Appendix

Thermodynamic forces

Let $\boldsymbol{T}^p$ and $\boldsymbol{Y}^p = \{Y^p, Z^p\}$ denote the thermodynamic forces conjugate to $\boldsymbol{F}^p$ and $\boldsymbol{Z}^p = \{\theta^p, \epsilon^p\}$, respectively, which are obtained by applying the chain rule in (0.16), as follows

$$\boldsymbol{T}^p = -\frac{\partial A}{\partial \boldsymbol{F}} \cdot \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{F}^p} - \frac{\partial A}{\partial \boldsymbol{F}^p} = \boldsymbol{F}^{eT} \boldsymbol{P} - A,_{\boldsymbol{F}^p}, \tag{0.1}$$

$$Y^p = p - p_c, \quad Z^p = \sigma - \sigma_c, \tag{0.2}$$

where

$$p = -\frac{dA}{d\theta^p} = \boldsymbol{T}^p \cdot \boldsymbol{N}^p \boldsymbol{F}^p, \quad p_c = \frac{\partial A}{\partial \theta^p}, \tag{0.3}$$

$$\sigma = -\frac{dA}{d\epsilon^p} = \boldsymbol{T}^p \cdot \boldsymbol{M}^p \boldsymbol{F}^p, \quad \sigma_c = \frac{\partial A}{\partial \epsilon^p}, \tag{0.4}$$

and in which p and σ are the effective pressure and the effective deviatoric stress, respectively; and p_c and σ_c are the flow pressure and the deviatoric flow stress, respectively. Substituting the free energy A, the elastic strain energy density W^e and the plastic stored energy W^p in (0.3) and (0.4) we obtain

$$p = -\frac{dA}{d\theta^p} = -\frac{\partial W^{e,\mathrm{vol}}}{\partial \theta^e}, \quad p_c = \frac{\partial W^{p,\mathrm{vol}}}{\partial \theta^p}, \tag{0.5}$$

$$\sigma = -\frac{dA}{d\epsilon^p} = -\frac{\partial W^{e,\mathrm{dev}}}{\partial e^e}, \quad \sigma_c = \frac{\partial W^{p,\mathrm{dev}}}{\partial \epsilon^p}, \tag{0.6}$$

where $e^e = \sqrt{\frac{2}{3}\boldsymbol{e}^e \cdot \boldsymbol{e}^e}$.

We assume that there exists a *viscous flow rule* that takes the form

$$\dot{\boldsymbol{F}}_i^v \boldsymbol{F}_i^{v-1} = \sum_{j=1}^{3} \dot{\epsilon}_{i,j}^v \boldsymbol{M}_{i,j}^v \otimes \boldsymbol{M}_{i,j}^v \quad (i = 1, \ldots, M), \tag{0.7}$$

where $\dot{\epsilon}^v_{i,j}$ and $\boldsymbol{M}^v_{i,j}$ are the eigenvalues and the eigenvectors of $\boldsymbol{d}^v_i = \dot{\boldsymbol{F}}^v_i \boldsymbol{F}^{v-1}_i$, respectively (null viscous spin is assumed). The viscous internal variables are

$$\boldsymbol{Z}^v_i = \left\{\epsilon^v_{i,1}, \epsilon^v_{i,2}, \epsilon^v_{i,3}\right\}, \tag{0.8}$$

where

$$\epsilon^v_{i,j} = \epsilon^v_{i,j}(0) + \int_0^t \dot{\epsilon}^v_{i,j}(\xi) d\xi. \tag{0.9}$$

The viscous driving forces $\boldsymbol{Y}^v_i = \{\sigma^v_{i,1}, \sigma^v_{i,2}, \sigma^v_{i,3}\}$ follow from (0.17) and the chain rule as

$$\sigma^v_{i,j} = -\frac{dA}{d\epsilon^v_{i,j}} = \boldsymbol{T}^v_i \cdot \boldsymbol{M}^v_{i,j} \boldsymbol{F}^v_i. \tag{0.10}$$

Substituting the free energy A, the viscous strain energy densities W^e_i in (0.20), the viscous principal stresses are obtained as

$$\sigma^v_{i,j} = -\frac{dA}{d\epsilon^v_{i,j}} = -\frac{\partial W^e_i}{\partial e^e_{i,j}}. \tag{0.11}$$

Isochoric viscous deformations may be obtained by enforcing the constraint $\dot{\theta}^v_i = \dot{\epsilon}^v_{i,1} + \dot{\epsilon}^v_{i,2} + \dot{\epsilon}^v_{i,3} = 0$ (Fancello et al., 2006), while purely elastic bulk behavior may be obtained by setting the volumetric viscosities to zero.

