Topics in Shear Flow

Chapter 5 – The Shear Layer

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Chapter 5

THE SHEAR LAYER

The shear layer or mixing layer is a more delicate analytical problem than those treated so far. The main application of this flow is at the edge of a jet or flow over a cavity. See high-bypass jet engines. An important special case is that of different densities. This is a prototype problem for coherent structures.

Roshko and some other investigators prefer the term organized structure to the term coherent structure, because of the meaning of the word coherent in optics and other wave phenomena. My own position is that coherent has another meaning, as in coherent speech, that is quite appropriate.

5.1 Plane laminar shear layer

The flow shown in FIGURE X is a plane shear layer between two parallel streams having constant velocities u_1 and u_2, where u_1 in a standard notation denotes the upper, higher-speed stream. The two streams are separated for $x < 0$ by a thin splitter plate or septum whose boundary layers are neglected in the analysis, although they can be a source of difficulty in practice. Especially when the velocity u_2 in the lower stream is small, the plane mixing layer is boundary-layer-like in its upper portion and jet-like in its lower portion.

5.1.1 Equations of motion

The laminar boundary-layer approximation in rectangular coordinates is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5.1}$$

$$\rho\left(\frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y}\right) = \mu \frac{\partial^2 u}{\partial y^2} \ . \tag{5.2}$$

The boundary conditions are that the pressure is constant everywhere and that the external streams are uniform;

$$u(x,\ \infty) = u_1 \ , \qquad u(x,\ -\infty) = u_2 \ . \tag{5.3}$$

The global parameters for the shear layer are u_1, u_2, ρ, μ. These parameters provide two characteristic velocities, u_1 and u_2, and corresponding lengths, ν/u_1 and ν/u_2, but no dimensionless combination except the velocity ratio u_2/u_1 itself, so that the mixing layer forms a one-parameter family of flows.

Within the boundary-layer approximation, the flow in the shear layer is not fully determined by the global parameters just listed. The root of the problem is the lack of symmetry in the boundary conditions, and the resolution of the problem is in some degree still open.

The standard first step is to test for the existence of an integral invariant. Let the momentum equation (5.2) be integrated formally from $-\infty$ to ∞ in y (this means from $-y$ to y, with $y \to \infty$ as a final step). The form obtained is

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{-\infty}^{\infty} uu \, \mathrm{d}y = -u_1 v_1 + u_2 v_2 \tag{5.4}$$

where

$$v_1 = v(x,\, \infty)\, , \qquad v_2 = v(x,\, -\infty) \ . \tag{5.5}$$

The same operation on the continuity equation (5.1) yields

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{\infty} u\,\mathrm{d}y = -v_1 + v_2 \ . \tag{5.6}$$

To avoid the difficulty that the integrals diverge, consider the identity

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{\infty}(u-u_1)(u-u_2)\mathrm{d}y = \frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{\infty} uu\,\mathrm{d}y - (u_1+u_2)\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{\infty} u\,\mathrm{d}y \ . \tag{5.7}$$

Substitution of equations (5.4) and (5.6) in (5.7) yields

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{\infty}(u-u_1)(u-u_2)\,\mathrm{d}y = -u_1 v_2 + u_2 v_1 \ . \tag{5.8}$$

The dimensionless form of this integral will be considered in SECTION X.

A different but equivalent approach is to treat equations (5.4) and (5.6) as linear algebraic equations for v_1 and v_2. Solution gives

$$v_1 = \frac{1}{(u_1-u_2)}\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{\infty} u\,(u_2-u)\,\mathrm{d}y \tag{5.9}$$

and

$$v_2 = \frac{1}{(u_1-u_2)}\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{\infty} u\,(u_1-u)\,\mathrm{d}y \ . \tag{5.10}$$

Substitution of these expressions in equation (5.4) leads back to equation (5.7).

5.1.2 Similarity

The appropriate affine transformation is

$$\begin{aligned} x &= a\widehat{x} \\ y &= b\widehat{y} \\ \psi &= c\widehat{\psi} \\ \rho &= d\widehat{\rho} \\ \mu &= e\widehat{\mu} \\ u_1 &= p\widehat{u}_1 \\ u_2 &= q\widehat{u}_2 = p\widehat{u}_2 \\ u &= r\widehat{u} = \frac{c}{b}\widehat{u} \\ v &= s\widehat{v} = \frac{c}{a}\widehat{v} \ . \end{aligned} \tag{5.11}$$

Some of the scaling factors in equation (5.4) are redundant, as indicated in the third column. First, introduction of a stream function defined by $\vec{u} = \operatorname{grad}\psi \times \operatorname{grad} z$ or equivalently by

$$u = \frac{\partial \psi}{\partial y} \ , \qquad v = -\frac{\partial \psi}{\partial x} \tag{5.12}$$

leads to

$$r\widehat{u} = \frac{c}{b}\frac{\partial \widehat{\psi}}{\partial \overline{y}} \ , \qquad s\widehat{v} = -\frac{c}{a}\frac{\partial \widehat{\psi}}{\overline{x}} \tag{5.13}$$

and thus to the equalities $r = c/b$, $s = c/a$. Second, the boundary conditions (5.3) are transformed to

$$r\widehat{u}(a\widehat{x},\ \infty) = p\widehat{u}_1, \qquad r\widehat{u}(a\widehat{x},\ -\infty) = q\widehat{u}_2 \ . \tag{5.14}$$

Because $a\widehat{x}$ is read "any value of $\widehat{x}$," the value of a is immaterial, so that $r = p = q$. These results are incorporated in the group (5.11). They yield one invariant of the mapping, representing the boundary conditions. I take u_1 as fundamental because it is by definition never zero, and rewrite the equivalence $r = c/b = p$ as

$$\frac{c}{bp} = 1 \ . \tag{5.15}$$

Transformation of the momentum equation (5.2) yields a second invariant,

$$\frac{bdc}{ae} = 1 \ . \tag{5.16}$$

After isolation of c and b, these invariants become

$$\frac{c^2 d}{aep} = 1, \qquad \frac{b^2 dp}{ae} = 1 \tag{5.17}$$

and thus lead to the preliminary ansatz

$$A\frac{\psi}{(U\nu x)^{1/2}} = f\left[B\left(\frac{U}{\nu x}\right)^{1/2} y\right] = f(\xi) \tag{5.18}$$

where the quantity U is a generic global velocity whose essential property is that it transforms like u_1 or u_2; that is, $U = p\widehat{U}$. The term generic is appropriate because any value of U on either side of equation (5.18) can be replaced by another value by a suitable choice of the normalizing constants A and B.

