \ , \tag{1.115}$$

$$\rho - \rho_0 = -\beta\rho_0(T - T_0) \ . \tag{1.116}$$

Define dimensionless variables according to the schedule

$$\begin{aligned} \vec{u} &= U\overline{\vec{u}} \\ \vec{x} &= L\overline{\vec{x}} \\ p - p_0 &= \rho_0 U^2 p^* \\ \rho &= \rho_0\overline{\rho} \\ T &= T_0\overline{T} \\ \mu &= \mu_0\overline{\mu} \\ k &= k_0\overline{k} \end{aligned} \tag{1.117}$$

where U and L are global scales. The relation for p already incorporates the conclusion reached in this introduction about the proper form in the limit $M \to 0$. There is no loss of force in taking the flow to be steady and in omitting the heat-addition term Q in the energy equation. In dimensionless form, equations (1.113)–(1.116) are

$$\frac{D\overline{\rho}}{D\overline{t}} = -\text{div } \overline{\vec{u}} \ , \tag{1.118}$$

$$\frac{D\overline{\vec{u}}}{D\bar{t}} = -\text{grad } p^* - \frac{(\overline{\rho}-1)}{Fr^2}\,\vec{\imath}_z + \frac{1}{Re}\text{ div } \overline{\mu}\ \underline{\text{grad } \overline{\vec{u}}}\ , \tag{1.119}$$

$$\frac{D\overline{\rho}}{D\bar{t}} = \frac{1}{Pr Re}\text{ div } \overline{k}\text{ grad } \overline{\rho}\ , \tag{1.120}$$

$$(\overline{\rho}-1) = -\beta\, T_0(\overline{T}-1)\ , \tag{1.121}$$

where

$$Fr^2 = \frac{U^2}{gL}\ . \tag{1.122}$$

The objective of the exercise here is to reduce all of the essential terms to order unity, free of dimensionless parameters that may be either large or small. In the viscous and heat-conduction terms, this is the business of the boundary-layer approximation, where μ_0 and κ_0 are incorporated into the independent variables. The issue here is the buoyancy term. The form of this term suggests a change in the dependent variable, namely putting

$$\rho^* = \frac{\overline{\rho}-1}{Fr^2} \tag{1.123}$$

or

$$\overline{\rho} = 1 + Fr^2\,\rho^*\ . \tag{1.124}$$

When this change is made, the equations become

$$Fr^2\,\frac{D\rho^*}{D\bar{t}} = -\text{div } \overline{\vec{u}}\ , \tag{1.125}$$

$$\frac{D\overline{\vec{u}}}{D\bar{t}} = -\text{grad } p^* - \rho^*\vec{\imath}_z + \frac{1}{Re}\text{ div } \overline{\mu}\ \underline{\text{grad } \overline{\vec{u}}}\ , \tag{1.126}$$

$$\frac{D\rho^*}{D\bar{t}} = \frac{1}{Pr Re}\text{ div } \overline{k}\text{ grad } \rho^*\ , \tag{1.127}$$

$$\rho^* = -\beta\, T_0\, T^*\ , \tag{1.128}$$

where $T^* = (\overline{T}-1)/Fr^2$. The Boussinesq approximation appears as the limit $Fr \to 0$, for which the left side of equation (1.125) vanishes, leaving

$$\text{div } \overline{\vec{u}} = 0\ , \tag{1.129}$$

together with equations (1.126)–(1.128). The parameters $\overline{\mu}$ and $\overline{k}$ can be expressed in parallel with (1.124) for ρ, and therefore can be replaced by μ_0 and κ_0.

In short, the limit $Fr = 0$ means that density differences are required to be small, of order U^2/gL, for ρ^* to be of order unity. Put another way, velocities in such a flow should be small compared with the velocity associated with free fall under gravity through a distance L. If $L = 1$ meter, for example, U should be small compared with 3 meters per second. If not, it is necessary to revert to the full equations. (Read White, viscous flow, on Boussinesq approximation.)

1.2.6 Coordinate systems

1.3 Operational prologue

1.3.1 Stream functions

The stream function is a device used to simplify the form of the equations of motion by satisfying the continuity equation identically. Assume that the fluid is incompressible. The equation

$$\text{div } \vec{u} = 0 \tag{1.130}$$

can be satisfied by at least two methods. One invokes a vector identity (1.14) from SECTION 1.2.1; div curl $\vec{a} = 0$, and puts

$$\vec{u} = \text{curl } \vec{a} \ , \tag{1.131}$$

where $\vec{a}$ is sometimes called the acceleration potential. The dependent variables of the problem are now the three components of the vector $\vec{a}$, and there is no obvious simplification in substituting these for the three components of the velocity itself.

The second device invokes two vector identities (1.17) and (1.19) from SECTION 1.2.1, div $(\vec{a} \times \vec{b}) = \vec{b} \cdot \text{curl } \vec{a} - \vec{a} \cdot \text{curl } \vec{b}$ and curl (grad α) $= 0$, and puts

$$\vec{u} = \text{grad } \psi \times \text{grad } \Psi \ . \tag{1.132}$$

There are now two scalar stream functions, ψ and Ψ, for a general three-dimensional problem. Because $\vec{u}$ in equation (1.132) is normal to grad ψ and to grad Ψ, the velocity vector lies in the intersection of the surfaces $\psi =$ constant and $\Psi =$ constant. The most effective use of equation (1.132) therefore occurs for plane or axisymmetric flows in which the velocity vectors always lie in one family of coordinate surfaces, say $\beta =$ constant for coordinates $(\alpha,\ \beta,\ \gamma)$. It is then sufficient to take $\Psi = \beta$.

All of the flows taken up in this monograph are two-dimensional in the sense just defined and also have a preferred direction. The velocity in this direction will be uniformly taken as the component u of the velocity vector $(u,\ v,\ w)$. The other two components should follow a right-hand convention. In rectangular coordinates $(x,\ y,\ z)$, for example, suppose the flow lies in the planes $z =$ constant. Then the velocities are related to the stream function by

$$\vec{u} = \text{grad } \psi \times \text{grad } z = \left(\frac{\partial \psi}{\partial y},\ -\frac{\partial \psi}{\partial x},\ 0 \right) = (u,\ v,\ w)\ . \tag{1.133}$$

Other coordinate systems require other results. In cylindrical polar coordinates $(r,\ \theta,\ z)$, suppose the flow is axial and radial and thus lies in planes $\theta =$ constant. Then

$$\vec{u} = \text{grad } \psi \times \text{grad } \theta = \left(-\frac{1}{r}\frac{\partial \psi}{\partial z},\ 0,\ \frac{1}{r}\frac{\partial \psi}{\partial r} \right) = (u,\ v,\ w)\ . \tag{1.134}$$

1.3.2 Boundary-layer approximations

The reason for the plural will become apparent shortly. For an incompressible fluid, the Navier-Stokes equations of motion for steady laminar two-dimensional flow (u, v) in rectangular coordinates are the continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\ , \tag{1.135}$$

and two momentum equations,

$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = -\frac{1}{\rho}\,\frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)\ ; \tag{1.136}$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) . \tag{1.137}$$

There are three dependent variables, u, v, p, whether the viscous terms are present (Navier-Stokes equations) or not (Euler equations). In the latter case, the order of the equations is lowered, and a boundary condition has to be omitted. Normally this is the no-slip condition at any walls that bound the flow. If the viscosity is not zero, the viscous terms are present and the no-slip condition has to be enforced or the walls must move with a velocity and direction that accommodates the flow.