Variational formulation of the rate problem

Consider a body $B \subset \mathbb{R}^3$ undergoing a motion described by the mapping $\boldsymbol{\varphi} : B \times [t_1, t_2] \to \mathbb{R}^3$. Assume that the boundary ∂B, with unit normal $\bar{\boldsymbol{N}}$, is the union of a displacement boundary $\partial_1 B$, where boundary displacements $\bar{\boldsymbol{\varphi}} : \partial_1 B \times [t_1, t_2] \to \mathbb{R}^3$ are prescribed, and a traction boundary $\partial_2 B$, where tractions $\bar{\boldsymbol{T}} : \partial_2 B \times [t_1, t_2] \to \mathbb{R}^3$ are applied ($\partial_1 B \cap \partial_2 B = \emptyset$). Let also $\boldsymbol{B} : B \times [t_1, t_2] \to \mathbb{R}^3$ be the body force. Furthermore, for every $t \in [t_1, t_2]$ we introduce the power functional

$$\Phi[\dot{\boldsymbol{\varphi}}, \dot{\boldsymbol{Z}}^p, \boldsymbol{M}^p, \boldsymbol{N}^p, \dot{\boldsymbol{Z}}_i^v, \boldsymbol{M}_{i,j}^v] =$$

$$\int_B \left[\dot{A} + \psi^* + \sum_{i=1}^{M} \phi_i^* - \left(\frac{\partial L}{\partial \boldsymbol{F}^p} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{F}}^p} \right) \cdot \dot{\boldsymbol{F}}^p \right] dV - \int_B \rho_0 (\boldsymbol{B} - \ddot{\boldsymbol{\varphi}}) \cdot \dot{\boldsymbol{\varphi}}\, dV - \int_{\partial_2 B} \bar{\boldsymbol{T}} \cdot \dot{\boldsymbol{\varphi}}\, dS, \tag{0.12}$$

where $\boldsymbol{F}^p$, $\boldsymbol{Z}^p$, $\boldsymbol{M}^p$, $\boldsymbol{N}^p$ $\boldsymbol{Z}_i^v$ and $\boldsymbol{M}_{i,j}^v$ are now regarded as fields over B; $\dot{\boldsymbol{F}}^p$ is determined by $\dot{\boldsymbol{Z}}^p$, $\boldsymbol{M}^p$ and $\boldsymbol{N}^p$ through the flow rule (0.13)

$$\dot{\boldsymbol{F}}^p \boldsymbol{F}^{p-1} = \dot{\theta}^p \boldsymbol{N}^p + \dot{\epsilon}^p \boldsymbol{M}^p, \tag{0.13}$$

, and $\dot{\boldsymbol{F}}_i^v$ is determined by $\dot{\boldsymbol{Z}}_i^v$ and $\boldsymbol{M}_{i,j}^v$ through the viscous flow rule (0.14)

$$\dot{\boldsymbol{F}}_i^v \boldsymbol{F}_i^{v-1} = \sum_{j=1}^{3} \dot{\epsilon}_{i,j}^v \boldsymbol{M}_{i,j}^v \otimes \boldsymbol{M}_{i,j}^v \quad (i = 1, \ldots, M), \tag{0.14}$$