Substitution of the ansatz (5.18) in the momentum equation (5.2) leads to the Blasius differential equation

$$2ABf''' + ff'' = 0 \tag{5.19}$$

with the boundary conditions

$$f'(\infty) = \frac{A}{B}\frac{u_1}{U} \ , \qquad f'(-\infty) = \frac{A}{B}\frac{u_2}{U} \ . \tag{5.20}$$

Note that only two boundary conditions have been established for a third-order ordinary differential equation. For all of the other laminar shear flows considered in this monograph, a natural origin for the y-coordinate and a natural third boundary condition, usually in the form $\psi = 0$ on $y = 0$, or $f(0) = 0$, are provided either by an explicit symmetry condition or by an implicit symmetry condition associated, for example, with the presence of a plane wall bounding the flow on the high-speed side. For the shear layer, this condition is still appropriate in the upstream region where the two fluids are physically separated by the septum. Downstream from the trailing edge, the symmetry condition is replaced by the concept of the dividing streamline, defined as the locus where $\psi = 0$ or $f = 0$ for $x > 0$. The two fluids, although they are assumed to have identical physical properties, are still separated by a hypothetical surface that begins at the trailing edge. **(The problem of the shear layer for two incompressible and immiscible fluids having different densities and/or viscosities was treated by Keuligan (ref) and Lock (ref), both thinking of wind over water.)** The lack of symmetry makes it unlikely that the dividing streamline $\psi = 0$ coincides with $y = 0$ or $\xi = 0$, where ξ is the argument of f in equation (5.23). At the same time, similarity requires ξ to be constant on the dividing streamline. **(Why?)**

Successive differentiation of equation (5.18) shows that the correspondence between physical variables and dimensionless similarity variables is $\psi \sim f$, $u \sim f'$, $v \sim \xi f' - f$, $\tau \sim f''$, $\partial\tau/\partial y \sim f'''$. The Blasius equation (5.19) requires $f''' = 0$ when $f = 0$. Consequently, the dividing streamline coincides with a maximum in the shearing stress and with an inflection point in the velocity profile, as was first pointed out by (**ref**). On practical grounds, it therefore seems preferable to move the origin for the dimensionless y-coordinate to the dividing streamline. That is, put

$$\eta = \xi + C \ . \tag{5.21}$$

The ansatz (5.18) should be revised to read

$$A\frac{\psi}{(U\nu x)^{1/2}} = f\left\{B\left(\frac{U}{\nu x}\right)^{1/2} y + C\right\} = f(\eta) \ . \tag{5.22}$$

The differential equation is still (5.19), the first two boundary conditions are still (5.20), and the third boundary condition is now

$$f(0) = 0 \tag{5.23}$$

on the dividing streamline, which for $x > 0$ is the parabola $B(U/\nu x)^{1/2}y = -C$, with $C > 0$. (Typical dependent variables for one value of u_2/u_1 are shown in figure x). The boundary condition (5.23) does not necessarily fix the position of the dividing streamline in physical space, because there may not be enough information to determine the constant C.

Point out somewhere the role of C in making a connection with Stewartson's limiting separating boundary layer and with the blow-off condition for the boundary layer with mass transfer.)

5.1.3 Normalization

The normalization used throughout this monograph for laminar plane flows requires putting

$$2AB = 1 \tag{5.24}$$

in equation (5.19) in order to obtain the standard Blasius operator $f''' + ff''$. One further condition is needed to determine A and B. The velocity parameter U stands in the way. Recall that the scaling parameter p in the affine group (5.11) can refer to u_1 or u_2, which transform in the same way, or to any suitable combination, not necessarily linear, of u_1 and u_2. Consider the case of a moving observer who starts a clock as he passes the station $x = 0$. The basic diffusion process provides an estimate $\delta^2 \sim \nu t$ for the layer thickness seen by the observer. A plausible choice for the velocity of the observer is the arithmetic mean of u_1 and u_2, and his position is then $x \sim (u_1 + u_2)t$. He therefore sees a thickness

$$\delta \sim \left(\frac{\nu x}{u_1 + u_2} \right)^{1/2} . \tag{5.25}$$

If the constant of proportionality does not depend on u_2/u_1, the rate of growth is decreased by a factor of $\sqrt{2}$, other things being equal, as u_2 increases from zero to u_1. **(Check the literature for calculations.)**

An independent condition on δ is implicit in the form of the dimensionless variable η in equation (5.22). The thickness δ represents an increment in y, and there is a corresponding increment in η which is not dependent on the value of C. It is enough to write the proportionality

$$\delta \sim \frac{1}{B}\left(\frac{\nu x}{U}\right)^{1/2} . \tag{5.26}$$

The last two equations are consistent if the equality holds,

$$U = u_1 + u_2 . \tag{5.27}$$

The boundary conditions (5.20) keep their parallel form. Since now $B = 1/2A$,

$$f'(\infty) = 2A^2\left(\frac{u_1}{u_1+u_2}\right) , \qquad f'(-\infty) = 2A^2\left(\frac{u_2}{u_1+u_2}\right) . \tag{5.28}$$

Finally, for no better reason than that the values $f'(\infty) = 1$, $f'(-\infty) = 0$ seem well suited to the special case $u_2 = 0$, I take

$$2A^2 = 1 \tag{5.29}$$

so that $A = B = 2^{-1/2}$.