To fix the ideas, consider flow along a wall placed along the positive x-axis, at $y = 0$, $x > 0$. In the absence of separation, what is expected for small viscosity is the appearance of a thin layer near the wall in which the velocity changes rapidly from zero to some finite value characteristic of the inviscid flow. The viscous terms in this layer present themselves as large derivatives multiplied by a small coefficient, with the product having a magnitude comparable with that of other terms in the equations. The structure of this thin layer is the business of the boundary-layer approximation, which was first formulated analytically by PRANDTL (1905) and was first applied by Prandtl's student BLASIUS (1908) to the case of flow over a semi-infinite flat plate at constant pressure.

Prandtl's original description of his concept was so terse that it was probably almost unintelligible to his audience of mathematicians at Heidelberg. GOLDSTEIN (1969) writes that in 1928 he asked Prandtl why this was so, "and he replied that he had been given ten minutes for his lecture at the congress and that, being still quite young [he was 29], he had thought that he could publish only what he had had time to say." Neither was Prandtl's paper well placed to reach the applied mathematicians and aerodynamicists (in today's terminology) for whom it was intended. Historical accounts by SCHLICHTING (1960) and TANI (1977) make a point of the fact that the power of boundary-layer theory was not appreciated outside Germany for nearly twenty years. LAMB, in the 1916 and 1924 editions of his monumental treatise *Hydrodynamics*, disposed of the work by Prandtl and Blasius in a dozen lines, with the remark "the results obtained, and presented graphically, are interesting." Lamb did not himself present the results, graphically or otherwise, until the 1932 edition. The NACA translation of Prandtl's 1905 paper was published only in 1928 and refers only to the 1927 reprint, without mentioning the original publication. The NACA translation of the paper by Blasius did not appear until 1950.

The essence of Prandtl's idea, as described by Blasius, is to represent the thickness of the viscous layer by a parameter ϵ, where ϵ, like ν, is a small quantity. In general, the level of rigor in what follows is not high, insofar as no attention is being paid to dimensions or to the formal machinery of singular-perturbation theory. In particular, the parameter ϵ is never defined precisely. It is not a boundary-layer thickness because it is not necessarily a length, and in any case it does not depend on x.

The boundary-layer approximation first expands the thickness of the viscous layer so as to make structural details easily visible. Define a new variable

$$\overline{y} = \frac{y}{\epsilon} \tag{1.138}$$

that is of order unity inside the boundary layer, and rewrite the equations, except the pressure terms, as

$$\frac{\partial u}{\partial x} + \frac{1}{\epsilon}\,\frac{\partial v}{\partial \overline{y}} = 0 \ ; \tag{1.139}$$

$$u\,\frac{\partial u}{\partial x} + \frac{v}{\epsilon}\,\frac{\partial u}{\partial \overline{y}} = -\frac{1}{\rho}\,\frac{\partial p}{\partial x} + \frac{\nu}{\epsilon^2}\left(\epsilon^2\,\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial \overline{y}^2}\right) \ ; \tag{1.140}$$

$$u\,\frac{\partial v}{\partial x} + \frac{v}{\epsilon}\,\frac{\partial v}{\partial \overline{y}} = -\frac{1}{\rho}\,\frac{\partial p}{\partial y} + \frac{\nu}{\epsilon^2}\left(\epsilon^2\,\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial \overline{y}^2}\right) \ . \tag{1.141}$$

In the pending limit $\epsilon \to 0$, the continuity equation and the transport operator will be pathological unless the v-component of velocity is also modified by putting

$$\overline{v} = \frac{v}{\epsilon} \ . \tag{1.142}$$

When this is done, the equations become

$$\frac{\partial u}{\partial x} + \frac{\partial \overline{v}}{\partial \overline{y}} = 0 \ ; \tag{1.143}$$

$$u\,\frac{\partial u}{\partial x} + \overline{v}\,\frac{\partial u}{\partial \overline{y}} = -\frac{1}{\rho}\,\frac{\partial p}{\partial x} + \frac{\nu}{\epsilon^2}\left(\epsilon^2\,\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial \overline{y}^2}\right) \; ; \tag{1.144}$$

$$\epsilon\left(u\,\frac{\partial \overline{v}}{\partial x} + \overline{v}\,\frac{\partial \overline{v}}{\partial \overline{y}}\right) = -\frac{1}{\rho}\,\frac{\partial p}{\partial y} + \epsilon\,\frac{\nu}{\epsilon^2}\left(\epsilon^2\,\frac{\partial^2 \overline{v}}{\partial x^2} + \frac{\partial^2 \overline{v}}{\partial \overline{y}^2}\right) \; . \tag{1.145}$$

If the continuity equation is satisfied by the introduction of a stream function, with

$$u = \frac{\partial \psi}{\partial y} \; , \qquad v = -\frac{\partial \psi}{\partial x} \; , \tag{1.146}$$

use of boundary-layer variables leads to

$$u = \frac{1}{\epsilon}\,\frac{\partial \psi}{\partial \overline{y}} \; , \qquad \overline{v} = -\frac{1}{\epsilon}\,\frac{\partial \psi}{\partial x} \; . \tag{1.147}$$

Thus it is also necessary to modify the stream function by putting

$$\overline{\psi} = \frac{\psi}{\epsilon} \; . \tag{1.148}$$

The variables x and u are deliberately left undisturbed by these manipulations, on the premise that the external flow is not much affected by the presence of a thin boundary layer (in the absence of flow separation, as emphasized by Prandtl in his original paper).

The final step is to take the formal inner or boundary-layer limit $\epsilon \to 0$, with the boundary-layer independent variables x and $\overline{y}$ held constant. During the limit process the viscous layer and its structure are preserved, along with the no-slip condition, when viewed in boundary-layer coordinates $(x, \overline{y})$. At the same time, the viscous layer shrinks to zero thickness when viewed in outer coordinates (x, y).

This final step has three important consequences. One is that the first viscous term vanishes in each of the momentum equations (1.144) and (1.145). The physical effect is to suppress diffusion of vorticity in the x-direction, so that the boundary conditions have no upstream influence. The mathematical effect is to change the type of the boundary-layer equations from elliptic to parabolic and to introduce real double characteristics $x =$ constant.

The second consequence of the boundary-layer approximation is that it imposes the condition

$$\epsilon \sim \nu^{1/2} \tag{1.149}$$

when the surviving viscous term $(\nu/\epsilon^2)\,\partial^2 u/\partial \overline{y}^2 = \nu \partial^2 u/\partial y^2$ in equation (1.144) is required to be of the same order as the transport terms in the limit. At this point the definition of ϵ becomes irrelevant, because the argument can be repeated with ϵ replaced everywhere by $\nu^{1/2}$. Although few constraints have so far been put on the nature of solutions to the boundary-layer equations, the condition (1.149) already controls the form of dimensionless similarity variables. For the Blasius problem, for example, the mandatory combinations y^2/ν and ψ^2/ν have the dimensions of time and length^2/time, respectively. The only combinations of x and u_∞ having these same dimensions are x/u_∞ and $u_\infty x$. The dimensionless variables $y(u_\infty/\nu x)^{1/2}$ and $\psi/(u_\infty \nu x)^{1/2}$ thus follow automatically.

The third consequence is that the pressure is moved from the list of dependent variables to the list of boundary conditions. The order of the system of equations is thereby reduced from fourth order to third order in ψ. In the y-momentum equation (1.145), $\partial p/\partial y$ is at most of order ϵ and thus is negligible in the limit, as are the other terms in this equation, which can therefore be discarded. In any application of the x-momentum equation (1.144), $\partial p/\partial x = \mathrm{d}p/\mathrm{d}x$ is typically specified at the outset and is sometimes tailored to generate certain special classes of solutions (see, for example, SECTION 4.2.2).

