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# Note on Green function formalism and topological invariants

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**ABSTRACT:** It has been discovered previously that the topological order parameter could be identified from the topological data of the Green function, namely the (generalized) TKNN invariant in general dimensions, for both non-interacting and interacting systems. In this note, we show that this phenomena has a clear geometric derivation. This proposal could be regarded as an alternative proof for the identification of the corresponding topological invariant and topological order parameter.

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## 1 Introduction

Recent researches in condensed matter community show novel applications of quantum field theory method in topological materials, which are not typically distinguished by local ordered parameters, but by topological quantum numbers which could be described and computed using topological field theories in the low energy limit (for review of those progress, for instance, see [1, 2], and also a standard book [3]).

A basic example is topological insulator (see [1–7]). Topological insulator is a special kind of quantum material, where the corresponding bulk band gap is similar with an ordinary insulator, but the boundary (edge or surface) of the material has been equipped with a state which is protected by the topological numbers of the bulk geometry. As a simplest example, the (2+1)-dimensional noninteracting topological insulator is described by the Thouless-Kohmoto-Nightingale-den Nijs (TKNN) invariant [8], defined by the first Chern class of complex bundle over the Brillouin zone. In the Chern-Simons theory description, one can identify the TKNN invariant with the level of the Chern-Simons action, which is proportional to the Hall conductivity. This identification could be proven by exact computation (for instance,

see [8]) or geometric argument (for instance, see [9]).

On the other hand, given the action of the Chern-Simons effective theory, one could derive an inversion formula from the field contents to the level (Hall conductivity) of the Chern-Simons action. This inversion formula is typically evaluated from a corresponding one-loop Feynman diagram [10–12], as an integration of combination of Green functions over all spacetime dimension, which we called topological order parameter. Thus, based on the knowledge of both the identification between TKNN invariant and the level of Chern-Simons theory, and the identification between the level and the topological order parameter, we arrive at the conclusion that the TKNN invariant has been identified with the topological order parameter, which could also be shown by straightforward calculation. The topological order parameter is very useful and has a strong computational power when generalized to interacting case, which only requires the Green function data instead of a complex bundle over Hilbert space for TKNN. In the past research [13–15], people show that the identification between topological order parameter and a generalized version of TKNN still holds in the interacting theory, by using smooth homotopy from interacting Green function to a non-interacting theory (see also, several related discussions [16–19]).

In this note, we will describe a geometric proof for identification between (generalized) TKNN invariant and the topological order parameter without explicit computation. This identification does not need the Hall conductivity (level of Chern-Simons action) in the middle, and it could form a clear geometric picture while those two quantities are equal based on knowledge of algebraic topology.

The organization of this note is given as following. In Section 2 we will review some preparations towards our main result. In Section 3 we will provide the geometric proof. In Section 4 we will provide a no-go theorem and an extension based on the identification in the previous discussions. In Section 5 we will arrive at a conclusion and outlook. In Appendix A, we will review some basic knowledge of Chern-Simons forms for completeness.

## 2 Preparation

### 2.1 Green function on momentum manifold

In this section we will write down the setup of the problem. We consider a even ( $2n$ ) spatial dimensional material, with the momentum space defined as the manifold  $M$  with the coordinate  $k$ . The functions over momentum space is often reduced to Brillouin zone because of periodicity of the crystal, which should be understood as quotient of momentum space following the argument from Bloch theorem. Thus as standard knowledge, the overall manifold  $M$  is often regarded as torus. However, here our statement could be generic, compact manifold  $M$ .

To understand the physical property of electronic dynamics in a material, we often define the Matsubara Green function as (thermal) two point correlations. Here, we are working in zero temperature, mathematically speaking, the Matsubara Green function  $G(i\omega, k)$  is a mapping from momentum spacetime  $M \times \mathbb{R}$  to the group  $\text{GL}(N, \mathbb{C})$ , where  $N$  is the number of bands, where here we assume it to be finite (here, the notation  $\omega$  we use here for frequency is often  $i\omega$  in other literatures),

$$G(\omega, k) : M \times \mathbb{R} \mapsto \text{GL}(N, \mathbb{C}) \quad (2.1)$$

In the non-interacting theory, the Green function is written as

$$G(\omega, k) = \frac{1}{\omega - H(k)} \quad (2.2)$$

where  $H(k)$  is the Hamiltonian, the mapping from manifold  $M$  to Hermitian matrices in  $\text{GL}(N, \mathbb{C})$ . In the electronic system, the Green function could be given from the Kallen-Lehmann form

$$G_{\alpha\beta}(\omega, k) = \sum_m \left[ \frac{\langle 0 | c_{k\alpha} | m \rangle \langle m | c_{k\beta}^\dagger | 0 \rangle}{\omega - (E_m - E_0)} + \frac{\langle m | c_{k\alpha} | 0 \rangle \langle 0 | c_{k\beta}^\dagger | m \rangle}{\omega + (E_m - E_0)} \right] \quad (2.3)$$

where  $m$  labels the eigenvector of  $H$  (or generically,  $H - \mu N_p$  with chemical potential  $\mu$  and particle number  $N_p$ ) with vacuum 0, and  $c, c^\dagger$ 's are annihilation and creation operators of fermions, labeled by momentum  $k$  and band  $\alpha$ .

Ideally, there might be infinitely large number of bands and we should take an infinite sum. However, practically people will take a cutoff and choose a periodic boundary condition. In this case, we have a finite abelian group, while finite choices of  $\alpha$  means generators of it. That's why we could say that  $N$  is the number of bands, because there is only finite number of Brillouin zones.