With this normalization, the ansatz (5.22) becomes

$$\frac{\psi}{[2(u_1+u_2)\nu x]^{1/2}} = f\left[\left(\frac{u_1+u_2}{2\nu x}\right)^{1/2} y + C\right] = f(\eta) . \tag{5.30}$$

The function f satisfies the Blasius equation

$$f''' + f\,f'' = 0 \tag{5.31}$$

with two boundary conditions

$$f'(\infty) = \frac{u_1}{u_1+u_2} , \qquad f'(-\infty) = \frac{u_2}{u_1+u_2} . \tag{5.32}$$

The solutions of equation (5.31) form a single-parameter family, and the parameter, u_2/u_1, say, appears explicitly only in the boundary conditions (5.32).

Integration of equation (5.31) can proceed formally without regard to the constant C or the global velocity U. To fix the ideas, and to avoid clutter, suppose that $A = B = 1$ in the ansatz (5.22). Then the function f satisfies the ordinary differential equation

$$2f''' + f\,f'' = 0 \tag{5.33}$$

with the boundary conditions

$$f(0) = 0 \ , \qquad f'(\infty) = \frac{u_1}{U} \ , \qquad f'(-\infty) = \frac{u_2}{U} \ . \tag{5.34}$$

The solutions will form a one-parameter family in u_2/u_1.

Integration, say by a shooting method, is complicated by the fact that both $f'(0)$ and $f''(0)$ must be properly chosen before the conditions at $\pm\infty$ can be satisfied (**what about U?**). An ingenious procedure was proposed by Töpfer, who observed (in the context of the Blasius boundary-layer problem) that if $f(\eta)$ is a solution of equation (5.33), so is $g(\eta) = af(a\eta)$, where a is a constant. This property can be proved directly, but it can also be connected with the normalization procedure. Relabel the dependent variable f in equation (5.22) as g. Then g satisfies

$$2AB\,g''' + g\,g'' = 0 \tag{5.35}$$

with the boundary conditions

$$g(0) = 0 \ , \qquad g'(\infty) = \frac{A}{B}\frac{u_1}{U} \ , \qquad g'(-\infty) = \frac{A}{B}\frac{u_2}{U} \ . \tag{5.36}$$

Equation (5.35) is the same as equation (5.33) if $AB = 1$. The boundary conditions become

$$g(0) = 0 \ , \qquad g'(\infty) = a^2\frac{u_1}{U} \ , \qquad g'(-\infty) = a^2\frac{u_2}{U} \tag{5.37}$$

and the problem takes Töpfer's form. Thus choose a value of u_2/u_1 and a value of U, which depends somehow on u_1 and u_2. Choose also a value of $g'(0)$ and iterate $g''(0)$ until the condition $g'(-\infty)/g'(\infty) = u_2/u_1$ is satisfied. The parameter a then follows from the relation

$$a^2 = \frac{u_1}{U}\frac{1}{g'(\infty)} \quad . \tag{5.38}$$

Since $f'(0) = a^2 g'(0)$ and $f''(0) = a^3 g''(0)$, the function $f(\eta)$ can be evaluated immediately by a final integration.

It remains to consider entrainment. The easiest way to determine v_1 and v_2 is through the definition $v = -\partial\psi/\partial x$ applied directly to the ansatz (5.30). The result is

$$v_1 \quad = -\left[\frac{(u_1+u_2)\nu}{2x}\right]^{1/2} \lim_{\eta\to\infty}\left[f - \left(\frac{u_1}{u_1+u_2}\right)\eta\right] \tag{5.39}$$

$$v_2 = -\left[\frac{(u_1+u_2)\nu}{2x}\right]^{1/2} \lim_{\eta\to-\infty}\left[f - \left(\frac{u_2}{u_1+u_2}\right)\eta\right] \quad . \tag{5.40}$$

Under certain common experimental conditions, the problem of evaluating the constant C solves itself. Suppose that $v_1 = 0$. This will be the case if there is a parallel wall above the shear layer, as shown in FIGURE X. A related configuration is the round jet, for which the core flow downstream from the exit can be expected to be uniform and at constant pressure. If the Reynolds number is large, the shear layer will be thin, and the effect of lateral curvature can be neglected, at least close to the exit. For such cases, equation (5.39) gives (**check again**)

$$C = \lim_{\eta\to\infty}(\eta - f) \quad . \tag{5.41}$$

This result is noted in FIGURE X. Integration of the Blasius equation will lead to a value for C **(check)**. The effect is very like the effect of the displacement thickness for a boundary layer, except for the direction of the deflection. **(Integrate in both directions from dividing streamline?)**

The conclusion (5.41) can also be argued from the continuity equation in a form that applies for both laminar and turbulent flow.

In FIGURE X, continuity requires for the contour $ABCD$, which is bounded in part by the wall $y = Y$ and in part by the dividing streamline $\eta = 0$ (**rethink this**),

$$\int_0^Y u_1 \,\mathrm{d}y = \int_{(\eta=0)}^Y u \,\mathrm{d}y \tag{5.42}$$

or

$$\int_0^Y (u_1 - u)\mathrm{d}y = \int_{(\eta=0)}^0 u \,\mathrm{d}y \ . \tag{5.43}$$

Thus the two shaded regions in the figure have equal areas. In similarity form, equation (5.43) is

$$\int_C^{(Y)} (1 - f')\mathrm{d}\eta = \int_0^C f'\mathrm{d}\eta = f(C) - f(0) \tag{5.44}$$

from which, if $f(0) = 0$,

$$\lim_{\eta \to \infty} (\eta - f) = C \ . \tag{5.45}$$

This conclusion does not depend on the parameter u_2/u_1 (**look at argument by Dimotakis; point out connection with displacement concept; upper wall may also be a parabola; note lines** $x =$ constant **are characteristics**).

