Topics in Shear Flow

Chapter 8 – The Round Jet

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Chapter 8

THE ROUND JET

The classical laminar or turbulent round jet issues from a point source into a stagnant fluid. An extreme case is shown in the photograph.[1] An important reason for the popularity of the turbulent round jet as a subject for fundamental experimental study is economy. The geometry is simple, and a large part of the energy supplied to the fluid appears as turbulent motion before being dissipated in heat. An easily accessible and important variation on the basic problem of momentum transport is simultaneous transport of heat or mass.

A jet may be surrounded by a moving fluid, as in a rocket or jet engine. In such cases the problem can sometimes be linearized, giving a point of contact with the round wake. Jets are components of many practical devices such as torches and sprays. A jet may also be enclosed in a shroud or housing to produce a jet pump or ejector. Several jets may interact, or a non-circular jet may relax toward axial symmetry, or a jet may have a component normal to an ambient flow, as in thrust-control devices. Swirl is sometimes used to enhance mixing. Finally, a jet of one fluid into another is a nice problem in entrainment, mixing, and relaxation.

[1] It is not clear what photograph was meant to be included here.

8.1 Laminar round jet into fluid at rest

8.1.1 Preview

The steady laminar flow associated with a point momentum source in a viscous incompressible fluid is one of the few known exact solutions of the Navier-Stokes equations. The reason that this solution came long ago to be known is not the usual one, which is that a particularly simple geometry has reduced the number of independent variables, as in pipe flow. The reason is dimensional in a different way, having to do with the number and nature of global parameters. A kindred case is the laminar sink flow in a wedge-shaped channel, treated earlier in SECTION X. These two flows have in common that the exact solution is known in closed form for all Reynolds numbers (if elliptic functions qualify as closed form). An important difference is that entrainment is an essential feature of the boundary-layer approximation for the round jet, but not for the channel flow. Hence the round jet is unique in providing an opportunity to practice the technique of matched asymptotic expansions to arbitrarily high order. As far as I know, this opportunity has never been exploited, and I will not attempt here to rise above the level of first-order boundary-layer theory in SECTIONS X, Y, and Z.

The organization of the next (**four**) sections is shown schematically in Figure 8.1. The box at the left, called "NS (Navier-Stokes) equations," is the core element for the diagram. My first objective is to show that the two paths from "NS equations" to "BL (boundary-layer) solution" are precisely equivalent, as are the two paths from "NS equations" to "potential flow." My second objective is to illustrate by example the use of the powerful technique of matched asymptotic expansions to construct a "composite expansion" that is not exact, but cannot for practical purposes be distinguished from the exact solution at the Reynolds number of the present exercise. These derivations lay a foundation for operations that build on boundary-layer theory in cases where no exact solution is known; e.g., the plane laminar jet, or any turbulent flow.

After some dimensional preliminaries, the exact solution is de-

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Figure 8.1: Caption for Figure with label Fig15-7 (missing)

rived in SECTION 8.1.3. Most of the material there is not original, and can be found in ROSENHEAD, "Laminar Boundary Layers" (1963) or in BATCHELOR, "An Introduction to Fluid Dynamics" (1967). The original sources are papers by SLEZKIN (1934), LANDAU (1944), and SQUIRE (1951).

8.1.2 Dimensional argument

An appropriate coordinate system for the problem of the laminar round jet is the spherical polar system (r, θ, ϕ) shown at the left in FIGURE 8.2. For comparison and contrast, the corresponding plane jet from a line momentum source is also sketched in a cylindrical polar system, (R, θ, z) at the right. These coordinates are deliberately chosen so that one coordinate in the two-dimensional reduced system is dimensionless.

The global parameters for the point momentum source (round jet) or the line momentum source (plane jet) are the fluid properties ρ and μ (or ρ and $\nu = \mu/\rho$) and the specified momentum flux J, whose dimensions for the round jet are momentum per unit time and for the plane jet are momentum per unit time per unit length. This small descriptive difference has large consequences. In terms

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Figure 8.2: Caption for Figure with label Fig15-6 (missing)

of mass, length, and time as fundamental units, the three global parameters have dimensions[2]

$$
\begin{array}{ll}
\text{Point source} & \text{Line source} \\
[J] = \dfrac{\mathbf{ML}}{\mathbf{T}^2} \ , & [J] = \dfrac{\mathbf{M}}{\mathbf{T}^2} \ , \\
[\rho] = \dfrac{\mathbf{M}}{\mathbf{L}^3} \ , & [\rho] = \dfrac{\mathbf{M}}{\mathbf{L}^3} \ , \\
[\nu] = \dfrac{\mathbf{L}^2}{\mathbf{T}} \ , & [\nu] = \dfrac{\mathbf{L}^2}{\mathbf{T}} \ .
\end{array}
\tag{8.1}
$$

Rearrangement to isolate the characteristic scales **M**, **L**, and **T** in each column yields quite different results for the two problems;

$$
\begin{array}{ll}
\left[\dfrac{J}{\rho\nu^2}\right] = 0 \ . & \left[\dfrac{\rho^4\nu^6}{J^3}\right] = \mathbf{M} \ , \\
 & \left[\dfrac{\rho\nu^2}{J}\right] = \mathbf{L} \ , \\
 & \left[\dfrac{\rho^2\nu^3}{J^2}\right] = \mathbf{T} \ .
\end{array}
\tag{8.2}
$$

[2]In this section equations on the left refer to the point source (round jet) and on the right the line source (plane jet).

For the point source on the left, no characteristic scales can be defined. It is this fact that makes an exact solution possible. Such a solution can be expected to depend on one dimensionless parameter, $(J/\rho\nu^2)^{1/2}$, having the nature of a Reynolds number. For the line source on the right, the situation is quite otherwise. The characteristic scales are well defined, but there is no dimensionless parameter. Hence there can be only one solution. Note that this argument is not based on the equations of motion, but only on the physical parameters for each flow.

Let a suitable solution be anticipated in terms of a single stream function ψ, where

$$[\psi] = \frac{L^3}{T} \quad . \qquad\qquad [\psi] = \frac{L^2}{T} \quad . \tag{8.3}$$

If L and T can be defined, these relations can be written in dimensionless form as equalities. To mark the profound change in content for the symbols, a different font is used;

$$\frac{\psi T}{L^3} = \text{fn}\left(\frac{r}{L}\ ,\ \theta\right) \quad . \qquad\qquad \frac{\psi T}{L^2} = \text{fn}\left(\frac{R}{L}\ ,\ \theta\right) \quad . \tag{8.4}$$

When T is eliminated using $T = L^2/\nu$, these become

$$\frac{\psi}{\nu L} = \text{fn}\left(\frac{r}{L}\ .\ \theta\right) \quad . \qquad\qquad \frac{\psi}{\nu} = \text{fn}\left(\frac{R}{L}\ .\ \theta\right) \quad . \tag{8.5}$$

In the left column, L can not be defined and therefore cannot appear. The only rational action is to replace L by r. In the right column, L can be defined as $\rho\nu^2/J$. Consequently, from one of equations (8.2).

$$\frac{\psi}{\nu r} = \text{fn}\,(1\ ,\ \theta) \quad . \qquad\qquad \frac{\psi}{\nu} = \text{fn}\left(\frac{RJ}{\rho\nu^2}\ ,\ \theta\right) \quad . \tag{8.6}$$

The result for the round jet on the left is much more than an accidental separation of variables. The result states that $\psi/\nu r$ depends only on θ, and must therefore be obtainable by solving an ordinary differential equation (with $J/\rho\nu^2$ as parameter). Such a conclusion normally requires a much more elaborate similarity argument based on the transformation properties of the equations of motion. It may also require good judgment in choosing an appropriate system of coordinates.