For future convenience, we will obtain the following decomposition formula for the Green function. We can decompose the Green function as

$$G = \frac{G + G^\dagger}{2} + i \frac{G - G^\dagger}{2i} \quad (2.4)$$

where two terms, without the factor  $i$ , are both Hermitian. Using Kallen-Lehmann form we have

$$\begin{aligned} G(\omega, k) &= \sum_m \left[ \frac{\text{Re}(\omega) - (E_m - E_0)}{|\omega - (E_m - E_0)|^2} u_m^\dagger u_m + \frac{\text{Re}(\omega) + (E_m - E_0)}{|\omega + (E_m - E_0)|^2} v_m^\dagger v_m \right] \\ &\quad - i \sum_m \left[ \frac{\text{Im}(\omega)}{|\omega - (E_m - E_0)|^2} u_m^\dagger u_m + \frac{\text{Im}(\omega)}{|\omega + (E_m - E_0)|^2} v_m^\dagger v_m \right] \end{aligned} \quad (2.5)$$

here  $u_m = (u_{m,\alpha}) = \langle m | c_{k\alpha}^\dagger | 0 \rangle$  and  $v_m = (v_{m,\alpha}) = \langle 0 | c_{k\alpha}^\dagger | m \rangle$  are vectors labelled by  $\alpha$ . Collect terms we have

$$G = \bar{\omega} \sum_m \left[ \frac{u_m^\dagger u_m}{|\omega - (E_m - E_0)|^2} + \frac{v_m^\dagger v_m}{|\omega + (E_m - E_0)|^2} \right] + \sum_m \left[ \frac{(E_m - E_0)v_m^\dagger v_m}{|\omega + (E_m - E_0)|^2} - \frac{(E_m - E_0)u_m^\dagger u_m}{|\omega - (E_m - E_0)|^2} \right] \quad (2.6)$$

Thus, this formula shows that we could decompose the Green function as

$$G(\omega, k) = \bar{\omega}A(\omega, k) + B(\omega, k) \quad (2.7)$$

where

$$\begin{aligned} A(\omega, k) &\equiv \frac{u_m^\dagger u_m}{|\omega - (E_m - E_0)|^2} + \frac{v_m^\dagger v_m}{|\omega + (E_m - E_0)|^2} \\ B(\omega, k) &\equiv \frac{(E_m - E_0)v_m^\dagger v_m}{|\omega + (E_m - E_0)|^2} - \frac{(E_m - E_0)u_m^\dagger u_m}{|\omega - (E_m - E_0)|^2} \end{aligned} \quad (2.8)$$

and we observe the fact that  $u_m u_m^\dagger$  and  $v_m v_m^\dagger$  are Hermitian and positive semidefinite. So we get the desired decomposition of  $G$  with the property that  $A(\omega, k)$  and  $B(\omega, k)$  are Hermitian and  $A(\omega, k)$  is positive definite. In fact, we claim that  $A$  is positive definite rather than being only positive semidefinite. The argument is, the Green function is non-degenerate for any  $(\omega, k) \in \mathbb{R} \times M$ , suppose that  $A$  is degenerate at some  $(\omega, k)$  for some vector  $r$ , namely,  $A(\omega, k) \cdot r = 0$ , then it's obvious that  $r$  is orthogonal to every  $u_m$  and  $v_m$ , by definition of  $A$ , hence  $B(\omega, k) \cdot r = 0$ , by definition of  $B$ , which contradicts the fact that  $G$  is non-degenerate.

## 2.2 Topological order parameter

In this part we will define topological numbers we will use. Firstly, on a  $2n$  dimensional compact manifold  $M$ , with a Matsubara Green function  $G$ , the topological order parameter is defined by

$$\mathcal{N}_{2n} = \left( \frac{1}{2\pi i} \right)^{n+1} \frac{n!}{(2n+1)!} \int_{M \times \mathbb{R}} \mathcal{G}^* \text{Tr}(\eta^{2n+1}) \quad (2.9)$$

where  $\eta$  is the fundamental one form on the Lie group  $\text{GL}(N, \mathbb{C})^1$ , namely,  $\eta_g = g^{-1}dg$  and  $\mathcal{G}$  is the inverse of the Matsubara Green function.

Now we are going to show that the integral in the above definition is convergent. To proceed, we need the following observation for the behaviour of the Green function when  $\omega$  approaching

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<sup>1</sup>In fact,  $\text{Tr}(\eta^{2n+1})$  can be identified with the generator of the rational homotopy group  $\pi_{2n+1}(\text{GL}(N, \mathbb{C}))_{\mathbb{Q}} = \mathbb{Q}$  when  $N$  is large enough (more precisely  $N > n$ ), or equivalently the generator  $x_{2n+1}$  in the cohomology  $H^*(\text{GL}(N, \mathbb{C}), \mathbb{Q}) = \Lambda^*[x_1, x_3, \dots, x_{2N-1}]$ .

infinity. Following the previous expansion formula, and as we claimed before, there is only a finite sum in the expansion so that we can extract  $1/|\omega|^2$  from the fraction and control the rest by  $\mathcal{O}(\omega^{-2})$ , we have

$$G(\omega, k) = \frac{1}{\omega} A_0(k) + \mathcal{O}\left(\frac{1}{\omega^2}\right) \quad (2.10)$$

where

$$A_0(k) = \sum_m u_m^\dagger u_m + v_m^\dagger v_m \quad (2.11)$$

is positive definite, and then apply the differential operator we have

$$dG(\omega, k) = \frac{1}{\omega} dA_0(k) + \mathcal{O}\left(\frac{1}{\omega^2}\right) dk - \frac{A_0(k)}{\omega^2} d\omega + \mathcal{O}\left(\frac{1}{\omega^3}\right) d\omega \quad (2.12)$$

where  $dk$  is the short hand notation for differential forms in  $M$ .