The most challenging element of the analytical problem is the subtlety of the required third boundary condition (check refs to see who was clear about this first). As a practical matter, the laminar mixing layer is very unstable, and any experimental information is likely to be incidental to work on the instability (Sato). However, the problem needs to be considered here because it also comes up for turbulent flow, and should not be blamed on the presence of turbulence.

Read Lu Ting and other papers. The presence of a wall or of axial symmetry removes the difficulty. Note singularity at $x = 0$

because equations are parabolic and characteristics are $x =$ constant. *Note use of rectangular coordinates; mention Kaplun on optimal coordinates. The essence is the final additive constant in f. Note also the need for an additive constant in η to get the shear layer from Stewartson's solution at the origin. Represent flow by distributed sources and/or doublets? Must avoid pressure force on splitter plate. Is streamline displacement at infinity symmetric or antisymmetric or neither? Do control-surface argument.*

The nonlinear equation (5.24) with the boundary conditions (5.16) and (5.17) has no known solution in closed form, and was first solved numerically by (**refs.**). The solution for f is fixed only within an additive constant in η, pending resolution of the third boundary condition.

5.2 Plane turbulent mixing layer

This flow is second only to the turbulent boundary layer in the volume of literature it has generated (**pipe flow?**). Much of the more recent work has aimed at the problem of chemistry, including the dominant role of coherent structure in turbulent mixing. The latter work also contributes substantially to the body of information on mean properties.

The turbulent mixing layer grows rapidly in the downstream direction. For a fixed ratio of the two constant external velocities, the growth is known to be very nearly linear and nearly independent of Reynolds number (see SECTION X). I will therefore not consider the laminar stresses, particularly since the problem of mixed similarity rules is beyond the state of my art. However, I can and will attempt to avoid some limitations of the usual boundary-layer approximation. The conical property suggests that cylindrical polar coordinates could be used, but experimenters move their probes in rectangular coordinates, and so will I. The equations of motion are

$$\rho\left(\frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(-\rho\overline{u'u'}) + \frac{\partial}{\partial y}(-\rho\overline{u'v'}) \qquad (5.46)$$

$$\rho\left(\frac{\partial uv}{\partial x} + \frac{\partial vv}{\partial y}\right) = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}(-\rho\overline{u'v'}) + \frac{\partial}{\partial y}(-\rho\overline{v'v'}) \ . \tag{5.47}$$

No information can be obtained about the remaining Reynolds stress $-\rho\overline{w'w'}$, which is involved only indirectly in the dynamics of the mean flow. I am obliged to use boundary conditions of boundary-layer type in what amounts to a hybrid formulation,

$$u(x,\infty) = u_1 \ , \qquad u(x,-\infty) = u_2 \ , \tag{5.48}$$

$$p(x,\infty) = p(x,-\infty) = p_\infty \ . \tag{5.49}$$

In addition, the Reynolds stresses are all assumed to vanish outside the shear layer. **(Is the pressure condition correct?)**

Let these equations and boundary conditions be subjected to the affine transformation

$$
\begin{aligned}
x &= a\widehat{x} \\
y &= b\widehat{y} \\
\psi &= c\widehat{\psi} \\
\rho &= d\widehat{\rho} \\
\overline{u'u'} &= f\widehat{(\overline{u'u'})} \\
\overline{u'v'} &= g\widehat{(\overline{u'v'})} \\
\overline{v'v'} &= h\widehat{(\overline{v'v'})} \\
p &= i\widehat{p} \\
p_\infty &= j\widehat{p}_\infty = i\widehat{p}_\infty \\
u &= p\widehat{u} = \frac{c}{b}\,\widehat{u} \\
v &= q\widehat{v} = \frac{c}{a}\,\widehat{v} \\
u_1 &= r\widehat{u}_1 \\
u_2 &= s\widehat{u}_2 = r\widehat{u}_2\ .
\end{aligned}
\tag{5.50}
$$

The definitions $u = \partial\psi/\partial y$ and $\widehat{u} = \partial\widehat{\psi}/\partial\widehat{y}$ require $p = c/b$. (**Mention** v**.**) Transformation of the boundary conditions on u leads, as in the laminar problem, to the relation $p = r = s$ and to an invariant which I take as

$$\frac{c}{br} = 1 \ . \tag{5.51}$$

By inspection, the boundary condition (5.49) on p leads to the relation $i = j$. Invariance of the equations of motion implies

$$\frac{c^2}{abg} = 1 \ , \qquad \frac{bi}{adg} = 1 \ , \qquad \frac{fb}{ag} = 1 \ , \tag{5.52}$$

$$\frac{c^2}{a^2h} = 1 \ , \qquad \frac{i}{dh} = 1 \ , \qquad \frac{bg}{ah} = 1 \ . \tag{5.53}$$

Note from the second of equations (5.53) that $i/d = h$. From the second of equations (5.52) and the third of equations (5.53), it follows that $h = g$ and thus that

$$\frac{b}{a} = 1 \ . \tag{5.54}$$

Finally, from the third of equations (5.52), $f = g$. Thus equations (5.52) and (5.53), which are a necessary condition for similarity, imply that the layer grows linearly if the pressure perturbation and the three surviving Reynolds stresses all transform in the same way, and conversely.

