A VARIATIONAL APPROACH TO NAVIER-STOKES

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Abstract. We present a variational resolution of the incompressible Navier-Stokes system by means of stabilized Weighted-Inertia-Dissipation-Energy (WIDE) functionals. The minimization of these parameter-dependent functionals corresponds to an elliptic-in-time regularization of the system. By passing to the limit in the regularization parameter along subsequences of WIDE minimizers one recovers a classical Leray-Hopf weak solution.

Contents

1. Introduction

This note is concerned with the Navier-Stokes system

$$\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = 0, \quad \operatorname{div} u = 0, \tag{1.1}$$

describing the flow velocity $u : \Omega \times (0, \infty) \to \mathbb{R}^3$ of an incompressible viscous fluid in a container $\Omega \subset \mathbb{R}^3$. Note that we use the classical notation $u \cdot \nabla u = u_j\, \partial_{x_j} u$ (sum over repeated indices). We advance a variational approach to the existence of classical *Leray-Hopf* weak solutions [48] to system (1.1) by minimizing the functionals I^ε on entire trajectories

$$I^\varepsilon(u) = \int_0^\infty \int_\Omega \mathrm{e}^{-t/\varepsilon} \left\{ \frac{1}{2} |\partial_t u + u \cdot \nabla u|^2 + \frac{\sigma}{2} |u \cdot \nabla u|^2 + \frac{\nu}{2\varepsilon} |\nabla u|^2 \right\} dx\, dt \tag{1.2}$$

under the incompressibility constraint $\operatorname{div} u = 0$ and for given initial and boundary conditions. Here, $\varepsilon > 0$ is a small parameter, eventually tending to 0.

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The relation between the minimization of I^ε and the Navier-Stokes system (1.1) is revealed by formally computing the Euler-Lagrange equation for I^ε at a critical point u^ε, namely

$$\begin{aligned} 0 &= \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \nu \Delta u^\varepsilon + \nabla p \\ &\quad + \varepsilon\big(- \partial_t(\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon) - \mathrm{div}((\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon) \otimes u^\varepsilon) + (\nabla u^\varepsilon)^\top (\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon)\big) \\ &\quad + \varepsilon\sigma\big(- \mathrm{div}((u^\varepsilon \cdot \nabla u^\varepsilon) \otimes u^\varepsilon) + (\nabla u^\varepsilon)^\top (u^\varepsilon \cdot \nabla u^\varepsilon)\big). \end{aligned} \tag{1.3}$$

Note that the first line above is nothing but the first equation in (1.1). The term ∇p is the Lagrangian multiplier corresponding to the constraint $\mathrm{div}\, u = 0$. The second and third lines in (1.3) feature terms premultiplied by the small parameter ε. By taking $\varepsilon \to 0$ in the Euler-Lagrange equations (1.3) one formally recovers a solution of (1.1). This paper is devoted to make this program rigorous. Our aim is to prove that

> The functional I^ε admits minimizers u^ε for all $\varepsilon > 0$ (Proposition 4.1) and, up to subsequences, such minimizers converge to a Leray-Hopf solution of the Navier-Stokes system as $\varepsilon \to 0$ (Theorem 3.2).

The ε-dependent part of the Euler-Lagrange equations (1.3) features the term $-\varepsilon\, \partial_t^2 u^\varepsilon$ as well. The minimization of I^ε hence corresponds to performing an *elliptic regularization* in time of the Navier-Stokes system (1.1). In particular, the minimizers u^ε of I^ε are more regular than Leray-Hopf solutions.

Elliptic regularizations of evolution problems have been introduced by Lions [28] and then used by Kohn & Nirenberg [23] and Oleinĭk [37] in order to discuss regularity issues. The reader is referred to the book by Lions & Magenes [31] for an account of results in the linear setting. Nonlinear systems, yet under stronger growth assumptions on the viscosity, including an application to Navier-Stokes are investigated by in Lions in [29, 30]. By way of contrast to our set-up, for the elliptic regularizations in these contributions no variational structure is available.

The novelty of our contribution is that of directly moving from a global-in-time variational perspective. The ε-dependent functionals I^ε correspond to the *Weighted Inertia-Dissipation-Energy* (WIDE) functionals for viscous-fluid flows. They are obtained as the weighted sum of inertia, dissipation, and energy of the fluid. We present an account of this derivation in Section 2 below. Let us however stress that, in addition to the above-mentioned classical terms, we include in the analysis a σ stabilization term. This is instrumental in proving a-priori estimates and, as it is apparent from inspecting the Euler-Lagrange equations (1.3), has no influence on the limit system. We note that stabilization is a standard tool in numerical methods for compressible and incompressible flows (cf., e.g., [21] for a review). The specific regularization employed in this work is referred to as 'streamline-upwind/Petrov-Galerkin' (SUPG) regularization in the numerical literature and was first introduced in [8].