Now we could claim that the integral

$$\left(\frac{1}{2\pi i}\right)^{n+1} \frac{n!}{(2n+1)!} \int_{M \times \mathbb{R}} \mathcal{G}^* \text{Tr}(\eta^{2n+1}) \quad (2.13)$$

is convergent, where  $\mathcal{G}$  is the inverse of  $G$ , and the integral is taken by integrating on  $M$  slice first. To show this, it's not harmful to use  $G$  rather than  $\mathcal{G}$  in the integrand, which only changes the sign (because  $g^{-1}dg = -d(g^{-1})g$ ). Now in large  $\omega$  limit, from the result above, we have asymptotic expansion

$$G^{-1}dG = A_0^{-1}dA_0 + \mathcal{O}\left(\frac{1}{\omega}\right) dk - \frac{d\omega}{\omega} + \mathcal{O}\left(\frac{1}{\omega^2}\right) d\omega \quad (2.14)$$

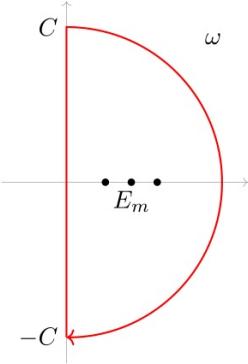
Since terms involving  $d\omega/\omega^s$  for  $s > 1$  are automatically convergent, we only need to look at  $d\omega/\omega$  terms, which are proportional to

$$\int_{|\omega| \geq C} \frac{d\omega}{\omega} \int_M \text{Tr}[(A_0^{-1}dA_0)^{2n}] \quad (2.15)$$

But  $\text{Tr}[(A_0^{-1}dA_0)^{2n}]$  vanishes because  $\text{Tr}(AB) = \text{Tr}(BA)$  and we can move the first  $A_0^{-1}dA_0$  term to the tail and get a minus sign

$$\text{Tr}[(A_0^{-1}dA_0)^{2n}] = -\text{Tr}[(A_0^{-1}dA_0)^{2n}] \quad (2.16)$$

Thus the result is convergent.



**Figure 1.** The integration contour for  $\omega$  in the complex plane  $\mathbb{C}$ .

### 2.3 Passing to contour integral

As is mentioned above, Eq.2.9 looks like a topological invariant, but it's defined on a non-compact manifold, and it is not compactly supported, so we need to find a substitute to characterize it in a more familiar way. The substitution is given in the following. We pick a cutoff by integrating  $\omega$  in a segment  $[-C, C]$  ( $C$  is sufficiently large to avoid hitting on  $E_m$ ), and then close the segments by connecting  $C$  with  $-C$  via a semicircle in the complex plane, as is shown in the Figure 1. A possible issue is that whether the analytical continued Green function is still non-degenerate along this semicircle or not. This is guaranteed by the large  $\omega$  expansion of  $G$

$$G(\omega, k) = \frac{1}{\omega} A_0(k) + \mathcal{O}\left(\frac{1}{\omega^2}\right) \quad (2.17)$$

Although we only show the expansion in the case that  $\omega$  is restricted in the imaginary line, there is no difficulty to extend the argument to the case that  $\omega$  runs through a large circle. So for large enough  $C$ , the analytical continued Green function is non-degenerate along the semicircle.

We will also claim that the integral along the semicircle will vanish as  $C \rightarrow \infty$ . This can be proved by looking at the asymptotic behaviour of integrand at infinity. Note that for extension of  $dG$ , there will be some  $d\bar{\omega}$  terms, but they are suppressed by  $1/|\omega|^2$ . Then we can apply the same arguments from the convergence statement to conclude that the integral along the big circle will approach zero as  $C$  goes to infinity.

Thus, we can safely reduce the original integral which involves  $\pm i\infty$  to a integral on compact manifold  $M \times T$  ( $T$  for torus). An important consequence is that, this is a well-defined topological invariant, so it is a constant as we move the circle to the infinity, which converges to the original integral, thus we can effectively use a contour to replace the imaginary line.

## 2.4 Smooth homotopy to non-interacting theory

Since the integral of pull-back cocycle on the compact manifold  $M \times T$  is stable under the homotopy of maps  $G_u : M \times T \times [0, 1] \rightarrow \mathrm{GL}(N, \mathbb{C})$  with  $u \in [0, 1]$ , we can deform the original Green function to be more manageable. In fact, it is shown in [13, 14] that we can always deform it to a Green function associated to a non-interacting theory. We shall explain this below.

A non-interacting theory has the standard Green function

$$\frac{1}{\omega - H(k)} \quad (2.18)$$

where  $H(k)$  is the Hamiltonian. Now we can define a system with effective Hamiltonian

$$H(k) \equiv G^{-1}(0, k) \quad (2.19)$$

so that the associated Green function is

$$\tilde{G}(\omega, k) = \frac{1}{\omega + G^{-1}(0, k)} \quad (2.20)$$

Here the inverse can be taken because  $\omega$  is either on the imaginary line or  $|\omega| = C$  being large so that  $\omega + G^{-1}(0, k)$  has nonzero eigenvalue. Connect  $G$  with  $\tilde{G}$  via a smooth homotopy

$$G_u(\omega, k) = (1 - u)G(\omega, k) + u\tilde{G}(\omega, k) \quad (2.21)$$

We claim that  $\forall u \in [0, 1]$ ,  $G_u(\omega, k) \in \mathrm{GL}(N, \mathbb{C})$ . This is obvious when  $\omega = 0$ . When  $\omega$  has nonzero image part, take any vector  $r$  with unit norm, and compute

$$\begin{aligned} \mathrm{Im}(\langle r | G_u(\omega, k) | r \rangle) &= \mathrm{Im}((1 - u)\langle r | G(\omega, k) | r \rangle) + \mathrm{Im}\left(u\langle r | \frac{1}{\omega + G^{-1}(0, k)} | r \rangle\right) \\ &= \mathrm{Im}(\bar{\omega})\left((1 - u)\langle r | A(\omega, k) | r \rangle + \frac{u}{|\omega|^2 + |\langle r | G^{-1}(0, k) | r \rangle|^2}\right) \end{aligned} \quad (2.22)$$

since  $A$  is positive definite. For  $\omega = C \in \mathbb{R}$ , this comes from asymptotic expansion

$$G_u(C, k) = \frac{1 - u}{C}A_0(k) + \frac{u}{C}\mathrm{Id} + \mathcal{O}\left(\frac{1}{C^2}\right) \quad (2.23)$$

where  $\mathrm{Id}$  is the identity. So we conclude that

$$\mathcal{N}_{2k} = \left(\frac{1}{2\pi i}\right)^{n+1} \frac{n!}{(2n+1)!} \int_{M \times T} \tilde{\mathcal{G}}^* \mathrm{Tr}(\eta^{2n+1}) \quad (2.24)$$

where  $\tilde{\mathcal{G}}(\omega, k) = \omega + G^{-1}(0, k)$ .

