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Discrete Euler–Poincaré and Lie–Poisson equations

Jerrold E Marsden†, Sergey Pekarsky† and Steve Shkoller‡

† Control and Dynamical Systems, California Institute of Technology, 107–81, Pasadena, CA 91125, USA

‡Department of Mathematics, University of California at Davis, Davis, CA 95616, USA

E-mail: marsden@cds.caltech.edu, sergey@cds.caltech.edu and shkoller@math.ucdavis.edu

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Abstract. In this paper, discrete analogues of Euler–Poincaré and Lie–Poisson reduction theory are developed for systems on finite dimensional Lie groups G with Lagrangians $L : TG \to \mathbb{R}$ that are G-invariant. These discrete equations provide 'reduced' numerical algorithms which manifestly preserve the symplectic structure. The manifold $G \times G$ is used as an approximation of TG, and a discrete Langragian $\mathbb{L} : G \times G \to \mathbb{R}$ is constructed in such a way that the G-invariance property is preserved. Reduction by G results in a new 'variational' principle for the reduced Lagrangian $\ell : G \to \mathbb{R}$, and provides the discrete Euler–Poincaré (DEP) equations. Reconstruction of these equations recovers the discrete Euler–Lagrange equations developed by Marsden *et al* (Marsden J E, Patrick G and Shkoller S 1998 *Commun. Math. Phys.* **199** 351–395) and Wendlandt and Marsden (Wendlandt J M and Marsden J E 1997 *Physica* D **106** 223–246) which are naturally symplectic-momentum algorithms. Furthermore, the solution of the DEP algorithm immediately leads to a discrete Lie–Poisson (DLP) algorithm. It is shown that when $G = \mathrm{SO}(n)$, the DEP and DLP algorithms for a particular choice of the discrete Lagrangian $\mathbb{L}$ are equivalent to the Moser–Veselov scheme for the generalized rigid body.

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1. Introduction

The goal of this paper is to develop structure-preserving numerical integrators on the reduced space of a mechanical system whose configuration space is a Lie group G and whose Lagrangian $L : TG \to \mathbb{R}$ is either left or right invariant by the group action. In particular, we shall develop the discrete analogue of Euler–Poincaré theory by following the variational approach introduced by Marsden, Patrick and Shkoller [MPS 98] for the construction of discrete Euler–Lagrange equations that naturally preserve the symplectic structure and the momentum mappings of the Lagrangian system.

In our setting, the results of [MPS 98] may be described as follows. Given a Lagrangian $L : TG \to \mathbb{R}$ form the action S on curves $g : [a, b] \to G$ defined in a chart by

$$S(g(t)) = \int_a^b L(g^i(t), \dot{g}^i(t))\mathrm{d}t.$$

Allowing for arbitrary variations δg, not constrained to vanish on $\{a, b\}$, a computation of the first variation of S leads to

$$\mathrm{d}S\big(g(t)\big) \cdot \delta g(t) = \int_a^b \delta g^i \left(\frac{\partial L}{\partial g^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{g}^i}\right)\mathrm{d}t + \frac{\partial L}{\partial \dot{g}^i}\delta g^i\bigg|_a^b. \tag{1.1}$$

The last term of (1.1) is a linear pairing of $\partial L/\partial \dot{g}^i$, a function of g^i and $\dot{g}^i$, with the tangent vector δg^i. Thus, one may consider it to be a 1-form $\theta_L = (\partial L/\partial \dot{q}^i)\mathrm{d}q^i$ on TG, and the symplectic structure is then defined by

$$\omega_L = -\mathrm{d}\theta_L.$$

Applying the operator $\mathrm{d}^2 = 0$ to S, restricted to the space of solutions of the Euler–Lagrange equations, shows that the flow F_t of the Euler–Lagrange equations conserves the symplectic form; namely, $F_t^*\omega_L = \omega_L$. Next, let $\mathfrak{g}$ denote the Lie algebra of G and define the momentum mapping $J_\xi : TG \to \mathbb{R}$ for each $\xi \in \mathfrak{g}$ corresponding to the tangent lift of the right (or left) action of G on itself by $J_\xi \equiv \xi_{TG} \lrcorner \theta_L$, where ξ_{TG} is the infinitesimal generator of $\xi \in \mathfrak{g}$ on TG. Then, the variational principle, together with the infinitesimal invariance of the action restricted to the space of solutions, immediately leads to the fact that $F_t^* J_\xi = J_\xi$. See [MPS 98] for details.

Hence, this variational approach can be used to obtain a symplectic-momentum integrator by discretizing TG and forming a discrete action sum. For every choice of discretization, a unique discrete symplectic structure is obtained, and the algorithm given by the discrete Euler–Lagrange equations is guaranteed to preserve this structure as well as the momentum mappings associated with it. Our goal is to apply the reduction procedure in this discrete setting, restrict the Lagrangian to the reduced space, and derive the algorithm which preserves the induced structure.

Our procedure results in the discrete Euler–Poincaré equation which defines an algorithm on the reduced space that is shown to be equivalent to the discrete Euler–Lagrange equations in the sense of reconstruction. This reduced algorithm is used together with the coadjoint action to advance points in $\mathfrak{g}^* \cong T^*G/G$ and thus approximate the Lie–Poisson dynamics. In subsequent papers, we shall make the extension to the more general setting of Lagrangian reduction of a G-invariant system on TQ (see, for example, Cendra, Marsden and Ratiu [CMR 98]), for a general manifold Q, as well as to the case of dynamical systems defined on Lie algebras.

2. The discrete Euler–Poincaré algorithm

In this section we develop the discrete Euler–Poincaré reduction of a Lagrangian system on TG. We approximate TG by $G \times G$ and form a discrete Lagrangian $\mathbb{L} : G \times G \to \mathbb{R}$ from the original Lagrangian $L : TG \to \mathbb{R}$ as

$$\mathbb{L}(g_k, g_{k+1}) = L(\kappa(g_k, g_{k+1}), \mathcal{X}(g_k, g_{k+1})),$$

where κ and $\mathcal{X}$ are functions of (g_k, g_{k+1}) which approximate the current configuration $g(t) \in G$ and the corresponding velocity $\dot{g}(t) \in T_gG$, respectively. We choose particular discretization schemes so that the discrete Lagrangian $\mathbb{L}$ inherits the symmetries of the original Lagrangian L: $\mathbb{L}$ is G-invariant on $G \times G$ whenever L is G-invariant on TG. In particular, the induced right (left) lifted action of G onto TG corresponds to the diagonal right (left) action of G on $G \times G$.

Having specified the discrete Lagrangian, we form the *action sum*

$$\mathbb{S} = \sum_{k=0}^{N-1} \mathbb{L}(g_k, g_{k+1})$$

and obtain the discrete Euler–Lagrange (DEL) equations

$$D_2\mathbb{L}(g_{k-1}, g_k) + D_1\mathbb{L}(g_k, g_{k+1}) = 0, \tag{2.1}$$

as well as the discrete symplectic form $\omega_{\mathbb{L}}$ given in coordinates on $G \times G$ by

$$\omega_{\mathbb{L}} = \frac{\partial^2 \mathbb{L}}{\partial g_k^i \partial g_{k+1}^j} \mathrm{d}g_k^i \wedge \mathrm{d}g_{k+1}^j, \tag{2.2}$$

by extremizing $\mathbb{S} : G^{N+1} \to \mathbb{R}$ with arbitrary variations. It is shown in [MPS 98] that the flow $\mathbb{F}_t$ of the DEL equations preserves this discrete symplectic structure. We remark here that the original canonical symplectic form ω is also preserved by this flow. Indeed, as the discrete Legendre transformations define a local symplectomorphism, we obtain that $\omega(t) = F\mathbb{L}^{-1}(\omega_{\mathbb{L}}(t)) = F\mathbb{L}^{-1}(\omega_{\mathbb{L}}(0)) = \omega(0)$.