When the original variables are restored after taking the limit, the boundary-layer approximation in rectangular coordinates is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \;; \tag{1.150}$$

$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = -\frac{1}{\rho}\,\frac{\mathrm{d}p}{\mathrm{d}x} + \nu\,\frac{\partial^2 u}{\partial y^2} \;; \tag{1.151}$$

$$p = p(x) \;. \tag{1.152}$$

These equations correspond to the limit $\nu \to 0$ or $Re \to \infty$. However, they are normally applied for Reynolds numbers that are large compared to unity but by no means infinite. Once a solution has been obtained, it is a tangible advantage of the formal boundary-layer method that the magnitude of terms discarded or retained, and thus the degree of approximation, can be estimated *a posteriori*.

The argument just given is not universal. Several plane laminar flows with the property of similarity are treated within the boundary-layer approximation (1.150)–(1.152) in various parts of this monograph. These flows include the boundary layer in SECTION 4.2.2, the shear layer in SECTION 5.1.2, the plane plume in CHAPTER 11, the plane jet in SECTION 9.1.2, the plane wall jet in SECTION 10.1.2, and the asymptotic suction later in this section. The first two of these flows fit the pattern just described, but the last four do not, as demonstrated in TABLE 1.3.[5]

The obvious difference is the presence or absence of an external stream. For the boundary layer and the shear layer, there is an external stream, and the boundary-layer approximation includes a matching condition $u \to u_\infty$ applied at the outer edge of the boundary-layer flow. For the plume, the jet, and the wall jet, all in a stagnant fluid, there is no external stream, and thus no matching condition. The condition involving u_∞ is replaced by an integral constraint on thermal or momentum flux, and the form of this constraint is different in each case. The flows listed in the table are ordered according to the exponents attached to the viscosity ν, but I have not discovered any deep significance in this order.

As an example of one such flow, consider the laminar plane jet into fluid at rest. Suppose that the boundary-layer equations (1.150)–(1.152) are valid, and in particular that the pressure is everywhere constant. The momentum equation (1.151) can then be integrated across the jet (see SECTION 9.1.1 *et seq.* for details) to

[5] *No Table numbered 1.2 appears in the manuscript.*

Table 1.3

Boundary-layer variables
for laminar plane flows with similarity

Flow	$\overline{x}$	$\overline{y}$	$\overline{\psi}$	$\overline{u}$	$\overline{v}$
Boundary layer	x	$\dfrac{y}{\nu^{1/2}}$	$\dfrac{\psi}{\nu^{1/2}}$	u	$\dfrac{v}{\nu^{1/2}}$
Shear layer	x	$\dfrac{y}{\nu^{1/2}}$	$\dfrac{\psi}{\nu^{1/2}}$	u	$\dfrac{v}{\nu^{1/2}}$
Plane plume	x	$\dfrac{y}{\nu^{3/5}}$	$\dfrac{\psi}{\nu^{2/5}}$	$u\nu^{1/5}$	$\dfrac{v}{\nu^{2/5}}$
Plane jet	x	$\dfrac{y}{\nu^{2/3}}$	$\dfrac{\psi}{\nu^{1/3}}$	$u\nu^{1/3}$	$\dfrac{v}{\nu^{1/3}}$
Plane wall jet	x	$\dfrac{y}{\nu^{3/4}}$	$\dfrac{\psi}{\nu^{1/4}}$	$u\nu^{1/2}$	$\dfrac{v}{\nu^{1/4}}$
Suction layer	x	$\dfrac{y}{\nu}$	ψ	$u\nu$	v

obtain an integral invariant,

$$J = \int_{-\infty}^{\infty} \rho\, u\, u \, \mathrm{d}y = \text{constant} \quad , \tag{1.153}$$

that dominates the further analysis. The global constant J is the momentum flux per unit span in the jet, or equivalently the reaction force per unit span at the jet source. Require this quantity J to be conserved as $\epsilon \to 0$ in the boundary-layer limit. Then equation (1.153) should be written in boundary-layer variables as

$$J = \int_{-\infty}^{\infty} \rho \overline{u}\, \overline{u} \, \mathrm{d}\overline{y} = \text{ constant} \quad , \tag{1.154}$$

where $\overline{y} = y/\epsilon$ as before, but now

$$\overline{u} = u\epsilon^{1/2} \quad , \tag{1.155}$$

rather than $\overline{u} = u$. With this information in hand, rewrite the full Navier-Stokes equations (1.135)–(1.137) as

$$\frac{\partial \overline{u}}{\partial x} + \frac{1}{\epsilon^{1/2}} \frac{\partial v}{\partial \overline{y}} = 0 \ ; \tag{1.156}$$

$$\overline{u}\, \frac{\partial \overline{u}}{\partial x} + \frac{v}{\epsilon^{1/2}} \frac{\partial \overline{u}}{\partial \overline{y}} = -\frac{\epsilon}{\rho} \frac{\partial p}{\partial x} + \frac{\nu}{\epsilon^{3/2}} \left(\epsilon^2 \frac{\partial^2 \overline{u}}{\partial x^2} + \frac{\partial^2 \overline{u}}{\partial \overline{y}^2} \right) \ ; \tag{1.157}$$

$$\frac{\overline{u}}{\epsilon^{1/2}} \frac{\partial v}{\partial x} + \frac{v}{\epsilon} \frac{\partial v}{\partial \overline{y}} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\nu}{\epsilon^{3/2}} \left(\frac{\epsilon^2}{\epsilon^{1/2}} \frac{\partial^2 v}{\partial x^2} + \frac{1}{\epsilon^{1/2}} \frac{\partial^2 v}{\partial \overline{y}^2} \right) \ . \tag{1.158}$$

Evidently it is now necessary to take

$$\overline{v} = \frac{v}{\epsilon^{1/2}} \quad , \tag{1.159}$$

rather than $\overline{v} = v/\epsilon$. The equations of motion become

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial \overline{y}} = 0 \ ; \tag{1.160}$$

$$\overline{u}\, \frac{\partial \overline{u}}{\partial x} + \overline{v}\, \frac{\partial \overline{u}}{\partial \overline{y}} = -\frac{\epsilon}{\rho} \frac{\partial p}{\partial x} + \frac{\nu}{\epsilon^{3/2}} \left(\epsilon^2 \frac{\partial^2 \overline{u}}{\partial x^2} + \frac{\partial^2 \overline{u}}{\partial \overline{y}^2} \right) \ ; \tag{1.161}$$

$$\epsilon\,\overline{u}\,\frac{\partial\overline{v}}{\partial x}+\overline{v}\,\frac{\partial\overline{v}}{\partial\overline{y}}=-\frac{\epsilon}{\rho}\,\frac{\partial p}{\partial y}+\epsilon\,\frac{\nu}{\epsilon^{3/2}}\left(\epsilon^2\,\frac{\partial^2\overline{v}}{\partial x^2}+\frac{\partial^2\overline{v}}{\partial\overline{y}}^{2}\right)\ , \qquad (1.162)$$

instead of equations (1.143)–(1.145). The stream function should be taken in its turn as

$$\overline{\psi}=\frac{\psi}{\epsilon^{1/2}}\ , \qquad (1.163)$$

rather than ψ/ϵ.

Two of the consequences of taking the limit $\epsilon\to 0$ have changed. First, equation (1.149) is replaced for the plane jet by

$$\epsilon\sim\nu^{2/3}\ , \qquad (1.164)$$

so that the boundary-layer variables take the form displayed in the fourth line of TABLE 1.3. Second, all that can be said about the pressure gradient $\partial p/\partial y$ within the viscous layer is that it is at most of order unity rather than of order ϵ, which is to say that $\partial p/\partial\overline{y}$ is of order ϵ. The difference here is connected with the phenomenon of entrainment and with the associated fact that the streamlines in the outer part of the jet are strongly curved (see, for example, FIGURE 9.4), unlike the streamlines in a boundary layer. During the limit process $\epsilon\to 0$, the viscous layer and its structure are again preserved in boundary-layer coordinates $(x,\ \overline{y})$. In outer coordinates $(x,\ y)$, the jet thickness shrinks to zero while the jet velocity u becomes infinite in such a way that the momentum flux J is conserved. Finally, the question of the proper dimensionless form for y and ψ is again settled. The mandatory combinations y^3/ν^2 and ψ^3/ν have the dimensions time^2/length and length^4/time^2, respectively. The only combinations of the remaining variables J/ρ and x with these same dimensions are $\rho x^2/J$ and Jx/ρ. The appropriate similarity variables are therefore $y\,(J/\rho\nu^2x^2)^{1/3}$ and $\psi(\rho/J\nu x)^{1/3}$.