## 2.5 Generalized TKNN invariant

Generically, the generalized TKNN invariant will include the topological description for interacting system [13–15]. Here we define it and make a brief introduction.

Firstly, we notice that Green function with zero frequency  $G(0, k)$  is Hermitian and non-degenerate, so we can split the total space into a direct sum of two parts

$$E(k) = E_+(k) \oplus E_-(k) \quad (2.25)$$

where  $E(k)$  is the total space and  $E_{\pm}(k)$  corresponds to subspaces spanned by plus or minus eigenvalues. As we vary the momentum  $k \in M$ , those  $E_{\pm}(k)$  patch together to form two complex vector bundles on  $M$ , we shall call them  $E_{\pm}$ , and there is a corresponding splitting of bundles

$$E = E_+ \oplus E_- \quad (2.26)$$

Note that  $E$  is a trivial bundle (a function on  $M$  with vector values). In the original papers [13, 14], they are called R-space ( $E_+$ ) and L-space ( $E_-$ ) because they are formed by zeroes in the right hand side and left hand side of the origin.

Now we define the generalized TKNN invariant  $C_n$  to be the  $n$ 'th Chern character of  $E_+$

$$C_n := \langle \text{ch}_n(E_+), [M] \rangle \quad (2.27)$$

## 3 Main proof

In this section, we will provide a geometric proof on the theorem

$$\mathcal{N}_{2n} = C_n \quad (3.1)$$

as a simple equation between generalized TKNN invariant and topological order parameter. The proof will be organized in the following steps. Firstly, we can understand the formula Eq.2.24 in terms of an integration over a Chern-Simons form on  $M \times T$ , and in turn a characteristic class of a complex vector bundle on  $M \times T \times T$ . Secondly, we should go to another perspective that we view this bundle as a direct sum of two bundles, where each of them comes from a tensor product of a bundle on  $M$  with a line bundle on  $T \times T$ . Finally, we will do some calculations using the property of Chern character.

The first step is basically contained in the Appendix A. According to example in this appendix, the topological order parameter  $\mathcal{N}_{2n}$  equals to the minus of  $(n+1)$ 'th Chern character of the twisted bundle on  $M \times T \times T$ , which is constructed by gluing the trivial bundle  $\underline{\mathbb{C}}^N$  on  $M \times T \times \{0\}$  and  $M \times T \times \{1\}$  via  $\tilde{\mathcal{G}}(\omega, k) = \omega + G(0, k)$ . We will denote this bundle  $\tilde{E}$ , so

we claim that the topological order parameter is given by the following simple characteristic class

$$\mathcal{N}_{2n} = -\langle \text{ch}_{n+1}(\tilde{E}), [M \times T \times T] \rangle \quad (3.2)$$

We can also continuously deform the gluing map  $\tilde{\mathcal{G}}(\omega, k)$  and obtain an isomorphic twisted bundle because of following isomorphisms:

$$\begin{aligned} & \left\{ P \in \text{Bun}_{U(N)}(M \times T \times T) \mid P \text{ trivial on } M \times T \times \{0\} \right\} \\ & \simeq [(M \times T \times T, M \times T \times \{0\}), (BU(N), *)] \\ & \simeq [S(M \times T), BU(N)] \\ & \simeq [M \times T, U(N)] \end{aligned} \quad (3.3)$$

The isomorphism is the definition of classifying space, the second comes from the fact that  $\pi_1(BU(N))$  is trivial so that we can contract  $M \times T \times \{0\} \vee T$  and obtain the suspension of  $M \times T$ , and the third isomorphism is just  $\Omega BG \simeq G$ .

### 3.1 Continuous deformation of $\tilde{\mathcal{G}}$

Now we know that diagonalization of the Green function gives a splitting

$$\tilde{\mathcal{G}} \in \Gamma(M \times T, \text{GL}(E'_+) \oplus \text{GL}(E'_-)) \quad (3.4)$$

where  $E'_{\pm}$  is the pull-back of  $E_{\pm}$  on  $M$ . Let's parametrize the torus  $T$  by  $\theta \in [-1, 1]$ , and we claim that

$$\tilde{\mathcal{G}} \simeq \text{Id}_{E'_+} \oplus e^{-i\pi\theta} \text{Id}_{E'_-} \quad (3.5)$$

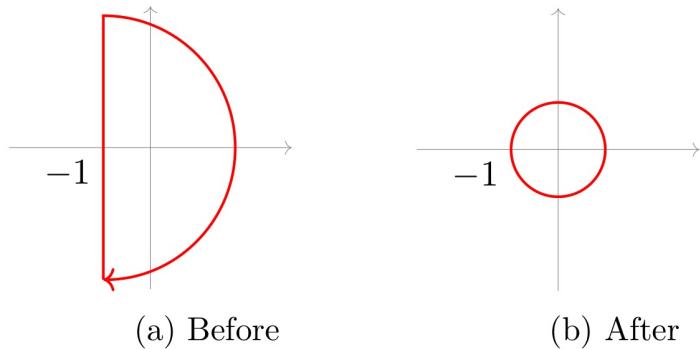
and this homotopy is taken in  $\Gamma(M \times T, \text{GL}(E'_+) \oplus \text{GL}(E'_-))$ .

In fact, we firstly notice that if  $\omega$  lies on the semicircle contour, then  $\omega + A$  and  $\omega + B$  are homotopic if  $A$  and  $B$  are both positive definite or negative definite: consider the path  $\omega + (1 - \lambda)A + \lambda B$  and any vector  $\langle r |, \langle r | \omega + (1 - \lambda)A + \lambda B | r \rangle \neq 0$  whenever  $\omega$  is pure imaginary, and if  $\omega$  has large modulus, this is also nonzero because of the domination of  $\omega$ , so  $\omega + (1 - \lambda)A + \lambda B$  is non-degenerate for any  $\lambda \in [0, 1]$ .