The study of WIDE functionals started from the case of gradient flows (no inertia, quadratic dissipation) and has to be traced back at least to Ilmanen [22], who used a global-in-time variational method to tackle existence and partial regularity of the Brakke mean-curvature flow of varifolds. An application to existence of periodic solutions for gradient flows is given by Hirano in [20]. The WIDE approach is even mentioned in the classical textbook by Evans [12, Problem 3, p. 487]. Two examples of relaxation related with micro-structure evolution have been provided in [9] and the case of mean-curvature evolution of Cartesian surfaces is in [46]. The analysis of the WIDE approach for abstract gradient flows for λ-convex and nonconvex energies is in [36, 5] in the Hilbertian case and in [42, 41] in the metric case. Melchionna

[32] extended the theory to classes of nonpotential perturbations and Bögelein, Duzaar, & Marcellini [7] recently used this variational approach to prove the existence of variational solutions to the equation $u_t - \nabla \cdot f(x, u, \nabla u) + \partial_u f(x, u, \nabla u) = 0$ where the field f is convex in $(u, \nabla u)$.

Doubly nonlinear parabolic evolution equations can be tackled by the WIDE variational formalism as well. Rate-independent processes are discussed in [34], see also the subsequent [35], and an application to crack-front propagation in brittle materials is in [24]. The rate-dependent case has been analyzed in [1, 2, 3, 4]. See also Liero & Melchionna [25] for a stability result via Γ-convergence [10] and Melchionna [33] for an application to the study of symmetries of solutions.

In the dynamic case, De Giorgi conjectured in [11] that the WIDE functional procedure could be implemented in the setting of semilinear waves. This has been ascertained in [47] (for the finite-time case) and in Serra & Tilli [44] (for the infinite-time case). The possibility of following this same variational approach in other hyperbolic situations has also been pointed out in [11]. Indeed, extensions to mixed hyperbolic-parabolic semilinear equations [27], to different classes of nonlinear energies [26, 45], and to nonhomogeneous equations [49] are also available. Here, we further develop De Giorgi's approach by showing its applicability to fluids. When formulated in terms of the fluid motion $\varphi : \Omega \times [0, \infty) \to \Omega$, fluid dynamics is hyperbolic. The corresponding WIDE functional (2.6) is derived in Section 2. Such functional is then recast in form of I^ε by changing variables from the Lagrangian motion φ to the Eulerian velocity field $u = \dot{\varphi} \circ \varphi^{-1}$.

The literature on the Navier-Stokes system is huge and we shall not attempt to review it here. The reader is referred to [40, 43, 48] for a collection of results. On the other hand, global-in-time variational approaches to (1.1) are just a few. A complete variational resolution has been provided by Ghoussoub within the theory of self-dual Lagrangians [17]. In particular, in [16, 18] the Navier-Stokes system is reformulated in terms of a so-called null-minimization principle, namely the attainment the value 0 of a specific nonnegative functional inspired by Fenchel's duality. Such attainment is then ascertained in [16, 18], giving rise to a complete existence theory for periodic-in-time solutions.

In [38] Pedregal characterizes weak solutions as those u such that $E(v) = 0$ for a given error functional E computed on the function $v = Tu$ where the mapping T is defined by minimizing a second functional parametrized by u. This approach provides an existence theory in 2D only.

Solutions of the Navier-Stokes system are related by Gomes [14, 15] to critical points of a stochastic control problem on the group of area-preserving diffeomorphisms, see also [6, 50] for some related discussion. Note however that the analysis in [14, 15] assumes sufficient smoothness. In particular, it does not directly originate an existence theory.

Moving from the transport-diffusion scheme by Pironneau [39], Gigli & Mosconi [13] present a variational time discretization based on minimizing movements and prove convergence as the fineness of the time-partition goes to 0. A saddle-point formulation in space-time is investigated from the numerical viewpoint by Guberovic, Schwab, & Stevenson [19].

2. Weighted Inertia-Dissipation-Energy functionals for viscous fluids

We devote this section to discuss the Weighted Inertia-Dissipation-Energy (WIDE) variational approach to general rate problems as well as its application to the motion of incompressible viscous fluids. This brings to a justification of the particular form of the functional I^ε.

Many physical systems are described by models in rate form: The state of the system is described by a trajectory $\varphi : [0,\infty) \to H$ in the Hilbert space H solving the nonlinear differential inclusion

$$0 \in \rho\ddot{\varphi}(t) + \partial_{\dot{\varphi}}\Psi(\varphi(t),\dot{\varphi}(t)) + \partial_{\varphi}E(t,\varphi(t)), \tag{2.1a}$$

$$\varphi(0) = \varphi_0, \quad \dot{\varphi}(0) = \varphi^1. \tag{2.1b}$$

Here, dots represent time derivation, $\rho > 0$, $\Psi : H \times H \to [0,\infty]$ is the dissipation potential which is convex in the second argument, and $E : [0,\infty)\times H \to \mathbb{R}\cup\{\infty\}$ is the time-dependent energy function. Relation (2.1a) results from the balance between the system of inertial forces $\rho\ddot{\varphi}$, the system of dissipative forces $\partial_{\dot{\varphi}}\Psi$, and that of conservative forces $\partial_{\varphi}E$, where ∂ stands for the partial (sub)differential. The trajectory $t \mapsto \varphi(t)$ of the system is the result of this balance and of the initial conditions (2.1b).