Two invariants of the transformation are given by equations (5.51) and (5.54). But these and the first of equations (5.52) or (5.53) imply

$$\frac{g}{r^2} = 1 \tag{5.55}$$

and three similar equations, given $f = g = h = i/d$. These combinations require the pressure and the Reynolds stresses $-\rho\overline{u_i u_j}$ to scale like ρu_1^2. Since these stresses must vanish when $u_2 = u_1$, it is reasonable to adapt this conclusion to read that the Reynolds stresses must scale like $\rho(u_1 - u_2)^2$. **(Why not $u_1^2 - u_2^2$?)** In full, the ansatz for the plane turbulent shear layer is **(introduce U)**

$$A\frac{\psi}{Ux} = f\Big(B\frac{y}{x} + C\Big) = f(\eta) \tag{5.56}$$

$$\overline{u'u'} = (u_1 - u_2)^2 F(\eta) \tag{5.57}$$

$$\overline{u'v'} = (u_1 - u_2)^2 G(\eta) \tag{5.58}$$

$$\overline{v'v'} = (u_1 - u_2)^2 H(\eta) \tag{5.59}$$

$$p - p_\infty = \rho(u_1 - u_2)^2 P(\eta) \tag{5.60}$$

where the constant C, as in the laminar case, supports the boundary condition $f(0) = 0$ by locating the dividing streamline $\psi = 0$ in the downstream flow as the straight line $y/x = -C/B$. The functions F and H are necessarily positive, and G is expected to be negative.

Substitution of the appropriate derivatives of equations (5.56)-(5.60) into the momentum equations (5.46) and (5.47) gives

$$-U^2\frac{B^2}{A^2}\frac{f\,f''}{(u_1-u_2)^2} = (\eta - C)P' + (\eta - C)F' - BG' \tag{5.61}$$

$$-U^2\frac{B}{A^2}\frac{(\eta - C)f\,f''}{(u_1-u_2)^2} = -BP' + (\eta - C)G' - BH' \ . \tag{5.62}$$

5.2.1 The boundary-layer approximation

A short digression is needed here to put into evidence the result that would be obtained if the boundary-layer approximation had been made in the beginning. The single momentum equation to be transformed is a truncated form of equation (5.46),

$$\rho\left(\frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y}\right) = \frac{\partial}{\partial y}(-\rho\overline{u'v'}\,) \tag{5.63}$$

with the boundary conditions (5.48). The invariants of the mapping are

$$\frac{c^2}{abg} = 1 \ , \qquad \frac{c}{ar} = 1 \ . \tag{5.64}$$

If the Reynolds shearing stress transforms like $(u_1 - u_2)^2$, equation (5.55) again applies;

$$\frac{g}{r^2} = 1 \ . \tag{5.65}$$

When the first two relationships are used to isolate b and c, the main invariants are unchanged;

$$\frac{c}{br} = 1 \ , \qquad \frac{b}{a} = 1 \ , \qquad \frac{g}{r^2} = 1 \ . \tag{5.66}$$

Thus the appropriate ansatz is the subset (5.56) and (5.58). The implied similarity equation can be derived directly or by dropping terms multiplied by $(\eta - C)$ in equation (5.61). The same approximation in equation (5.62) gives $P' + H' = 0$ or

$$p + \rho\overline{v'v'} = p_\infty = \text{constant} \ . \tag{5.67}$$

The full equations yield a more complicated equation for the pressure. Multiply equation (5.61) by $(\eta - C)$ and equation (5.62) by B and subtract to obtain

$$[B^2 + (\eta - C)^2]P' = -(\eta - C)^2 F' + 2B(\eta - C)G' - B^2 H' \ . \tag{5.68}$$

With $(\eta - C) = By/x$, this becomes in physical variables

$$\left(1 + \frac{y^2}{x^2}\right)\frac{\partial(p - p_\infty)/\rho}{\partial y} = -\frac{\partial\overline{v'v'}}{\partial y} + 2\frac{y}{x}\frac{\partial\overline{u'v'}}{\partial y} - \frac{y^2}{x^2}\frac{\partial\overline{u'u'}}{\partial y} \ . \tag{5.69}$$

The three Reynolds stresses have similar shapes and comparable magnitudes (see section x). At least the second term on the right in equation (5.69) is not negligible since the coefficient $2y/x$ is typically about 0.2 in regions where the derivatives are appreciable. Note that equation (5.69) does not involve the constants A, B, and especially C in equation (5.56). **(Integrate by parts.)** Equation (5.69) will be tested experimentally and compared with the conventional boundary-layer approximation (5.67) in SECTION X. **(Note that $f = 0$ when $G' = 0$. Does $p(\infty) = p(-\infty)$?)**

To eliminate the pressure from the problem, multiply equation (5.61) by B and equation (5.62) by $(\eta - C)$ and add to obtain

(give boundary-layer approximation, comment on eddy viscosity)

$$\begin{aligned}&\left[B^2 - (\eta - C)^2\right]G' \\ &= \frac{U^2}{(u_1 - u_2)^2}\frac{B}{A^2}\left[B^2 + (\eta - C)^2\right] f\, f'' + B(\eta - C)(F' - H') \end{aligned} \tag{5.70}$$

This expression is useful for comparing values of G measured directly with those inferred from measurements of the other quantities; namely f, F, and H. The effect of making the boundary-layer approximation in equation (5.70) is less conspicuous than in the case of equation (5.68), because the difference $(F' - H')$ is much smaller than F' or H' alone. In physical variables, equation (5.70) becomes

$$\left(1 + \frac{y^2}{x^2}\right)\psi\frac{\partial u}{\partial y} = x\left[\frac{y}{x}\frac{\partial}{\partial y}\left(\overline{v'v'} - \overline{u'u'}\right) + \left(1 - \frac{y^2}{x^2}\right)\frac{\partial}{\partial y}\left(\overline{u'v'}\right)\right] \ . \tag{5.71}$$

Within the boundary-layer approximation, the dividing mean streamline $\psi = 0$ corresponds, as in the laminar case, to the condition $\partial\tau/\partial y = 0$, although not necessarily to an inflection point in the mean velocity profile. This condition should not be much in error for the full equations.