Applying this observation to  $\tilde{\mathcal{G}}|_{E'_+}$  and  $\tilde{\mathcal{G}}|_{E'_-}$  separately, we conclude that

$$\tilde{\mathcal{G}}(\omega, k) \simeq (\omega + 1)\text{Id}_{E'_+} \oplus (\omega - 1)\text{Id}_{E'_-} \quad (3.6)$$

where  $\omega$  runs through the semicircle contour described above. Then we can shrink the semicircle associated to  $\omega + 1$  to a constant function 1, and shrink the semicircle associated to  $\omega - 1$  to the unit circle  $e^{-i\pi\theta}, \theta \in [-1, 1]$  (we can not shrink the contour to a constant like the  $E'_+$  sector because the contour can not pass through the singularity at zero), as is shown in the following Figure 2. Hence the claim is proved.



**Figure 2.** Deformation for  $E'_-$  sector.

### 3.2 Another perspective for $\tilde{E}$

Now  $\tilde{E}$  is constructed by gluing the trivial bundle  $\mathbb{C}^N$  on  $M \times T \times \{0\}$  and  $M \times T \times \{1\}$  via

$$\mathrm{Id}_{E'_+} \oplus e^{-i\pi\theta} \mathrm{Id}_{E'_-} \quad (3.7)$$

There is a simple observation: the twisting data, namely, the gluing function is only relevant to the coordinate of  $T$ , which is a clue that we can *split* the base manifold  $M$  from the construction. As a matter of fact, we could claim that, the twisted bundle  $\tilde{E}$  can be equivalently constructed by taking the trivial bundle  $\underline{\mathbb{C}}^N = E_+ \oplus E_-$  on  $M$ , and twisting the plus and minus components by tensoring with two different line bundles on 2-torus  $T^2$ , namely

$$\widetilde{E} \simeq E_+ \boxtimes \mathbb{C} \oplus E_- \boxtimes \mathcal{L} \quad (3.8)$$

where  $\underline{\mathbb{C}}$  is the trivial line bundle, and  $\mathcal{L}$  is defined by gluing the boundary of trivial line bundle on  $T \times [0, 1]$  via  $e^{-i\pi\theta}$ .

In fact, since  $\text{Id}_{E'_+} \oplus e^{-i\pi\theta}\text{Id}_{E'_-}$  leaves the two components  $E'_+$  and  $E'_-$  invariant, we have a splitting

$$\tilde{E} = \tilde{E}_+ \oplus \tilde{E}_- \quad (3.9)$$

where  $\tilde{E}_+$  is the gluing of the bundle  $E'_+$  on  $M \times T$  with a  $\mathbb{C}$  on  $T$ , but  $E'_+$  is the tensor product of  $E_+$  with a  $\mathbb{C}$  on  $T$ , so

$$\tilde{E}_+ \equiv E_+ \boxtimes \mathbb{C} \quad (3.10)$$

On the other hand,  $\widetilde{E}_-$  is the twist of  $E'_-$  by  $e^{-i\pi\theta}$  at point  $(\theta, k)$ , or equivalently, forming the line bundle  $\mathcal{L}$  on  $T^2$  first, and tensor product with  $E_-$ , which gives

$$\tilde{E}_- = E_- \boxtimes \mathcal{L} \quad (3.11)$$

thus we prove the claim.

### 3.3 Conclude the proof

To compute  $\mathcal{N}_{2n}$ , it suffices to calculate the Chern character of  $\tilde{E}$ , according to the previous results, we have

$$\begin{aligned}\text{ch}(\tilde{E}) &= \text{ch}(E_+ \boxtimes \underline{\mathbb{C}}) + \text{ch}(E_- \boxtimes \mathcal{L}) \\ &= \text{ch}(E_+) \cup \text{ch}(\underline{\mathbb{C}}) + \text{ch}(E_-) \cup \text{ch}(\mathcal{L}) \\ &= p^* \text{ch}(E_+) + \text{ch}(E_-) \cup \text{ch}(\mathcal{L})\end{aligned}\tag{3.12}$$

Here  $p$  is the projection from  $M \times T^2$  to  $M$ . After contracting with the fundamental cycle  $[M \times T^2]$ , according to Künneth theorem, only the  $\langle \text{ch}_n(E_-) \cup \text{ch}_1(\mathcal{L}), [M \times T^2] \rangle$  survives, which in turn equals to

$$\langle \text{ch}_n(E_-), [M] \rangle \langle \text{ch}_1(\mathcal{L}), [T^2] \rangle\tag{3.13}$$

Moreover,  $E_+ \oplus E_-$  is a trivial bundle on  $M$ , so

$$\begin{aligned}\mathcal{N}_{2n} &= -\langle \text{ch}_n(E_-), [M] \rangle \langle \text{ch}_1(\mathcal{L}), [T^2] \rangle \\ &= \langle \text{ch}_n(E_+), [M] \rangle \langle \text{ch}_1(\mathcal{L}), [T^2] \rangle \\ &= \langle \text{ch}_1(\mathcal{L}), [T^2] \rangle C_n\end{aligned}\tag{3.14}$$

Thus we need to show that

$$\langle \text{ch}_1(\mathcal{L}), [T^2] \rangle = 1\tag{3.15}$$

In fact, recall the construction of  $\mathcal{L}$ , it is the gluing of the boundary of a trivial line bundle on  $T \times [-1, 1]_\theta$  via  $e^{-i\pi\theta}$ . Use the example in the Appendix A again, we conclude that

$$\begin{aligned}\langle \text{ch}_1(\mathcal{L}), [T^2] \rangle &= -\left(\frac{1}{2\pi i}\right) \int_{-1}^1 e^{i\pi\theta} de^{-i\pi\theta} \\ &= \frac{1}{2} \int_{-1}^1 d\theta \\ &= 1\end{aligned}\tag{3.16}$$

which concludes the proof.