A time discretization of Problem (2.1) is given by

$$\inf_{\varphi_{n+1}\in H} F_{n+1}(\varphi_{n+1};\varphi_n), \quad n = 1, 2, \dots, \tag{2.2}$$

for $\varphi_n = \varphi(t_n)$, where

$$\begin{aligned} F_{n+1}(\varphi_{n+1};\varphi_n) = \tau\frac{\rho}{2}&\left\|\frac{\varphi_{n+1} - 2\varphi_n + \varphi_{n-1}}{\tau^2}\right\|_H^2 \\ &+ \Psi\left(\varphi_n, \frac{\varphi_{n+1}-\varphi_n}{\tau}\right) - \Psi\left(\varphi_{n-1}, \frac{\varphi_n - \varphi_{n-1}}{\tau}\right) \\ &+ \frac{E(t_{n+1},\varphi_{n+1}) - 2E(t_n,\varphi_n) + E(t_{n-1},\varphi_{n-1})}{\tau} \end{aligned}$$

and $\tau > 0$ is the time step. We verify that, indeed, the Euler-Lagrange equations of this problem are

$$0 \in \rho\,\frac{\varphi_{n+1} - 2\varphi_n + \varphi_{n-1}}{\tau^2} + \partial_{\dot{\varphi}}\Psi\left(\varphi_n, \frac{\varphi_{n+1}-\varphi_n}{\tau}\right) + \partial_{\varphi}E(t_{n+1},\varphi_{n+1})$$

which may be regarded as a central-difference/backward-Euler time discretization of (2.1a). The causal nature of Problem (2.1) is reflected in the fact that the minimizations (2.2) are solved *sequentially*: problem $n = 1$ is solved first with initial conditions φ_0 and $\varphi_1 = \varphi_0 + \tau\,\varphi^1$ in order to compute φ_2; subsequently, problem $n = 2$ is solved to compute φ_3, taking the solution φ_2 of the preceding problem and φ_1 as initial conditions; and so on. Alternatively, by following [34] we proceed to formulate a single minimum problem for the entire trajectory $\varphi = \{\varphi_2, \varphi_3, \dots\}$ by *stringing together* all the incremental problems (2.2) with Pareto weights $\mathrm{e}^{-t_n/\varepsilon}$, where $\varepsilon > 0$ is a small parameter. The resulting functional is

$$\begin{aligned} F^{\varepsilon}(\{\varphi_2,\varphi_3,\dots\};\tau) = \sum_{n=1}^{\infty}\mathrm{e}^{-t_{n+1}/\varepsilon}&\left\{\frac{\rho}{2}\left\|\frac{\varphi_{n+1} - 2\varphi_n + \varphi_{n-1}}{\tau^2}\right\|_H^2\right. \\ &+ \frac{1}{\tau}\left[\Psi\left(\varphi_n, \frac{\varphi_{n+1}-\varphi_n}{\tau}\right) - \Psi\left(\varphi_{n-1}, \frac{\varphi_n-\varphi_{n-1}}{\tau}\right)\right] \\ &\left.+ \frac{E(t_{n+1},\varphi_{n+1}) - 2E(t_n,\varphi_n) + E(t_{n-1},\varphi_{n-1})}{\tau^2}\right\}\tau. \end{aligned} \tag{2.3}$$

In the *causal limit* of $\varepsilon \to 0$, the exponential weights accord disproportionately larger importance to the first incremental problem relative to the second; to the second incremental problem relative to the third, and so on, as required by *causality*. The functional (2.3) may formally be regarded as a time discretization of the continuous-time functional

$$F^\varepsilon(\varphi) = \int_0^\infty \mathrm{e}^{-t/\varepsilon} \left\{ \frac{\rho}{2}\|\ddot\varphi(t)\|_H^2 + \frac{d}{dt}\Psi\big(\varphi(t),\dot\varphi(t)\big) + \frac{d^2}{dt^2}E\big(t,\varphi(t)\big) \right\} dt$$

and integration by parts gives

$$\begin{aligned} F^\varepsilon(\varphi) = \int_0^\infty \mathrm{e}^{-t/\varepsilon} &\left\{ \frac{\rho}{2}\|\ddot\varphi(t)\|_H^2 + \frac{1}{\varepsilon}\Psi\big(\varphi(t),\dot\varphi(t)\big) + \frac{1}{\varepsilon^2}E\big(t,\varphi(t)\big) \right\} dt \\ &+ \Big[\mathrm{e}^{-t/\varepsilon}\Big(\Psi\big(\varphi(t),\dot\varphi(t)\big) + \dot E\big(t,\varphi(t)\big)\Big)\Big]_0^\infty + \frac{1}{\varepsilon}\Big[\mathrm{e}^{-t/\varepsilon}E\big(t,\varphi(t)\big)\Big]_0^\infty . \end{aligned}$$

