PACIFIC JOURNAL OF MATHEMATICS
Vol. 60, No. 1, 1975

COUNTABLE ORDINALS AND THE ANALYTICAL HIERARCHY, I

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The following results are proved, using the axiom of Projective Determinacy: (i) For $n \geqq 1$, every Π^1_{2n+1} set of countable ordinals contains a Δ^1_{2n+1} ordinal, (ii) For $n \geqq 1$, the set of reals Δ^1_{2n} in an ordinal is equal to the largest countable Σ^1_{2n} set and (iii) Every real is Δ^1_n inside some transitive model of set theory if and only if $n \geqq 4$.

In general we shall use the terminology and notation of **[3]**. In particular letters $i, j, k, \cdots$ are used as variables over $\omega = \{0, 1, 2, \cdots\}$ and $\alpha, \beta, \gamma, \cdots$ as variables over ${}^\omega\omega$ (= the set of *reals*). For a collection of sets of reals Γ, *Determinacy* (Γ) abbreviates the statement that every set in Γ is determined and *Projective Determinacy* (PD) is the axiom that every projective set is determined.

1. An ordinal basis theorem.

A well known boundedness result in recursion theory asserts that if $WO = \{\alpha : \leqq_\alpha \text{ is a wellordering}\}$, where $\leqq_\alpha = \{(m, n) : \alpha(2^m \cdot 3^n) = 0\}$ and $A \subseteq WO$ is Σ^1_1, then $\sup\{|\alpha| : \alpha \in A\} < \delta^1_1$, where for $\alpha \in WO$, $|\alpha| = \text{length}(\leqq_\alpha)$ and $\delta^1_n = \sup\{|\alpha| : \alpha \in \Delta^1_n \& \alpha \in WO\}$. We prove below a generalization of this fact to all odd levels of the analytical hierarchy.

THEOREM 1.1. *Assume Projective Determinacy, when $n \geqq 1$. If $A \subseteq WO$ is Σ^1_{2n+1} and $\sup\{|\alpha| : \alpha \in A\} < \aleph_1$, then $\sup\{|\alpha| : \alpha \in A\} < \delta^1_{2n+1}$.*

Proof. For notational simplicity let us take $n = 1$ as a typical case. Thus let $A \subseteq WO$ be Σ^1_3 and assume $\sup\{|\alpha| : \alpha \in A\} < \aleph_1$. Let $B \subseteq {}^\omega\omega$ be Π^1_2 and $f : {}^\omega\omega \to {}^\omega\omega$ recursive such that $f[B] = A$. Consider then the following game: Player I plays β, player II plays γ and II wins iff $\gamma \in WO \& (\beta \in B \to |f(\beta)| \leqq |\gamma|)$. Clearly player II has a winning strategy in this game. But his payoff set is Σ^1_2, so by a result of Moschovakis **[6]** he has a winning strategy τ which is Δ^1_3. Let $T = \{\beta * \tau : \beta \in {}^\omega\omega\}$, where $\beta * \tau$ is the result of II's moves following τ when I plays β. Then $T \subseteq WO$ and T is $\Sigma^1_1(\tau)$, so by the Boundedness Theorem

$$\sup\{|\gamma| : \gamma \in T\} < \delta^1_1(\tau) < \delta^1_3.$$

But clearly

$$\sup\{|\alpha|: \alpha \in A\} \leqq \sup\{|\gamma|: \gamma \in T\}.$$

It is not hard now to reformulate Theorem 1.1. into a basis theorem. Call a set of ordinals $X \subseteq \aleph_1$ Π^1_n if $\{\alpha \in WO: |\alpha| \in X\}$ is Π^1_n. Recall that a countable ordinal ξ is called Δ^1_n iff $\xi < \delta^1_n$.

THEOREM 1.2. *Assume Projective Determinacy and $n \geqq 1$. Every nonempty Π^1_{2n+1} set of ordinals contains a Δ^1_{2n+1} ordinal.*

Proof. Take again $n = 1$. Let $X \subseteq \aleph_1$ be Π^1_3 and consider $A = \{\alpha \in WO: |\alpha| \leqq \min X\}$. Then $\alpha \in A \Leftrightarrow \alpha \in WO \& \forall m(|\alpha_m| \notin X)$, where α_m is a real coding the restriction of $\leqq_\alpha$ to its initial segment determined by m. Clearly A is Σ^1_3 and $A \subseteq WO$, so by Theorem 1.1. $\sup\{|\alpha|: \alpha \in A\} = \min X < \delta^1_3$, thus X contains a Δ^1_3 ordinal.