It is important that the limit $\epsilon\to 0$ in equations (1.160)–(1.162) leads back to the original boundary-layer approximation (1.150)–(1.152), except for the minor reservation already stated for the pressure. The argument just given for the laminar plane jet is therefore circular, consistent, and closed. However, this argument is not the one usually found in textbooks and monographs on viscous flow. Of eight such volumes in my personal library, only those by LOITSYANSKII (1966, pp. 559–565) and TRITTON (1977) are faithful to the fourth line of TABLE 1.3. The others begin either with $\psi/\nu^{1/2}$, $y/\nu^{1/2}$, following GOLDSTEIN (1938, Vol. 1, pp. 145–146) or with ψ/ν, y, following ROSENHEAD (1963, pp. 254–256). The correct variables $\psi/\nu^{1/3}$, $y/\nu^{2/3}$ do eventually emerge, but only after the argument is adjusted to satisfy the integral invariant (1.153). The point of this discussion is that the thickness of a diffusing layer does not necessarily vary as the square root of the diffusivity. This property can be shown more directly by using the Lagrangian device of a moving observer, in the style of H.W. Liepmann. This device often allows the exponents to be established correctly without recourse to the governing equations except through integral constraints such as equation (1.153). Return to the case of the Blasius boundary layer or the mixing layer in a uniform half-flow. Suppose that an observer travels with constant free-stream velocity along the streamline (of the inviscid flow) that represents the site of the thin viscous layer. A clock carried by the observer shows a time $t \sim x/u_\infty$. Diffusion of vorticity requires a stationary clock at any station to show a time $t \sim \delta^2/\nu$, where $\delta(x)$ is the local thickness of the diffusing viscous layer. Consequently, a plausible estimate of δ can be obtained by equating times;

$$\delta(x) \sim \left(\frac{\nu x}{u_\infty}\right)^{1/2} . \qquad (1.165)$$

Now consider the plane jet into a stagnant fluid. An observer moving with the local fluid velocity along a streamline (of the viscous flow) in the plane of symmetry reads a time $t \sim x/u_c$ (c for center line). The thickness δ is governed by the same equation (1.165), with u_∞ replaced by $u_c(x)$;

$$\delta(x) \sim \left(\frac{\nu x}{u_c}\right)^{1/2} . \qquad (1.166)$$

But the integral invariant (1.153) requires $\delta u_c^2 \sim J/\rho =$ constant. Elimination of u_c then yields for δ, instead of equation (1.165), the power law

$$\delta(x) \sim \left(\frac{\rho \nu^2 x^2}{J}\right)^{1/3} , \qquad (1.167)$$

and the proper similarity variables are reached in another way.

1.3.3 Subcharacteristics

These qualitative differences for various boundary-layer flows are connected with the role of subcharacteristics in the boundary-layer approximation, as pointed out by KEVORKIAN and COLE (1981, pp. 370–373) for slightly different model equations. The important consideration is that the steady Euler equations; i.e., the inviscid approximation to the Navier-Stokes equations (1.135)–(1.137), have real characteristics. These are the streamlines, on which the stagnation pressure is constant and across which discontinuities in streamwise velocity are permitted. The text of boundary-layer theory is a palimpsest. The real characteristics of the Euler equations are imperfectly erased by addition of the viscous terms and by passage to the boundary-layer limit. They become subcharacteristics of the viscous problem and have subtle but important effects on the form of boundary-layer solutions.

Of various methods for finding characteristic curves for systems of partial differential equations, the one used here has the property that it automatically finds not only the characteristic curves, but also any invariants that may exist on these curves. When applied to isentropic supersonic flow, for example, it produces the Mach lines and the Riemann invariants. As an illustration of this method, consider the Euler equations of motion for steady, inviscid, two-dimensional, laminar flow of an incompressible fluid in rectangular coordinates. When the derivatives of highest order are written in terms of a stream function, with $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$, these equations are

$$u\,\frac{\partial^2\psi}{\partial x\partial y} + v\,\frac{\partial^2\psi}{\partial y^2} + \frac{1}{\rho}\,\frac{\partial p}{\partial x} = 0 \ ; \tag{1.168}$$

$$-\,u\,\frac{\partial^2\psi}{\partial x^2} - v\,\frac{\partial^2\psi}{\partial x\partial y} + \frac{1}{\rho}\,\frac{\partial p}{\partial y} = 0 \ . \tag{1.169}$$

To these, add three definitions for appropriate total differentials;

$$\mathrm{d}x\,\frac{\partial^2\psi}{\partial x\partial y} + \mathrm{d}y\,\frac{\partial^2\psi}{\partial y^2} = \mathrm{d}u \;\; ; \tag{1.170}$$

$$-\mathrm{d}x\,\frac{\partial^2\psi}{\partial x^2} - \mathrm{d}y\,\frac{\partial^2\psi}{\partial x\partial y} = \mathrm{d}v \;\; ; \tag{1.171}$$

$$\mathrm{d}x\,\frac{\partial p}{\partial x} + \mathrm{d}y\,\frac{\partial p}{\partial y} = \mathrm{d}p \;\; . \tag{1.172}$$

Let these five equations be viewed as a system of linear algebraic equations for the three second derivatives of ψ and the two first derivatives of p. The solution is readily worked out by the method of determinants and verified by substitution. It is

$$\frac{\partial^2\psi}{\partial x^2} = \frac{(\mathrm{d}y)^2\,\mathrm{d}p_0 - \rho\mathrm{d}\psi(\mathrm{d}v\mathrm{d}x + \mathrm{d}u\mathrm{d}y)}{\rho(\mathrm{d}s)^2\mathrm{d}\psi} \;\; ; \tag{1.173}$$

$$\frac{\partial^2\psi}{\partial x\partial y} = \frac{-\mathrm{d}x\mathrm{d}y\mathrm{d}p_0 + \rho\mathrm{d}\psi(\mathrm{d}u\mathrm{d}x - \mathrm{d}v\mathrm{d}y)}{\rho(\mathrm{d}s)^2\,\mathrm{d}\psi} \;\; ; \tag{1.174}$$

$$\frac{\partial^2\psi}{\partial y^2} = \frac{(\mathrm{d}x)^2\,\mathrm{d}p_0 + \rho\mathrm{d}\psi(\mathrm{d}v\mathrm{d}x + \mathrm{d}u\mathrm{d}y)}{\rho(\mathrm{d}s)^2\,\mathrm{d}\psi} \;\; ; \tag{1.175}$$

$$\frac{\partial p}{\partial x} = \frac{\mathrm{d}p\mathrm{d}x + \rho\mathrm{d}y(u\mathrm{d}v - v\mathrm{d}u)}{(\mathrm{d}s)^2} \;\; ; \tag{1.176}$$

$$\frac{\partial p}{\partial y} = \frac{\mathrm{d}p\mathrm{d}y - \rho\mathrm{d}x(u\mathrm{d}v - v\mathrm{d}u)}{(\mathrm{d}s)^2} \;\; , \tag{1.177}$$

where $(\mathrm{d}s)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2$ and $p_0 = p + \rho(u^2 + v^2)/2$. The given data on the right-hand sides include the velocity, $\vec{u} = (u,\, v)$; the direction of the derivative, $\mathrm{d}\vec{x} = (\mathrm{d}x,\ \mathrm{d}y)$; and the associated differentials, $\mathrm{d}p$ and $\mathrm{d}\vec{u} = (\mathrm{d}u,\ \mathrm{d}v)$.