Assume that the terms $\mathrm{e}^{-t/\varepsilon}(\Psi(\varphi(t),\dot\varphi(t)) + \dot E(t,\varphi(t)))$ and $\mathrm{e}^{-t/\varepsilon}E(t,\varphi(t))$ vanish at $t=\infty$ and drop them at $t=0$, for these just depend on the fixed initial conditions and do not affect the minimization. This leads to the general WIDE functional

$$F^\varepsilon(\varphi) = \int_0^\infty \mathrm{e}^{-t/\varepsilon} \left\{ \frac{\rho}{2}\|\ddot\varphi(t)\|_H^2 + \frac{1}{\varepsilon}\Psi\big(\varphi(t),\dot\varphi(t)\big) + \frac{1}{\varepsilon^2}E\big(t,\varphi(t)\big) \right\} dt. \tag{2.4}$$

Assuming sufficient differentiability, the Euler-Lagrange equation of F^ε is

$$\frac{d^2}{dt^2}\Big(\mathrm{e}^{-t/\varepsilon}\rho\ddot\varphi\Big) + \mathrm{e}^{-t/\varepsilon}\frac{1}{\varepsilon}\partial_\varphi\Psi(\varphi,\dot\varphi) - \frac{d}{dt}\left(\mathrm{e}^{-t/\varepsilon}\frac{1}{\varepsilon}\partial_{\dot\varphi}\Psi(\varphi,\dot\varphi)\right) + \mathrm{e}^{-t/\varepsilon}\frac{1}{\varepsilon^2}\partial_\varphi E(t,\varphi) \ni 0$$

which, upon simplification, reduces to

$$\begin{aligned} 0 \in\ & \rho\ddot\varphi + \partial_{\dot\varphi}\Psi(\varphi,\dot\varphi) + \partial_\varphi E(t,\varphi) \\ & - \varepsilon\big[2\rho\dddot\varphi - \partial_\varphi\Psi(\varphi,\dot\varphi) + \partial_\varphi\partial_{\dot\varphi}\Psi(\varphi,\dot\varphi)\dot\varphi + \partial^2_{\dot\varphi}\Psi(\varphi,\dot\varphi)\ddot\varphi\big] + \varepsilon^2\rho\ddddot\varphi . \end{aligned}$$

The original problem (2.1a) is formally recovered in the limit of $\varepsilon\to 0$ as the terms in the second line drop. Note that the term $\varepsilon^2\rho\ddddot\varphi$ qualifies the latter as an elliptic regularization of relation (2.1a).

As mentioned in the Introduction, the above described variational procedure has been conjectured by De Giorgi [11] to be amenable for general functionals of the calculus of variations of the form (2.4) (actually $\Psi=0$ is assumed in [11]). Our results confirm this possibility in the specific case of incompressible fluids. Let $\varphi:\Omega\times[0,\infty)\to\Omega\subset\mathbb{R}^3$ be the motion of a Newtonian viscous fluid in a container Ω. For simplicity, we suppose that the fluid is free of body forces (these can easily be added) and the deformation mapping φ vanishes at the boundary $\partial\Omega$ of the container (no slip boundary conditions). Let $\rho:\Omega\to(0,\infty)$ denote the density of the fluid at time $t=0$ and assume Newtonian viscosity, i. e., that the viscous part σ^v of the Cauchy stress tensor, and the rate of deformation tensor

$$d = \mathrm{sym}(\dot F F^{-1}) = \frac{1}{2}\big((\dot F F^{-1}) + (\dot F F^{-1})^\top\big)$$

obey the relation

$$\sigma^v = \lambda\,\mathrm{tr}(d) I + 2\mu\,\mathrm{dev}(d) \tag{2.5}$$

where $F=\nabla\varphi$ is the deformation gradient, $\mathrm{dev}(d) = d - \mathrm{tr}(d)I/3$ is the deviatoric part of d, λ and μ are the bulk and the shear viscosity parameters, and I is the identity 2-tensor. The viscosity law (2.5) may be expressed explicitly as a function of $\dot F$ and F in the form

$$\sigma^v(F,\dot F) = \lambda\mathrm{tr}(\dot F F^{-1})I + \mu\,\mathrm{dev}(\dot F F^{-1} + F^{-\top}\dot F^\top).$$