REMARK. If $X = \{\xi < \aleph_1: \omega_0 < \xi$ is admissible$\}$ then X is Π^1_1 but contains no Δ^1_1 ordinal.

2. The set of reals Δ^1_n in an ordinal. A real α is called Δ^1_n in an ordinal $\xi < \aleph_1$ if α is Δ^1_n in every real $\beta \in WO$ such that $|\beta| = \xi$. A simple argument shows that a real is Δ^1_1 in an ordinal iff it is Δ^1_2 in an ordinal iff it is constructible. Martin and Solovay [4] proved that under PD, the set of reals Δ^1_{2n+1} in an ordinal, when $n \geqq 1$, is exactly the set Q_{2n+1} (see [4] or [3].) [The set Q_{2n+1} can be defined in many equivalent ways. One of the most suggestive ones is the following: $Q_{2n+1} = \{\alpha: \alpha$ belongs to every model of $ZFC + PD$ for which Σ^1_{2n} formulas are absolute$\}$]. We shall identify below the set of reals which are Δ^1_{2n} in an ordinal. Before doing this though we shall give as an application of Theorem 1.1. a new proof of the result of Martin and Solovay. Their original proof used forcing.

THEOREM 2.1. (Martin-Solovay [4]). *Assume Projective Determinacy and $n \geqq 1$. Then $Q_{2n+1} = \{\alpha: \alpha$ is Δ^1_{2n+1} in an ordinal$\}$.*

Proof. Let $n = 1$ again. From the results of [3] (especially the Lemma before Theorem (3B-3)) we can easily see that every real in Q_3 is Δ^1_3 in an ordinal. Conversely, if α is Δ^1_3 in an ordinal, then by Theorem 1.2, α is Δ^1_3 in an ordinal ξ which is Δ^1_3 in α. Thus $A = \{\alpha: \alpha$ is Δ^1_3 in an ordinal$\}$ is Π^1_3. By Theorem (3B-3) of [3] the set Q_3 is characterized as the largest Π^1_3-bounded set, where a set $B \subseteq \omega^\omega$ is Π^1_3-bounded if for all predicates $P(\alpha, \beta)$ in Π^1_3 the predicate $\exists \alpha \in BP(\alpha, \beta)$ is also Π^1_3. To complete the proof it is thus enough to show that A is Π^1_3 bounded. For any Π^1_3 predicate $P(\alpha, \beta)$ put

$$X_\beta = \{\xi < \aleph_1 : \forall\, \gamma(\gamma \in WO \& |\gamma| = \xi \to \exists\, \alpha \in \Delta^1_3(\gamma)[P(\alpha,\beta)\& a \in A].\}$$

Then X_β is Π^1_3 in β and $\exists\, \alpha \in AP(\alpha,\beta) \Rightarrow X_\beta \neq \varnothing$, so $\exists \alpha \in AP(\alpha,\beta) \Rightarrow X_\beta$ contains an ordinal Δ^1_3 in $\beta \Rightarrow \exists \alpha \in \Delta^1_3(\beta)\ [P(\alpha,\beta)\&\alpha \in A]$. Thus $\exists\, \alpha \in AP(\alpha,\beta) \Leftrightarrow \exists\, \alpha \in \Delta^1_3(\beta)\ [\alpha \in A\&P(\alpha,\beta)]$ and we are done.

We now proceed to identify the set of reals Δ^1_{2n} in an ordinal. Our result generalizes the fact that $\{\alpha : \alpha$ is Δ^1_2 in an ordinal$\} = \{\alpha : \alpha$ is constructible$\} = C_2 =^{\text{def}}$ the largest countable Σ^1_2 set of reals (we assume here that there are only countably many constructible reals).

THEOREM 2.2. *Assume Projective Determinacy and let C_{2n} = the largest countable Σ^1_{2n} set of reals. Then $C_{2n} = \{\alpha : \alpha$ is Δ^1_{2n} in an ordinal$\}$.*

Proof. For notational simplicity take $n = 2$. By Theorem (1C-3) of [3], $C_4 \subseteq \{\alpha : \alpha$ is Δ^1_4 in an ordinal$\} =^{\text{def}} S$. Since S is countable it is enough to prove that S is Σ^1_4.