The discrete reduction of a right-invariant system proceeds as follows. The induced group action on $G \times G$ is simply right multiplication in each component:

$$\bar{g} : (g_k, g_{k+1}) \mapsto (g_k \bar{g}, g_{k+1}\bar{g}),$$

for all $\bar{g}, g_k, g_{k+1} \in G$. Then the quotient map is given by

$$\pi : G \times G \to (G \times G)/G \cong G, \qquad (g_k, g_{k+1}) \mapsto g_k g_{k+1}^{-1}. \tag{2.3}$$

We note that one may alternatively use $g_{k+1} g_k^{-1}$ instead of $g_k g_{k+1}^{-1}$; our choice is consistent with other literature (see, for example, [MPS 98]). The projection map (2.3) defines the *reduced discrete Lagrangian* $\ell : G \to \mathbb{R}$ for any G-invariant $\mathbb{L}$ by $\ell \circ \pi = \mathbb{L}$, so that

$$\ell\left(g_k g_{k+1}^{-1}\right) = \mathbb{L}(g_k, g_{k+1}),$$

and the *reduced action sum* is given by

$$s = \sum_{k=0}^{N-1} \ell(f_{kk+1}),$$

where $f_{kk+1} \equiv g_k g_{k+1}^{-1}$ denote points in the quotient space. A reduction of the DEL equations results in the **discrete Euler–Poincaré** (DEP) equations. We state this as the following theorem.

Theorem 2.1. *Let $\mathbb{L}$ be a right invariant Lagrangian on $G \times G$ and let $\ell : (G \times G)/G \cong G \to \mathbb{R}$ be the restriction of $\mathbb{L}$ to G given by $\ell(g_1 g_2^{-1}) = \mathbb{L}(g_1, g_2)$. For any integer $N \geqslant 3$, let $\{(g_k, g_{k+1})\}_{k=0}^{N-1}$ be a sequence in $G \times G$ and define $f_{kk+1} \equiv g_k g_{k+1}^{-1}$ to be the corresponding sequence in G. Then, the following are equivalent.*

(1) The sequence $\{(g_k, g_{k+1})\}_{k=0}^{N-1}$ is an extremum of the action sum $\mathbb{S} : G^{N+1} \to \mathbb{R}$ for arbitrary variations $\delta g_k = (\mathrm{d}/\mathrm{d}\epsilon)|_0 g_k^{\epsilon}$ where for each k, $\epsilon \mapsto g_k^{\epsilon}$ is a smooth curve in G such that $g_k^0 = g_k$.

*(2) The sequence $\{(g_k, g_{k+1})\}_{k=0}^{N-1}$ satisfies the **discrete Euler–Lagrange equations** (2.1).*

(3) The sequence $\{f_{kk+1}\}_{k=0}^{N-1}$ is an extremum of the reduced action sum $s : G \to \mathbb{R}$ with respect to variations δf_{kk+1}, induced by the variations δg_k, and given by

$$\delta f_{kk+1} = TR_{f_{kk+1}}(\delta g_k g_k^{-1} - \mathrm{Ad}_{f_{kk+1}} \cdot \delta g_{k+1} g_{k+1}^{-1}).$$

*(4) The sequence $\{f_{kk+1}\}_{k=0}^{N-1}$ satisfies the **discrete Euler–Poincaré equations***

$$-\ell'(f_{k-1k})\mathrm{Ad}_{f_{k-1k}} TR_{f_{k-1k}} + \ell'(f_{kk+1}) TR_{f_{kk+1}} = 0 \tag{2.4}$$

for $k = 1, \ldots, N-1$, where the operators act on variations of the form $\vartheta_k = \delta g_k g_k^{-1}$.

Proof. We begin with the proof that (1) and (2) are equivalent following [MPS 98] and [WM 97]. One computes the first variation of the discrete action $\mathbb{S}$ with variations that vanish on the set $k = \{0, N\}$. Thus,

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \mathbb{S}(g_k^\epsilon) = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \sum_{k=0}^{N-1} \mathbb{L}(g_k^\epsilon, g_{k+1}^\epsilon) &= \sum_{k=0}^{N-1} D_1\mathbb{L}(g_k, g_{k+1})\delta g_k + \sum_{k=0}^{N-1} D_2\mathbb{L}(g_k, g_{k+1})\delta g_{k+1} \\ &= \sum_{k=1}^{N-1} D_1\mathbb{L}(g_k, g_{k+1})\delta g_k + \sum_{r=1}^{N-1} D_2\mathbb{L}(g_{r-1}, g_r)\delta g_r \\ &= \sum_{k=1}^{N-1} \left(D_1\mathbb{L}(g_k, g_{k+1}) + D_2\mathbb{L}(g_{k-1}, g_k)\right)\delta g_k,\end{aligned}$$

where we have used the discrete analogue of integration by parts which simply shifts the sequence $g_k \mapsto g_r$ where $r = k + 1$. Since for each $k = 1, \ldots, N - 1$, the variations δg_k are arbitrary, this establishes the DEL algorithm. We remark that choosing variations which do not vanish at $k = 0$ and $k = N$ defines two 1-forms whose exterior derivative is the unique symplectic 2-form given in (2.2).

To see that (1) is equivalent to (3), notice that since $\mathbb{L} = \ell \circ \pi$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_0 s(f_{kk+1}^\epsilon) = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_0 \mathbb{S}(g_k^\epsilon).$$

Now, for (3) $\Leftrightarrow$ (4), we compute

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_0 \sum_{k=0}^{N-1} \ell\left(g_k^\epsilon g_{k+1}^{\epsilon\,-1}\right)$$

and find that

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} s(f_{kk+1}^\epsilon) &= \sum_{k=0}^{N-1} \ell'(f_{kk+1})\left[\delta g_k g_{k+1}^{-1} - g_k g_{k+1}^{-1}\delta g_{k+1} g_{k+1}^{-1}\right] \\ &= \sum_{k=1}^{N-1} \ell'(f_{kk+1})\delta g_k g_k^{-1} g_k g_{k+1}^{-1} - \sum_{r=1}^{N-1} \ell'(f_{r-1r}) g_{r-1} g_r^{-1}\delta g_r g_r^{-1},\end{aligned}$$

where again we have used discrete integration by parts shifting the sequence $g_k \to g_r$ with $r = k + 1$, and the fact that $\delta g_0 = \delta g_N = 0$. Defining $\vartheta_k \equiv \delta g_k g_k^{-1}$, we obtain the discrete Euler–Poincare equations (2.4) for all variations of this form. □

Remark 2.1. In the case that $\mathbb{L}$ is left invariant, the discrete Euler–Poincaré equations take the form

$$-\ell'(f_{kk-1})TR_{f_{kk-1}} + \ell'(f_{k+1k})\mathrm{Ad}_{f_{k+1k}}TR_{f_{k+1k}} = 0, \tag{2.5}$$

where $f_{k+1k} \equiv g_{k+1}^{-1}g_k$ is in the left quotient $(G \times G)/G$, and the operators act on variations of the form $\vartheta_k = g_k^{-1}\delta g_k$.

We may associate to any C^1 function F on $G \times G$ its Hamiltonian vector field X_F satisfying $X_F \,\lrcorner\, \omega_{\mathbb{L}} = \mathrm{d}F$. The symplectic structure $\omega_{\mathbb{L}}$ naturally defines a Poisson structure $\{\cdot, \cdot\}_{G\times G}$ on $G \times G$ by the relation

$$\{F, H\}_{G\times G} = \omega_{\mathbb{L}}(X_F, X_H). \tag{2.6}$$

Theorem 2.2. *If the action of G on $G \times G$ is proper, then the algorithm on G defined by the discrete Euler–Poincaré equations (2.4) preserves the induced Poisson structure $\{\cdot, \cdot\}_G$ on G given by*

$$\{f, h\}_G \circ \pi = \{f \circ \pi, h \circ \pi\}_{G\times G} \tag{2.7}$$

for any C^1 functions $f, h : (G \times G)/G \cong G \to \mathbb{R}$.