The corresponding viscous part of the first Piola-Kirchhoff stress tensor is

$$P^v(F,\dot F) = J\sigma^v F^{-\top},$$

where $J = \det(F)$. The total stress is then,

$$P(F, \dot{F}) = -\pi I + P^v(F, \dot{F})$$

where π is the hydrostatic pressure of the fluid. A simple calculation reveals that the Newtonian viscosity law possesses the potential structure

$$P^v = \partial_{\dot{F}} \psi(F, \dot{F}),$$

where the viscous potential per unit undeformed volume is

$$\psi(F, \dot{F}) = J \left\{ \frac{\lambda}{2} \mathrm{tr}(d)^2 + \mu \, |\mathrm{dev}(d)|^2 \right\}$$

For later reference we also introduce the viscous potential per unit deformed volume as

$$\psi(d) = J^{-1} \psi(F, \dot{F}) = \frac{\lambda}{2} \mathrm{tr}(d)^2 + \mu \, |\mathrm{dev}(d)|^2,$$

which has the property that

$$\sigma^v = \partial \psi(d).$$

The viscous potential reads hence

$$\Psi(\varphi, \dot{\varphi}) = \int_\Omega \psi(\nabla \varphi, \nabla \dot{\varphi}) \, dx.$$

On the other hand, the energy is given by

$$E(\varphi) = \int_\Omega e(J) \, dx$$

where $e(J)$ is the internal energy density as a function of specific volume.

Let us now restrict our attention to incompressible fluids. In this case, the internal energy reduces to

$$e(J) = \begin{cases} 0 & \text{if } J = 1 \text{ a.e. in } \Omega, \\ \infty & \text{otherwise} \end{cases}$$

and we get $\mathrm{tr}(d) = 0$ in (2.5), and let $\rho = \rho_0$ constant and $\nu = \mu/\rho_0$ be the kinematic viscosity coefficient. By using the Eulerian velocity field $v = \dot{\varphi} \circ \varphi^{-1}$ we can transform the representation of the motion from Lagrangian to Eulerian. The incompressibility constraint $J = 1$ along the flow corresponds via Jacobi identity to $\mathrm{div}\, v = 0$ and we can rewrite the viscous dissipation potential as

$$\int_\Omega \psi(d) \, dx = \mu \int_\Omega |\mathrm{dev}(\mathrm{sym} \nabla v)|^2 \, dx = \mu \int_\Omega |\mathrm{sym} \nabla v|^2 \, dx.$$

Hence, the WIDE functional

$$F^\varepsilon(\varphi) = \int_0^\infty \mathrm{e}^{-t/\varepsilon} \int_\Omega \left\{ \frac{\rho}{2} |\ddot{\varphi}|^2 + \frac{1}{\varepsilon} \psi(\nabla \varphi, \nabla \dot{\varphi}) + \frac{1}{\varepsilon^2} e(J) \right\} \, dx \, dt \tag{2.6}$$

can be rewritten as

$$\frac{1}{\rho_0} F^\varepsilon(\varphi) \equiv I_0^\varepsilon(v) := \begin{cases} \displaystyle\int_0^\infty \mathrm{e}^{-t/\varepsilon} \int_\Omega \left\{ \frac{1}{2} |\partial_t v + v \cdot \nabla v|^2 + \frac{\nu}{\varepsilon} |\mathrm{sym} \nabla v|^2 \right\} \, dx \, dt \\ \qquad \text{if } \ \mathrm{div}\, v = 0 \ \text{ a.e. in } \ \Omega \times (0, \infty), \\ \infty \qquad \text{otherwise.} \end{cases}$$

Assuming sufficient differentiability, the corresponding Euler-Lagrange equations read as

$$
\begin{aligned}
&-\partial_t\left(\mathrm{e}^{-t/\varepsilon}\rho_0(\partial_t v_i + v_j\,\partial_{x_j} v_i)\right) + \mathrm{e}^{-t/\varepsilon}\rho_0(\partial_t v_k + v_j\,\partial_{x_j} v_k)\partial_{x_i} v_k\\
&\quad -\partial_{x_k}\left(\mathrm{e}^{-t/\varepsilon}\rho_0(\partial_t v_i + v_j\,\partial_{x_j} v_i)v_k\right) - \frac{1}{\varepsilon}\mathrm{e}^{-t/\varepsilon}(\mu\Delta v_i - \partial_{x_i}\pi) = 0
\end{aligned}
$$

(repeated indices to be summed) for $i = 1, \ldots, n$, which, upon simplification, reduces to

$$
\begin{aligned}
0 =&(\partial_t v_i + v_j\,\partial_{x_j} v_i) - \nu\Delta v_i + \partial_{x_i} p\\
&+\varepsilon\left[-\partial_t\left((\partial_t v_i + v_j\,\partial_{x_j} v_i)\right) + (\partial_t v_k + v_j\,\partial_{x_j} v_k)\,\partial_{x_i} v_k - \partial_{x_k}\left((\partial_t v_i + v_j\,\partial_{x_j} v_i)v_k\right)\right]
\end{aligned}
$$

where $p = \pi/\rho_0$ is the pressure per unit mass. The incompressible Navier-Stokes system (1.1) can hence be formally recovered in the causal limit of $\varepsilon \to 0$.