It remains to assign values to the scaling parameters A and B in the defining similarity equation (5.56), or better in its derivative (note that $\eta = 0$ is the dividing streamline).

$$\frac{A}{B}\frac{u}{U} = f'\left(B\frac{y}{x} + C\right) = f'(\eta) \ . \tag{5.72}$$

As in the laminar problem, it is reasonable to take **(explain)**

$$\frac{A}{B} = 1 \tag{5.73}$$

and thereby to fix two of the boundary conditions as

$$f'(\infty) = \frac{u_1}{U} \ , \qquad f'(-\infty) = \frac{u_2}{U} \ . \tag{5.74}$$

Again, the physical parameter u_2/u_1 of the problem appears only in the lower boundary condition. The normal component of velocity outside the shear layer is the limit of $-\partial\psi/\partial x$, with ψ given by equation (5.56). The boundary conditions (5.9) and (5.10) lead to

$$v_1 = -\frac{u_1}{A} \lim_{\eta\to\infty} \left[f - (\eta - C)\right] \ , \tag{5.75}$$

$$v_2 = -\frac{u_1}{A} \lim_{\eta\to-\infty} \left[f - \frac{u_2}{u_1}(\eta - C)\right] \ . \tag{5.76}$$

If the condition $v_1 = 0$ is enforced by an upper wall or by axial symmetry with the faster stream near the axis, then again

$$C = \lim_{\eta\to\infty} (\eta - f) \ . \tag{5.77}$$

The present state of the art of normalization is based on a more empirical similarity approach suggested by FIGURE X. A thickness δ, commonly called the vorticity thickness, can be tentatively defined in terms of the maximum slope $\partial u/\partial y$, which no longer necessarily occurs on the dividing streamline. Within the boundary-layer approximation, the shearing stress in turbulent flow should have a maximum on the dividing streamline, because $Du/Dt = 0$ **(explain)**. For a particular value of the parameter u_2/u_1, the profile in FIGURE X can be represented in another similarity form, starting with

$$\frac{u - u_2}{u_1 - u_2} = g'\left(\frac{y^*}{\delta}\right) \tag{5.78}$$

where y^* is measured from the dividing streamline. The relationship between g' and f' follows from equations (5.53), (5.54), and (5.58);

$$\frac{u}{u_1} = f'\left(\frac{By^*}{x}\right) = \frac{u_2}{u_1} + \left(1 - \frac{u_2}{u_1}\right) g'\left(\frac{y}{\delta}\right) \ . \tag{5.79}$$

To preserve the benefits of the affine argument, it is necessary to have

$$B = \frac{x}{\delta} \ . \tag{5.80}$$

The notation σ for x/δ was introduced by Görtler (**ref**). This parameter σ, which is the same as my B, can be expected to depend on u_2/u_1. For the most thoroughly studied case $u_2/u_1 = 0$, σ is about 11, or δ/x is about 1/9. I prefer not to use Görtler's notation on the hard ground that a different definition of δ might prove more useful, and on the softer ground that the parameter σ seems to me to be defined upside down; the ratio δ/x has a greater graphic and mnemonic value. **(Is there an integral of $(u_1 - u)(u - u_2)$?)**

The boundary conditions for g' from equation (5.101)[1] are

$$g'(\infty) = 1, \qquad g'(-\infty) = 0 \ . \tag{5.81}$$

Integration of (5.79) yields

$$f\left(\frac{y}{\delta}\right) = \frac{u_2}{u_1}\frac{y}{\delta} + \left(1 - \frac{u_2}{u_1}\right) g\left(\frac{y}{\delta}\right) \tag{5.82}$$

with $g(0) = 0$ if $f(0) = 0$ (**where was the latter done?**).

The parameter u_2/u_1 has thus disappeared from the boundary conditions and appeared in the defining equation (5.82). An associated result, completely empirical at present, is based on the fact that the profile $g'(y/\delta)$ is a monotonic transition from one constant value to another. It is only a small step to the proposition that this profile is for practical purposes universal; i.e., it is the same function of y/δ for all values of u_2/u_1. This proposition will be tested experimentally in section x. (**Look also at laminar case. Do entrained flow, both cases. Calculate $v(-\infty)$.**)

The dependence of B or δ/x or σ on u_2/u_1 was first studied by SABIN (1965), who worked with plane shear layers at quite low Reynolds numbers and who chose to plot x/δ against u_2/u_1 (**check**). It is now more common to see δ/x plotted against $(u_1 - u_2)/(u_1 + u_2)$. The reason is that the latter dependence is found to be very nearly linear;

$$\frac{\delta}{x} \sim \left(\frac{u_1 - u_2}{u_1 + u_2}\right) \ . \tag{5.83}$$

I suspect that this linear dependence is somehow implicit in the equations, especially equation (5.79), but I have not found a valid argument.

[1] Possibly an incorrect reference.

5.2.2 Structure of the shear layer

Some guidance on normalization is provided by the device of the moving observer. For the case of a turbulent mixing layer, this device is both real and important, because it introduces the subject of coherent structure, and thus requires another digression.

At the level of eddy viscosity or mixing length, the turbulent mixing layer was thought to be a featureless wedge of turbulence, perhaps with a trivially irregular boundary. This view changed drastically with the work of BROWN and ROSHKO (1971, 1974) and WINANT and BROWAND (1974). It is now recognized that the turbulent mixing layer is inhabited by, or more properly is constructed from, large spanwise vortex structures that grow both by entrainment and by coalescence during the evolution of the flow. The structures originate in an inviscid Kelvin-Helmholtz instability that operates in both laminar and turbulent flow. In this coherent-structure model of turbulence, each structure is assumed to move as a unit, preserving its geometry and operational properties between coalescence events, while the ambient flow accommodates itself to the kinematic and dynamic needs of the structure. According to this model, the translational velocity of the structures in laboratory coordinates is well defined. It is is often called convection velocity and occasionally phase velocity, although I will use the term celerity and the notation c.