No corresponding reduction appears for the line source on the right. This flow will be discussed in SECTION 9.1.2.

8.1.3 The exact solution

Take the velocity for the point momentum source to be $\vec{u} = (u,\ v,\ w)$ in spherical polar coordinates $(r,\ \theta,\ \phi)$. Consider steady axisymmetric flow without swirl; i.e., $\partial/\partial t = \partial/\partial\phi = w = 0$. The equations of motion (**reference**) are then

$$\frac{1}{r^2}\,\frac{\partial u r^2}{\partial r} + \frac{1}{r\sin\theta}\,\frac{\partial v\sin\theta}{\partial\theta} = 0\ , \tag{8.7}$$

$$u\,\frac{\partial u}{\partial r} + \frac{v}{r}\,\frac{\partial u}{\partial\theta} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u - 2\,\frac{u}{r^2} - \frac{2}{r^2\sin\theta}\,\frac{\partial v\sin\theta}{\partial\theta}\right)\ , \tag{8.8}$$

$$u\,\frac{\partial v}{\partial r} + \frac{v}{r}\,\frac{\partial v}{\partial\theta} + \frac{uv}{r} = -\frac{1}{\rho r}\,\frac{\partial p}{\partial\theta} + \nu\left(\nabla^2 v + \frac{2}{r^2}\,\frac{\partial u}{\partial\theta} - \frac{v}{r^2\sin^2\theta}\right)\ , \tag{8.9}$$

where the Laplace operator is

$$\nabla^2\alpha = \frac{1}{r^2}\,\frac{\partial}{\partial r}\,r^2\,\frac{\partial\alpha}{\partial r} + \frac{1}{r^2\sin\theta}\,\frac{\partial}{\partial\theta}\,\sin\theta\,\frac{\partial\alpha}{\partial\theta}\ . \tag{8.10}$$

It is convenient first to eliminate the pressure by working with the vorticity, $\vec{\Omega} = \operatorname{curl}\vec{u} = (\xi,\ \eta,\ \zeta)$, which has only a ϕ-component;

$$\zeta = \frac{1}{r}\,\frac{\partial r v}{\partial r} - \frac{1}{r}\,\frac{\partial u}{\partial\theta}\ . \tag{8.11}$$

The velocity components are derivable from a stream function ψ using the definition $\vec{u} = \operatorname{grad}\psi \times \operatorname{grad}\phi$;

$$u = \frac{1}{r^2\sin\theta}\,\frac{\partial\psi}{\partial\theta}\ , \qquad v = -\frac{1}{r\sin\theta}\,\frac{\partial\psi}{\partial r}\ , \tag{8.12}$$

and consequently, from equation (8.11),

$$\zeta = -\frac{1}{r\sin\theta}\,\nabla^2\psi + \frac{2}{r\sin\theta}\,(u\cos\theta - v\sin\theta)\ . \tag{8.13}$$

The equation satisfied by ζ, from equations (8.8) and (8.9), is

$$u\,\frac{\partial\zeta}{\partial r}+\frac{v}{r}\,\frac{\partial\zeta}{\partial\theta}=\frac{\zeta}{r\sin\theta}\,(u\sin\theta+v\cos\theta)+\nu\left(\nabla^2\zeta-\frac{\zeta}{r^2\sin^2\theta}\right)\ . \tag{8.14}$$

The first term on the right-hand side represents vortex stretching of a trivial kind. The meaning of this term emerges if it is noted that $r\sin\theta=R$, say, is the perpendicular distance from any point in the flow to the polar axis of symmetry. Moreover,

$$(u\sin\theta+v\cos\theta)=\vec{u}\cdot\nabla R=DR/D\ . \tag{8.15}$$

In the absence of diffusion due to viscosity, therefore, the vorticity obeys the equation (**need to display formula for grad?**)

$$\frac{1}{\zeta}\,\frac{D\zeta}{Dt}=\frac{1}{R}\,\frac{DR}{Dt}\ . \tag{8.16}$$

Hence ζ/R is constant following an element of the fluid; the strength of any (circular) vortex filament varies directly with the filament diameter. This observation is obviously not limited to the problem of the point momentum source, but is valid for any axially symmetric motion.

It has already been argued in equation (8.6) that the stream function must have the form $\psi=\nu\,r\,f(\theta)$. A convenient variant is

$$\psi=\nu\,r\,f(\cos\theta)=\nu\,r\,f\,(\xi)\ , \tag{8.17}$$

where $\xi=\cos\theta$. It follows from equations (8.12) and (8.11) that

$$u=-\frac{\nu}{r}\,f'\ , \tag{8.18}$$

$$v=-\frac{\nu}{r}\,\frac{f}{\sin\theta}\ , \tag{8.19}$$

$$\zeta=-\frac{\nu}{r^2}\,f''\sin\theta\ , \tag{8.20}$$

where f' means $\mathrm{d}f/\mathrm{d}\cos\theta=\mathrm{d}f/\mathrm{d}\xi$. When these are substituted in equation (8.14), f satisfies

$$3f'f''+ff'''+4\xi\ f'''-(1-\xi^2)f''''=0\ . \tag{8.21}$$

This equation is fourth-order because the pressure is still a variable, although it has been formally suppressed by using the curl operator. Two successive integrals of equation (8.21) are

$$f'\,f' + f\,f'' - 2f' + 2\xi\,f'' - (1-\xi^2)\,f''' = C_1 \ , \tag{8.22}$$

$$f\,f' - 2f - (1-\xi^2)\,f'' = C_1\xi + C_2 \ . \tag{8.23}$$

The boundary condition of axial symmetry is expressed by taking $f = 0$ on $\theta = 0$ and $\theta = \pi$, or $f(1) = f(-1) = 0$. At $\xi = 1$, equation (8.23) requires $C_1 + C_2 = 0$, and at $\xi = -1$ it requires $-C_1 + C_2 = 0$. Hence $C_1 = C_2 = 0$. A further integration gives

$$f^2 - 4\xi\,f - 2(1-\xi^2)\,f' = C_3 = 0 \ , \tag{8.24}$$

where the same symmetry condition requires $C_3 = 0$. Finally, the substitution

$$h(\xi) = \frac{f(\xi)}{(1-\xi^2)} \tag{8.25}$$

transforms equation (8.24) into

$$h' = \frac{h^2}{2} \ . \tag{8.26}$$

The final exact solution (retransformed) is therefore

$$f(\xi) = \frac{2(1-\xi^2)}{1-\xi+c} \tag{8.27}$$

or

$$f(\cos\theta) = \frac{2\sin^2\theta}{1-\cos\theta+c} \ , \tag{8.28}$$

where $(1+c)/2$ is a constant of integration. The stream function, velocity components, and azimuthal vorticity are obtained from equa-

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Figure 8.3: Caption for Figure with label Fig15-8 (missing)

tions (8.17)–(8.20) as

$$\psi = 2\nu r \, \frac{\sin^2\theta}{1-\cos\theta + c} \ , \tag{8.29}$$

$$u = \frac{2\nu}{r} \, \frac{\left[2c\cos\theta - (1-\cos\theta)^2\right]}{(1-\cos\theta+c)^2} \ , \tag{8.30}$$

$$v = -\frac{2\nu}{r} \, \frac{\sin\theta}{1-\cos\theta+c} \ , \tag{8.31}$$

$$\zeta = \frac{4\nu}{r^2} \, c(c+2) \, \frac{\sin\theta}{(1-\cos\theta+c)^3} \ . \tag{8.32}$$

Some typical streamline patterns are shown in FIGURE 8.3 [for?] values of the parameter c. The sense of the figure is borrowed from Batchelor, p. 208. The chief difference is that the stream function and other variables are here put in dimensionless form with the aid of a trick, which is the introduction of a length **L** that is never defined. Lagerstrom used to call **L** the length of the blackboard.