Proof. Theorem 4.1 of [MPS 98] guarantees that the DEL algorithm preserves the symplectic structure $\omega_{\mathbb{L}}$ on $G \times G$; hence, by (2.6), the DEL algorithm preserves the Poisson structure on $G \times G$. Since the action of G on $G \times G$ is proper, the general Poisson reduction theorem [MR 94] states that the projection $\pi : G \times G \to G$ is a Poisson map.

By theorem 2.1, the projection of the DEL algorithm,

$$\pi \circ (g_{k-1}, g_k) \mapsto \pi \circ (g_k, g_{k+1}),$$

is equivalent to the DEP algorithm on G, $f_{k-1k} \mapsto f_{kk+1}$. Therefore, as the Poisson structure on G is induced by π and as π is Poissonian, we have proven the theorem. □

As we shall prove in the following theorem, reconstruction of the DEP algorithm (2.4) on G reproduces the DEL algorithm on $G \times G$.

Theorem 2.3. *The discrete Euler–Lagrange algorithm governed by $\mathbb{L}$ and the discrete Euler–Poincaré algorithm governed by ℓ are related as follows. The canonical projection of a solution of DEL gives a solution of DEP, while the reconstruction of a solution of the DEP equations results in a solution of the DEL equations.*

Proof. The first assertion follows by construction. For the second assertion, using the definition $f_{kk+1} = g_k g_{k+1}^{-1}$, the DEL algorithm can be reconstructed from the DEP algorithm by

$$(g_{k-1}, g_k) \mapsto (g_k, g_{k+1}) = \left(f_{k-1k}^{-1} \cdot g_{k-1}, f_{kk+1}^{-1} \cdot g_k\right), \tag{2.8}$$

where f_{kk+1} is the solution of (2.4). Indeed, $f_{kk+1}^{-1} \cdot g_k$ is precisely g_{k+1}. Thus, at each increment, one needs only to compute $f_{kk+1}^{-1} \cdot g_k$ since $g_k = f_{k-1k}^{-1} \cdot g_{k-1}$ is already known.

Similarly one shows that in the case of a left G action, the reconstruction of the DEP equations (2.5) is given by

$$(g_{k-1}, g_k) \mapsto (g_k, g_{k+1}) = \left(g_{k-1} \cdot f_{kk-1}^{-1}, g_k \cdot f_{k+1k}^{-1}\right). \tag{2.9}$$

□

Remark 2.2. Let us denote by $\overline{\pi}$ the quotient map $\overline{\pi} : TG \to TG/G \cong \mathfrak{g}$ mapping $\dot{g} \in T_g G$ to $\dot{g}g^{-1} \in \mathfrak{g}$. In the limit as the time step $h \to 0$, the DEL algorithm converges to the flow of the EL equations.

We denote the reconstruction of the flow of the Euler–Lagrange equations from the flow of the Euler–Poincaré equations by $\mathfrak{R}_{EP}$. Similarly, we denote the reconstruction of the DEL algorithm from the DEP algorithm provided by theorem 2.3 by $\mathfrak{R}_{DEP}$. The following noncommutative diagram shows these relations.

$$\begin{tikzcd} G \times G \arrow[r, "h\to 0"] \arrow[d, "\pi"] & TG \arrow[d, "\overline{\pi}"] \\ G & \mathfrak{g} \end{tikzcd} \qquad \begin{tikzcd} DEL \arrow[r, "h\to 0"] & EL \\ DEP \arrow[u, "\mathfrak{R}_{DEP}"'] & EP \arrow[u, "\mathfrak{R}_{EP}"'] \end{tikzcd}$$

where $G \times G \to TG$ as $h \to 0$ in the following sense. Locally, $G \times G = F\mathbb{L}^*(T^*G)$ and as $h \to 0$, $F\mathbb{L} \to FL$ which pulls back T^*G to TG. Thus, the DEP algorithm approximates the flow of the Euler–Poincaré equations if properly interpreted by means of reconstruction.

3. The discrete Lie–Poisson algorithm

In addition to reconstructing the dynamics on $G \times G$, we may use the coadjoint action to form a discrete Lie–Poisson algorithm approximating the dynamics on $\mathfrak{g}^*$. Recall that in the Lie–Poisson reduction setting, for $m \in T^*_g G$, the momentum corresponding to the velocity vector $\dot{g} \in T_g G$, we define

$$m_c = L^*_g m \in \mathfrak{g}^*, \qquad m_s = R^*_g m \in \mathfrak{g}^*$$

to be the *body* and *spatial* momentum vectors, respectively, with the relation

$$m_s = \mathrm{Ad}^*_{g^{-1}} m_c.$$

For the right invariant system, the first Euler theorem states that $(\mathrm{d}/\mathrm{d}t)m_c = 0$ (see theorem 4.4 of Arnold and Khesin [AK 98]) so that the body momentum is a constant of the motion. For convenience, we denote the constant m_c by μ_0 and $m_s(t)$ by $\mu(t)$ so that

$$\mu(t) = \mathrm{Ad}^*_{g^{-1}(t)} \cdot \mu_0. \tag{3.1}$$

Now, let $\mathcal{O} \subset \mathfrak{g}$ be a coadjoint orbit; that is, the orbit of a point under the coadjoint action of G on $\mathfrak{g}^*$. Then $\mathcal{O}$ is a symplectic manifold with unique Kirillov–Kostant forms $\omega^{\pm}$ as the coadjoint orbit symplectic structures (see, for example, theorem 14.4.1 in [MR 94]). Lemma 14.4.2 of [MR 94] states that for any $g \in G$, $\mathrm{Ad}^*_{g^{-1}} : \mathcal{O} \to \mathcal{O}$ preserves $\omega^{\pm}$. On the other hand, there are natural Lie–Poisson $\{\cdot,\cdot\}^{\pm}$ structures on $\mathfrak{g}^*$ (coming from Lie–Poisson reduction on T^*G) which induce $(\pm)$ symplectic forms on each symplectic leaf in $\mathfrak{g}^*$. These induced symplectic structures coincide with the coadjoint orbit symplectic structures on each coadjoint orbit (see Kostant [K 66]); hence, the coadjoint action preserves the Lie–Poisson structures.

Using the evolution equation (3.1) along with the sequence $\{f_{kk+1}\}$ obtained by the DEP algorithm, we find that

$$\mu_{k+1} = \mathrm{Ad}^*_{g^{-1}_{k+1}} \mu_0 = \mathrm{Ad}^*_{(f^{-1}_{kk+1} \cdot g_k)^{-1}} \mu_0 = \mathrm{Ad}^*_{f_{kk+1}} \cdot \mathrm{Ad}^*_{g^{-1}_k} \mu_0 = \mathrm{Ad}^*_{f_{kk+1}} \mu_k.$$

Thus, we have proven the following.

Proposition 3.1. *An algorithm, called the* ***discrete Lie–Poisson*** *(DLP) algorithm, on $\mathfrak{g}^*$ defined along the sequence $\{f_{kk+1}\}$ provided by the DEP algorithm on G and given by*

$$\mu_{k+1} = \mathrm{Ad}^*_{f_{kk+1}} \cdot \mu_k \tag{3.2}$$

is Lie–Poisson, i.e. it preserves the $(+)$ Lie–Poisson structure on $\mathfrak{g}^*$.