For each countable ordinal $\xi \geqq \omega$ consider the space ${}^\omega\xi$ with the product topology, where ξ has the discrete topology. If $f \in {}^\omega\xi$ let $\leqq_f$ be the relation on ω given by $m \leqq_f n \Leftrightarrow f(m) \leqq f(n)$. Write $\alpha \in \Delta^1_4(f)$ iff $\alpha \in \Delta^1_4(\leqq_f)$. We shall prove that if

$$P = \{\alpha : (\exists\, \xi)(\omega \leqq \xi < \aleph_1\ \&\ \{f \in {}^\omega\xi : \alpha \in \Delta^1_4(f)\} \text{ is not meager})\},$$

then $P \in \Sigma^1_4$ and $P = S$. Note first that

$$\alpha \in P \Leftrightarrow (\exists\, \sigma)(\sigma \in WO^*\ \&\ \{f \in {}^\omega|\sigma| : \alpha \in \Delta^1_4(f)\} \text{ is not meager}),$$

where $\alpha \in WO^* \Leftrightarrow \alpha \in WO\& |\alpha| \geqq \omega \& \leqq_\alpha$ has field ω. If $\sigma \in WO^*$ let $h_\sigma : \omega \to |\sigma|$ be the bijection such that $m \leqq_\sigma n \Leftrightarrow h_\sigma(m) \leqq h_\sigma(n)$ and let $h^*_\sigma : {}^\omega\omega \to {}^\omega|\sigma|$ be given by $h^*_\sigma(\alpha) = h_\sigma \circ \alpha$. Then h^*_σ is a homeomorphism of ${}^\omega\omega$ with ${}^\omega|\sigma|$. Thus

$$\begin{aligned}\alpha \in P \Leftrightarrow & (\exists\sigma)(\sigma \in WO^*\& h^{*-1}_\sigma[\{f \in {}^\omega|\sigma| : \alpha \in \Delta^1_4(f)\}] \\ & \text{is not meager}) \\ \Leftrightarrow & (\exists\sigma)(\sigma \in WO^*\&\{\beta : \alpha \in \Delta^1_4(h_\sigma \circ \beta)\} \\ & \text{is not meager}) \\ \Leftrightarrow & (\exists\sigma)(\sigma \in WO^*\&\{\beta : \alpha \in \Delta^1_4(\leqq_{h_\sigma \circ \beta})\} \\ & \text{is not meager}).\end{aligned}$$

But for $\sigma \in WO^*$,

$$\begin{aligned} m \leqq_{h_\sigma \circ \beta} n &\Leftrightarrow h_\sigma(\beta(m)) \leqq h_\sigma(\beta(n)) \\ &\Leftrightarrow \beta(m) \leqq_\sigma \beta(n) \\ &\Leftrightarrow H(\beta, \sigma)(2^m \cdot 3^n) = 0 \end{aligned}$$

where $H: {}^\omega\omega \times {}^\omega\omega \to {}^\omega 2$ is recursive and $H(\beta, \sigma)(k) = 1$, if $\forall m \forall n (k \neq 2^m \cdot 3^n)$. So

$$\alpha \in P \Leftrightarrow (\exists \sigma)(\sigma \in WO^* \;\&\; \{\beta : \alpha \in \Delta^1_4(H(\beta, \sigma))\} \text{ is not meager})$$

which by Theorem 2.2.5(b) of **[2]** shows that P is Σ^1_4.

We prove now that $S = P$. Clearly $S \subseteq P$. For the converse let $\alpha \in P$ and find $\xi \geqq \omega$ such that $\{f \in {}^\omega\xi : \alpha \in \Delta^1_4(f)\}$ is not meager. Let $\sigma \in WO$ be such that $|\sigma| = \xi$. We shall show that $\alpha \in \Delta^1_4(\sigma)$. Without loss of generality we can assume that $\leqq_\sigma$ has field all of ω (i.e., $\sigma \in WO^*$). Then as before $\{\beta : \alpha \in \Delta^1_4(H(\sigma, \beta))\}$ is not meager, thus $\{\beta : \alpha \in \Delta^1_4(\sigma, \beta)\}$ is not meager. But then by Theorem 3.1.2.(b) of **[2]**, $\alpha \in \Delta^1_4(\sigma)$.