Using identities (0.15) through (0.20)

$$\boldsymbol{P} = \frac{\partial A}{\partial \boldsymbol{F}}, \tag{0.15}$$

$$\boldsymbol{Y}^p = -\frac{dA}{d\boldsymbol{Z}^p}, \tag{0.16}$$

$$\boldsymbol{Y}_i^v = -\frac{dA}{d\boldsymbol{Z}_i^v}. \tag{0.17}$$

$$\boldsymbol{T}^p = -\frac{\partial A}{\partial \boldsymbol{F}} \cdot \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{F}^p} - \frac{\partial A}{\partial \boldsymbol{F}^p} = \boldsymbol{F}^{eT} \boldsymbol{P} - A_{,\boldsymbol{F}^p}, \tag{0.18}$$

$$Y^p = p - p_c, \quad Z^p = \sigma - \sigma_c, \tag{0.19}$$

$$\sigma^v_{i,j} = -\frac{dA}{d\epsilon^v_{i,j}} = \boldsymbol{T}^v_i \cdot \boldsymbol{M}^v_{i,j}\boldsymbol{F}^v_i. \tag{0.20}$$

and the flow rules (0.13) and (0.14), (0.12) may be rewritten as

$$\begin{aligned}
&\Phi[\dot{\boldsymbol{\varphi}}, \dot{\boldsymbol{Z}}^p, \boldsymbol{M}^p, \boldsymbol{N}^p, \dot{\boldsymbol{Z}}^v_i, \boldsymbol{M}^v_{i,j}] = \\
&\int_B \left(\boldsymbol{P} \cdot \text{Grad}\dot{\boldsymbol{\varphi}} - \boldsymbol{Y}^p \cdot \dot{\boldsymbol{Z}}^p - \sum_{i=1}^{M} \boldsymbol{Y}^v_i \cdot \dot{\boldsymbol{Z}}^v_i + \psi^* + \sum_{i=1}^{M} \phi^*_i \right) dV - \\
&\int_B \rho_0(\boldsymbol{B} - \ddot{\boldsymbol{\varphi}}) \cdot \dot{\boldsymbol{\varphi}}\, dV - \int_{\partial_2 B} \bar{\boldsymbol{T}} \cdot \dot{\boldsymbol{\varphi}}\, dS,
\end{aligned} \tag{0.21}$$

where $\boldsymbol{F} = \text{Grad}\boldsymbol{\varphi}$ has been introduced. The rates $\dot{\boldsymbol{\varphi}}$, $\dot{\boldsymbol{Z}}^p$, $\dot{\boldsymbol{Z}}^v_i (i = 1, ..., M)$ and the directions of plastic and viscous flows $\boldsymbol{M}^p$, $\boldsymbol{N}^p$, $\boldsymbol{M}^v_{i,j}$ $(j = 1, 2, 3)$ at the generic time $t \in [t_1, t_2]$ are found by solving the minimization problem

$$\Phi^{\text{eff}}[\dot{\boldsymbol{\varphi}}] = \inf_{\dot{\boldsymbol{Z}}^p, \boldsymbol{M}^p, \boldsymbol{N}^p, \dot{\boldsymbol{Z}}^v_i, \boldsymbol{M}^v_{i,j}} \Phi\left[\dot{\boldsymbol{\varphi}}, \dot{\boldsymbol{Z}}^p, \boldsymbol{M}^p, \boldsymbol{N}^p, \dot{\boldsymbol{Z}}^v_i, \boldsymbol{M}^v_{i,j}\right], \tag{0.22}$$

subject to constraints (0.23) and (0.24)

$$\text{tr}(\boldsymbol{M}^p) = 0, \quad \boldsymbol{M}^p \cdot \boldsymbol{M}^p = \frac{3}{2}, \quad \boldsymbol{N}^p = \pm\frac{1}{3}\boldsymbol{I}. \tag{0.23}$$

$$\dot{\epsilon}^p \geq 0\ , \ \dot{\theta}^p \geq 0. \tag{0.24}$$

The material velocity field follows from the minimization of the reduced power functional (0.22)

$$\inf_{\dot{\boldsymbol{\varphi}}} \Phi^{\text{eff}}[\dot{\boldsymbol{\varphi}}], \quad \dot{\boldsymbol{\varphi}} = \dot{\bar{\boldsymbol{\varphi}}} \text{ on } \partial_1 B. \tag{0.25}$$