Thus write, in what might be called virtual variables,

$$\Psi = \frac{\psi}{4\nu \mathbf{L}} \ , \tag{8.33}$$

$$R = \frac{r}{\mathbf{L}} \ , \tag{8.34}$$

$$\Theta = \theta \ . \tag{8.35}$$

so that equation (8.29) becomes

$$\Psi = R \frac{\sin^2 \Theta}{2(1 - \cos\Theta + c)} \ . \tag{8.36}$$

(**Define coordinates used in figure**). Since ψ varies linearly with r at constant θ, one streamline suffices to define each flow, with other streamlines obtained by a zoom transformation. Also at constant θ, the velocities u and v vary inversely with r, according to equations (8.30) and (8.31). As the Reynolds number increases (the parameter c decreases toward zero) on the one hand, a strong narrow jet emerges along the polar axis. The remainder of the flow represents fluid motion induced by this jet. At values of c that are large compared with unity, on the other hand, c dominates the denominator of equation (8.27), giving a Stokes flow with streamlines that are symmetrical upstream and downstream.

There is no mass flux from the singularity at the origin. Consider an integral over a sphere of fixed radius r about the origin, with $\mathrm{d}S = 2\pi\, r \sin\theta\, r\, \mathrm{d}\theta$. The net flux for the sphere is

$$\iint \rho \vec{u} \cdot \vec{n} \mathrm{d}S = 2\pi\rho r^2 \int_0^\pi u \sin\,\theta\, \mathrm{d}\theta = 2\pi\rho\nu r \int_{-1}^{1} f'(\xi)\, \mathrm{d}\xi = 0 \ . \tag{8.37}$$

To compute the corresponding momentum flux, observe that the component of any vector $\vec{a} = (a_1,\ a_2,\ 0)$ along the polar axis (the

x-axis in FIGURE 8.2) is $(a_1 \cos\theta - a_2 \sin\theta) = a_x$ (say). Then

$$
\begin{aligned}
J &= \left[\iiint \rho\,\vec{F}\,\mathrm{d}V\right]_x \\
&= \left[\iint \rho\,\vec{u}(\vec{u}\cdot\vec{n})\,\mathrm{d}S\right]_x - \left[\iint (-p\underline{I} + \mu\underline{\mathrm{def}\ \vec{u}}\,\vec{n}\,\mathrm{d}S\right]_x \\
&= \iint \rho\,(u\cos\theta - v\sin\theta)u\,\mathrm{d}S + \iint p\cos\,\theta\,\mathrm{d}S \qquad (8.38)\\
&- \mu \iint \left[2\frac{\partial u}{\partial r}\cos\,\theta - \left(\frac{1}{r}\,\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}\right)\sin\,\theta\right]\mathrm{d}S\ .
\end{aligned}
$$

A brief digression is necessary to calculate the pressure p from the two non-zero components of the momentum equations (8.8) and (8.9) above. For the exact solution $\psi = \nu r f(\xi)$, these become

$$
\begin{aligned}
\frac{1}{\rho}\frac{\partial p}{\partial r} &= \frac{\nu^2}{r^3}\left[2\xi f'' - (1-\xi^2)f''' + f'f' + f\,f'' + \frac{f^2}{(1-\xi^2)}\right] \\
&= \frac{\nu^2}{r^3}\left[2f' + \frac{f^2}{(1-\xi^2)}\right] \qquad (8.39)
\end{aligned}
$$

and

$$
\frac{1}{\rho r}\frac{\partial p}{\partial \theta} = -\frac{\nu^2}{r^3}\left[f'' + \frac{f\,f'}{(1-\xi^2)} + \frac{\xi f^2}{(1-\xi^2)^2}\right]\ , \qquad (8.40)
$$

after use of equation (8.22) with $C_1 = 0$ in equation (8.39). The first equation can be integrated to obtain

$$
\frac{p}{\rho} = -\frac{\nu^2}{2r^2}\left[2f' + \frac{f^2}{(1-\xi^2)}\right] + g(\xi)\ . \qquad (8.41)
$$

Differentiation then shows that the equation (8.40) is satisfied if $g(\xi)$ = constant = p_0/ρ (say). Given that ξ, $f(\xi)$, etc. are of order unity, the difference between p/ρ and p_0/ρ for large r is of order $(\nu/r)^2$, as are the squared velocities from equations (8.30) and (8.31). Another form of equation (8.41) that may be useful is

$$
\frac{p}{\rho} + \frac{v^2}{2} - \frac{\nu u}{r} = \frac{p_0}{\rho}\ . \qquad (8.42)
$$

On the jet axis, where $f(1) = 0$ and $f'(1) = -4/c$ from equation (8.27), it follows from equation (8.41) that the static pressure slightly exceeds the stagnation pressure p_0/ρ (why not Bernoulli equation far from jet?); (**is this so?**)

$$\frac{p}{\rho} = \frac{p_0}{\rho} + \frac{4\nu^2}{r^2 c} \ . \tag{8.43}$$

This expression already suggests that c should approach zero like ν^2 as the Reynolds number increases.

Use of equation (8.41) for p and equations (8.30) and (8.31) for u and v in equation (8.38) leads eventually to

$$J = 2\pi\rho\nu^2 \int_{-1}^{1} \left[f' f' - \frac{f^2}{2(1-\xi^2)} - 3f' \right] \xi \, \mathrm{d}\xi \ . \tag{8.44}$$

After substitution for $f(\xi)$ from equation (8.27) and evaluation of the integrals, there is obtained

$$\frac{J}{8\pi\rho\nu^2} = \frac{8}{3}\,\frac{(c+1)}{c(c+2)} + 2(c+1) - (c+1)^2 \ln\left(\frac{c+2}{c}\right) \ , \tag{8.45}$$

which shows precisely how the exact solution depends on the single parameter $J/\rho\nu^2$. Particularly useful for what follows is an expansion for small c,

$$\frac{J}{8\pi\rho\nu^2} = \frac{4}{3c} + \ln\, c + \left(\frac{8}{3} - \ln\, 2\right) + 2c\,\ln\, c + 2c\left(\frac{7}{12} - \ln\, 2\right) + \\ + c^2 \ln\, c - c^2\left(\frac{17}{24} + \ln\, 2\right) + \ \dots \ , \tag{8.46}$$

where $\dots$ stands for the third and higher powers of c. There are no more logarithms. The series evidently converges for $0 < c < 2$. (**check**) The leading term represents the boundary-layer approximation (see below);

$$c = \frac{32}{3}\,\pi\,\frac{\rho\nu^2}{J} \ . \tag{8.47}$$

(**Jet out of wall? Do θ_0 here?**)