3. Ordinals and reals Δ^1_n in models of set theory. Call a countable ordinal ξ *almost* Δ^1_n if there is a countable transitive model M of ZFC such that $\xi \in M$ and $M \models \xi$ is Δ^1_n. For $n = 1, 2$ it is trivial to see that ξ is almost Δ^1_n iff ξ is Δ^1_n. By using a simple Solovay type game one can prove easily the following. (We call a transitive model M of ZFC *Σ^1_n-correct* if Σ^1_n formulas are absolute for M.)

PROPOSITION 3.1. *Assume Projective Determinacy. For any $n \geqq 1$ and any countable ordinal ξ there is a model M of ZFC + Determinacy ($\mathbf{\Delta}^1_{2n}$) which is Σ^1_{2n}-correct, $\xi \in M$ and $M \models \xi$ is Δ^1_{2n+1}.*

Call now a real α *almost* Δ^1_n if there is a transitive countable model M of ZFC such that $\alpha \in M$ and $M \models \alpha$ is Δ^1_n. Abbreviate by D_n the statement $\forall \alpha (\alpha$ is almost $\Delta^1_n)$. As with ordinals, α is almost Δ^1_n iff α is Δ^1_n, for $n = 1, 2$, so D_1, D_2 fail. Unlike the case of the ordinals D_3 also fails.

THEOREM 3.2. *Assume Projective Determinacy and let $D_n \Leftrightarrow \forall \alpha (\alpha$ is almost $\Delta^1_n)$. Then D_n holds iff $n \geqq 4$.*

Proof. Clearly $\{\alpha : \alpha$ is not almost $\Delta^1_n\}$ is a Π^1_2 set, so if not empty it has a Δ^1_4 solution, a contradiction if $n \geqq 4$. To complete the proof we show that D_3 fails. Let α_0 be a real which codes a countable transitive model N of $ZFC + Determinacy$ $(\mathbf{\Delta}^1_2)$ which is Σ^1_2-correct. We show that α_0 is not almost Δ^1_3. If not, let M be a countable transitive model of ZFC such that $\alpha_0 \in M$ and $M \models \alpha_0$ is Δ^1_3. Then $N \in M$ and Σ^1_2 formulas are absolute from M to N, so, since $N \models$Determinacy $(\mathbf{\Delta}^1_2)$, clearly $M \models$Determinacy (Δ^1_2) (in general we cannot conclude that $M \models$Determinacy $(\mathbf{\Delta}^1_2)$). By the arguments in [1] or [5] the class of Π^1_3 relations on ω has the prewellordering property, *in M*. But then by Moschovakis [7], $M \models$"There is a Δ^1_2 game in which I has a winning strategy and α_0 is recursive in *every strategy* of this game". Since $N \models$Determinacy $(\mathbf{\Delta}^1_2)$ at least one of these strategies is in N, a contradiction.

Theorem 3.2 exposes a weak phenomenon which for the first time happens in the fourth level of the analytical hierarchy. Hopefully the discovery of more and especially stronger such phenomena will result in the understanding of the structural differences between the third and the fifth level of the analytical hierarchy. For that purpose it seems that what is needed is a deeper understanding of the role of uncountable ordinals and models of set theory in the analytical hierarchy.

REFERENCES

1. J. W. Addison and Y. N Moschovakis, *Some consequences of the axiom of definable determinateness*, Proc. Nat. Acad. Sci., USA **59** (1968), 708–712.

2. A. S. Kechris, *Measure and category in effective descriptive set theory*, Annals of Math. Logic, **5** (1973), 337–384.

3. ———, *The theory of countable analytical sets*, Trans. of Amer. Math. Soc., **202** (1975), 259–297.

4. D. A. Martin and R. M. Solovay, *Basis theorems for* Π^1_{2k} *sets of reals*, to appear.

5. D. A. Martin, *The axiom of determinateness and reduction principles in the analytical hierarchy*, Bull. Amer. Math. Soc., **74** (1968), 687–689.

6. Y. N. Moschovakis, *Analytical definability in a playful universe*, P. Suppes et al., eds. Logic, Methodology and Philosophy of Science IV, North Holland, 1973, 77–85.

7. ———, *Proof of a conjecture of Martin*, mimeographed notes.

Received August 29, 1974. Research partially supported by NSF grant GP 27964.

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