The functional $\Phi[\dot{\boldsymbol{\varphi}}, \dot{\boldsymbol{Z}}^p, \boldsymbol{M}^p, \boldsymbol{N}^p, \dot{\boldsymbol{Z}}^v_i, \boldsymbol{M}^v_{i,j}]$ does not depend on spatial derivatives of its fields, therefore the minimization (0.22) may be obtained locally as

$$\Phi^{\text{eff}}[\dot{\boldsymbol{\varphi}}] = \int_B \phi(\text{Grad}\dot{\boldsymbol{\varphi}}) dV - \int_B \rho_0 (\boldsymbol{B} - \ddot{\boldsymbol{\varphi}}) \cdot \dot{\boldsymbol{\varphi}} dV - \int_{\partial_2 B} \bar{\boldsymbol{T}} \cdot \dot{\boldsymbol{\varphi}} dS, \tag{0.26}$$

where

$$\begin{aligned} \phi(\dot{\boldsymbol{F}}) &= \inf_{\dot{\boldsymbol{Z}}^p, \boldsymbol{M}^p, \boldsymbol{N}^p, \dot{\boldsymbol{Z}}_i^v, \boldsymbol{M}_{i,j}^v} f\left(\dot{\boldsymbol{F}}, \dot{\boldsymbol{Z}}^p, \boldsymbol{M}^p, \boldsymbol{N}^p, \dot{\boldsymbol{Z}}_i^v, \boldsymbol{M}_{i,j}^v\right), \\ f\left(\dot{\boldsymbol{F}}, \dot{\boldsymbol{Z}}^p, \boldsymbol{M}^p, \boldsymbol{N}^p, \dot{\boldsymbol{Z}}_i^v, \boldsymbol{M}_{i,j}^v\right) &= \boldsymbol{P} \cdot \dot{\boldsymbol{F}} - \boldsymbol{Y}^p \cdot \dot{\boldsymbol{Z}}^p - \sum_{i=1}^{M} \boldsymbol{Y}_i^v \cdot \dot{\boldsymbol{Z}}_i^v + \psi^* + \sum_{i=1}^{M} \phi_i^*. \end{aligned} \tag{0.27}$$

The Euler-Lagrange equations of the power functional Φ with respect to $\dot{\boldsymbol{Z}}^p$ and $\dot{\boldsymbol{Z}}_i^v$ are the kinetic relations, and its Euler-Lagrange equations with respect to $\boldsymbol{M}^p$, $\boldsymbol{N}^p$ and $\boldsymbol{M}_{i,j}^v$ result in the optimal directions of plastic and viscous flows, as indicated in subsection .

In can be shown via the kinetic relations (0.28)

$$\boldsymbol{Y}^p = \frac{\partial \psi^*}{\partial \dot{\boldsymbol{Z}}^p}, \quad \boldsymbol{Y}_i^v = \frac{\partial \phi_i^*}{\partial \dot{\boldsymbol{Z}}_i^v}. \tag{0.28}$$

and the flow rules (0.13),(0.14) (Yang et al., 2006; Weinberg et al., 2006; Ortiz and Stainier, 1999; Fancello et al., 2006) that the Euler-Lagrange equations corresponding to the minimization problem (0.25) are the equations of motion

$$\text{Div}\boldsymbol{P} + \rho_0 \boldsymbol{B} = \rho_0 \ddot{\boldsymbol{\varphi}} \text{ in } B, \quad \boldsymbol{P} \cdot \bar{\boldsymbol{N}} = \bar{\boldsymbol{T}} \text{ on } \partial_2 B. \tag{0.29}$$

Algorithm

The time integration of the constitutive equations within a generic time interval $[t_k, t_{k+1}]$ is effected by recourse to an incremental variational update. Assume that $\boldsymbol{F}_k^p$, $\boldsymbol{Z}_k^p$, $\boldsymbol{Z}_{i,k}^v$ $(i = 1, \ldots, M)$, $\dot{\theta}_k^p$ and $\ddot{\theta}_k^p$ are known at time t_k and that the deformation gradient $\boldsymbol{F}_{k+1}$ and the temperature T_{k+1} at time t_{k+1} are given. A discrete version of problem (0.22) is obtained by introducing the effective incremental strain-energy density (Weinberg et al., 2006)