## 4 No-go theorem and extension

Based on the technology we have built above, we could discuss some extensions of the identification theorem. An obvious attempt to extend this topological order parameter is looking for similar construction such as

$$\int_{M \times \mathbb{R}} \mathcal{G}^* (\text{Tr}(\eta^{n_1}) \text{Tr}(\eta^{n_2}) \cdots \text{Tr}(\eta^{n_l})) , n_1 + n_2 + \cdots + n_l = 2n + 1\tag{4.1}$$

$$\begin{array}{ccc}
M \times T^2 & \xrightarrow{\pi_1} & M \times T \\
& \searrow p & \downarrow \pi_2 \\
& & M
\end{array}$$

**Figure 3.** A commutative diagram.

or more formally, the pull-back via  $\mathcal{G}$  of other elements in  $H^{2n+1}(GL(N, \mathbb{C}))$ . It can be shown in a similar way with the method in this essay that they are convergent. However, it turns out that they are trivial. Before we show that, we shall define a refined version of theorem 3.1 first, which relates the Chern character of  $E_+$  (or equivalently  $E_+$ ) to pull-back of  $H^*(GL(N, \mathbb{C}))$ .

Consider the following commutative diagram 3. Use formula Eq.3.12 and the result Eq.4 we obtain

$$\begin{aligned}
p_!(ch(\tilde{E})) &= p_!(p^*ch(E_+) + ch(E_-) \cup ch(\mathcal{L})) \\
&= ch(E_-)
\end{aligned} \tag{4.2}$$

on the other hand, the commutative diagram and the nature of Chern-Simons form gives

$$\begin{aligned}
p_!(ch(\tilde{E})) &= \pi_{2!} \circ \pi_{1!}(ch(\tilde{E})) \\
&= -\pi_{2!} \left( \mathcal{G}^* \sum_{i \geq 0} x_{2i+1} \right)
\end{aligned} \tag{4.3}$$

Here the lower shriek symbol means integrating along fibers, and  $x_{2i+1}$  is the generator of  $H^*(GL(N, \mathbb{C}))$  defined by  $\text{Tr}(\eta^{2i+1})$  (up to some constant, which is not important here). Thus we have proved the following refined version of the main theorem: The Chern character of  $E_+$  is related to  $\mathcal{G}^*H^*(GL(N, \mathbb{C}))$  by

$$ch(E_+) = \pi_{2!} \left( \mathcal{G}^* \sum_{i \geq 0} x_{2i+1} \right) \tag{4.4}$$

or more transparently,

$$ch_l(E_+) = \left( \frac{1}{2\pi i} \right)^l \frac{l!}{(2l+1)!} \int_{-i\infty}^{i\infty} d\omega \mathcal{G}^* \text{Tr}(\eta^{2l+1}) \tag{4.5}$$

This equation holds in the cohomology  $H^{2l}(M)$  (not necessarily equals as differential forms).

Another conclusion from the above diagram is that

$$\begin{aligned}
\mathcal{G}^* \sum_{i \geq 0} x_{2i+1} &= -\pi_{1!}(\text{ch}(\tilde{E})) \\
&= -\pi_{1!}(\pi_1^* \pi_2^* \text{ch}(E_-) \cup \text{ch}(\mathcal{L})) \\
&= -\pi_2^* \text{ch}(E_-) \cup \pi_{1!} \text{ch}(\mathcal{L})
\end{aligned} \tag{4.6}$$

We have already known from the conclusion Eq. that  $\text{ch}(\mathcal{L}) = 1 + \alpha_1 \cup \alpha_2$  where  $\alpha_{1,2}$  is the generator of  $H^1(T, \mathbb{Z})$  for the first and second 1-torus respectively, so this gives us the inverse map of the above theorem

$$\left( \frac{1}{2\pi i} \right)^l \frac{l!}{(2l+1)!} \tilde{\mathcal{G}}^* \text{Tr}(\eta^{2l+1}) = -\alpha_2 \cup \pi_2^* \text{ch}_l(E_-) \tag{4.7}$$

This equation holds in the cohomology  $H^{2l+1}(M \times T)$  (not necessarily equals as differential forms). Then the promised no-go theorem is that, the naive generalization to the topological order parameter

$$\int_{M \times i\mathbb{R}} \mathcal{G}^* (\text{Tr}(\eta^{n_1}) \text{Tr}(\eta^{n_2}) \cdots \text{Tr}(\eta^{n_l})) , n_1 + n_2 + \cdots + n_l = 2n + 1 \tag{4.8}$$

for  $l > 1$  are trivial, following from the fact that every cup product of two *trace-blocks*

$$\mathcal{G}^* \text{Tr}(\eta^{n_1}) \cup \mathcal{G}^* \text{Tr}(\eta^{n_2}) \tag{4.9}$$

is trivial, because there are two  $\alpha_2$  involved, and they square to zero.

To see a nontrivial generalization, instead of taking wedge product (or cup product in cohomology) first and doing integration along imaginary line, we can integrate out  $\omega$  first, and take wedge product in  $M$ . Then we could claim that, there exist generalizations to the topological order parameter

$$N'_{j_1, j_2 \dots j_l} := \left( \frac{1}{2\pi i} \right)^n \left( \prod_l \frac{j_l!}{(2j_l+1)!} \right) \int_M \left( \prod_l \int_{-i\infty}^{i\infty} d\omega_l \mathcal{G}(\omega_l, n)^* \text{Tr}(\eta^{2j_l+1}) \right) \tag{4.10}$$

which is related to the topology of the band structure by

$$N'_{j_1, j_2 \dots j_l} = (-1)^l \langle \text{ch}_{j_1}(E_-) \text{ch}_{j_2}(E_-) \cdots \text{ch}_{j_l}(E_-), [M] \rangle \tag{4.11}$$

## 5 Conclusion and outlook

In this paper, we mainly discuss the identification proof of topological order parameter and (generalized) TKNN invariant motivated by topological condensed matter physics. The main part of this proof is a geometric identification of the non-interacting theory from topological order parameter to TKNN invariant defined over filled bands. This proof is an alternative way

to find an inversion formula of Chern-Simons level using classical field data from Feynman diagram, combining the argument that TKNN invariant from band theory could be identified as level of Chern-Simons theory, or direct computations. Through this proof, we build up the intuition from statements of algebraic topology that what happens between those two quantities and what happens to form such a connection. Although we are motivated by condensed matter physics, this is a statement which is intrinsically from Chern-Simons theory. Thus, based on such a construction, we revisit and formalize the smooth contour deformation that is used in [13, 14] by order estimation of the Green function by spectral decomposition. The deformation will make the identification theorem to be more general, from non-interacting theory to interacting theory. Moreover, we discuss some possible extensions over this identification, to make the picture more fruitful based on the technology we use.