8.1.4 The boundary-layer approximation

Suppose now that the exact solution of the Navier-Stokes equations is not known. The round jet into fluid at rest can also be approached from the outset as a boundary-layer problem of classical type, and was approached in this way by SCHLICHTING (1933, **check**) before the exact solution was discovered. The essential assumption is that the jet is concentrated near the polar axis, as indicated in FIGURE 8.2. A suitable magnified boundary-layer variable in spherical polar coordinates is evidently

$$\overline{\theta} = \frac{\theta}{\epsilon} \ , \tag{8.48}$$

where the small quantity ϵ is specified to be dimensionless and independent of r and θ, with a magnitude chosen to make $\overline{\theta} = O(1)$ in the body of the jet. By assumption, $\epsilon \to 0$ as $\nu \to 0$ or $\mathrm{Re} \to \infty$. The continuity equation (8.7) with $\sin\theta \sim \theta$ becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}ur^2 + \frac{1}{\epsilon r\overline{\theta}}\frac{\partial}{\partial\overline{\theta}}v\overline{\theta} = 0 \ . \tag{8.49}$$

The two terms must be of the same order, and the v-velocity must also be magnified by a factor $1/\epsilon$,

$$\overline{v} = \frac{v}{\epsilon} \ , \tag{8.50}$$

to give

$$\frac{1}{r^2}\frac{\partial}{\partial r}ur^2 + \frac{1}{r\overline{\theta}}\frac{\partial}{\partial\overline{\theta}}\overline{v}\overline{\theta} = 0 \ . \tag{8.51}$$

Now introduce a stream function in the usual way, putting

$$u = \frac{1}{r^2\overline{\theta}}\frac{\partial\overline{\psi}}{\partial\overline{\theta}} \ , \qquad \overline{v} = -\frac{1}{r\overline{\theta}}\frac{\partial\overline{\psi}}{\partial r} \ , \tag{8.52}$$

where the stream function ψ is magnified according to its own rule;

$$\overline{\psi} = \frac{\psi}{\epsilon^2} \ . \tag{8.53}$$

The azimuthal vorticity, defined by equation (8.11) becomes

$$\zeta = \frac{\epsilon}{r}\frac{\partial r\overline{v}}{\partial r} - \frac{1}{\epsilon r}\frac{\partial u}{\partial \overline{\theta}} \tag{8.54}$$

and leads to

$$\overline{\zeta} = \frac{\zeta}{\epsilon} \ . \tag{8.55}$$

After these preliminaries, the radial and azimuthal momentum equations (8.8) and (8.9) become, respectively (**check these carefully**),

$$u\frac{\partial u}{\partial r} + \frac{\overline{v}}{r}\frac{\partial u}{\partial \overline{\theta}} - \frac{\epsilon^2\overline{v}^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\nu}{\epsilon^2}\left(\frac{\epsilon^2}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial u}{\partial r} + \frac{1}{r^2\overline{\theta}}\frac{\partial}{\partial \overline{\theta}}\overline{\theta}\frac{\partial u}{\partial \overline{\theta}} - 2\frac{\epsilon^2 u}{r^2} - \frac{2\epsilon^2}{r^2\overline{\theta}}\frac{\partial \overline{v}\overline{\theta}}{\partial \theta}\right) \ , \tag{8.56}$$

$$\epsilon^2\left(u\frac{\partial \overline{v}}{\partial r} + \frac{\overline{v}}{r}\frac{\partial \overline{v}}{\partial \overline{\theta}} + \frac{u\overline{v}}{r}\right) = -\frac{1}{\rho r}\frac{\partial p}{\partial \overline{\theta}} + \frac{\nu}{\epsilon^2}\left(\frac{\epsilon^4}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial \overline{v}}{\partial r} + \frac{\epsilon^2}{r^2\overline{\theta}}\frac{\partial}{\partial \overline{\theta}}\overline{\theta}\frac{\partial \overline{v}}{\partial \overline{\theta}} + \frac{2\epsilon^2}{r^2}\frac{\partial u}{\partial \overline{\theta}} - \frac{\epsilon^2\overline{v}}{r^2\overline{\theta}^2}\right) \ . \tag{8.57}$$

The equations of motion are now ready for the boundary-layer approximation or inner limit. This is the limit $\epsilon \to 0$ with r and $\overline{\theta} = \theta/\epsilon$ fixed and $O(1)$, so that points in the body of the jet remain in the jet, even in the limit as the body of the jet becomes the polar axis.

Because at least one of the viscous terms in the first equation (8.56) must survive in the limit $\epsilon \to 0$, it is necessary that

$$\epsilon \sim \nu^{1/2} \ . \tag{8.58}$$

Each of the terms in the second equation (8.57), except possibly the pressure term, is at most $O(\epsilon^2)$, and this must therefore also be true of the pressure term. Hence $\partial p/\partial\overline{\theta} = O(\epsilon^2)$, or

$$\frac{\partial p}{\partial \theta} = O(\epsilon) = O(\nu^{1/2}) \ . \tag{8.59}$$

This estimate can be confirmed from equation (8.41) (**check**). To this order, the pressure is constant across the body of the jet, although the constant may depend on r. However, the ambient fluid has been stipulated to be at rest. Hence the centripetal-acceleration term and the pressure-gradient term can be dropped entirely. When physical variables are restored, the boundary-layer problem is defined by

$$u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} = \nu\left(\frac{1}{r^2\theta}\frac{\partial}{\partial \theta}\theta\frac{\partial u}{\partial \theta}\right) \tag{8.60}$$

with

$$u = \frac{1}{r^2\theta}\frac{\partial \psi}{\partial \theta}\ , \quad v = -\frac{1}{r\theta}\frac{\partial \psi}{\partial r}\ . \tag{8.61}$$

The pressure is no longer a dependent variable, but is determined as part of the boundary conditions. The order of the governing equations is reduced by one.

As is usual with boundary-layer problems, the validity of the argument just given can be tested *a posteriori* by using the boundary-layer solution in the full equations. **(Expand on this.)**

8.1.5 The boundary-layer Solution

The original dimensional argument for the form of the solution made no use of equations and is unchanged by the boundary-layer approximation leading to equation (8.60). It should be possible to argue this form using an affine transformation together with the associated invariants (a scheme which is equivalent to a dimensional argument), but I am not satisfied with the analysis at present. **(Do this)**. The absence of characteristic scales again requires the ansatz

$$\psi = \nu r\, g(\theta) \tag{8.62}$$

where θ now means θ and not $\cos\theta$ as in the exact solution. The velocity components from equation (8.12) become

$$u = \frac{\nu}{r\theta}\, g'\ , \qquad v = -\frac{\nu}{r\theta}\, g\ , \tag{8.63}$$

with

$$\zeta = -\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\nu}{r}\left(\frac{g''}{\theta} - \frac{g'}{\theta^2}\right) \ . \tag{8.64}$$

Substitution for u and v in equation (8.60) yields

$$-\frac{g'g'}{\theta} + \frac{gg'}{\theta^2} - \frac{gg''}{\theta} - \frac{g'}{\theta^2} + \frac{g''}{\theta} - g''' = 0 \ . \tag{8.65}$$

This equation is third order (where equation (8.21) was fourth order) because the absence of the pressure in the boundary-layer approximation makes it unnecessary to take the curl. A first integration gives

$$-\frac{gg'}{\theta} + \frac{g'}{\theta} - g'' = C_1 = 0 \ . \tag{8.66}$$

To show that the constant C_1 is zero, note from the first of equations (8.63) that a power-series expansion for $g(\theta)$ must begin with a term in θ^2. A second and non-trivial integration, with the same condition at $\theta = 0$, gives

$$-\frac{g^2}{2\theta} + \frac{2g}{\theta} - g' = C_2 = 0 \tag{8.67}$$

and finally

$$g = \frac{4\theta^2}{\theta^2 + 2c} \ , \tag{8.68}$$

where $2c$ is an undetermined constant of integration.