$$W_k(\boldsymbol{F}_{k+1}, T_{k+1}) = \min_{\boldsymbol{Z}^p_{k+1}, \boldsymbol{M}^p, \boldsymbol{N}^p, \boldsymbol{Z}^v_{i,k+1}, \boldsymbol{M}^v_{i,j}} f_k(\boldsymbol{F}_{k+1}, T_{k+1}, \boldsymbol{Z}^p_{k+1}, \boldsymbol{M}^p, \boldsymbol{N}^p, \boldsymbol{Z}^v_{i,k+1}, \boldsymbol{M}^v_{i,j}), \quad (0.30)$$

where f_k is the incremental objective function

$$f_k(\boldsymbol{F}_{k+1}, T_{k+1}, \boldsymbol{Z}^p_{k+1}, \boldsymbol{M}^p, \boldsymbol{N}^p, \boldsymbol{Z}^v_{i,k+1}, \boldsymbol{M}^v_{i,j}) = W^e(\boldsymbol{\epsilon}^e_{k+1}, T_{k+1}) + W^p(\boldsymbol{Z}^p_{k+1}, T_{k+1}) + \sum_{i=1}^{M} W^e_i(\boldsymbol{\epsilon}^e_{i,k+1}, T_{k+1}) + \rho_0 C_v T_{k+1}\left(1 - \log\frac{T_{k+1}}{T_0}\right) + \Delta t\left(\psi^*_{k+1} + \sum_{i=1}^{M}\phi^*_{i,k+1}\right) + \beta\Delta t^2 B_{k+1}, \quad (0.31)$$

with $\Delta t = t_{k+1} - t_k$, and

$$\boldsymbol{\epsilon}^e_{k+1} = \frac{1}{2}\log({\boldsymbol{F}^e_{k+1}}^T \boldsymbol{F}^e_{k+1}), \qquad \boldsymbol{\epsilon}^e_{i,k+1} = \frac{1}{2}\log({\boldsymbol{F}^e_{i,k+1}}^T \boldsymbol{F}^e_{i,k+1}), \quad (0.32)$$

$$\psi^*_{k+1} = \psi^*\left(\frac{\Delta\boldsymbol{Z}^p}{\Delta t}, J^p_{k+1}, T_{k+1}\right), \qquad \phi^*_{i,k+1} = \phi^*_i\left(\frac{\Delta\boldsymbol{Z}^v_i}{\Delta t}, T_{k+1}\right), \quad (0.33)$$

$$\Delta\boldsymbol{Z}^p = \boldsymbol{Z}^p_{k+1} - \boldsymbol{Z}^p, \qquad \Delta\boldsymbol{Z}^v_i = \boldsymbol{Z}^v_{i,k+1} - \boldsymbol{Z}^v_{i,k}, \quad (0.34)$$

$$B_{k+1} = \frac{3\rho_{v_0}}{2}\left(\frac{b_{k+1} - b^{\text{pre}}_{k+1}}{\beta\Delta t^2}\right), \qquad b^{\text{pre}}_{k+1} = b_k + \Delta t\dot{b}_k + \left(\frac{1}{2} - \beta\right)\Delta t^2\ddot{b}_k, \quad (0.35)$$

where $\beta \in (0, \frac{1}{2})$. Eqn (0.35) defines a Newmark predictor for b_{k+1}, which is regarded as a function of J^p_{k+1}. $\boldsymbol{F}^p_{k+1}$ and $\boldsymbol{F}^v_{i,k+1}$ $(i = 1, \ldots, M)$ are computed through the following discrete versions of the flow rules (0.13) and (0.14)

$$\boldsymbol{F}^p_{k+1} = \exp(\Delta\epsilon^p\boldsymbol{M}^p + \Delta\theta^p\boldsymbol{N}^p)\boldsymbol{F}^p_k, \quad (0.36)$$

$$\boldsymbol{F}^v_{i,k+1} = \exp\left(\sum_{j=1}^{3}\Delta\epsilon^v_{i,j}\boldsymbol{M}^v_{i,j}\otimes\boldsymbol{M}^v_{i,j}\right)\boldsymbol{F}^v_{i,k}. \quad (0.37)$$

The minimum problem (0.30) returns the updated values of the internal variables $\boldsymbol{Z}^p_{k+1}$, $\boldsymbol{M}^p$, $\boldsymbol{N}^p$,