As an outlook, for future application of this work, we observe that this proof is technically pretty similar with proofs in historical discussions in the structure of gauge and gravitational anomalies [20]. It is possible to find more physical consequences of this work and possible extensions from high energy theory to condensed matter physics.

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## A On the Chern-Simons forms

We briefly review the Chern-Weil homomorphism and Chern-Simons forms based on the original paper [21] for completeness, and write a crucial example that could be used in the main text.

For a manifold  $X$  with a principal  $G$ -bundle  $P$  on  $X$ , and a connection  $\omega$  on  $P$  with curvature  $F$ , we can use  $G$ -invariant polynomials on the Lie algebra  $g$  to construct characteristic differential forms. Namely, pick a  $S \in \text{Sym}^r(g^*)^G$ , we can view  $S(F)$  as a differential  $2r$ -form via sending a (antisymmetrized) tensor product of  $2r$  vectors to  $g \otimes \cdots \otimes g$  first, and then acted on by  $S$ . One can show that  $S(F) \in \Omega^{2r}(P)$  is closed and invariant under local gauge transformation, so it can descend to a closed  $2r$ -form on the base manifold  $X$ . More importantly, the cohomology class associated to  $S(F)$  (on  $X$ ) is independent of the connection. This is called the Chern-Weil construction.

Suppose that there are two connections  $\omega_0$  and  $\omega_1$  on  $P$  with curvature  $F_0$  and  $F_1$ , then the cohomology class of  $S(F_1) - S(F_0)$  is trivial, by Hodge theory there exists a  $2r - 1$ -form with exterior differential equals to  $S(F_1) - S(F_0)$ . It was shown by Chern and Simons that this  $(2r - 1)$ -form can be constructed in the following canonical way

Take the product space  $X \times [0, 1]$  and extend the bundle trivially, put the connection  $\omega_0$  and  $\omega_1$  on two slices  $X \times \{0\}$  and  $X \times \{1\}$  respectively, and extend them via a smooth curve  $\omega_t$ , for example  $\omega_t = (1 - t)\omega_0 + t\omega_1$ , so the curvature on  $P \times [0, 1]$  is

$$F = F_t + \frac{\partial \omega_t}{\partial t} dt \quad (\text{A.1})$$

where  $F_t$  is the curvature on each slice. From basic calculus

$$S(F_1) - S(F_0) = \int_0^1 dt \mathcal{L}_{\partial t} S(F) \quad (\text{A.2})$$

so we find the desired  $(2r - 1)$ -form which is called the Chern-Simons form

$$\text{CS}_{2r-1}(S; \omega_0, \omega_1) := \int_0^1 dt i_{\partial t} S(F) \quad (\text{A.3})$$

it can be easily shown that Chern-Simons form is local gauge invariant thus descending to the base  $X$  and still satisfy the defining property

$$S(F_1) - S(F_0) = d\text{CS}_{2r-1}(S; \omega_0, \omega_1) \quad (\text{A.4})$$

Actually, Chern-Simons form is independent of the smooth curve  $\omega_t$ , up to a coboundary term. This can be easily seen from the fact that Chern-Weil homomorphism is independent of the connection, up to a coboundary term, so a second curve  $\omega'_t$  gives another  $S(F')$  on  $X \times [0, 1]$  but the difference between them is  $d\eta$ , where  $\eta \in \Omega^{2r-1}(P)^G$  (or equivalently,  $\Omega^{2r-1}(X)$ ). The difference between Chern-Simons forms is

$$\int_0^1 dt i_{\partial t} d\eta = \eta_1 - \eta_0 - d \left( \int_0^1 dt i_{\partial t} \eta \right) \quad (\text{A.5})$$

but  $\eta_1 = \eta_0 = 0$  because the starting and ending points are fixed.

So without losing information at the cohomology level, we can choose a special curve

$$\omega_t = (1 - t)\omega_0 + t\omega_1 \quad (\text{A.6})$$

to obtain a more explicit formula

$$r \cdot \int_0^1 dt S(\omega_1 - \omega_0, F_t, \dots, F_t) \quad (\text{A.7})$$

and from the construction, we also have this additivity property for Chern-Simons forms

$$\text{CS}_{2r-1}(S; \omega_0, \omega_2) = \text{CS}_{2r-1}(S; \omega_0, \omega_1) + \text{CS}_{2r-1}(S; \omega_1, \omega_2) \quad (\text{A.8})$$

If the base manifold  $X$  has odd dimension  $2n - 1$ , then  $\text{CS}_{2r-1}(S; \omega_0, \omega_1)$  is a top form and can be integrated on  $X$  and gives

$$\int_X \text{CS}_{2r-1}(S; \omega_0, \omega_1) = \int_{X \times [0,1]} S(F) \quad (\text{A.9})$$

If moreover that  $\omega_1$  is a gauge transformation of  $\omega_0$ , namely,  $\omega_0^h$  for some  $G$ -valued function  $h$ , so that we can glue the bundle at  $X \times \{0\}$  with  $X \times \{1\}$  via the gauge transformation  $h$ , then the above integral formula has a simple topological interpretation: the integration of  $\text{CS}_{2r-1}(S; \omega_0, \omega_1)$  on  $X$  gives a characteristic number of the twisted bundle (mapping torus  $\mathcal{M}_P$ ) on  $X \times T$ . This will be the key to the proof of the main theorem in this paper.