When the averaging process introduced by REYNOLDS (1895) is stopped at second order, which is to say at the first revelation of the closure problem, all information about scale and phase of the turbulent motions is lost. Various methods are available to recover some of this information. If data are available at two points, for example, some phase information can be rescued by the technique of space-time correlation. The condition of optimum time delay then provides an imperfect measure of phase velocity or celerity. Without filtering, the results indicate that the celerity is not constant through the thickness of a given plane flow, but is biased in the direction of the mean-velocity profile. The bias can be reduced by retaining only the low-frequency or large-scale content of the signals, as demonstrated for the boundary layer by FAVRE, GAVIGLIO, and DUMAS (Phys. Fluids **10**, Supplement, S138-S145, 1967). Early experimenters who used this technique were not in any doubt about the meaning of their work, as is evident from the fact that Favre and Kovasznay chose the French word célérité to describe their findings quantitatively, rather than the more conventional French word vitesse. (The technique of time-space correlation could in principle be applied to existing numerical solutions of the Navier-Stokes equations.) Much more accurate measures have been obtained by flow visualization, particularly for the mixing layer, where the large structures are two-dimensional in the mean.

FIGURE X is a sequence of frames from a motion picture of a mixing layer (Roshko, private communication; **what flow?**). FIGURE Y is a corresponding x-t diagram showing trajectories of the recognizable features marked by + symbols in the first frame. Coalescence events occur quickly, like punctuation marks in the text of turbulence. Similar figures have been published by DAMMS and KÜCHEMANN (RAE Tech Rep 72139, 1972), BROWN and ROSHKO (JFM **64**, 775, 1974), and ACTON (JFM **98**, 1, 1980). Each of these observers chose a particular local feature, not necessarily the same feature, in order to assign values to the variable $x(t)$ and hence to the celerity $\mathrm{d}x/\mathrm{d}t$, and each was successful in exposing the phenomenon of coalescence in the mixing layer.

These observations play a central role in normalization. FIGURE X (COLES 1981) is a cartoon of the instantaneous mean streamlines in the turbulent mixing layer, as seen by an observer moving with the celerity c. **(Look up Brown, thesis, Univ. Missouri, 1978.)** In the model, the vortices are stationary, the flow is inviscid, and the layer grows in time rather than in space. The essence of figure x is the topology, which consists of saddle points (stagnation points) alternating with stable foci (vortices). Fluid flowing toward each saddle point along the converging separatrices (instantaneous streamlines) must arrive at the saddle point with the same stagnation pressure. Suppose for the moment that Mach numbers are small but the streams have different densities. Far from the mixing layer, where the two streams have the same static pressure, Bernoulli's equation requires

$$\frac{\rho_1}{2}(u_1 - c)^2 = \frac{\rho_2}{2}(u_2 - c)^2 \ . \tag{5.84}$$

When this equation is solved for c, the result is

$$c = \frac{(\rho_1)^{1/2}u_1 + (\rho_2)^{1/2}u_2}{(\rho_1)^{1/2} + (\rho_2)^{1/2}} \ . \tag{5.85}$$

Note that if the two densities are very different, the celerity approaches the velocity of the denser stream. Thus equation (5.85) contains more information about the mixing process than might have been anticipated. If the fluids are compressible, still with the same p outside the mixing layer and the same p_o at the saddle points, it follows from equation (1.82) of the introduction that equations (5.84) and (5.85) are unchanged, provided only that the flow along the separatrices is isentropic. If the velocity of one or both streams is supersonic relative to the large structures, shock waves and expansion waves may appear and may intersect the separatrices. The effect on equation (5.85) is at present an open question.

I first used the relation (5.85) in a survey paper (COLES 1985) that attempted to collect some important results that can be obtained using the concept of coherent structure and cannot be obtained without it. The same equation was derived independently by DIMOTAKIS (AIAA Paper 84-0368) and perhaps by others. There is persuasive evidence (WANG) that equation (5.85) predicts quite accurately the effect of density ratio on celerity for low-speed flow.

The calculation just made depends on the topological simplicity of the mixing layer. No comparable result has so far been obtained for any other turbulent shear flow, presumably because the large structures in other flows arise from more complex instabilities, are less dominant in the mixing process, and are almost certainly three-dimensional. Moreover, in any morphology of coherent structure, an important distinction arises between flows containing large-scale mean vorticity of only one sense and flows containing large-scale mean vorticity of both senses. The mixing layer is unique among the classical plane flows in that it is the only flow that is driven naturally toward a two-dimensional structure.

Now return to the problem of normalization. For the case of equal densities, the constant celerity of the large structures, and thus the proper velocity of the observer, is

$$U = c = \frac{u_1 + u_2}{2} \ . \tag{5.86}$$

It is a little ironic that this estimate is less of a guess than the same estimate (5.27) for laminar flow. It defines the global velocity U in the ansatz (5.56) and gives the position of an observer moving with the structures as

$$x \sim (u_1 + u_2)\ t \ . \tag{5.87}$$

For the laminar problem, another measure for the parameter t was obtained from the diffusive model $\delta \sim (\nu t)^{1/2}$, which does not apply when the flow is turbulent. DIMOTAKIS (1991) has proposed a different relation that involves δ and meets the need. The vorticity thickness δ, or more accurately the maximum-slope thickness, is defined by

$$\delta = \frac{u_1 - u_2}{(\partial u/\partial y)_{\max}} \ . \tag{5.88}$$

Since δ varies like x and like t, the reciprocal of the quantity $(\partial u/\partial y)_{\max}$ is a plausible time scale;

$$\delta \sim (u_1 - u_2)\ t \ . \tag{5.89}$$

Immediately, therefore,

$$\frac{\delta}{x} \sim \frac{(u_1 - u_2)}{(u_1 + u_2)} \ . \tag{5.90}$$

Equation (5.90) is a genuine scaling law only if the implied constant of proportionality is independent of the velocity ratio u_2/u_1. The derivation here supposes and suggests that it is, although the argument, like many arguments in science, illustrates the principle that it helps to know the answer. Note that negative values for u_2/u_1, which can occur for base or cavity flows, are permitted by the scaling law, with δ/x varying from zero to infinity as u_2/u_1 varies from $+1$ to -1. This question will be taken up in SECTION X. ABRAMOVICH (19xx) proposed equation (5.90) for the mixing layer, but stipulated some further dependence on u_2/u_1. SABIN (19xx) proposed equation (5.90) in a different form as a scaling law based on his own measurements at relatively low Reynolds numbers. ROSHKO in 19xx collected the experimental data available at that time and demonstrated an essentially linear relationship between the two sides of equation (5.90), with a constant of proportionality on the right of about xxx. There was considerable scatter in the data, especially for the case $u_2/u_1 = 0$.