$\boldsymbol{Z}^v_{i,k+1}$ and $\boldsymbol{M}^v_{i,j}$ $(i = 1, .., M; j = 1, 2, 3)$. The first Piola-Kirchhoff stress and consistent tangent can now be computed (Weinberg et al., 2006) as

$$\boldsymbol{P}_{k+1} = \frac{\partial W_k}{\partial \boldsymbol{F}_{k+1}}, \quad D\boldsymbol{P}_{k+1} = \frac{\partial^2 W_k}{\partial \boldsymbol{F}_{k+1} \partial \boldsymbol{F}_{k+1}}. \tag{0.38}$$

The symmetry of the consistent tangent is a direct consequence of the potential structure of the incremental problem.

By adopting a predictor-corrector strategy based on logarithmic elastic strains to solve the variational problem (0.30), the constitutive update is reduced to small-strains and purely kinematic steps (Cuitino and Ortiz, 1992; Ortiz and Stainier, 1999; Weinberg et al., 2006; Fancello et al., 2006). The corresponding elastic logarithmic strains at time t_{k+1} are

$$\boldsymbol{\epsilon}^e_{k+1} = \boldsymbol{\epsilon}^{e,\text{pre}}_{k+1} - \Delta\epsilon^p \boldsymbol{M}^p - \Delta\theta^p \boldsymbol{N}^p, \tag{0.39}$$

$$\boldsymbol{\epsilon}^e_{i,k+1} = \boldsymbol{\epsilon}^{e,\text{pre}}_{i,k+1} - \sum_{j=1}^{3} \Delta\epsilon^v_{i,j} \boldsymbol{M}^v_{i,j} \otimes \boldsymbol{M}^v_{i,j}, \tag{0.40}$$

where $\boldsymbol{M}^v_{i,j}$ are also the eigenvectors of $\boldsymbol{\epsilon}^{e,\text{pre}}_{i,k+1}$, and

$$\boldsymbol{\epsilon}^{e,\text{pre}}_{k+1} = \frac{1}{2} \log(\boldsymbol{F}^{p-T}_k \boldsymbol{C}_{k+1} \boldsymbol{F}^{p-1}_k), \tag{0.41}$$

$$\boldsymbol{\epsilon}^{e,\text{pre}}_{i,k+1} = \frac{1}{2} \log(\boldsymbol{F}^v_{i,k}{}^{-T} \boldsymbol{C}_{k+1} \boldsymbol{F}^v_{i,k}{}^{-1}), \tag{0.42}$$

with $\boldsymbol{C}_{k+1} = \boldsymbol{F}^T_{k+1} \boldsymbol{F}_{k+1}$. Eqns. (0.39)-(0.42) follow from the co-linearity between $\boldsymbol{M}^p$ and $\boldsymbol{\epsilon}^{e,\text{pre}}_{k+1}$ (Weinberg et al., 2006); the optimization of f_k with respect to the viscous flow directions $\boldsymbol{M}^v_{i,j}$ (Fancello et al., 2006); and the assumption of null incremental plastic and viscous deformations in the predictor phase.

Minimization with respect to $\boldsymbol{M}^p$, $\boldsymbol{N}^p$

Optimization of f_k with respect to $\boldsymbol{M}^p$, $\boldsymbol{N}^p$ yields, after some algebraic manipulation

$$\boldsymbol{M}^p = \frac{3\boldsymbol{s}_{k+1}}{2\sigma_{k+1}}, \quad \boldsymbol{N}^p = \frac{1}{3}\text{sgn}(p^{\text{pre}}_{k+1})\boldsymbol{I} \tag{0.43}$$

where

$$\boldsymbol{s}_{k+1} = \frac{\partial W^e}{\partial \boldsymbol{\epsilon}^e_{k+1}} = \text{dev}\left(\frac{\partial W^{e,\text{dev}}}{\partial \boldsymbol{e}^e_{k+1}}\right), \tag{0.44}$$

$$\sigma_{k+1} = \sqrt{\frac{3}{2}\boldsymbol{s}_{k+1} \cdot \boldsymbol{s}_{k+1}}, \tag{0.45}$$

