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A STUDY OF THE SECOND-DIFFERENCE METHOD FOR RADIAL MAGNET ALIGNMENT

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Contents

I Introduction.

Preliminary estimates indicate that the cost of surveying the main ring of the proposed 300-BeV cascade synchrotron will be quite high, because of the large diameter (~ 1-1/2 miles) and small tolerances (~ 0.005 inch). Courant and Sands have suggested replacing conventional surveying techniques by a scheme in which the only measurements to be made are local, relating each magnet to its neighbors. In principle, these measurements would enable one to determine the magnet positions (to within an overall translation and/or rotation) much more simply than by conventional techniques. Furthermore, the procedure of measurement and adjustment could be carried out completely automatically, if desired.

The principle difficulty with the method lies in the accumulation of errors from the large number of observations. In order to study this phenomenon, we have considered in some detail a model scheme of this type. In Section II we describe the scheme, in Section III we outline the solution in the absence of measurement errors, and in Section IV we consider the effect of such errors.

II Model Scheme.

We assume the ideal configuration consists of N points (magnets) P_i equally spaced around the circumference of a circle of radius R. In fact, the points will not lie exactly on the circle; let x_i denote the radial error measured outwards. That is, x_i is the distance from the circle outwards to P_i. We ignore the possibility of tangential displacements, as these are much less serious in an actual accelerator.

Let s_i' be the radial distance, or sagitta, measured outwards to P_i from the line joining P_{i-1} and P_{i+1}. An elementary calculation gives

$$s_i' = x_i + R - 2 \left(\cos \frac{2\pi}{N}\right) \frac{(R + x_{i-1})(R + x_{i+1})}{2R + x_{i-1} + x_{i+1}}$$

$$= x_i + R\left(1 - \cos \frac{2\pi}{N}\right) - \left(\cos \frac{2\pi}{N}\right) \left[\frac{x_{i-1} + x_{i+1}}{2} - \frac{(x_{i+1} - x_{i-1})^2}{4R} + \ldots\right] \qquad (1)$$

In the case of a 300-BeV AGS, $\frac{x}{R} \sim \frac{0.005 \text{ inch}}{1300 \text{ m}} \sim 10^{-7}$, so we shall neglect higher order terms in the square brackets of (1). Also, $N \gg 2\pi$ for the type of scheme under consideration, say $N \sim 10^3$. Then, $1 - \cos \frac{2\pi}{N} \sim 2 \times 10^{-5}$, and we may approximate (1) by

$$s_i' = R\left(1 - \cos \frac{2\pi}{N}\right) + x_i - \frac{1}{2}\left(x_{i+1} + x_{i-1}\right) \tag{2}$$

The first term in (2) is a distance ~ 1 inch which is the sagitta that would be measured if the x_i were all zero. We shall let s_i denote the sagitta with this constant removed, i.e.,

$$s_i = x_i - \frac{1}{2}\left(x_{i+1} + x_{i-1}\right) \tag{3}$$

We suppose that some device exists which measures the s_i, and the problem is to translate this information into values for the x_i, so that the magnets may be repositioned on the ideal trajectory.

It is clear from (3) that if the same constant is added to each x_i, the s_i are all unaffected. Thus we can only hope to return the magnets to <u>some</u> circle, perhaps with an incorrect radius. Closely connected with this fact is the identity

$$\sum_{i=1}^{N} s_i = 0 \tag{4}$$

which follows from (3). There are thus only $N - 1$ independent s_i, and we cannot hope to determine all N of the x_i; we can only determine them to within an overall additive constant.

This property results from the second approximation we made in going from (1) to (2), namely

$$1 - \cos \frac{2\pi}{N} \approx 1$$

If we had not made this approximation, (3) would have become

$$s_i = x_i - \frac{1}{2}\left(x_{i+1} + x_{i-1}\right) \cos \frac{2\pi}{N}$$

from which we deduce

$$\sum_{i=1}^{N} s_i = (1 - \cos \frac{2\pi}{N}) \sum_{i=1}^{N} x_i$$

instead of (4). If N is 10^3, and each x_i were one foot, each s_i would be only 2×10^{-4} inch. Under these circumstances, this method is clearly not suited to measuring the machine radius. If N were smaller by a factor of 10, and s_i were still capable of measurement to an accuracy of several thousandths of an inch, one could determine the overall radius to $\sim \frac{1}{2}$ inch.