The boundary-layer approximation to the momentum integral (8.38) is **(connect this with momentum integral for plane jet)**

$$J = 2\pi\rho\epsilon^2 \int_0^{\pi} u^2 r^2 \overline{\theta}\, \mathrm{d}\overline{\theta} \ . \tag{8.69}$$

Note that the axial or x-component of velocity, $u\cos\theta - v\sin\theta = u - \epsilon^2\overline{v}\overline{\theta}$, is indistinguishable from the radial component for $\epsilon \to 0$. A similar statement holds for the radial and axial coordinates r and z. Consequently,

$$J = \epsilon^2 2\pi\rho\overline{\nu}^2 \int_0^{\infty} \frac{g'g'}{\overline{\theta}} \mathrm{d}\,\overline{\theta} \ . \tag{8.70}$$

Substitution from equation (8.68) for g and integration give a relation between J and c,

$$\frac{J}{2\pi\rho\nu^2} = \frac{16}{3c} \quad . \tag{8.71}$$

8.1.6 The inner limit

The operational diagram in FIGURE X [3] identifies the boundary-layer solution with the inner limit of the exact solution. With the preliminary approximation $\sin\theta \sim \theta$, $\cos\theta \sim 1 - \theta^2/2$ for small θ, the exact solution from equations (8.29)–(8.31) is reduced to

$$\psi = 4\,\nu r\,\frac{\theta^2}{(\theta^2 + 2c)} \quad , \tag{8.72}$$

$$u = \frac{16\nu c}{r}\,\frac{1}{(\theta^2 + 2c)^2} \quad , \tag{8.73}$$

$$v = -\frac{4\nu}{r}\,\frac{\theta}{(\theta^2 + 2c)} \quad , \tag{8.74}$$

$$\zeta = \frac{4\nu}{r^2}c(c+2)\frac{\theta}{(\theta^2/2 + c)^3} \quad . \tag{8.75}$$

In the virtual variables defined by equations (8.33)–(8.35), the boundary-layer approximation for the stream function is

$$\Psi = R\frac{\Theta^2}{\Theta^2 + 2c} \quad . \tag{8.76}$$

These streamlines are plotted in FIGURE 15.x for a value $c = 0.005$ (**check**).[4] Again there is only one streamline, with others derived from this by a zoom transformation. The fictitious spherical stream surfaces at the left are generated by equation (8.76) when $\Theta \gg c$, so that $\Psi = R$, approximately. This behavior is an artifact of the spherical polar coordinate system and the fact that the boundary-layer

[3] It is not known what figure this is meant to refer to.

[4] Possibly refers to a missing Figure 8.4

solution has no meaning outside the boundary layer. The characteristics of the boundary-layer equations are the lines $r =$ constant. There is no upstream diffusion of vorticity, and upstream here means directed inward toward the origin along a radius in FIGURE 15.x. SQUIRE (1955) encountered this behavior in his third paper, whose subject was conical laminar jets in spherical polar coordinates. He questioned the behavior, but did not resolve it.

Equations (8.xx) are identical with the boundary-layer solution given by equations (8.62), (8.63), and (8.68). Finally, equation (8.71) in the form

$$\frac{J}{2\pi\rho\nu^2} = \frac{16}{3c} \tag{8.77}$$

is seen to be the leading term in the expansion (8.46). In short, the inner limit of the exact solution coincides in all respects with the solution of the inner limit of the exact equations, as originally claimed for the operational diagram, FIGURE 8.1.

8.1.7 The outer limit

As a fluid element is entrained in the jet, it first undergoes a rapid acceleration. This is followed by a slow deceleration, as the element finds itself close to the jet axis, where $u \sim 1/r$. Each stream tube therefore first converges and then diverges. A convenient measure for the angle $\overline{\theta}$, say $\overline{\theta} = \overline{\theta}_0$, is provided by the point of closest approach to the axis (other nearly equivalent measures can be defined). In boundary-layer variables, as indicated in the sketch (**comment on outer flow**),

$$\overline{\psi} = \frac{4\bar{\nu} r \overline{\theta}^2}{\overline{\theta}^2 + 2\bar{c}} = \frac{4\bar{\nu} z\, \overline{R}^2}{\overline{R}^2 + 2\bar{c} z^2} \ . \tag{8.78}$$

Along a streamline, therefore,

$$\frac{\mathrm{d}\overline{R}}{\mathrm{d}z} = \frac{\overline{R}}{4\bar{c}\, z^3}\left(2\bar{c}\, z^2 - \overline{R}^2\right) \ . \tag{8.79}$$

The derivative vanishes when

$$\frac{\overline{R}_0^2}{z_0^2} = \theta_0^2 = 2\bar{c} \ . \tag{8.80}$$

The parameter

$$\theta_0 = (2c)^{1/2} = \left(\frac{32}{3}\frac{2\pi\rho\nu^2}{J}\right)^{1/2} \tag{8.81}$$

suggests itself *a posteriori* as a suitable quantitative choice for the dimensionless parameter ϵ, with all of the correct properties, beginning with the property $\bar{\theta} = \mathrm{O}(1)$. In fact, according to equation (8.48), where ϵ was first introduced, this choice amounts to putting $\overline{\theta}_0 = 1$. (*Is there an equivalent for other flows? Why stick to spherical polar coordinates? Note that the thickness of the laminar round jet varies like ν, not $\nu^{1/2}$: mention subcharacteristics. Figure out what this means for $\overline{J}$ and $\overline{\nu}$.*)

(*Need to consider laminar round jet out of wall; see p. 19 of 1981 notes and paper by Squire.*)

Outside the jet; i.e., for $\theta \gg \theta_0$, the stream function and velocity components from equations (8.72)–(8.74) approach (**mention circle, Squire**)

$$\overline{\psi} = 4\,\overline{\nu}r \qquad \text{or} \qquad \psi = 4\,\nu r \ , \tag{8.82}$$

$$u = \frac{16\overline{\nu c}}{\overline{\theta}^4} \qquad \text{or} \qquad u = \frac{16\nu c}{\theta^4} \ , \tag{8.83}$$

$$\overline{v} = -\frac{4\overline{\nu}}{r\overline{\theta}} \qquad \text{or} \qquad v = -\frac{4\nu}{r\theta} = -\frac{4\nu}{R} \ . \tag{8.84}$$

It is worth noting that the streamwise velocity u approaches zero for large θ algebraically, like θ^{-4}, rather than exponentially; and also that a power series expansion for $u(\theta)$ converges only for $\theta < \theta_0$, and thus for $u(\theta)/u(0) > 1/4$ from equation (8.73). The reason is the existence of simple poles at $\theta = \pm i\theta_0$ in the complex θ-plane (**mention Weyl?**). (**Exact solution has same property?**)