The road to normalization is now open, as much as any road in turbulence is ever open. If the argument $B\,y/x$ of the function f in equation (5.56) is to be equivalent to y/δ, then it is necessary to have

$$B = \frac{x}{\delta} = b\left(\frac{u_1 + u_2}{u_1 - u_2}\right) \tag{5.91}$$

where b is independent of u_2/u_1. Finally, as in the laminar case, I take

$$A = B \tag{5.92}$$

on esthetic rather than logical grounds. This normalization leads to the ansatz

$$\frac{2b\psi}{(u_1 - u_2)x} = f\left[b\left(\frac{u_1 + u_2}{u_1 - u_2}\right)\frac{y}{x} + C\right] \tag{5.93}$$

with the boundary conditions

$$f(0) = 0 \quad , \quad f'(\infty) = \frac{2u_1}{u_1 + u_2} \quad , \quad f'(-\infty) = \frac{2u_2}{u_1 + u_2} \quad . \tag{5.94}$$

The constant C is still unspecified.

From equation (5.94), with the boundary-layer result that f'' is a maximum on the dividing streamline $\eta = 0$ **(check)**,

$$\left(\frac{\partial u}{\partial y}\right)_{\max} = \frac{b(u_1 + u_2)^2}{2x(u_1 - u_2)}\, f''(0) \tag{5.95}$$

and thus, with equations (5.88) and (5.91),

$$f''(0) = \frac{2}{b}\left(\frac{u_1 - u_2}{u_1 + u_2}\right) \quad . \tag{5.96}$$

This result will be tested experimentally in SECTION X.

FIGURE X shows the experimental evidence for equation (5.83) as presented by Roshko, with a number of later measurements. The scatter, especially for the case $u_2 = 0$, is unreasonable. Several reasons have been proposed for the scatter. One, due to BATT (1975) is that non-uniqueness is a relict of varying initial conditions, especially the laminar or turbulent state of the boundary layer at the trailing edge of the splitter plate or septum. This conjecture has inspired a number of detailed and difficult studies (**refs**). Another, which I favor as an equally plausible source of scatter, is three-dimensionality. An easy measure is the aspect ratio, or the ratio of shear-layer thickness to the distance between the side plates usually provided to control the entrainment process. The effect should not be present for the axisymmetric shear layer, but another more systematic effect of lateral curvature is likely to be present instead, along with any effect of initial conditions.

Collect profile data for plane flow, $u_2 = 0$*. Plot* $d\delta/dx$ *(assign this to the mean* x*) against* δ/w*, where* w *is the spanwise width. Use tanh for a fit to the central part of the profile. Do not use* x/δ*, because the origin of* x *depends on linear growth, which is not yet proved (thus argument is circular). For round jet, use* $\delta/\pi D$*, where* D *is orifice diameter (neglect effect of growing diameter of dividing surface).*

For two-stream flows, comment on use of porous obstacle and two tailored nozzles, with a very delicate design condition. The direction of the dividing streamline may be seriously affected.

Note that the third boundary condition is still unspecified, as in the laminar problem.

Note also the singularity when $u_2 = -u_1$*; the layer becomes infinitely thick. The origin for* x *is not unique.* **(Who did crossed flows?)**

It might be more useful to work in terms of the variable (should *this be done during the transformation?)*

$$\frac{u - u_2}{u_1 - u_2} = 1 + \frac{u_1}{u_1 - u_2}\left(\frac{B}{A} f' - 1\right) \quad . \tag{5.97}$$

The left-hand side is unity when $u = u_1$ *and zero when* $u = u_2$. *Then*

$$\frac{B}{A} f'(\infty) = 1, \qquad \frac{B}{A} f'(-\infty) = \frac{u_2}{u_1} \tag{5.98}$$

suggesting $B = A$. *(Look at ansatz and at momentum equations.) A different scheme is to put*

$$F'(\eta) = \frac{u - u_2}{u_1 - u_2} = 1 + \frac{u_1}{u_1 - u_2}\left(\frac{B}{A} f' - 1\right) \quad . \tag{5.99}$$

Then

$$F(\eta) = \frac{u_1}{u_1 - u_2}\frac{B}{A} f(\eta) - \frac{u_2}{u_1 - u_2} + \text{constant} \quad . \tag{5.100}$$

The third boundary condition reappears.

In the absence of an integral invariant, the thickness of the turbulent shear layer is usually defined by an ad hoc *normalization. One easy definition is based on the maximum slope of the mean-velocity profile, as shown in figure x. The quantity* δ *so defined is called the vorticity thickness. Another definition seen in the literature uses more detail in the profile by measuring* δ *between points where specified fractions, say, 10 percent and 90 percent of the total velocity change are observed. I have no use for this definition.*

For a particular value of the parameter u_2/u_1, *the profile in figure x can be represented by the equation* **(fix this)**

$$\frac{u - u_2}{u_1 - u_2} = g'\left(\frac{y}{\delta}\right) \tag{5.101}$$

where y *is now measured from the dividing streamline.*