III Solution in the Absence of Errors.

For the moment, we shall neglect the possibility of measurement errors, and suppose that we know the N numbers s_i exactly. They must, of course, have zero sum, from (4). As remarked above, we cannot hope to determine the N numbers x_i, unless we impose a constraint. We shall arbitrarily require

$$\sum_{i=1}^{N} x_i = 0 \tag{5}$$

and re-emphasize the fact that the general solution will differ from ours by an arbitrary constant.

The solution of the difference equation (3), subject to (5), is

$$x_i = \sum_{j=1}^{N} C_{ij} s_j \tag{6}$$

where the symmetric matrix C_{ij} is given by

$$C_{ij} = \begin{cases} \dfrac{3\left[(i-j+1)^2 + (N+j-i+1)^2\right] - (2N^2+6N+7)}{6N} & (i \geq j) \\ \text{ditto with } i \longleftrightarrow j & (i \leq j) \end{cases}$$

In order to see more clearly the structure of C_{ij}, assume N large. Then,

$$C(\alpha) \sim N(\alpha^2 - \alpha + \frac{1}{6})$$

where

$$\alpha = \frac{|i - j|}{N}$$

Thus, C goes from a maximum of $\frac{N}{6}$ when $\alpha = 0$, to a minimum of $-\frac{N}{12}$ when $\alpha = 1/2$.

IV Effect of Measurement Errors.

We now imagine a situation opposite to that of the preceding section. We suppose the x_i are actually zero, but errors occur in measuring the s_i. More precisely, we suppose the s_i to be independent random variables, with zero mean and rms value ϵ. Using (6), we compute (erroneous) values for the x_i, as linear combinations of the s_i. We shall Fourier analyze the resulting x_i:

$$x_i = \frac{A_o}{2} + \sum_{j=1}^{N/2} (A_j \cos \frac{2\pi ji}{N} + B_j \sin \frac{2\pi ji}{N}) \tag{7}$$

where the A_j and B_j are given by

$$A_j = \frac{2}{N} \sum_{i=1}^{N} x_i \cos \frac{2\pi ji}{N}$$

$$B_j = \frac{2}{N} \sum_{i=1}^{N} x_i \sin \frac{2\pi ji}{N} \tag{8}$$

Combining (6) and (8), we have the Fourier coefficients A_j and B_j expressed as linear combinations of the random variables s_k:

$$A_j = \frac{2}{N} \sum_{i=1}^{N} \cos \frac{2\pi ji}{N} \sum_{k=1}^{N} C_{ik}\, s_k$$

$$= \frac{2}{N} \sum_{k=1}^{N} M_{jk}\, s_k \tag{9}$$

where

$$M_{jk} = \sum_{i=1}^{N} (\cos \frac{2\pi ji}{N})\; C_{ik}$$

Rather tedious algebra gives the surprisingly simple result

$$M_{jk} = \frac{\cos \frac{2\pi jk}{N}}{2 \sin^2 \frac{\pi j}{N}} \tag{10}$$

Similarly,

$$B_j = \frac{2}{N} \sum_{k=1}^{N} N_{jk} s_k \tag{11}$$

where

$$N_{jk} = \sum_{i=1}^{N} (\sin \frac{2\pi ji}{N}) \; C_{ik}$$

$$= \frac{\sin \frac{2\pi jk}{N}}{2 \sin^2 \frac{\pi j}{N}} \tag{12}$$

We must now consider the effect on the equilibrium orbit of the magnet displacements (7). ξ_i, the displacement of the orbit from the $i^{\underline{th}}$ magnet, is approximately given by[1)]

$$\xi_i = K \sum_{j=1}^{N/2} \left(\frac{j^2-1}{\nu^2-j^2}\right) (A_j \cos \frac{2\pi ji}{N} + B_j \sin \frac{2\pi ji}{N}) \tag{13}$$

where K is a number somewhat larger than 1 (say 2), and ν is the number of betatron wavelengths in the circumference.