If the bundle  $P$  is trivial, we can take the trivial connection as  $\omega_0$ , and any other connection  $A$  (as an element in  $\Omega^1(X, g)$ ) will generate a special Chern-Simons form

$$\text{CS}_{2r-1}(S; A) := \text{CS}_{2r-1}(S; 0, A) \quad (\text{A.10})$$

Now we introduce,

**Example:** Take  $G = U(N)$ , and a trivial bundle  $P$ . Consider the invariant polynomials

$$S_s(g) = \text{Tr}(g^s)/s! \quad (\text{A.11})$$

(by standard symmetric function argument, these polynomials generate the algebra

$$\text{Sym}^*(\text{gl}(N)^*)^{U(N)} \quad (\text{A.12})$$

then the associated characteristic classes are Chern characters<sup>2</sup>

$$\text{ch}_s(A) = \left(\frac{i}{2\pi}\right)^s \frac{\text{Tr}(F^s)}{s!} \quad (\text{A.13})$$

where  $F$  is the curvature form,

$$F = dA + A \wedge A \quad (\text{A.14})$$

Then the associated Chern-Simons form

$$\text{CS}_{2s-1}(A) := \text{CS}_{2s-1}(S_s; 0, A) \quad (\text{A.15})$$

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<sup>2</sup> Here we use the same  $i/2\pi$  convention with topologist.

is given by

$$\text{CS}_{2s-1}(A) = \left(\frac{i}{2\pi}\right)^s \frac{1}{(s-1)!} \int_0^1 \text{Tr} \left( A \wedge (tdA + t^2 A \wedge A)^{s-1} \right) dt \quad (\text{A.16})$$

For instance, for  $s = 2$  we obtain the ordinary Chern-Simons action for gauge field  $A$ ,

$$\left(\frac{i}{2\pi}\right)^2 \int_0^1 \text{Tr} \left( A \wedge (tdA + t^2 A \wedge A) \right) dt = \frac{-1}{4\pi^2} \text{Tr} \left( \frac{1}{2} A \wedge dA + \frac{1}{3} A \wedge A \wedge A \right) \quad (\text{A.17})$$

If  $A$  is a pure gauge, namely,  $A = g^{-1}dg$  for some  $G$ -valued function on  $X$ , then  $F = dA + A \wedge A = 0$  (Maurer-Cartan), and the Chern-Simons form can be simplified to

$$\begin{aligned} \text{CS}_{2s-1}(A) &= \left(\frac{i}{2\pi}\right)^n \frac{\text{Tr}(A^{2s-1})}{(s-1)!} \int_0^1 (t^2 - t)^{s-1} dt \\ &= - \left(\frac{1}{2\pi i}\right)^s \frac{(s-1)!}{(2s-1)!} \text{Tr}(A^{2s-1}) \end{aligned} \quad (\text{A.18})$$

If the dimension of the manifold  $X$  is  $2s-1$  ( $n = s$ ), the integration of this Chern-Simons form on  $X$ , follow the mapping torus argument above, equals to the  $s$ 'th Chern character of the mapping torus  $\mathcal{M}_P$  associated to the gauge transformation  $g$  on the boundary.

## References

- [1] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010) [arXiv:1002.3895 [cond-mat.mes-hall]].
- [2] X. L. Qi and S. C. Zhang, Rev. Mod. Phys. **83**, no. 4, 1057 (2011).
- [3] B. Bernevig and T. Hughes, *Topological Insulators And Topological Superconductors*, (Princeton University Press, 2013).
- [4] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, no. 22, 226801 (2005) [cond-mat/0411737 [cond-mat.mes-hall]].
- [5] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 146802 (2005) [cond-mat/0506581 [cond-mat.mes-hall]].
- [6] L. Fu, C. Kane and E. Mele, Phys. Rev. Lett. **98**, no. 10, 106803 (2007).
- [7] J. E. Moore and L. Balents, Phys. Rev. B **75**, no. 12, 121306 (2007) [cond-mat/0607314 [cond-mat.mes-hall]].
- [8] D. J. Thouless, M. Kohmoto, M. P. Nightingale and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982).
- [9] E. Witten, Riv. Nuovo Cim. **39**, no. 7, 313 (2016) [arXiv:1510.07698 [cond-mat.mes-hall]].
- [10] A. J. Niemi and G. W. Semenoff, Phys. Rev. Lett. **51**, 2077 (1983).
- [11] M. F. L. Golterman, K. Jansen and D. B. Kaplan, Phys. Lett. B **301**, 219 (1993) [hep-lat/9209003].

- [12] X. L. Qi, T. Hughes and S. C. Zhang, Phys. Rev. B **78**, 195424 (2008) [arXiv:0802.3537 [cond-mat.mes-hall]].
- [13] Z. Wang, X. L. Qi and S. C. Zhang, Physical Review Letters 105.25 (2010): 256803.
- [14] Z. Wang and S. C. Zhang, Physical Review X 2.3 (2012): 031008.
- [15] Z. Wang, X. L. Qi, and S. C. Zhang, Phys. Rev. B 85, 165126 (2012).
- [16] Z. Wang, X. L. Qi and S. C. Zhang, Phys. Rev. B **84**, 014527 (2011) [arXiv:1011.0586 [cond-mat.str-el]].
- [17] V. Gurarie, Phys. Rev. B 83, 085426 (2011).
- [18] S. Ryu and S. C. Zhang, Phys. Rev. B **85**, 245132 (2012) [arXiv:1202.4484 [cond-mat.str-el]].
- [19] A. Martn-Ruiz, M. Cambiaso and L. F. Urrutia, Phys. Rev. D **92**, no. 12, 125015 (2015) [arXiv:1511.01170 [cond-mat.other]].
- [20] L. Alvarez-Gaume and P. H. Ginsparg, Annals Phys. **161**, 423 (1985) Erratum: [Annals Phys. **171**, 233 (1986)].
- [21] S. S. Chern and J. Simons, Annals Math. **99**, 48 (1974).