The analysis of the functional I_0^ε is quite challenging, for the nonlinear term $\partial_t v + v\cdot\nabla v$ couples the time derivative $\partial_t v$ and the nonlinear convection term $v\cdot\nabla v$ and a separate control of these two terms seems not aivailable. We hence resort in *stabilizing* I_0^ε by augmenting it with the additional term

$$
\int_\Omega \frac{\sigma}{2}|v\cdot\nabla v|^2\,dx. \tag{2.7}
$$

Up to the addition of such stabilization term, the functional I^ε from (1.2) and the WIDE functional I_0^ε coincide (note that $|\mathrm{sym}\nabla v|^2 = 2|\nabla v|^2$ on divergence-free fields). The inclusion of the σ stabilizing term does not affect the limit $\varepsilon \to 0$ of the Euler-Lagrange equations. Indeed, any additional term of order one in ε which does not contain a time derivative will not appear to leading order in the Euler-Lagrange equations. In particular, such a term will vanish in the causal limit $\varepsilon \to 0$. Yet, as we will see, such stabilization allows for significantly improved a-priori estimates.

The specific form (2.7) for a stabilization term is not the only possibility. As will become clear from our analysis (cf. Proposition 6.1), other choices would lead to the same a-priori estimates and, eventually, to the same convergence result.

3. Statement of the main result

Let us start by setting up the functional frame. Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with Lipschitz boundary $\partial\Omega$. In the following, we indicate with $\|\cdot\|_E$ the norm of the Banach space E and by $\|\cdot\|$ the norm of a square integrable function, regardless of the number of its components. Let $\mathcal{V} := \{v \in C_c^\infty(\Omega;\mathbb{R}^3) : \mathrm{div}\, v = 0\}$ be given and introduce the Hilbert spaces

$$
\begin{aligned}
V_s &:= \text{the closure of } \mathcal{V} \text{ in } H^s(\Omega;\mathbb{R}^3)\cap H_0^1(\Omega;\mathbb{R}^3),\\
H &:= \text{the closure of } \mathcal{V} \text{ in } L^2(\Omega;\mathbb{R}^3),
\end{aligned}
$$

for $s \geq 1$. If $s = 1$ we simply write $V = V_s$. Identifying H with its dual H' we have the dense and continuous inclusions

$$
V_s \hookrightarrow V \hookrightarrow H \equiv H' \hookrightarrow V' \hookrightarrow V_s'.
$$

Suppressing s, the dual pairing between V_s' and V_s is denoted by $\langle\cdot,\cdot\rangle$: $V_s'\times V_s \to \mathbb{R}$. We denote by $A : V \to V'$ ($\hookrightarrow V_s'$, $s \geq 1$) the *Stokes* operator given by

$$
\langle Av, \psi\rangle = \int_\Omega \nabla v : \nabla\psi\,dx \qquad \forall\, v, \psi \in V, \tag{3.1}
$$

which satisfies

$$
\|Av\|_{V'} \leq C\|\nabla v\| \qquad \forall\, v \in V. \tag{3.2}
$$

Let us now define the quadratic functional $B : V \to V'$ as

$$\langle B(v), \psi \rangle := \int_\Omega [v \cdot \nabla v] \cdot \psi \, dx = - \int_\Omega (v \otimes v) : \nabla \psi \, dx \qquad \forall \, v, \, \psi \in V. \tag{3.3}$$

Note that $[v \cdot \nabla v] \cdot \psi \in L^{6/5}(\Omega)$ for all $v, \, \psi \in V$ and that we have

$$\|B(v)\|_{V'} \le C \| |v|^2 \| \le C \|v\|_V^2 \qquad \forall \, v \in V \tag{3.4}$$

and, for all $s > 3/2$

$$\|B(v)\|_{V_s'} \le C \|v \cdot \nabla v\|_{L^1(\Omega;\mathbb{R}^3)} \le C \|v\|_V^2 \qquad \forall \, v \in V_s \tag{3.5}$$

where we have used that $V_s \hookrightarrow L^\infty(\Omega;\mathbb{R}^3)$.