Remark 3.1. The corresponding discrete Lie–Poisson equations for the left invariant system is given by†

$$\Pi_{k+1} = \mathrm{Ad}^*_{f^{-1}_{k+1k}} \cdot \Pi_k, \tag{3.3}$$

where $\Pi_k := \mathrm{Ad}^*_{g_k} \pi_0$, the reduced variable $m_c(t)$ is denoted by $\Pi(t)$ and the constant m_s by π_0.

Thus, one can obtain a Lie–Poisson integrator by solving (2.4) for f_{kk+1} and then substituting it into (3.2) to generate the algorithm. This algorithm manifestly preserves the coadjoint orbits and hence the Poisson structure on $\mathfrak{g}^*$. In section 5, we shall show that this recovers the Moser–Veselov equations for generalized rigid-body dynamics on $\mathrm{SO}(n)$.

It is instructive to compare our discrete Lie–Poisson algorithm with that obtained by Ge and Marsden [GM 88] using the Lie–Poisson Hamilton–Jacobi equations. We now state their results which were obtained for the *left* action of a group G on itself. Let H be a G-invariant

† Henceforth, we shall use the notation $\mu \in \mathfrak{g}^*$ for the *right* invariant system and $\Pi \in \mathfrak{g}^*$ for the *left*.

Hamiltonian on T^*G and let H_L be the corresponding left reduced Hamiltonian on $\mathfrak{g}^*$. If a generating function $S : G \times G \to \mathbb{R}$ of canonical transformations is invariant, then there exists a unique function S_L such that $S_L(g^{-1}g_0) = S(g, g_0)$.

The left reduced Hamilton–Jacobi equation for the function $S_L : G \to \mathbb{R}$ is given by

$$\frac{\partial S_L}{\partial t} + H_L(-TR_g^* \cdot \mathrm{d}S_L(g)) = 0, \tag{3.4}$$

and is called the *Lie–Poisson Hamilton–Jacobi* equation. The Lie–Poisson flow of the Hamiltonian H_L is generated by its solution S_L; in particular, the flow $t \mapsto F_t$ of S_L taking initial data Π_0 to $\Pi(t)$ is Poissonian for each t in the domain of definition. Next, one defines $g \in G$ as the solution of

$$\Pi_0 = -TL_g^* \cdot D_g S_L \tag{3.5}$$

and then sets

$$\Pi = \mathrm{Ad}^*_{g^{-1}} \Pi_0. \tag{3.6}$$

Thus, one obtains a Lie–Poisson integrator by approximately solving (3.4), and then using (3.5) and (3.6) to generate the algorithm.

Note that (3.4) is the analogue of the usual Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + H\left(q^i, \frac{\partial S}{\partial q^i}\right) = 0$$

and that (3.5) and (3.6) are the analogues of the corresponding canonical transformations generated by a solution S which in a local chart is given by

$$p_{0i} = -\frac{\partial S}{\partial q_0^i} \qquad p_i = \frac{\partial S}{\partial q^i}.$$

It is interesting to compare the approach using the Lie–Poisson Hamilton–Jacobi equation (3.4) with that using the discrete Euler–Lagrange equations. The choice of discrete Lagrangian ℓ may be viewed as a choice of approximate solution to the Hamilton–Jacobi equation. Then the steps of solving (3.5) and (3.6) are parallel to the solution of equations (2.4) and (3.2). Namely, the DLP equation provides a time evolution map $\mu_k \mapsto \mu_{k+1}$ on $\mathfrak{g}^*$ using a *known* solution f_{kk+1}, while (3.6) advances the initial value Π_0 along the coadjoint orbit and requires at each time step the solution g of (3.5) that approximates the current ‘position’ $g(t)$.

4. Discretization using natural charts

In this section, we discretize TG by $G \times G$ and use the group exponential map at the identity, $\exp_e : \mathfrak{g} \to G$, to construct an appropriate discrete Lagrangian.

4.1. The general theory

For finite dimensional Lie groups G, $\exp_e$ is locally a diffeomorphism and thus provides a natural chart. Namely, there exists an open neighborhood U of $e \in G$ such that $\exp_e^{-1} : U \to \mathfrak{u} \equiv \exp_e^{-1}(U)$ is a C^∞ diffeomorphism (this is not in general true for infinite dimensional groups). Hence, the manifold structure is provided by right translation, so that a chart at $g \in G$ is given by

$$\psi_g = \exp_e^{-1} \circ R_{g^{-1}}. \tag{4.1}$$

We now define the *discrete Lagrangian*, $\mathbb{L} : G \times G \to \mathbb{R}$, by

$$\mathbb{L}(g_1, g_2) = L\left(\psi_g^{-1}\left[\frac{\psi_g(g_1) + \psi_g(g_2)}{2}\right], (\psi_g^{-1})_*\left[\frac{\psi_g(g_2) - \psi_g(g_1)}{h}\right]\right), \tag{4.2}$$

where $h \in \mathbb{R}_+$ is the given time step and $g_1, g_2 \in U_g \equiv R_g(U)$.

We shall assume that G has a right invariant Riemannian metric $\langle\cdot,\cdot\rangle$ obtained by right translating a positive bilinear form on $\mathfrak{g}$ over the entire group. We also assume that G has a regular quadratic Lie algebra, as in [GM 88].

For $K \subset G$ a compact set, we define the Riemannian distance function, $\mathrm{dist} : K \times K \to \mathbb{R}^+$ by

$$\mathrm{dist}(g_1, g_2) = \int_0^1 \langle\dot\gamma(t), \dot\gamma(t)\rangle \mathrm{d}t,$$

where $\gamma : [0, 1] \to G$ is the geodesic with $\gamma(0) = g_1$ and $\gamma(1) = g_2$. It is then clear that $\mathrm{diam}(U) = \mathrm{diam}(U_g)$ for all $g \in G$, so in order for (4.2) to be well defined we require that $\mathrm{dist}(g_1, g_2) < \mathrm{diam}(U)$. In other words, we require that (g_1, g_2) be close to the diagonal in $G \times G$. Our restriction on $\mathrm{dist}(g_1, g_2)$ in turn places a restriction on the timestep h.

Next, let

$$\eta = \frac{\psi_g(g_1) + \psi_g(g_2)}{2},$$

with corresponding group element

$$g' = \exp(\eta) \in U.$$

We denote the algebra element approximating the velocity $g^{-1}\dot g$ by

$$\zeta = \frac{\psi_g(g_2) - \psi_g(g_1)}{h}.$$

Using the standard formula for the derivative of the exponential (see, for example, Dragt and Finn [DF 76] or Channel and Scovel [CS 91]) given by

$$T_\eta \exp = T_e R_{g'} \cdot \mathrm{iex}(-\mathrm{ad}_\eta), \qquad \eta \in \mathfrak{g}, \qquad g' = \exp(\eta) \in U,$$

where iex is the function defined by

$$\mathrm{iex}(w) = \sum_{n=0}^{\infty} \frac{w^n}{(n+1)!}, \tag{4.3}$$

we may evaluate the push-forward of ψ_g^{-1} at η. We obtain the following expression for the discrete Lagrangian

$$\mathbb{L}(g_1, g_2) = L\left(\psi_g^{-1}(\eta), T_{g'}R_g \cdot T_e R_{g'} \cdot \mathrm{iex}(-\mathrm{ad}_\eta)(\zeta)\right).$$

Setting $q \equiv \psi_g^{-1}(\eta) = R_g g'$, the last formula is expressed as

$$\mathbb{L}(g_1, g_2) = L\left(q, T_e R_q \cdot \mathrm{iex}(-\mathrm{ad}_\eta)(\zeta)\right), \tag{4.4}$$

so that locally the Lagrangian is evaluated at the base point $q = \psi_g^{-1}(\eta) \in U_g \subset G$, and the Lie algebra (fibre) element $\mathrm{iex}(-\mathrm{ad}_\eta)(\zeta)$ is right translated to the tangent space at the point q, $T_q G$; as $h \to 0$, this fibre element converges to the group velocity $\dot g \in T_g G$.