The message of equation (8.74) is that rv outside the boundary layer is independent of r, and therefore that the action of the jet on the ambient fluid is essentially like that of a uniform line sink along the positive z-axis. To calculate the associated potential flow, first use the representation for a sink of strength Q at the origin in spherical polar coordinates;

$$u = -\frac{Q}{4\pi\rho r^2} \quad . \tag{8.85}$$

For a distribution of such sinks along the positive z-axis the velocity becomes

$$u = \int \mathrm{d}u = \int \cos(\theta' - \theta)\,\mathrm{d}u' = -\int \frac{\cos(\theta' - \theta)\mathrm{d}Q}{4\pi\rho r'^2} \tag{8.86}$$

in the notation shown in the sketch. For a uniform sink, put **(explain why there should be a constant A)**

$$\mathrm{d}Q = 4\pi\rho A\,\mathrm{d}z' \tag{8.87}$$

where A is a constant to be determined from the boundary conditions. Note from the sketch that

$$\frac{z - z'}{r\sin\theta} = \cot\theta' \ , \qquad \mathrm{d}z' = \frac{r\sin\theta\,\mathrm{d}\theta'}{\sin^2\theta'} \quad . \tag{8.88}$$

Finally, therefore,

$$u = -\frac{A}{r\,\sin\theta}\int_\theta^\pi \cos(\theta' - \theta)\,\mathrm{d}\theta' = -\frac{A}{r} \quad . \tag{8.89}$$

The outer stream function in spherical polar coordinates follows from the first of equations (8.12),

$$\psi = Ar\,\cos\theta + B(r) \ \ . \tag{8.90}$$

The boundary conditions are $\psi = 4\nu r$ at $\theta = 0$ and $\psi = 0$ at $\theta = \pi$. Consequently $A = 2\nu$ and $B(r) = 2\nu r$, and

$$\psi = 2\nu r(1 + \cos\theta) \ \ . \tag{8.91}$$

This expression is evidently the outer limit ($\epsilon \to 0$ or $c \to 0$ with r and θ fixed) of the exact solution (8.29). The streamlines described by equation (8.91) are confocal paraboloids of revolution, as shown in the sketch.[5] Write $\psi/2\nu = r + z$, take the square, and use $r^2 = R^2 + z^2$ to obtain

$$R^2 = \frac{\psi}{\nu}\left(\frac{\psi}{4\,\nu} - z\right) \ . \tag{8.92}$$

It is worth noting that the streamlines from the boundary-layer solution outside the boundary layer, $\psi = 4\,\nu r$, are concentric circles (see Squire). If the boundary-layer approximation had been made in cylindrical polar coordinates, there would have been obtained $\psi = 4\nu z = 4\nu r \cos\theta$. These streamlines are straight lines normal to the z-axis. Neither result is useful, because the boundary-layer approximation should not be relied on outside the boundary layer. The correct streamlines are the paraboloids (?) given by equation (8.91), $\psi = 2\,\nu r(1 + \cos\theta)$. **(Comment on the outer limit of the exact solution.)**

The rule for constructing the composite expansion (see Van Dyke 1975 and the sketch) is to add the inner and outer approximations and subtract the common part. The procedure is illustrated in the sketch. From equations (8.72) and (8.91),

$$\psi_c = 4\nu r \frac{\theta^2}{\theta^2 + \theta_0^2} + 2\nu r(1 + \cos\theta) - 4\nu r \ . \tag{8.93}$$

For $\theta \sim \theta_0$, the third term essentially cancels the second, and the first term dominates. For $\theta \gg \theta_0$, the third term essentially cancels the first, and the second term dominates. It is plausible that the expression (8.93) is a uniformly valid approximation to the exact solution of equation (8.29),

$$\psi_e = 2\nu r \frac{\sin^2\theta}{1 - \cos\theta + c} \ . \tag{8.94}$$

That is, the quantity

$$\left|\frac{\psi_e - \psi_c}{2\nu r}\right| \tag{8.95}$$

should be a bounded function of θ for sufficiently small c (actually for all c).

[5] It is not known what sketch this refers to.

8.1.8 Miscellaneous remarks

CHIN (1981) recently showed that confocal paraboloidal coordinates are optimal for the Squire-Landau problem in the sense defined by KAPLUN (1954); the boundary-layer solution includes the outer solution, although the boundary-layer solution is not exact. In my review of Chin's paper for another journal, I objected (unsuccessfully) that it is not necessary to derive and solve the boundary-layer equations in the paraboloidal system, since Kaplun's substitution theorem is more efficient.

WYGNANSKI (1970) has extended the original exact Squire-Landau solution in equations (8.29)–(8.32) to the case of flow with swirl by resorting to numerical methods. The minimum in the axial velocity on the axis for large swirl, the increased entrainment, and the approach of the outer flow to a viscous core/potential vortex motion are clearly brought out.

SQUIRE in a second paper (1952) considered the exact problem when there is a conical wall at $\theta = \Theta$, particularly $\Theta = \pi/2$, with a slip boundary condition at the wall. The issue is mainly the evaluation of the constants c_1, c_2, c_3 in equations (8.22)–(8.24) when these are not all zero, as well as the complications that set in during the final integration step. SCHNEIDER (1981) claims to find an exact solution for this case with a no-slip condition, thus taking into account the displacement effect on the outer flow of the boundary layer on the wall. I have not studied this paper closely enough to understand in what sense the solution is exact. GINEVSKII (1966) carried out the outer-flow approximation for a turbulent jet with a wall at $\theta = \Theta$, in the manner used to obtain equation (8.91). The entrainment velocity was estimated by using the polynomial mean-velocity profile. The two-dimensional case is treated similarly. In no case was a no-slip condition applied at the wall.

A third paper by SQUIRE (1955) attempted to treat a conical jet lying along a surface $\theta = \theta^*$, with the radial jet ($\theta^* = \pi/2$) as an important special case. The analysis uses a boundary-layer approximation, and the streamlines outside the jet show the same pathological behavior shown by equations (8.72) for $\theta^2 \gg 2c$. The outer flow and the composite expansion are not considered.

A few other references are cited in ROSENHEAD (1963), WYGNANSKI (1970), and SCHNEIDER (1981).

8.2 Laminar round jet into moving fluid

8.3 Transition

ANDRADE (1937)
DOMM et al (1955)
BECKER and MASSARO (1968)
SYMONS and LABUS (1971)
ZAUNER (1985)
TUCKER and ISLAM (1986)
PETERSEN et al (1988)
MEIBURG et al (1989)
LIEPMANN (1991)
BROZE and HUSSAIN (1994)
TONG and WARHAFT (1994)

References

Batchelor, G. K. 1967 “An Introduction to Fluid Dynamics,” Cambridge, 205–211.

Chin, W. C. 1981 *Optimal Coordinates for Squire’s Jet*. AIAA J. *19*, 123-124.

Kaplun, S. 1954 *The Role of Coordinate Systems in Boundary-Layer Theory.* ZAMP**5**, 111–135.

Landau, L. F 1944 *A New Exact Solution of the Navier Stokes Equations*. Doklady Acad. Sci. **43**, 286–288.

Rosenhead, L. (ed.) 1963 *Laminar Boundary Layers,* Oxford, 150–155.

Schneider, W. 1981 *Flow Induced by Jets and Plumes*. J. Fluid Mech. **108**, 55–65.

Slezkin, N. A. 1934 *On an Exact Solution of the Equations of Viscous Flow*. Scientific Papers, Moscow State University, No. 2 (I have not seen this paper).

Squire, H. B. 1951 *The Round Laminar Jet*. Quart. J. Mech. Appl. Math. **4**, 321–329.

Squire, H. B. 1952 *Some Viscous Fluid Flow Problems. I: Jet Emerging From a Hole in a Plane Wall*. Phil. Mag. (7) **43**, 942–945.