Characteristic curves are determined by the condition that the right-hand sides of the system (1.173)–(1.177) take on the form

0/0, so that the highest derivatives are not defined. In the last two equations, (1.176) and (1.177), the denominator never vanishes, and the static pressure is therefore continuous. In the first three equations, (1.173)–(1.175), the denominators vanish on streamlines, where $\mathrm{d}\psi = -v\mathrm{d}x + u\mathrm{d}y = 0$, or $v/u = \mathrm{d}y/\mathrm{d}x$. The numerators then vanish when $\mathrm{d}p_0 = 0$. It follows that the streamlines are characteristics of the Euler equations, and that the stagnation pressure p_0 is constant on streamlines (there are, of course, easier ways to prove this particular result). On a streamline, the five equations (1.168)–(1.172) are no longer linearly independent. The quantities u, v, $\mathrm{d}u$, $\mathrm{d}v$, $\mathrm{d}p$ cannot be chosen independently/ but must satisfy the condition $\mathrm{d}p + \rho u\mathrm{d}u + \rho v\mathrm{d}v = 0$.

At the next level of detail, it is necessary to consider whether the inviscid flow is rotational or irrotational. To distinguish the two cases, add equations (1.173) and (1.175) to construct the Laplacian,

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \frac{1}{\rho}\,\frac{\mathrm{d}p_0}{\mathrm{d}\psi} \ , \tag{1.178}$$

and recall from SECTION X[6], for flow of an incompressible fluid in two dimensions, that

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = -\,\zeta \ , \tag{1.179}$$

where ζ is the z-component of vorticity. Hence

$$\frac{1}{\rho}\,\frac{\mathrm{d}p_0}{\mathrm{d}\psi} = -\,\zeta \ , \tag{1.180}$$

and the numerators in equations (1.173)–(1.175), like the denominators, have $\mathrm{d}\psi$ as a factor, whether the flow is rotational or not. This factor can be cancelled unless $\mathrm{d}\psi = 0$.

To recapitulate: in the viscous problem, with ν small but not zero, a viscous layer may develop along a subcharacteristic; i.e., a characteristic of the inviscid equations (1.168)–(1.169). If it does, then the original boundary-layer approximation (1.150)–(1.152) is

[6] *Unclear what section is intended.*

appropriate, because there is an external stream, and the thickness of the viscous layer is proportional to $\nu^{1/2}$. If the viscous layer does not develop along a subcharacteristic, each case has to be treated separately. Three flows in TABLE 1.3 fall into the second class, because there is no external stream, and thus no subcharacteristics.

Because the same boundary-layer equations are ultimately obtained in the limit $\epsilon \to 0$, this discussion of subcharacteristics may seem to be irrelevant, or at least academic. The discussion is not academic, however, if higher-order solutions are wanted and basis functions are required for a systematic expansion. Moreover, at least two of the three flows in question can also appear as flows into a moving fluid, shifting the problem from the second class to the first one and altering the dimensionless similarity variables.

Two further examples reinforce these comments and demonstrate that the presence of a free stream is neither a necessary nor a sufficient condition for the thickness of a laminar viscous layer to vary like $\nu^{1/2}$. The first example is boundary-layer flow with suction or blowing at the wall. Although the flow has a free stream, streamlines of the inviscid flow pass through the wall. A special case mentioned in many texts on viscous flow is the laminar asymptotic suction layer. The asymptotic flow has no x-dependence, and the continuity equation therefore requires $v = v_w = \text{constant} < 0$. The velocity profile in the boundary layer is easily worked out as

$$\frac{u}{u_\infty} = 1 - exp(yv_w/\nu) \ . \tag{1.181}$$

The wall friction for this flow can be obtained either from the truncated momentum-integral equation or by calculating $\mu(\partial u/\partial y)_w$ directly for the profile (4.51).[7] In either case,

$$\tau_w = -\rho v_w u_\infty \ . \tag{1.182}$$

The flow is pathological in the sense that the wall friction does not depend on the viscosity of the fluid. It depends only on the product $(\rho u_\infty)(v_w)$, where ρu_∞ is the initial x-momentum per unit volume

[7] *See Section 4.2.3.*

of the fluid that is eventually removed at the wall, and $-v_w$ is the volume flow of this fluid per unit wall area per unit time. The proper entry for this flow in TABLE 1.3 is x, y/ν, ψ, $u\nu$, v.

The second example is the free-convection boundary layer on a vertical wall, or the wall plume, discussed in SECTION X. By inspection of the results **(check)**, the proper entry for this flow in TABLE 1.3 is x, $y/\nu^{1/2}$, $\psi/\nu^{1/2}$, u, $v/\nu^{1/2}$, exactly as for the Blasius boundary layer, despite the fact that the wall plume has no free stream.

1.3.4 Reynolds averaging

This monograph deals with motions that are turbulent, by which I mean motions that are locally three-dimensional, non-steady, rotational, and, for practical purposes, random, at least in the components of intermediate and small scales. Define any physical variable ϕ, such as a component of the velocity vector $\vec{u}$, as the sum of a mean value and a fluctuation. The mean value is traditionally denoted by an overbar, representing a time or a statistical average. For example,

$$\overline{\phi} = \lim_{T\to\infty} \frac{1}{T} \int_o^T \phi(t)\mathrm{d}t \quad \text{or} \quad \overline{\phi} = \int_{-\infty}^{\infty} \phi\, p(\phi)\, \mathrm{d}\phi \ , \tag{1.183}$$

where T is some suitable time interval. In the second definition, the probability density $p(\phi)$ refers to relative frequency of a particular value of ϕ in a large number of repeated observations. In this monograph I will seldom consider problems that are not steady in the mean.

Reynolds stresses, first defined in a seminal paper by REYNOLDS (1895), are generated by replacing each variable in the equations of motion by a mean value plus a fluctuation and then averaging. The method of averaging insures that fluctuations have zero mean, because

$$\phi = \overline{\phi} + \phi' \tag{1.184}$$

becomes

$$\overline{\phi} = \overline{\overline{\phi}} + \overline{\phi'} \ , \tag{1.185}$$

from which

$$\overline{\phi'} = 0 \ . \tag{1.186}$$

The averaging operation for a steady mean flow of an incompressible fluid yields

$$\text{div}\ \overline{\vec{u}} = 0 \ ; \tag{1.187}$$

$$\rho\ \text{div}\ (\underline{\overline{\vec{u}} \circ \overline{\vec{u}}}) = -\text{grad}\ \overline{p} + \text{div}\ \underline{\overline{\tau}} \ , \tag{1.188}$$

where $\overline{\underline{\mathrm{u} \circ \mathrm{u}'}} = 0$ and

$$\underline{\overline{\tau}} = \mu\ \text{div}\ \underline{\text{grad}\ \vec{u}} - \rho\,(\overline{\underline{\mathrm{u}' \circ \mathrm{u}'}}) \ . \tag{1.189}$$

The last term is the Reynolds stress tensor. (*Discuss kinetic theory,* $\tau \sim -\rho\ \overline{c_i c_j}$. *Check Reynolds; did he mention this? See Kennard, p 178.*)

Averaging of the dyadic product $\underline{\tilde{\mathrm{u}} \circ \tilde{\mathrm{u}}}$ generates the Reynolds stresses together with the product of the means, $\underline{\overline{\vec{u}} \circ \overline{\vec{u}}}$. Thus the continuity equation can be used if desired to recover the form of equation (1.32) for mean quantities. However, the general result of the averaging operation is best described as dry water. There are six new unknowns; only six because $\overline{\underline{\vec{u}' \circ \vec{u}'}}$ is a symmetric tensor. Unfortunately, there are no new equations. In simplistic terms, what is normally done about the six new variables is to reduce their number, by plausible arguments, or by brute force, until there is only one left, usually the shearing stress $\overline{u'v'}$. Then one additional equation is invented. The new equation has no physics in it, although it may (must, to be successful) have some insight. For example, Prandtl's mixing-length model shows profound insight, and I am quite willing to use it if there is nothing better. Not much is contributed to understanding of turbulence by analysis of what has been written down for the missing equation or equations, although the record of the two Stanford contests (   ) may be worth study.