The following lemma establishes that the discrete Lagrangian $\mathbb{L}$ inherits the G-invariance property from the original Lagrangian L, so that the discrete counterpart of the Euler–Poincare reduction is well-defined.

Lemma 4.1. *The discrete Lagrangian $\mathbb{L} : G \times G \to \mathbb{R}$ is right (left) invariant under the diagonal action of G on $G \times G$, whenever $L : TG \to \mathbb{R}$ is right (left) invariant.*

Proof. We fix the right action and consider $R^*_{\bar{g}}(\mathbb{L})$ for some $\bar{g} \in G$. By construction, $R_{\bar{g}}g_1, R_{\bar{g}}g_2 \in R_{\bar{g}}(U_g)$, whenever $g_1, g_2 \in U_g \equiv R_g(U)$, so that the chart is given by $\psi_{g\bar{g}} = \exp_e^{-1} \circ R_{(g\bar{g})^{-1}}$.

By definition, both η and ζ are always elements of a neighborhood of $0 \in \mathfrak{g}$, so it is clear that they are right invariant. Hence, using the explicit form of the chart $\psi_{g\bar{g}}$ together with the right invariance of the Lagrangian L, we obtain from (4.2) and (4.4) that

$$\begin{aligned}
\mathbb{L}(R_{\bar{g}}g_1, R_{\bar{g}}g_2) &= L\left(\psi_{g\bar{g}}^{-1}\left[\frac{\psi_{g\bar{g}}(g_1\bar{g}) + \psi_{g\bar{g}}(g_2\bar{g})}{2}\right], (\psi_{g\bar{g}})_*\left[\frac{\psi_{g\bar{g}}(g_2\bar{g}) - \psi_{g\bar{g}}(g_1\bar{g})}{h}\right]\right) \\
&= L\left(R_{\bar{g}} \cdot \psi_g^{-1}(\eta), T_q R_{\bar{g}} \cdot T_{g'} R_g \cdot T_e R_{g'} \cdot \operatorname{iex}(-\operatorname{ad}_\eta)(\zeta)\right) \\
&= L\left(R_{\bar{g}} \cdot q,\ T_q R_{\bar{g}} \cdot T_e R_q \cdot \operatorname{iex}(-\operatorname{ad}_\eta)(\zeta)\right) \\
&= \mathbb{L}(g_1, g_2).
\end{aligned}$$

In the case that the group action is on the left, we use $\phi_g = \exp_e^{-1} \circ L_{g^{-1}}$ as the chart, and proceed with the same argument. □

Corollary 4.1. *Using the discretization defined by (4.2), the reduced discrete Lagrangian ℓ determined by the projection map (2.3), $\ell(g_1 g_2^{-1}) = \mathbb{L}(g_1, g_2)$, can be expressed in terms of the continuous reduced Lagrangian l by*

$$\ell(g_1 g_2^{-1}) = l(\operatorname{iex}(-\operatorname{ad}_\eta)(\zeta)), \tag{4.5}$$

where $\eta = (\psi_g(g_1)+\psi_g(g_2))/2$, $\zeta = (\psi_g(g_2)-\psi_g(g_1))/h$, and l can be defined by translation to the identity of the arguments of the right invariant Lagrangian L, i.e. $l(\xi) = L(R_{g^{-1}}g, TR_{g^{-1}}\dot{g}) = L(e, \xi)$, where $\xi = TR_{g^{-1}}\dot{g} \in \mathfrak{g}$.

The proof of this corollary follows from expression (4.4), and the fact that the Lagrangian L is right invariant so that translation by q^{-1} to e gives (4.5).

The expressions (4.4) and (4.5) for the discrete Lagrangian in general require evaluation of the infinite series for the iex function given by (4.3); however, a simplification occurs when g is set to either g_k or g_{k+1}. This is due to the fact that when $g = g_k$ or $g = g_{k+1}$, one may easily verify that $\operatorname{ad}_\zeta \eta := [\zeta, \eta] = 0$, and hence that $\operatorname{iex}(-\operatorname{ad}_\eta)(\zeta) = \zeta$.

For example, with $g = g_{k+1}$, the discrete Lagrangian is simply

$$\mathbb{L}(g_k, g_{k+1}) = L\left(q, T_e R_q(\zeta)\right), \tag{4.6}$$

where

$$\eta = \tfrac{1}{2}\log(g_k g_{k+1}^{-1}), \qquad q \equiv \psi_{g_{k+1}}(\eta) = (g_k g_{k+1})^{1/2}, \qquad \zeta = \frac{1}{h}\log(g_k g_{k+1}^{-1})$$

and $\log \equiv \exp^{-1}$. Consequently, the reduced discrete Lagrangian is given by

$$\ell(f_{kk+1}) = l(\log(f_{kk+1})/h), \tag{4.7}$$

where $f_{kk+1} = g_k g_{k+1}^{-1}$.

Substituting the discrete Lagrangian (4.7) into the DEP equation (2.4), we obtain the following implicit algorithm on the Lie algebra

$$l'(\xi_{kk+1}/h) \cdot \chi(\operatorname{ad}_{\xi_{kk+1}}) = l'(\xi_{k-1k}/h) \cdot \chi(\operatorname{ad}_{\xi_{k-1k}}) \cdot \exp(\operatorname{ad}_{\xi_{k-1k}}), \tag{4.8}$$

where $\xi_{kk+1} \equiv \log f_{kk+1} \in \mathfrak{g}$ and the function χ is defined to be the inverse of the function iex defined by (4.3), $\chi(\operatorname{ad}_\xi) \cdot \operatorname{iex}(-\operatorname{ad}_\xi) = \operatorname{Id}_{\mathfrak{g}}$. The function χ in (4.8) arises from taking the derivative of the log function viewed as a map from the Lie group to its algebra. It is interesting to compare the above algorithm with the one obtained by Channel and Scovel [CS 91] using the Hamilton–Jacobi equation.

4.2. Generalized rigid body dynamics

We apply our DEP algorithm to the generalized rigid body problem. In this case, $G = \mathrm{SO}(n)$ with Lie algebra $\mathfrak{g} = \mathfrak{so}(n)$, and the left invariant Lagrangian is given by the kinetic energy

$$L_{RB}(g, \dot{g}) = \tfrac{1}{2}(\dot{g}, \dot{g})_g = \tfrac{1}{2}\langle \dot{g}, \mathbb{J}_g(\dot{g})\rangle = \tfrac{1}{2}\langle g^{-1}\dot{g}, \mathbb{J}(g^{-1}\dot{g})\rangle = \tfrac{1}{2}(g^{-1}\dot{g}, g^{-1}\dot{g}). \tag{4.9}$$

Here, $\langle\cdot,\cdot\rangle_g$ denotes the pairing between $T_g\mathrm{SO}(n)$ and its dual $T_g^*\mathrm{SO}(n)$ which we associate to the metric $(\cdot,\cdot)$ on $\mathrm{SO}(n)$ by

$$(X_g, Y_g)_g = \langle X_g, \mathbb{J}_g Y_g\rangle_g, \quad X_g, Y_g \in T_g\mathrm{SO}(n),$$

where $\mathbb{J}_g = (L_g^*)^{-1}\,\mathbb{J}\,(L_{g^{-1}})_*$ is the left translated inertia tensor, and $\mathbb{J} : \mathfrak{so}(n) \to \mathfrak{so}(n)^*$. On $\mathrm{SO}(n)$, $(L_{g^{-1}})_* \cdot \dot{g} = g^{-1}\dot{g}$.