Squire, H. B. 1955 *Radial Jets*. In *50 Jahre Grenzschichtforschung,* Vieweg, Braunschweig, 47–54.

Van Dyke, M. 1975 “Perturbation Methods in Fluid Mechanics,” Parabolic Press.

Wygnanski, I. 1970 *Swirling Axisymmetrical Laminar Jet*. Phys. Fluids **13**, 2455–2460.

8.4 Turbulent round jet into fluid at rest

[This section was found in a separate file and appears to belong here, though in its found form it repeated the first two paragraphs of the start of this chapter]

For the round jet, cylindrical polar coordinates (r, θ, z) are appropriate because experimenters move their probes in a plane z = constant. The velocity components are (u, v, w), and the jet motion is along the positive z-axis. The ambient fluid, which is here taken to be the same as the fluid in the jet, is nominally at rest, so that the pressure is nominally constant. If the mean motion is steady, axially symmetric, and free of swirl, the boundary-layer equations of motion are

$$\frac{1}{r}\frac{\partial ru}{\partial r} + \frac{\partial w}{\partial z} = 0 \ , \tag{8.96}$$

$$u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = \frac{1}{\rho r}\frac{\partial r\tau}{\partial r} \ , \tag{8.97}$$

$$\tau = \mu\frac{\partial w}{\partial r} - \rho\overline{u'w'} \ . \tag{8.98}$$

The first and second equations can be combined in the form

$$\frac{\partial ruw}{\partial r} + \frac{\partial rww}{\partial z} = \frac{1}{\rho}\frac{\partial r\tau}{\partial r} \tag{8.99}$$

and this expression can be integrated over a plane z = constant, with boundary conditions $u = 0$ at $r = 0$ and $w = \tau = 0$ at $r = \infty$, to obtain the momentum integral

$$J = 2\pi\rho\int_0^\infty rww\,dr = \text{constant} \ . \tag{8.100}$$

The parameters of the problem are J and ρ, with, from equation (8.100),

$$\left[\frac{J}{\rho}\right] = \frac{\mathbf{L}^4}{\mathbf{T}^2} = \mathbf{L}^2\mathbf{U}^2 \ . \tag{8.101}$$

There is nothing else to work with. The absence of a characteristic length means that the jet must grow conically. When a stream function is introduced such that

$$u = -\frac{1}{r}\frac{\partial \psi}{\partial z} \quad , \qquad w = \frac{1}{r}\frac{\partial \psi}{\partial r} \tag{8.102}$$

it is seen that

$$[\psi] = \mathbf{L}^2\mathbf{U} = \mathbf{L}\left[\frac{J}{\rho}\right]^{1/2} \quad . \tag{8.103}$$

(Profile is f'/η, not f'. Can this be fixed?) The profile similarity assumption is therefore to use z and J/ρ to form the non-dimensionalizing combination; thus

$$A\psi = z\left(\frac{J}{\rho}\right)^{1/2} f\left(B\frac{r}{z}\right) \tag{8.104}$$

where A and B are disposable dimensionless constants. **(Why not use spherical polar coordinates?)**

The mean velocity components are

$$u = -\frac{1}{Ar}\left(\frac{J}{\rho}\right)^{1/2}(f - \eta f') \ , \tag{8.105}$$

$$w = \frac{B}{Ar}\left(\frac{J}{\rho}\right)^{1/2} f' \ . \tag{8.106}$$

Papers with profiles can give both the growth rate and the centerline velocity decay. To begin with, use only data that include both sides of the profile (denoted by sym). Try 3 functions for fit near plane of symmetry. Look for constant in growth rate and distance required to achieve it after arbitrary initial condition. Apparent origins from $\delta(x)$ and $u_o/u_c(x)$ should be the same.

Papers with profiles:

VOORHEIS (1940)
REICHARDT (1942) 9A (sym)
ALBERTSON et al. (1948)

CORRSIN and UBEROI (1949)
HINZE and ZIJNEN (1949)
NOTTAGE (1951)
TAYLOR et al. (1951)
GAYLORD (1953) (sym)
CORRSIN and KISTLER (1954) 7N (sym)
LAURENCE (1956)
SUNAVALA et al. (1957)
HIDY (1962)
ROSLER (1962)
FARIS (1963)
KNYSTAUTAS (1964) 17A (sym)
LAWRENCE (1965)
DELLEUR et al. (1966) (sym)
SAMI (1967)
SAMI et al. (1967)
WHITE (1967)
KAMOTANI and WISKIND (1968) (sym)
WYGNANSKI and FIEDLER (1968)
CHUANG (1970)
CROW and CHAMPAGNE (1970)
EASTLAKE (1971) 17C
GOLDSCHMIDT et al. (1972) 17A
LABUS and SYMONS (1972) (sym)
WITZE (1974) 17G
ABBISS et al. (1975)
BRADBURY and KHADEM (1975)
HATTA and NOZAKI (1975) 17A
RODI (1975)
BARNETT and GIEL (1976)
EBRAHIMI and KLEINE (1977)
SHAUGHNESSY and MORTON (1977) (sym)
BORREGO and OLIVARI (1979) (sym)
MODARRESS et al. (1984)
CHUA and ANTONIA (1986)
OBOT and TRABOLD (1987)
SHLIEN (1987)

TAULBEE et al. (1987)
KASAGI et al. (1988)
KINDLER (1988)
HUSSEIN and GEORGE (1989)
DOWLING and DIMOTAKIS (1990)
KUHLMAN and GROSS (1990)
PANCHAPAKESAN and LUMLEY (1993)
HUSSEIN et al. (1994)

Some other papers that give growth rate and/or velocity decay, but not profiles, are

KISER (1963)
BECKER et al. (1965, 1967)
SINGAMSETTI (1965)
MONS and SFORZA (1971)
SFORZA and MONS (1978)
AHMED et al. (1988)
DRUBKA et al. (1989)

Substitution of equations (8.105) and (8.106) in equation (8.97) leads to

$$\frac{ff''}{\eta} + \frac{f'f'}{\eta} - \frac{ff'}{\eta^2} = \frac{A^2}{B^3}\,\frac{z^2}{J}\,\frac{\partial r\tau}{\partial r} \tag{8.107}$$

where

$$\eta = B\,\frac{r}{z} \ . \tag{8.108}$$

In the course of this operation the term $w\,\partial w/\partial z$ in equation (8.97) was cancelled by another term. This property means that the left side of equation (8.107) must be a perfect differential; it is in fact $(ff'/\eta)'$. On the right, if the laminar stress is neglected, the dimensions of τ are

$$\left[\frac{\tau}{\rho}\right] = U^2 = \frac{1}{L^2}\left[\frac{J}{\rho}\right] \ . \tag{8.109}$$

The only length available is z, so that it is necessary to put (**need another constant?**)

$$\frac{\tau}{\rho} = \frac{J}{\rho z^2}\,g(\eta) \ . \tag{8.110}$$

It follows that

$$\frac{\partial r\tau}{\partial r} = \frac{J}{z^2}(\eta g)' \qquad (8.111)$$

and therefore, from equation (8.107), that

$$\frac{ff'}{\eta} = \frac{A^2}{B^3}\eta g + c \ . \qquad (8.112)$$

Since f'/η is bounded and f goes to zero as $\eta \to 0$ (see formula for w; give B.C. separately), the constant c is zero. (**Something about eddy viscosity.**)