$$p^{\text{pre}}_{k+1} = k\left[\text{tr}(\boldsymbol{\epsilon}^{e,\text{pre}}_{k+1}) - \alpha(T_{k+1} - T_0)\right], \tag{0.46}$$

with $\boldsymbol{e}^e_{k+1} = \text{dev}(\boldsymbol{\epsilon}^e_{k+1})$. Eqn (0.43) determines $\boldsymbol{M}^p$ implicitly, which can be expressed as

$$m^p_j = \frac{3s_{j,k+1}}{2\sigma_{k+1}}, \quad j = 1, 2, 3 \tag{0.47}$$

where m^p_j and $s_{j,k+1}$ are the eigenvalues of $\boldsymbol{M}^p$ and $\boldsymbol{s}_{k+1}$, respectively.

Minimization with respect to θ^p_{k+1}, ϵ^p_{k+1}

Optimization of f_k with respect to θ^p_{k+1}, ϵ^p_{k+1} yields

$$\Delta\theta^p = 0, \quad \Delta\epsilon^p = 0 \tag{0.48}$$

if

$$p^{\text{pre}}_{k+1} \leq p_c(\theta^p_k, \epsilon^p_k, T_{k+1}), \quad \sigma^{\text{pre}}_{k+1} \leq \sigma_c(\theta^p_k, \epsilon^p_k, T_{k+1}), \tag{0.49}$$

or

$$p_{k+1}^{\text{pre}} - k\,\Delta\theta^p = p_{c,k+1} + \frac{\partial}{\partial\theta_{k+1}^p}[\Delta t\psi_{k+1}^* + \beta\Delta t^2 B_{k+1}], \tag{0.50}$$

$$\sigma_{k+1} = \sigma_{c,k+1} + \frac{\partial}{\partial\epsilon_{k+1}^p}[\Delta t\psi_{k+1}^* + \beta\Delta t^2 B_{k+1}], \tag{0.51}$$

otherwise, with

$$\sigma_{k+1}^{\text{pre}} = \sqrt{\frac{3}{2}\boldsymbol{s}_{k+1}^{\text{pre}} \cdot \boldsymbol{s}_{k+1}^{\text{pre}}}, \tag{0.52}$$

and

$$\boldsymbol{s}_{k+1}^{\text{pre}} = \frac{\partial W^{e,\text{dev}}}{\partial \boldsymbol{e}_{k+1}^{e,\text{pre}}} = \text{dev}\left(\frac{\partial W^e}{\partial \boldsymbol{\epsilon}_{k+1}^{e,\text{pre}}}\right), \tag{0.53}$$

where $\boldsymbol{e}_{k+1}^{e,\text{pre}}$ is the deviatoric part of $\boldsymbol{\epsilon}_{k+1}^{e,\text{pre}}$.

Eqns. (0.47), (0.50), (0.51) may be solved for the unknowns θ_{k+1}^p, ϵ_{k+1}^p, $m_j^p (j = 1, 2, 3)$ by recourse to a Newton-Raphson iteration, under the constraints

$$\Delta\theta^p \geq 0, \quad \Delta\epsilon^p \geq 0. \tag{0.54}$$

Minimization with respect to $\epsilon_{i,j,k+1}^v$

Optimization of f_k with respect to $\epsilon_{i,j,k+1}^v$ $(i = 1, \ldots, M;\ j = 1, 2, 3)$ leads to the system of equations

$$\sigma_{i,j,k+1}^v = \frac{\partial}{\partial\epsilon_{i,j,k+1}^v}(\Delta t\ \phi_i^*) \tag{0.55}$$

that can again be solved via a Newton-Raphson iteration. Furthermore, (0.55) may be recast as

$$\sigma_{i,j,k+1}^v = \frac{\partial W_i^e}{\partial\epsilon_{i,j,k+1}^e} \tag{0.56}$$

which defines $\sigma_{i,j,k+1}^v$ as a function of $\epsilon_{i,j,k+1}^v$, together with (0.40).