We discretize $T\mathrm{SO}(n)$ by $\mathrm{SO}(n)\times\mathrm{SO}(n)$ and construct the discrete Lagrangian following (4.6) as

$$\mathbb{L}_{RB}(g_k, g_{k+1}) = L_{RB}\left(q_{k+1k}, T_e L_{q_{k+1k}}(\zeta_{k+1k})\right),$$

where $q_{k+1k} = g_{k+1}\left(g_{k+1}^{-1}g_k\right)^{1/2}$ and $\zeta_{k+1k} = \frac{1}{h}\log\left(g_{k+1}^{-1}g_k\right)$. Using the left invariance of the metric, we may express the discrete rigid body Lagrangian as

$$\mathbb{L}_{RB}(g_k, g_{k+1}) = \tfrac{1}{2}(\zeta_{k+1k}, \zeta_{k+1k}) = \tfrac{1}{2}\langle \zeta_{k+1k}, \mathbb{J}(\zeta_{k+1k})\rangle. \tag{4.10}$$

The Lagrangian for the reduced system on $(\mathrm{SO}(n)\times\mathrm{SO}(n))/\mathrm{SO}(n) \cong \mathrm{SO}(n)$ is then given by

$$\ell_{RB}(f_{k+1k}) = \mathbb{L}_{RB}(g_k, g_{k+1}) = \frac{1}{2h^2}\langle \log f_{k+1k}, \mathbb{J}(\log f_{k+1k})\rangle, \tag{4.11}$$

where $f_{k+1k} \equiv g_{k+1}^{-1}g_k \in \mathrm{SO}(n)$ is an element of the reduced space and h is the time step.

The DEP equation (2.5) has the following implicit form

$$\zeta_{k+1k} = \mathbb{J}^{-1}\left(\mathrm{iex}(-\mathrm{ad}^*_{h\zeta_{k+1k}})\cdot\chi(\mathrm{ad}^*_{h\zeta_{kk-1}})\cdot\mathrm{Ad}^*_{\exp(-h\zeta_{kk-1})}\mathbb{J}(\zeta_{kk-1})\right). \tag{4.12}$$

5. Moser–Veselov discretization of the generalized rigid body

An alternative discretization approach may be taken if we first embed our group G into a linear space; for finite dimensional matrix groups, the linear ambient space is $\mathfrak{gl}(n)$. Then, summation of the group elements becomes a legitimate operation provided we project the result back onto the group G by using Lagrange multipliers.

In this section, we consider the *left* invariant generalized rigid body equations on $\mathrm{SO}(n)$. The corresponding Lagrangian is determined by a symmetric positive definite operator $J : \mathfrak{so}(n) \to \mathfrak{so}(n)$, defined by $J(\xi) = \Lambda\xi + \xi\Lambda$, where $\xi \in \mathfrak{so}(n)$ and Λ is a diagonal matrix satisfying $\Lambda_i + \Lambda_j > 0$ for all $i \neq j$. The left invariant metric on $\mathrm{SO}(n)$ is obtained by left translating the bilinear form at e given by

$$(\xi, \xi) = \tfrac{1}{4}\mathrm{Tr}\left(\xi^T J(\xi)\right).$$

The operator J, viewed as a mapping $\mathbb{J} : \mathfrak{so}(n) \to \mathfrak{so}(n)^*$, has the usual interpretation of the inertia tensor, and the Λ_i correspond to the sums of certain principal moments of inertia.

The rigid body Lagrangian is the kinetic energy of the system

$$L(g, \dot{g}) = \tfrac{1}{4}\langle g^{-1}\dot{g}, \mathbb{J}(g^{-1}\dot{g})\rangle = \tfrac{1}{4}\langle \xi, \mathbb{J}(\xi)\rangle, \tag{5.1}$$

where $\xi = g^{-1}\dot{g} \in \mathfrak{so}(n)$ and $\langle\cdot,\cdot\rangle$ is the pairing between the Lie group and its dual; hence, the Hamiltonian vector field of L is the geodesic spray on TG.

Using the definition of J we rewrite the Lagrangian (5.1) in the following form:

$$L = \tfrac{1}{4}\mathrm{Tr}\left(\xi^T J(\xi)\right) = \tfrac{1}{2}\mathrm{Tr}\left(\xi^T \Lambda \xi\right).$$

We now discretize the Lie algebra elements by $\xi = g^{-1}\dot{g}$

$$\xi \approx \frac{1}{h} g_{k+1}^T (g_{k+1} - g_k), \tag{5.2}$$

where h is the time step. Substituting (5.2) into the Lagrangian L (and using properties of the trace), we obtain the following expression for the discrete Lagrangian (modulo a constant):

$$\mathbb{L}(g_k, g_{k+1}) = -\frac{1}{h^2}\mathrm{Tr}\left(g_k \Lambda g_{k+1}^T\right).$$

We remark that exactly the same expression is obtained if we instead discretize ξ by $\frac{1}{h} g_k^T (g_{k+1} - g_k)$. Notice that up to a multiplier of $-1/h^2$, this is precisely the Lagrangian used by Moser and Veselov [MoV 91].

We scale the above Lagrangian and introduce matrix Lagrange multipliers λ_k, imposing the constraint $\Phi_k(g_k) = g_k g_k^T - \mathrm{Id} = 0$. By decomposing λ_k into symmetric and skew components, we see that the skew component of λ_k does not contribute to the action because the constraint Φ_k is symmetric; thus, we find that $\lambda_k = \lambda_k^T$. The action sum then takes the form

$$S = \sum_k \mathrm{Tr}\left(g_k \Lambda g_{k+1}^T\right) - \frac{1}{2}\sum_k \mathrm{Tr}\left(\lambda_k \left(g_k g_k^T - \mathrm{Id}\right)\right) \tag{5.3}$$

Notice that the discrete Lagrangian $\mathbb{L}$ is left invariant and can be reduced to a Lagrangian $\ell : G \to \mathbb{R}$ using the canonical projection $\pi : (g_k, g_{k+1}) \mapsto f_{k+1k} = g_{k+1}^{-1} g_k$ so that

$$\ell(f_{k+1k}) = \mathrm{Tr}(f_{k+1k}\Lambda).$$

Because the constraint, ensuring that each $g_k \in G$, is G-invariant, there exists a Lagrange multiplier $\bar{\lambda}_k$ in the conjugacy class of λ_k, i.e., $\bar{\lambda}_k = g^T \lambda_k g$ for all $g \in G$, so that $\bar{\lambda}_k = \bar{\lambda}_k^T$. Hence, computing the discrete variation of (5.3) with respect to g_k, we obtain the operator equation

$$-\ell'(f_{kk-1}) T R_{f_{kk-1}} + \ell'(f_{k+1k}) \mathrm{Ad}_{f_{k+1k}} T R_{f_{k+1k}} = \bar{\lambda}_k,$$

where the operators act on the variations $\vartheta_k = g_k^T \delta g_k$. Using the expression for the reduced Lagrangian ℓ, the DEP equation can then be written as

$$f_{k+1k}^T \Lambda + f_{kk-1}\Lambda = \bar{\lambda}_k.$$

Using the fact that $\bar{\lambda}_k^T = \bar{\lambda}_k$, we obtain the DEP algorithm on $\mathrm{SO}(n)$ as

$$f_{k+1k}^T \Lambda - \Lambda f_{k+1k} = \Lambda f_{kk-1}^T - f_{kk-1}\Lambda. \tag{5.4}$$

This is an implicit scheme to be solved for f_{k+1k} using the current value f_{kk-1}. The solution of (5.4) generates the *explicit* DLP algorithm on $\mathfrak{so}(n)^*$ given by

$$\Pi_{k+1} = \mathrm{Ad}_{f_{k+1k}^{-1}} \Pi_k = f_{k+1k} \Pi_k f_{k+1k}^T. \tag{5.5}$$