Various Reynolds stresses:

CORRSIN and UBEROI (1949)
LITTLE and WILBUR (1951)
ROSLER (1962)
BRADSHAW et al. (1964)
LAWRENCE (1965)
BECKER et al. (1967)
WYGNANSKI and FIEDLER (1968)
CHUANG (1970)
CROW and CHAMPAGNE (1970)
GOLDSCHMIDT et al. (1972)
ABBISS et al. (1975)
RODI (1975)
BARNETT and GIEL (1976)
SHAUGHNESSY and MORTON (1977)
CHEVRAY and TUTU (1978)
CHUA and ANTONIA (1986)
AHMED et al. (1988)
KASAGI et al. (1988)
HUSSEIN and GEORGE (1989)
HUSSEIN et al. (1993)
PANCHAPAKESAN and LUMLEY (1993)

Papers with intermittency data:

CORRSIN and KISTLER (1954) 7N

BECHER et al. (1965)
WYGNANSKI and FIEDLER (1968)
ANTONIA (1974)
ANTONIA et al. (1975)
SHAUGHNESSY and MORTON (1977)
CHEVRAY and TUTU (1978)

The entrainment is given by

$$(ru)_\infty = -\frac{1}{A}\left(\frac{J}{\rho}\right)^{1/2} f(\infty) \tag{8.113}$$

provided that the product $\eta f'$ (i.e., $r^2 w$) goes to zero at infinity, a condition that is plausible but experimentally unprovable. The entrainment velocity is independent of z.

For flow out of a wall at $z = 0$, the outer flow responds to a uniform distributed sink on the positive z-axis, and the streamlines are radial. If there is no wall, the outer-flow streamlines are confocal parabolas, as in the laminar case. (**Do outer stream function and composite flow.**)

Scale: use r for $w/w_c = \frac{1}{2}$, or closest approach of mean streamlines. **(Comment on problem with hot wire rectifying, or seeding problem with LDV.)**

A comment can be made about dilution. If the flow comes from an orifice of diameter d with uniform velocity w_0, as shown in the sketch, then

$$\frac{m_0}{\rho} = \frac{\pi}{4}\, w_0 d^2 \ , \tag{8.114}$$

$$\frac{J}{\rho} = \frac{\pi}{4}\, {w_0}^2 d^2 \ . \tag{8.115}$$

For turbulent flow, it follows from

$$\frac{m}{\rho} = 2\pi \int_0^\infty rw\, dr \tag{8.116}$$

and

$$w = \frac{1}{r}\, \frac{B}{A}\left(\frac{J}{\rho}\right)^{1/2} f'(\eta) \tag{8.117}$$

that

$$\frac{m}{\rho} = \frac{2\pi}{A}\left(\frac{J}{\rho}\right)^{1/2} f(\infty)z \ . \tag{8.118}$$

Now use equation (8.115) to eliminate J/ρ; thus **(check this)**

$$\frac{m}{m_0} = \frac{f(\infty)}{A}\,(16\pi)^{\frac{1}{2}}\frac{z}{d} \ . \tag{8.119}$$

The condition that $m/m_0 \gg 1$, if $f(\infty)$ and A are O(1) is, conservatively,

$$\frac{z}{d} \gg 1 \ . \tag{8.120}$$

Note that the spreading angle is not small, so that there is some uncertainty about the security of the boundary-layer approximation. However, the other turbulent stresses can be carried along. **(Say something about initial conditions.)**

(The round jet is a good flow for checking consistency of growth rate, because there are no side walls in the problem.) **Newman**. *Column 3 is not quite correlation. Figure 2 has laminar profile, two guessed profiles, and one empirical profile. Thickness is not always defined consistently. On p. 3, there is reasonable agreement in* $UL^{1/2}$. **Comment on effect of wall for plane jet (and absence of wall). Check first on momentum balance.**

Harsha. *Note large discrepancies in Reynolds stresses. Some of this may be real,and may depend on initial conditions. Such figures bound the accuracy that modelers can hope to get.*

Mention momentumless wake.

Put together derivation of turbulent energy equation.

Discuss scales, $\kappa - \epsilon$ *models.*

Notes on handout.

Round jet into moving fluid:

FORSTALL and SHAPIRO (1950)
TANI and KOBASHI (1951)
KOBASHI (1952)

PABST (1960)
BECKER et al. (1962, 1965)
ALPINIERI (1964)
CHIGIER and BEER (1964)
REICHARDT (1964, 1965)
CHAMPAGNE and WYGNANSKI (1970, 1971)
ROZENMAN and WEINSTEIN (1970)
RAZINSKY and BRIGHTON (1971)
DURAO and WHITELAW (1973)
MATSUMOTO et al. (1973)
HAMMERSLEY (1974)
ANTONIA and BILGER (1976)
OWEN (1976)
SMITH and HUGHES (1977)
DE WOLF and MUNNIKSMA (1980)
BINDER and KLAN (1983)
HUSAIN (1984)
SO and AHMED (1984)
KO and KWAN (1985)
KNUDSEN and WOOD (1986)
SUZUKI et al. (1987)
GORE and CROWE (1988)
KHODADI and VLACHOS (1989)
GLADNIK et al. (1990)
HENBEST and YACOUB (1991)
STRYKOWSKI and WILCOXON (1992)
HUANG and LIN (1994)

Round jet into different gas:

KEAGY et al. (1949)
WILSON and DANKWERTS (1964)
TOMBACH (1969)
TRENTACOSTE and SFORZA (1970)
LENZE (1976)
BIRCH et al. (1978)
CHEN et al. (1986)
BALLAL and CHEN (1987)

ZHU, SO, and OTUGEN (1988)
PANCHAPAKESAN and LUMLEY (1993)
SAUTET and STEPOWSKI (1995)
DJERIDANE et al. (1996)

Non-round jets:

Elliptic

TRENTACOSTE and SFORZA (1969)
HUSAIN (1984)
GUTMARK et al. (1985)
HUSAIN and HUSSAIN (1985)
HO and GUTMARK (1987)
QUINN (1989)
HUSAIN and HUSSAIN (1993)
BROWN et al. (1994)

Triangular

GUTMARK et al. (1985)
KOSHIGOE et al. (1989)
QUINN (1989, 1990)

Square

GRINSTEIN et al. (1995)

Chapter 9

THE PLANE JET

Consider a plane jet issuing into a stagnant fluid from a slit in a plane wall, as shown schematically in FIGURE 9.1. This configuration is often used in experimental work for the sake of its standard geometry. The most important quantity in any description of such a jet flow, whether laminar or turbulent, is J, the initial flux of momentum per unit span. As the flow develops in the downstream direction, this initial momentum is conserved as it is gradually transferred from the jet fluid to the ambient fluid by shearing stresses. The rate of momentum transfer will depend not only on the nature of these stresses, but also on the relative densities of the two fluids, if these are different, and on the effect of real rather than ideal initial and boundary conditions. In any case, the total rate of fluid flow in the jet will increase continuously in the downstream direction as external fluid is entrained. It is this entrainment process that dominates most practical problems.

For laminar flow, the solution of the boundary-layer problem is known in closed form, and the absence of a dimensionless parameter implies that all laminar plane jets are equivalent. So are all turbulent jets, for the same reason. However, for turbulent flow the growth rate is relatively rapid, and a boundary-layer approximation may not be appropriate.

A simple generalization of the classical flow is obtained if the wall is absent and the jet issues from a line momentum source into