To see what can be done, consider a turbulent shear flow that is two-dimensional and steady in the mean. The mean velocity is ($\overline{u}$, $\overline{v}$); the velocity fluctuations are (u', v', w'); and the Reynolds stress tensor in rectangular coordinates is

$$\underline{\tau} = -\rho \begin{pmatrix} \overline{u'u'} & \overline{u'v'} & \overline{u'w'} \\ \overline{v'u'} & \overline{v'v'} & \overline{v'w'} \\ \overline{w'u'} & \overline{w'v'} & \overline{w'w'} \end{pmatrix} \ . \tag{1.190}$$

Two of the six Reynolds stresses can be assumed to have zero mean, namely $\overline{u'w'}$ and $\overline{v'w'}$, on the ground of reflection symmetry. There is no correlation, as there is for $\overline{u'v'}$. The equations to be considered are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \ , \tag{1.191}$$

$$\begin{aligned} \rho\, u \frac{\partial u}{\partial x} + \rho\, v \frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x} \ + \\ + \ \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) &+ \frac{\partial}{\partial x} \left(-\rho \overline{u'u'} \right) + \frac{\partial}{\partial y} \left(-\rho \overline{u'v'} \right) \ , \end{aligned} \tag{1.192}$$

$$\begin{aligned} \rho\, u \frac{\partial v}{\partial x} + \rho\, v \frac{\partial v}{\partial y} &= -\frac{\partial p}{\partial y} \ + \\ + \ \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) &+ \frac{\partial}{\partial x} \left(-\rho \overline{v'u'} \right) + \frac{\partial}{\partial y} \left(-\rho \overline{v'v'} \right) \ , \end{aligned} \tag{1.193}$$

$$0 = \frac{\partial}{\partial z} \left(-\rho \overline{w'w'} \right) \ , \tag{1.194}$$

where mean values for u, v, p are now understood rather than expressed by an overbar, to simplify the notation.

Equation (1.194) is empty but is displayed as a reminder that $\overline{w'w'}$ is not zero. All the muscle is gone from Prandtl's order-of-magnitude argument for a boundary-layer approximation, because the order of magnitude of the turbulent terms is not known with any precision in the absence of experimental data. It is not possible to say that equations (1.191)–(1.193) are an elliptic or parabolic system. There may be no time-like or x-like characteristics of the system. Nevertheless, upstream influence in time or in x is not expected except for separation. There is little guidance about what to do in the turbulent case except by analogy.

Now consider channel flow, which like pipe flow is a special case because of the geometry. All x-derivatives are zero except for the pressure. In particular, $v = 0$. The equations of motion are reduced to

$$0 = -\frac{\partial p}{\partial x} - \frac{\partial}{\partial y}\left(\rho\,\overline{u'v'}\right) + \mu\frac{\partial^2 u}{\partial y^2} \ , \tag{1.195}$$

$$0 = -\frac{\partial p}{\partial y} - \frac{\partial}{\partial y}\left(\rho\,\overline{v'v'}\right) \ . \tag{1.196}$$

The first equation can also be written

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial y} \ , \tag{1.197}$$

where

$$\tau = \mu\,\frac{\partial u}{\partial y} - \left(\rho\,\overline{u'v'}\right) \ . \tag{1.198}$$

The second equation can be integrated rigorously to obtain

$$p + \rho\,\overline{v'v'} = p_w \ , \tag{1.199}$$

where w indicates a wall value. In channel flow this equation may be useful in estimating the mean pressure p away from the wall, this being more difficult to measure than the velocity fluctuation v'. Note that p_w is a function only of x and that $\overline{v'v'}$ is a function only of y, but that p is a function of both. Occasionally, an author will cite equation (1.199) for turbulent boundary-layer flow. The argument is that $v \ll u$ and $\partial/\partial x \ll \partial/\partial y$, so that the laminar boundary-layer approximation holds. Then equation (1.199) can be differentiated with respect to x and the result substituted in equation (1.192) to obtain

$$\begin{aligned}\rho\,u\frac{\partial u}{\partial x} + \rho\,v\frac{\partial u}{\partial y} &= -\frac{\partial p_w}{\partial x} + \\ &+ \frac{\partial}{\partial x}\left(\rho\,\overline{v'v'} - \rho\,\overline{u'u'}\right) - \frac{\partial}{\partial y}\left(\rho\,\overline{u'v'}\right) + \mu\frac{\partial}{\partial y}\,\frac{\partial u}{\partial y} \ .\end{aligned} \tag{1.200}$$

Then if $\overline{u'u'}$ and $\overline{v'v'}$ are very nearly equal (they are not) the only surviving Reynolds stress is the shearing stress $-\rho\overline{u'v'}$. I don't recommend this argument, since the boundary-layer version is not on solid ground. I prefer to be guided by the aphorism that the only truth about turbulence is experimental truth, including the results of careful computer simulations. Liepmann, who was no respecter of archaic traditions, once said that the invention of Reynolds stresses impeded research in turbulence in the same way that the invention of the vacuum tube impeded the development of the transistor. In fact, eighty years passed while the turbulence research community struggled with the absence of phase information in Reynolds' model, until the rise of computer-assisted instrumentation finally provided access to the revolutionary concept of coherent structure.

1.3.5 Dimensions and similarity

Fluid mechanics is characterized by an almost religious dedication to the concept of dimensionless variables. The rules of this discipline are sometimes applied at a primitive level, the objective being to discover dimensionless variables without necessarily discovering any bond between them except through experiment. Turbulent pipe flow is a good example. The spirit of this subject is the same one that prompts the use of semi-log or log-log coordinates to display exponential or power-law behavior and occasionally to show that different laws are operating in different ranges of the variables. A case in point is the plane jet from a long unshielded rectangular orifice into an unconfined fluid. Near the orifice, the flow behaves like a plane jet. Far downstream, the flow must behave like a round jet, with different similarity laws. For each of the flows discussed in this monograph it is assumed that the flow is determined by a finite number of global parameters, physical or geometrical, some important and some not. These parameters are the ostensible data of the problem. They are numbers that would have to be communicated from one observer to another in order for a particular experiment to be duplicated precisely.

The subject of dimensional analysis is so important in fluid mechanics that it has generated its own literature (SEDOV, more). Applications can be divided roughly into two categories; those that apply when the problem is defined by differential equations, and those that apply when it is not. Both categories will be illustrated in what follows.