Finally, reconstruction of the DEP algorithm recovers the DEL algorithm on $G \times G$ which, according to (2.9), is given by

$$(g_{k-1}, g_k) \mapsto (g_k, g_{k+1}) = (g_k, g_k \cdot f_{k+1k}^{-1}).$$

Theorem 5.1. *The above DEP and DLP algorithms given by (5.4) and (5.5), respectively, are equivalent to the Moser–Veselov equations*

$$\begin{aligned} &M_{k+1} \equiv \omega_{k-1} M_k \omega_{k-1}^{-1} \\ &M_k = \omega_k^T \Lambda - \Lambda \omega_k, \qquad \omega_k \in \mathrm{SO}(n), \end{aligned} \tag{5.6}$$

where (using the notation of [MoV 91]) $\omega_k = g_k^T g_{k-1} \in \mathrm{SO}(n)$ *is the discrete angular velocity,* $M_k = g_{k-1}^T m_k g_{k-1} = \omega_k^T \Lambda - \Lambda \omega_k \in \mathrm{so}(n)$ *is the discrete body angular momentum, and* $m_k = m_0$ *is the constant discrete spatial angular momentum.*

Proof. Comparing the definitions of $f_{kk-1} = g_k^T g_{k-1}$ and $\omega_k = g_k^T g_{k-1}$, we see that $f_{kk-1} \equiv \omega_k$. Similarly, comparing the definitions of $\Pi_k = \mathrm{Ad}_{g_k}^* \pi_0$ and

$$M_k = g_{k-1}^T m_k g_{k-1} = g_{k-1}^T m_o g_{k-1} = \mathrm{Ad}_{g_{k-1}}^* m_0,$$

we conclude that $\Pi_{k-1} \equiv M_k$ and $\pi_0 \equiv m_0$. Hence, the first equation in (5.6) is precisely the DLP algorithm (5.5).

Substituting the second equation of (5.6) into the first, results in the following expression:

$$\omega_{k+1}^T \Lambda - \Lambda \omega_{k+1} = \Lambda \omega_k^T - \omega_k \Lambda,$$

which is precisely the DEP equation (5.4) when the above identifications are invoked. □

The Moser–Veselov algorithm (5.6) has an an obvious geometric mechanical interpretation. The first equation can be viewed as a discretization of the left Lie–Poisson equation

$$M_k = g_{k-1}^T m_0 g_{k-1} = \mathrm{Ad}_{g_{k-1}}^* m_0,$$

rewritten in terms of the ω_k and this corresponds to the DLP algorithm (5.5). The second equation is a discrete version of the relation between the angular momentum and angular velocity, as it is obtained by substitution of (5.2) into $M = J(\xi) = \Lambda\xi + \xi\Lambda$.

The DEP algorithm (5.4) provides an equivalent alternative to the Moser–Veselov scheme (5.6), the difference being that the former is an algorithm on G only, while the latter is a combined algorithm on G and $\mathfrak{g}^*$ and schematically can be represented by the mappings $\mathfrak{g}^* \mapsto G \mapsto \mathfrak{g}^* \mapsto G; \quad M_k \mapsto \omega_k \mapsto M_{k+1} \mapsto \omega_{k+1}$.

In the proof of theorem 5.1, we identified Π_{k-1} with M_k in order to establish the equivalence with the Moser–Veselov algorithm; however, without any such identification, we exactly obtain the algorithm given by equation (4.1) in Lewis and Simo [LS 96] which we write in our notation as

$$\begin{aligned} g_{k+1} &= g_k f_{k+1k}^T, \\ \Pi_{k+1} &= f_{k+1k} \Pi_k f_{k+1k}^T, \\ \Delta t \Pi_k &= 2\mathrm{skew}(g_k \Lambda). \end{aligned} \tag{5.7}$$

The first equation of (5.7) corresponds to our reconstruction algorithm (2.9), the second equation of (5.7) corresponds to our DLP algorithm (3.3), and the third equation of (5.7) is our DEP algorithm (5.4). To see this, simply note that

$$g_k^T \text{ ([LS 96], Equation 4.5) } g_k = \text{Equation (5.4) (i.e. DEP)}.$$

It is worthwhile to make a few remarks at this point. Although it is claimed in [LS 96] that a computation of the first variation of the action $\sum_k \mathrm{Tr}\left(g_k \Lambda g_{k+1}^T\right)$ leads to the algorithm (5.7), we have shown that only constrained variations of the action function (5.3) lead to this algorithm. Furthermore, the algorithm (5.7) is obtained by constraining the iterates of the momentum to be equal; this constraint is superfluous as the discrete Euler–Lagrange equations necessarily conserve the momentum. Finally, if we choose $f_{k+1k} = \mathrm{cay}(\xi_{k+1k})$ where $\mathrm{cay} : \mathrm{so}(n) \to \mathrm{SO}(n)$ is the Cayley transform given by $\mathrm{cay}(\xi) = (1 + \frac{1}{2}\xi)(1 - \frac{1}{2}\xi)^{-1}$ for any $\xi \in \mathrm{so}(n)$, then the rigid-body algorithm for ξ_{k+1k} is second-order accurate, as proven in [LS 96]. It is not clear, however, whether the second-order accuracy can be maintained in the absence of the Cayley transform.

6. A comparison of DEP/DLP algorithms with splitting methods

For the purpose of comparison, we shall now describe the Hamiltonian splitting methods for generating Lie–Poisson integrators on $\mathfrak{g}^*$, the dual of the Lie algebra of a group G. The basic idea behind the construction of such an algorithm follows from the fact that many Lie–Poisson systems are governed by reduced Hamiltonians h which can be written as a sum $h^1 + \cdots + h^N$, where each h^i can be exactly integrated. Letting ϕ_t^i denote the flow of the Hamiltonian system h^i, we see that to first-order in the time-step Δt, the flow ϕ_t generated by h may be expressed as

$$\phi_{\Delta t} = \phi^1_{\Delta t} \circ \ldots \circ \phi^N_{\Delta t}.$$

As each of the maps $\phi^i_{\Delta t}$ is a Poisson map, hence symplectic on each leaf, the composition must also preserve the Poisson structure. Consequently, all Casimirs are also preserved by this splitting algorithm. Furthermore, one may construct this splitting algorithm to any order of accuracy in Δt. (For example, the leapfrog method $\phi_{\frac{1}{2}\Delta t}\phi^{-1}_{-\frac{1}{2}\Delta t}$ is a second-order accurate scheme (see, for example, [McS 96]).)

Whereas the DEP/DLP algorithms manifestly preserve the Poisson structure and all of the corresponding Casimirs as well, they do much more. First, the reduced algorithms may be used in both the Lagrangian and Hamiltonian sides, in that computation of the discrete Euler–Poincaré trajectory immediately leads to the discrete Lie–Poisson trajectory on $\mathfrak{g}^*$. More importantly, the discrete Lie–Poisson or Euler–Poincaré dynamics may be reconstructed to obtain symplectic-momentum integrators on TG, for example. Conservation of momentum ensures that the reconstructed discrete trajectory lies in an n-dimensional submanifold of the full $2n$-dimensional space $G \times G$, approximating TG. This n-dimensional submanifold is the level set of the discrete momentum mapping. For a small enough time step Δt, $G \times G$ is locally diffeomorphic to TG through the discrete Legendre transform, and hence we ensure that our discrete reconstructed trajectory is conserving the actual momentum.