Substituting (9) and (11) into (13) gives

$$\xi_i = K \sum_{j=1}^{N/2} \left(\frac{j^2-1}{\nu^2-j^2}\right) \left[\frac{2}{N} \sum_{k=1}^{N} M_{jk} s_k \cos \frac{2\pi ji}{N} + \frac{2}{N} \sum_{k=1}^{N} N_{jk} s_k \sin \frac{2\pi ji}{N}\right]$$

$$= K \sum_{k=1}^{N} \alpha_{ik} s_k \tag{14}$$

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1) R. Hulsizer, "Magnet Positioning Problems for a 300 GeV Proton Synchrotron", California Institute of Technology Synchrotron Laboratory Report CTSL-11 (1960).

where

$$\alpha_{ik} = \frac{2}{N} \sum_{j=1}^{N/2} \left(\frac{j^2-1}{\nu^2-j^2}\right) \left(M_{jk} \cos \frac{2\pi ji}{N} + N_{jk} \sin \frac{2\pi ji}{N}\right)$$

$$= \frac{1}{N} \sum_{j=1}^{N/2} \left(\frac{j^2-1}{\nu^2-j^2}\right) \frac{1}{\sin^2 \frac{\pi j}{N}} \cos \frac{2\pi j(i-k)}{N} \tag{15}$$

Equation (14) is the crucial result; it expresses the $i^{\underline{th}}$ deviation ξ_i as a linear combination of the independent random variables s_k. The central limit theorem tells us that each ξ_i has a Gaussian distribution, with mean zero and mean square

$$< \xi^2 > = K^2 \sum_{k=1}^{N} \alpha_{ik}^2 < s_k^2 > = K^2 \epsilon^2 \sum_{k=1}^{N} \alpha_{ik}^2 \tag{16}$$

Substituting (15) into (16) gives our final result

$$\frac{< \xi^2 >}{\epsilon^2} = \frac{K^2}{2N} \sum_{j=1}^{N/2} \left[\frac{j^2-1}{\nu^2-j^2} \frac{1}{\sin^2 \frac{\pi j}{N}}\right]^2 = K^2 \Xi^2 \tag{17}$$

In Table I below, we give values of Ξ for various values of N and ν.

TABLE I

Values of $\Xi(N,\nu)$

N \ ν	30.25	35.25	40.25	45.25
100	7.16	6.89	6.91	7.19
250	21.8	19.1	17.0	15.5
500	59.6	51.4	45.2	40.4
1000	167	144	126	112

For large N, an approximate expression is

$$\Xi \approx (0.15) \frac{N^{3/2}}{\nu}$$

It is of some interest to examine in detail the sum in (17). It turns out that, for values of N and ν given in Table I, about 80% of the sum comes from the single term in which j is closest to ν. In Fig. 1, the summand