Stokes. My first example is a dimensional argument that uses group theory to solve a linear problem, the unsteady laminar motion produced in an incompressible fluid by the impulsive motion of an infinite flat plate in its own plane. The differential equation in question is

$$\frac{\partial u}{\partial t} = \nu \, \frac{\partial^2 u}{\partial y^2} \ , \tag{1.201}$$

with the boundary conditions

$$\begin{aligned} u(y,\ t) &= 0 \qquad t < 0 \ , \\ u(0,\ t) &= U \qquad t > 0 \ , \\ u(\infty,\ t) &= 0 \qquad \text{all } t \ , \end{aligned} \tag{1.202}$$

where U is a constant velocity. The resulting motion is shown in FIGURE 1.2. The fluid problem was first solved by STOKES (19XX), who had at the time reservations about the validity of the no-slip condition. One method of solution that is both physical and general is group theory, because it can serve for problems that are linear or non-linear. Consider an affine transformation of all of the variables and parameters of problem (1.201)–(1.202). Let unmarked quantities refer to one flow, and let quantities marked by a circumflex refer to another flow obtained by the transformation

$$\begin{aligned} y &= a\widehat{y} \ , \\ t &= b\widehat{t} \ , \\ u &= c\widehat{u} \ , \\ U &= d\widehat{U} \ , \\ \nu &= e\widehat{\nu} \ , \end{aligned} \tag{1.203}$$

where the scaling parameters a, b, $\ldots$ are finite dimensionless numbers. No distinction is made in this formulation between independent variables, dependent variables, and parameters, whether entering through the equation or the boundary conditions. Everything

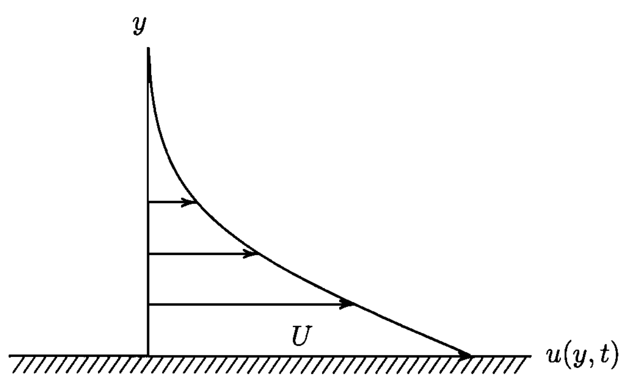

Figure 1.2: The laminar velocity profile for impulsive motion of an infinite flat plate in its own plane.

is included. Equations (1.203) form a group. Application of the transformation to the differential equation (1.201) and the boundary conditions (1.202) gives

$$\frac{c}{b}\frac{\partial\widehat{u}}{\partial\widehat{t}} = \frac{ce}{a^2}\,\widehat{\nu}\,\frac{\partial^2\,\widehat{u}}{\partial\widehat{y}^{\,2}} \quad , \tag{1.204}$$

together with

$$\begin{aligned} c\widehat{u}\,(a\widehat{y},\ \ b\widehat{t}) &= 0 \ , & b\widehat{t} &< 0 \ , \\ c\widehat{u}\,(0,\ \ b\widehat{t}) &= d\widehat{U} \ , & b\widehat{t} &> 0 \ , \\ c\widehat{u}\,(\infty,\ \ b\widehat{t}) &= 0 \ , & \text{all } & b\widehat{t} \ . \end{aligned} \tag{1.205}$$

The conditions that make the differential equation (1.204) and the middle boundary condition (1.205) invariant to the transformation are evidently

$$\frac{a^2}{be} = 1, \qquad \frac{c}{d} = 1 \ . \tag{1.206}$$

These are the key relationships. Substitution in equations (1.206) for a, b, c, ... from equations (1.203) yields

$$\frac{y^2}{\widehat{y}^2}\,\frac{\widehat{t}}{t}\,\frac{\widehat{\nu}}{\nu} = 1 \ , \qquad \frac{u}{\widehat{u}}\,\frac{\widehat{U}}{U} = 1 \ , \tag{1.207}$$

or better

$$\eta = \frac{y}{(\nu t)^{1/2}} = \frac{\widehat{y}}{(\widehat{\nu}\,\widehat{t})^{1/2}} = \widehat{\eta} \ , \tag{1.208}$$

$$f = \frac{u}{U} = \frac{\widehat{u}}{\widehat{U}} = \widehat{f} \ . \tag{1.209}$$

At this point the material of the problem has been reduced to two dimensionless combinations, η and f. It is natural to assume that one combination must be a function of the other;

$$\frac{u}{U} = f\left(\frac{y}{\sqrt{vt}}\right) \ . \tag{1.210}$$

A rigorous argument that this assumption is correct can be found in SECTION 9.1.2 of Chapter 9. The function in question satisfies the ordinary differential equation

$$f'' + \frac{1}{2}\eta f' = 0 \ . \tag{1.211}$$

With the boundary conditions (1.202) and use of a table of integrals, the solution emerges as the complementary error function,

$$f(\eta) = 1 - \frac{1}{\sqrt{\pi}} \int\limits_0^\eta e^{-x^2} \mathrm{d}x \ . \tag{1.212}$$

This group method will be applied throughout this monograph to a variety of flows, laminar and turbulent. Similarity solutions are important. Nature tends to them carefully, and they may play a role in non-similar problems as well.

Kolmogorov. My second example of dimensional analysis is a celebrated prediction by KOLMOGOROV (1941) of a property of homogeneous isotropic turbulence called the $-5/3$ power law. Note that a periodic function calls for a Fourier series. A function that vanishes at infinity calls for a Fourier integral. Turbulence is neither of these. It requires a new kind of mathematics called generalized harmonic analysis. This subject was introduced to the fluid mechanics community in three papers by G. I. TAYLOR ( ) and later in a monograph by BATCHELOR ( ). The primary elements are the two-point covariance and the spectral density as Fourier transforms of each other. (**Check**)

Consider an energy density $E(\kappa)$, defined by

$$\int_0^\infty E\left(\kappa\right) \mathrm{d}\kappa = \frac{1}{2}\left(\overline{u'u'} + \overline{v'v'} + \overline{w'w'}\right) \ , \tag{1.213}$$

where κ is wave number or waves per unit length. The dimensional units of E are energy per unit mass and per unit wave number. The function $E(\kappa)$ may depend on numerous parameters, including various properties of the mechanism that is stirring the fluid.

Kolmogorov's first assumption is that only two parameters are important. These are ϵ, the rate of energy dissipation per unit mass and unit time, and ν, the kinematic viscosity of the fluid. Assume, therefore, that there exists for some range of κ a relationship

$$E = E\left(\kappa;\ \epsilon,\ \nu\right)\ . \tag{1.214}$$

The dimensions of the independent variable κ and the dependent variable E are, respectively,

$$[\kappa] = \frac{1}{\text{length}} = \frac{1}{L}\ , \tag{1.215}$$

$$[E] = \frac{\text{energy} \cdot \text{length}}{\text{mass}} = U^2 L\ . \tag{1.216}$$

These relationships are not equations in the normal sense. The notation $[\ldots] =$ is read "the dimensions of $[\ldots]$ are." The two parameters ϵ and ν have dimensions

$$[\,\epsilon\,] = \frac{\text{energy}}{\text{mass} \cdot \text{time}} = \frac{U^2}{T} = \frac{U^3}{L}\ , \tag{1.217}$$

$$[\,\nu\,] = \text{velocity} \cdot \text{length} = UL\ . \tag{1.218}$$

Suppose that E and κ are to be made dimensionless with ϵ and ν. It is almost self-evident that in the present problem there are just enough dimensional relationships to make this possible. Schematically, look for dimensionless combinations

$$[\kappa\epsilon^a\,\nu^b\,] = 0 = \frac{1}{L}\left(\frac{U^3}{L}\right)^a (UL)^b\ , \tag{1.219}$$

$$\left[E\,\epsilon^c\,\nu^{\mathrm{d}}\right] = 0 = U^2 L\left(\frac{U^3}{L}\right)^c (UL)^{\mathrm{d}}\ . \tag{1.220}$$

These statements can be satisfied only if $a = -1/4$, $b = 3/4$ and $c = -1/4$, $\mathrm{d} = -5/4$. The desired dimensionless form of the relationship (1.214) is therefore

$$\frac{E}{\epsilon^{1/4}\,\nu^{5/4}} = F\left(\frac{\kappa\nu^{3/4}}{\epsilon^{1/4}}\right) = F(\eta)\ ,\ \text{say}\ . \tag{1.221}$$

Kolmogorov introduced the notion of what is now called the inertial subrange. His first hypothesis that the spectral density $E(\kappa)$ is uniquely determined by the quantities ν and ϵ is securely fomulated in the dimensional argument. The largest eddies in a flow are the scale at which it is stirred. The smallest eddies are at a scale where dissipation dominates. If the two scales are well separated, the important quantity becomes the rate that energy is supplied to the system at large scales and removed at small scales. The problem is assumed to be stationary. The energy flows through a series of scales through a mechanism that is called a cascade but is difficult to describe.