Now recall that for right invariant systems, we have used the variable m_s to denote the solution of the Lie–Poisson equation, from which we obtain that $m_c(t) \equiv \mathrm{Ad}^*_{g(t)} m_s(t)$ is conserved. Using our DEP algorithm, we may compute the discrete trajectory $\{(m_s)_{kk+1}\}$, reconstruct to find g_k, and find that $(m_c)_{kk+1} = \mathrm{Ad}^*_{g_k}(m_s)_{kk+1}$ is conserved. On the other hand, the splitting method does not provide an algorithm for reconstructing the motion on T^*G in such a way as to ensure conservation of momentum; thus, there is no obvious way to define the discrete analogue of m_c, let alone check that it is conserved.

Nevertheless, there are some computational advantages to using the splitting method; the fact that the splitting method leads to an explicit scheme is perhaps the most important of these advantages. An efficient explicit algorithm for the $\mathrm{SU}(n)$ model of 2-dimensional hydrodynamics on a torus is constructed in [Mc 93]. The author presents a Poisson integrator of complexity $\mathrm{O}(N^3 \log N)$ which preserves $N - 1$ Casimirs.

7. Addendum: relation to other works

It is very interesting to compare the above constructions and algorithms to the recent results of Bobenko and Suris [BS 99]. In this paper they consider the theory of discrete time Lagrangian mechanics on Lie groups and, more specifically, address the issue of discrete Lagrangian reduction using left or right trivializations of the (co)tangent bundles of Lie groups. They adopt a somehow broader point of view when the symmetry group of a system defined on a Lie group G is a subgroup of G. Hence, it includes the Lie–Poisson case as a special case. Below we shall demonstrate that the reduced discrete equations obtained in [BS 99] agree with our DEP/DLP algorithms when the symmetry group is taken to be the full group G. Here we summarize their results choosing for consistency and simplicity the case of right trivialization and refer the reader to [BS 99] for details of proofs and notations.

Let the discrete Lagrangian $\mathbb{L}(g_k, g_{k+1}) : G \times G \to \mathbb{R}$ define a discrete system with the corresponding DEL equations. Consider the map

$$(g_k, w_k) \in G \times G \ \mapsto\ (g_k, g_{k+1}) \in G \times G, \tag{7.1}$$

where

$$g_{k+1} = w_k g_k \ \Leftrightarrow\ w_k = g_{k+1} g_k^{-1}.$$

Consider also the right trivialization of the cotangent bundle T^*G:

$$(g_k, m_k) \in G \times \mathfrak{g}^* \ \mapsto\ (g_k, \Pi_k) \in T^*G,$$

where

$$\Pi_k = R^*_{g_k^{-1}} m_k \ \Leftrightarrow\ m_k = R^*_{g_k} \pi_k.$$

Denote the pull-back of the Lagrange function under (7.1) by

$$\mathbb{L}^{(r)}(g_k, w_k) = \mathbb{L}(g_k, g_{k+1}).$$

Proposition 3.5 of [BS 99] gives the DEL equations in these coordinates:

$$\begin{aligned} &\mathrm{Ad}^*_{w_k} m_{k+1} = m_k + d_g \mathbb{L}^{(r)}(g_k, w_k), \\ &g_{k+1} = w_k g_k, \end{aligned} \tag{7.2}$$

where

$$m_k = d_w \mathbb{L}^{(r)}(g_{k-1}, w_{k-1}) \in \mathfrak{g}^*.$$

Assume that for some $\zeta \in \mathfrak{g}$, $\mathbb{L}^{(r)}$ is invariant under the action of a subgroup $G^{[\zeta]} \equiv \{h \in G | \mathrm{Ad}_h \zeta = \zeta\} \subset G$ on $G \times G$ induced by right translations on G:

$$\mathbb{L}^{(r)}(gh, w) = \mathbb{L}^{(r)}(g, w), \qquad h \in G^{[\zeta]}.$$

Define the reduced Lagrange function $\Lambda^{(r)} : \mathfrak{g}_\zeta \times G \mapsto \mathbb{R}$ as

$$\Lambda^{(r)}(a, w) = \mathbb{L}^{(r)}(g, w), \qquad a = \mathrm{Ad}_g \zeta \in \mathfrak{g}_\zeta$$

here $\mathfrak{g}_\zeta$ is the adjoint orbit of ζ.

Then, Proposition 3.7 of [BS 99] states that under the reduction by $G^{[\zeta]}$, the reduced Euler–Lagrange equations become

$$\begin{aligned} &\mathrm{Ad}^*_{w_k} m_{k+1} = m_k - \mathrm{ad}^*_{a_k} \nabla_a \Lambda^{(r)}(a_k, w_k), \\ &a_{k+1} = \mathrm{Ad}_{w_k} a_k, \end{aligned} \tag{7.3}$$

where

$$m_k = d_w \Lambda^{(r)}(a_{k-1}, w_{k-1}) \in \mathfrak{g}^*.$$

In (7.3) the following notations are used (see [BS 99]). For a function $f : G \mapsto \mathbb{R}$, its left and right Lie derivatives, $\mathrm{d}f(g) : G \mapsto \mathfrak{g}^*$ and $\mathrm{d}' f(g) : G \mapsto \mathfrak{g}^*$, are defined via

$$\langle \mathrm{d}f(g), \eta \rangle = \frac{\mathrm{d}}{\mathrm{d}\epsilon} f(\mathrm{e}^{\epsilon\eta} g)|_{\epsilon=0}, \quad \forall\, \eta \in \mathfrak{g},$$

$$\langle \mathrm{d}' f(g), \eta \rangle = \frac{\mathrm{d}}{\mathrm{d}\epsilon} f(g\mathrm{e}^{\epsilon\eta})|_{\epsilon=0}, \quad \forall\, \eta \in \mathfrak{g}.$$

Then, the gradient $\nabla f : G \mapsto T^*G$ is related to the above derivatives via

$$\nabla f(g) = R^*_{g^{-1}} \mathrm{d}f(g) = L^*_{g^{-1}} \mathrm{d}' f(g).$$

Notice that in the Lie–Poisson case, when the symmetry group is G itself, the reduced space is simply the group G represented by $w_k = g_{k+1} g_k^{-1}$, and equations (7.3) become

$$\mathrm{Ad}^*_{w_k} m_{k+1} = m_k \tag{7.4}$$

with

$$m_k = \mathrm{d}\Lambda^{(r)}(w_{k-1}) \in \mathfrak{g}^*. \tag{7.5}$$

Comparing the above notations with the results in our paper, we immediately see that w_k corresponds to the other choice for the quotient map (2.3) $\pi : (g_k, g_{k+1}) \mapsto f_{kk+1} \equiv g_k g_{k+1}^{-1}$, i.e. $f_{kk+1} = w_k^{-1}$. Similarly, the reduced Lagrangian $\Lambda^{(r)}(w_k)$ corresponds to $\ell(f_{kk+1})$ in our notations. Finally, using the definitions of the Lie derivatives above we obtain for the angular momentum (7.5)

$$m_k = \mathrm{d}\Lambda^{(r)}(w_{k-1}) = R^*_{w_{k-1}} \nabla \Lambda^{(r)}(w_{k-1}) = R^*_{f^{-1}_{k-1k}} \ell'(f_{k-1k}),$$

where we have substituted our notations. Hence, (7.4) can be written as

$$\mathrm{Ad}^*_{f^{-1}_{kk+1}} R^*_{f^{-1}_{kk+1}} \ell'(f_{kk+1}) = R^*_{f^{-1}_{k-1k}} \ell'(f_{k-1k}).$$

The last expression is precisely the DEP algorithm (2.4) after rewriting it with the adjoints of the above operators acting on the variation $\vartheta_k = \delta g_k g_k$ (see section 2). It is interesting to note that the second equation in (7.2) corresponds to our reconstruction equation (2.8). Similar correspondence can be established for the case of left trivialization considered in [BS 99].

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