For a given initial condition $u_0 \in H$ we choose a sequence $u_0^\varepsilon \in V$ with

$$u_0^\varepsilon \to u_0 \text{ in } H \qquad \text{and} \qquad \|\nabla u_0^\varepsilon\|^2 + \varepsilon \|u_0^\varepsilon \cdot \nabla u_0^\varepsilon\|^2 \le C_0 \varepsilon^{-1} \tag{3.6}$$

for some constant $C_0 > 0$ and define the admissible set of trajectories as

$$U^\varepsilon := \{ u \in L^2_{\rm loc}(0,\infty;V) : \partial_t u + u \cdot \nabla u, \ \sigma \, v \cdot \nabla u \in L^2_{\rm loc}(0,\infty; L^2(\Omega;\mathbb{R}^3)), u(0) = u_0^\varepsilon \}.$$

Note that, for all $u \in U^\varepsilon$, from $u \in L^2_{\rm loc}(0,\infty;V)$ one has that $u \cdot \nabla u \in L^1_{\rm loc}(0,\infty;L^{3/2}(\Omega;\mathbb{R}^3))$. Hence $\partial_t u \in L^1_{\rm loc}(0,\infty;L^{3/2}(\Omega;\mathbb{R}^3))$ as well, $t \mapsto u(t)$ is continuous in $L^{3/2}(\Omega;\mathbb{R}^3)$, and the initial condition $u(0) = u_0^\varepsilon$ in the definition of U^ε makes sense.

Definition 3.1 (Leray-Hopf solutions)**.** We say that $u \in L^2(0,\infty;V)$ is a *Leray-Hopf solution* of the Navier-Stokes system (1.1) if $\partial_t u \in L^1(0,\infty;V')$ and

$$\partial_t u + B(u) + \nu A u = 0 \quad \text{in} \quad V', \quad \text{a.e. in } (0,T), \tag{3.7}$$

$$u(0) = u_0 \quad \text{in} \quad H. \tag{3.8}$$

Existence of Leray-Hopf solutions is rather classical [48, Thm. III.3.1]. These are actually weakly continuous from $[0,T]$ to H [48, Thm. III.3.1] so that the initial condition (3.8) makes sense. Moreover, the energy inequality (3.9) can be guaranteed to hold true. In three dimensions they belong to $L^r_{\rm loc}(0,\infty;L^s(\Omega;\mathbb{R}^3))$ if $2/r + 3/s = 3/2$ and $2 \le s \le 6$ [40, Lemma 3.5], with $\partial_t u \in L^{4/3}_{\rm loc}(0,\infty;V') \cap L^2(0,\infty;V'_{3/2})$, $B(u) \in L^{4/3}_{\rm loc}(0,\infty;V')$ [48, Thm. III.3.3], and $u \in L^r_{\rm loc}(\delta,\infty;W^{2,s}(\Omega;\mathbb{R}^3)) \cap W^{1,r}_{\rm loc}(\delta,\infty;L^s(\Omega;\mathbb{R}^3))$ for any $\delta > 0$ if $2/r + 3/s = 4$ [43, Thm. V.6.11]. Note that the pressure p plays the role of the Lagrange multiplier corresponding to the incompressibility constraint $u(t) \in V$ and as such does not show up in (3.7). Equivalently, one could reformulate the problem in the pair (u,p), see [48, Rem. I.1.4 and Eq. (III.3.129)]. Leray-Hopf solutions are unique in two dimensions [48, Thm. III.3.2]. In three dimensions, if $u \in L^r_{\rm loc}(0,\infty;L^s(\Omega;\mathbb{R}^3))$ with $2/r + 3/s = 1$ and $s > 3$ and $u_0 \in V$, then u is unique among Leray-Hopf solutions satisfying the energy inequality [40, Thm. 8.19].

For $\sigma \ge 0$ and $\nu > 0$ we consider the WIDE functionals $I^\varepsilon : U^\varepsilon \to [0,\infty)$

$$I^\varepsilon(u) = \int_0^\infty {\rm e}^{-t/\varepsilon} \left\{ \frac{1}{2} \|\partial_t u + u \cdot \nabla u\|^2 + \frac{\sigma}{2} \|u \cdot \nabla u\|^2 + \frac{\nu}{2\varepsilon} \|\nabla u\|^2 \right\} dt,$$

taking values in $[0,\infty]$. (Note that any divergence-free Sobolev functions on which I^ε is finite necessarily belongs to U^ε.)

In Proposition 4.1 below we will see that I^ε indeed admits a minimizer in U^ε for all $\sigma \ge 0$. For $\sigma > 0$ we compute the corresponding Euler-Lagrange equations in Lemma 5.1 below. By assuming further that $\sigma > 1/8$, one can check the validity of the a-priori bounds of Proposition 6.1. These are instrumental for passing to the limit for $\varepsilon \to 0$. Our main result is the convergence of minimizers up to subsequences to Leray-Hopf solutions.