Once $\boldsymbol{Z}_{k+1}^p$, $\boldsymbol{M}^p$, $\boldsymbol{N}^p$, $\boldsymbol{Z}_{i,k+1}^v$, $\boldsymbol{M}_{i,j}^v$ are determined, the updated equilibrium and viscous stresses follow from

$$\boldsymbol{\sigma}_{k+1} = p_{k+1}\boldsymbol{I} + \boldsymbol{s}_{k+1}, \tag{0.57}$$

$$\boldsymbol{\sigma}^v_{i,k+1} = p^v_{i,k+1}\boldsymbol{I} + \sum_{j=1}^{3} s^v_{i,j,k+1}\boldsymbol{M}^v_{i,j} \otimes \boldsymbol{M}^v_{i,j}, \tag{0.58}$$

with $\boldsymbol{s}_{k+1}$ given by (0.44), and

$$p_{k+1} = \frac{\partial W^{e,\mathrm{vol}}}{\partial \theta^e_{k+1}}, \tag{0.59}$$

$$p^v_{i,k+1} = \left(\sigma^v_{i,1,k+1} + \sigma^v_{i,2,k+1} + \sigma^v_{i,3,k+1}\right)/3, \tag{0.60}$$

$$s^v_{i,j,k+1} = \sigma^v_{i,j,k+1} - p^v_{i,k+1}. \tag{0.61}$$

Due to the variational structure of the update, the stresses and strains satisfy the potential relations

$$\boldsymbol{\sigma}_{k+1} = \frac{\partial W_k}{\partial \boldsymbol{\epsilon}^e_{k+1}}, \quad \boldsymbol{\sigma}^v_{i,k+1} = \frac{\partial W_k}{\partial \boldsymbol{\epsilon}^{e,i}_{k+1}}. \tag{0.62}$$

The first Piola-Kirchhoff stress follows from

$$\boldsymbol{P}_{k+1} = \boldsymbol{P}^\infty_{k+1} + \boldsymbol{P}^{ve}_{i,k+1} \tag{0.63}$$

with

$$\boldsymbol{P}^\infty_{k+1} = \frac{\partial W_k}{\partial \boldsymbol{\epsilon}^p_{k+1}} \cdot \frac{\partial \boldsymbol{\epsilon}^e_{k+1}}{\partial \boldsymbol{C}_{k+1}} \cdot \frac{\partial \boldsymbol{C}_{k+1}}{\partial \boldsymbol{F}_{k+1}}, \tag{0.64}$$

$$\boldsymbol{P}^{ve}_{i,k+1} = \frac{\partial W_k}{\partial \boldsymbol{\epsilon}^e_{i,k+1}} \cdot \frac{\partial \boldsymbol{\epsilon}^e_{i,k+1}}{\partial \boldsymbol{C}_{k+1}} \cdot \frac{\partial \boldsymbol{C}_{k+1}}{\partial \boldsymbol{F}_{k+1}}. \tag{0.65}$$

$\boldsymbol{P}^\infty_{k+1}$ and $\boldsymbol{P}^{ve}_{i,k+1}$ can also be expressed in indicial notation as (index $k+1$ not shown for convenience)

$$(\boldsymbol{P}^{\infty})_{jH} = (\boldsymbol{\sigma})_{AB} D\log(\boldsymbol{C}^{e,pre})_{ABCD}(\boldsymbol{F}_k^{p-1})_{HC}(\boldsymbol{F}_k^{p-1})_{LD}F_{jL}, \tag{0.66}$$

$$(\boldsymbol{P}_i^{ve})_{jH} = (\boldsymbol{\sigma}^{v})_{AB} D\log(\boldsymbol{C}_i^{e,pre})_{ABCD}(\boldsymbol{F}_{i,k}^{v}{}^{-1})_{HC}(\boldsymbol{F}_{i,k}^{v}{}^{-1})_{LD}F_{jL}, \tag{0.67}$$

with $\boldsymbol{C}^{e,\text{pre}} = \boldsymbol{F}_k^{p-T}\boldsymbol{C}_{k+1}\boldsymbol{F}_k^{p-1}$, and $\boldsymbol{C}_i^{e,\text{pre}} = \boldsymbol{F}_{i,k}^{v}{}^{-T}\boldsymbol{C}_{k+1}\,\boldsymbol{F}_{i,k}^{v}{}^{-1}$. The consistent may be obtained by following (Weinberg et al., 2006; Ortiz et al., 2001).

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