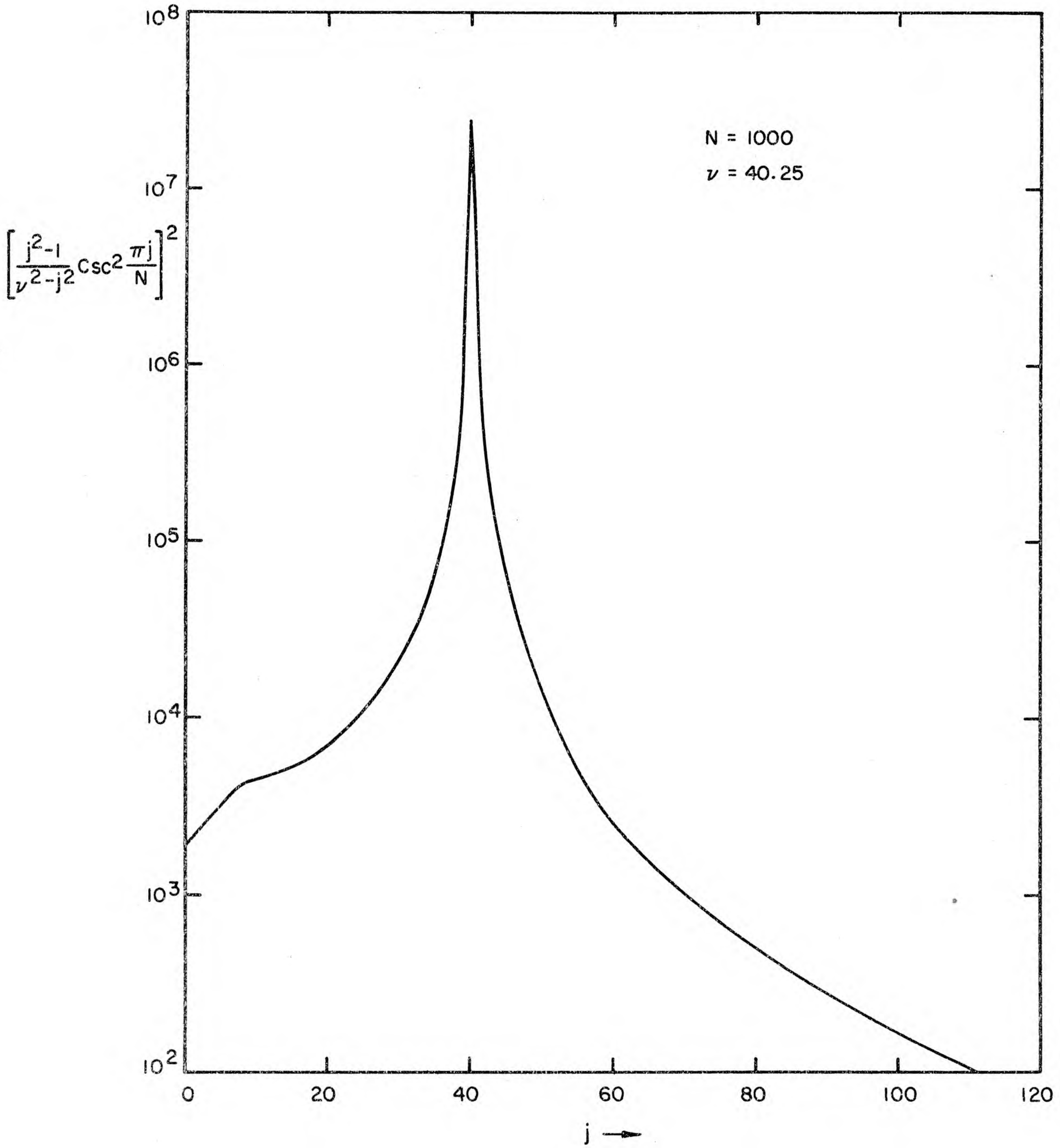
$$\left[\left(\frac{j^2-1}{\nu^2-j^2} \right) \frac{1}{\sin^2 \frac{\pi j}{N}} \right]^2$$

is plotted against j for $N = 1000$, $\nu = 40.25$.

V Conclusions.

It would appear from the numbers in Table I that a "follow your nose" alignment technique of the sort considered in this note may well be feasible, but accuracy requirements will be severe. For example, let $N = 1000$, $\nu = 40.25$, and $K = 2$. If the machine radius is about 1300 meters, the measuring stations will be about 8 meters apart. If we require the rms value of the displacement ξ to be less than 1/2 inch, we must require the rms error in measuring sagittas to be less than 0.002 inch.

Since the error is dominated by a single Fourier component, the rms value of the maximum displacement will be $\sqrt{2}$ times the rms displacement.



$\left[\frac{j^2-1}{\nu^2-j^2}\operatorname{Csc}^2\frac{\pi j}{N}\right]^2$
10^8
10^7
10^6
10^5
10^4
10^3
10^2
N = 1000
ν = 40.25
0
20
40
60
80
100
120
j →

FIGURE 1