Kolmogorov's second hypothesis is that in the inertial subrange $E(\kappa)$ is uniquely determined by the quantity ϵ and does not depend on ν. That is, there are intermediate scales, not small enough for viscosity to be important, but not large enough to have significant memory of the process that supplies energy to the flow. In the inertial subrange, the hypothesis is that $\partial E/\partial\nu = 0$. Equation (1.221) then implies

$$\frac{5}{3}\,F + \eta\,\frac{dF}{d\eta} = 0 \ , \tag{1.222}$$

and thus immediately

$$F = c\,\eta^{-5/3} \ , \tag{1.223}$$

where c is a constant of integration. In dimensional form,

$$E(\eta) = c\,\epsilon^{2/3}\,\kappa^{-5/3} \ . \tag{1.224}$$

A formally equivalent but physically more transparent argument illustrates a different approach to the same problem. In this argument, quantities like M, L, U, T are no longer the dimensions of a quantity, but physical scales that are characteristic of the process. The distinction will be made by using bold-face symbols for such quantities. In the present instance, equations (1.217) and (1.218) are replaced by

$$\epsilon = \frac{\mathbf{U}^3}{\mathbf{L}} \ , \tag{1.225}$$

$$\nu = \mathbf{UL} \ , \tag{1.226}$$

where the absence of the notation $[\ldots]$ means that these equations now *define* physical scales **U** and **L**. When the last two equations are solved for **U** and **L**, the result is

$$\mathbf{U} = (\nu\,\epsilon)^{1/4} \ , \tag{1.227}$$

$$\mathbf{L} = \left(\frac{\nu^3}{\epsilon}\right)^{1/4} \ . \tag{1.228}$$

Normalization of E and κ in the form

$$\frac{E}{\mathbf{U}^2\mathbf{L}} = F\,(\kappa\,\mathbf{L}) \tag{1.229}$$

leads again to equation (1.221). The advantage of the second approach is that it creates an image of fluid elements of size **L** moving at velocity **U**. These characteristic scales are related to each other and to the global parameters by

$$\frac{\mathbf{UL}}{\nu} = 1 \ , \tag{1.230}$$

$$\frac{\epsilon\,\mathbf{L}}{\mathbf{U}^3} = 1 \ , \tag{1.231}$$

so that the characteristic Reynolds number is unity.

The argument just given is in one sense unique. If my understanding is correct, equation (1.224) is the only prediction about turbulent flow that was made before there were any reliable measurements. The ideas of Kolmogorov are not of themselves central to this monograph. However, there is a close connection, developed in SECTION X below, with an idea that *is* central, namely, the existence of a logarithmic law for the mean-velocity profile in flow near a wall.

Readers who consult Kolmogorov's paper may be surprised to discover that he did not use the variables E and κ, but rather the double correlation of velocity fluctuations measured at different points in the fluid. Consequently, Kolmogorov's assumptions are tailored to the argument. If there were another scale, say the mesh size of a grid in a wind tunnel, the argument might break down. There would also be the solidity of the grid, which is already a dimensionless number.

P L A C E H O L D E R

Figure 1.3: Caption for Figure with label Fig1-10 (missing).

FIGURE 1.3 is a collection of spectral measurements from a variety of sources found in nature or manufactured in a laboratory. These include boundary layer, channel, pipe, jet, wake, tidal channel, and grid turbulence. In general, these flows are neither homogeneous nor isotropic. I have not vetted this figure, but the display may not be as simple as it seems. What is measured is usually frequency, which is not directly convertible to wave number without a knowledge of phase velocity. The critical quantity fueling equation (1.224), the dissipation ϵ, is difficult to measure except perhaps in grid turbulence.

Two flows that are not represented in figure 1.3 are the mixing layer and the vortex street. These two flows have several features in common, and their mechanisms can be tested by descriptions that are at present more than conjecture but less than conviction. They appear in laminar and turbulent versions and tend to a two-dimensional form. The topology is also similar. There is a fundamental instability leading to periodic vorticity concentrations. When the motion is viewed in coordinates moving at the celerity of the large eddies, the pattern is as sketched in the two cartoons in FIGURE 1.4, which is reproduced from COLES (1985). In each case the pattern consists of alternating centers and saddles. An unexpected common property of the two flows is the location of the region of maximum turbu-

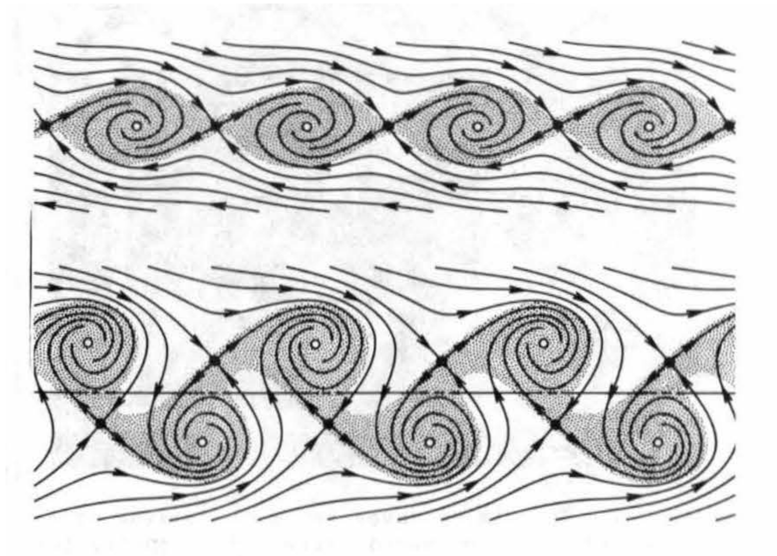

Figure 1.4: Topological cartoons of the vortex street and the mixing layer (caption by K. Coles).

lence production, defined as $\overline{\underline{\tau} \cdot \text{grad}\, \vec{u}}$. It was shown experimentally for the mixing layer by HUSSAIN ( ) and for the vortex street by CANTWELL and COLES ( ) that this maximum is closer to the saddle points than it is to the large turbulent centers. Each saddle lies in a novel structure, called a braid, that is the locus of contact between irrotational fluids arriving from the two sides of the mixing layer or vortex street. The braid contains the two outgoing separatrices of the saddle. These carry entrained fluid in both directions to the two nearest centers. In this process, vortex tubes, called ribs, are formed in the braid by another instability that is not yet completely understood, although the potential flow very near a two-dimensional saddle point may be eligible to undergo Taylor-Görtler instability. Vortex tubes of alternate sign are shown in the elegant view of a mixing layer in FIGURE 1.5 from a paper by METCALF *et al.* (1987). This paper records elaborate numerical solutions of the Navier-Stokes equations for the laminar mixing layer. The emphasis is on vorticity rather than velocity, so the figure does not display the

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Figure 1.5: Caption for Figure with label Fig1-12 (missing).

alleged saddle points directly.