Theorem 3.2 (Variational approach to Navier-Stokes)**.** *Let $\sigma > 1/8$ and $u^\varepsilon \in U^\varepsilon$ be a minimizer of I^ε. Then there exists a subsequence (not relabeled) such that*

$$u^\varepsilon \rightharpoonup u \quad \text{in } L^2(0,\infty;V), \qquad \partial_t u^\varepsilon \rightharpoonup \partial_t u \quad \text{in } L^2(0,\infty;V_s'),$$

$s > 5/2$, for some $u \in L^2(0,\infty;V) \cap L^\infty(0,\infty;H)$ with $u(0) = u_0$ where u is a Leray-Hopf solution of the Navier-Stokes system (1.1)*. Moreover, u satisfies the energy inequality*

$$\|u(T)\|^2 + 2\nu \int_0^T \|\nabla u(t)\|^2\,dt \le \|u_0\|^2 \quad \text{for a.e. } T > 0. \tag{3.9}$$

The proof of the statement relies on a-priori estimates on the Euler-Lagrange equations of I^ε. After deriving the Euler-Lagrange equations in Section 5, we prove the a-priori bounds in Section 6. Eventually, the proof of Theorem 3.2 is in Section 7. An interesting open problem raised by our analysis is, if convergence to solutions of the Navier-Stokes system can also be obtained in the unstabilized case $\sigma = 0$.

Let us now comment on some possible variants of the statement. First of all, the convergence holds more generally for critical points u^ε of I^ε, as long as $I^\varepsilon(u^\varepsilon) \le C\varepsilon^{-1}$. In case $u_0 \in V$ with $u_0 \cdot \nabla u_0 \in H$ we may choose $u_0^\varepsilon = u_0$ and no approximation as in (3.6) is needed. For general u_0 and $\sigma > 0$ this is not possible since $u \in U^\varepsilon$ implies that $u \in L^2(0,1;V)$ and $\partial_t u \in L^2(0,1;H)$ and hence $u(0) \in H^{1/2}(\Omega;\mathbb{R}^3)$. The occurrence of external forces can be considered as well.

A caveat on notation: In the following, we use the same symbol C for a positive constant, possibly depending on σ, ν, and C_0 but independent of ε and T. We warn the reader that the value of C may change from line to line.

4. Existence of minimizers

We start by checking that I^ε admits minimizers on U^ε for all $\sigma \ge 0$ and $\varepsilon > 0$, which are assumed to be fixed throughout this section.

Proposition 4.1 (Existence of minimizers)**.** *Let $\sigma \ge 0$. Then, there exists a minimizer u^ε of I^ε in U^ε and $I(u^\varepsilon) \le C\varepsilon^{-1}$.*

Proof. The set U^ε in not empty, as the constant-in-time trajectory $u \equiv u_0^\varepsilon$ belongs to it, see assumption (3.6). In this case one has that

$$\begin{aligned} I^\varepsilon(u) &= \int_0^\infty \mathrm{e}^{-t/\varepsilon} \Big\{ \frac{1+\sigma}{2} \|u_0^\varepsilon \cdot \nabla u_0^\varepsilon\|^2 + \frac{\nu}{2\varepsilon} \|\nabla u_0^\varepsilon\|^2 \Big\}\,dt \\ &\le \frac{(1+\sigma)\varepsilon}{2} \|u_0^\varepsilon \cdot \nabla u_0^\varepsilon\|^2 + \frac{\nu}{2}\|\nabla u_0^\varepsilon\|^2 \le C\varepsilon^{-1}. \end{aligned}$$

In particular, $\inf I^\varepsilon \le C\varepsilon^{-1}$. Assume now $u_k \in U^\varepsilon$ to be a minimizing sequence for I^ε. By letting $v_k = \partial_t u_k + u_k \cdot \nabla u_k$ we have that $\mathrm{e}^{-t/2\varepsilon} v_k$ are bounded in $L^2(0,\infty;L^2(\Omega;\mathbb{R}^3))$ and $\mathrm{e}^{-t/2\varepsilon} u_k$ are bounded in $L^2(0,\infty;V)$. One can hence extract not relabeled subsequences such that $\mathrm{e}^{-t/2\varepsilon} v_k \rightharpoonup \mathrm{e}^{-t/2\varepsilon} v$ in $L^2(0,\infty;L^2(\Omega;\mathbb{R}^3))$ and $\mathrm{e}^{-t/2\varepsilon} u_k \rightharpoonup \mathrm{e}^{-t/2\varepsilon} u$ in $L^2(0,\infty;V)$.

Fix any $T > 0$ and note that u_k are bounded in $L^2(0,T;L^6(\Omega;\mathbb{R}^3))$ as $V \subset L^6(\Omega;\mathbb{R}^3)$. This entails that $u_k \cdot \nabla u_k$ are bounded in $L^1(0,T;L^{3/2}(\Omega;\mathbb{R}^3))$. As v_k are bounded in $L^2(0,T;L^2(\Omega;\mathbb{R}^3))$, we conclude that $\partial_t u_k$ are bounded in $L^1(0,T;L^{3/2}(\Omega;\mathbb{R}^3))$ as well. Hence, u_k are bounded in $C([0,T];L^{3/2}(\Omega;\mathbb{R}^3))$ and, by interpolation, in $L^4(0,T;L^{12/5}(\Omega;\mathbb{R}^3))$. Eventually, both $u_k \cdot \nabla u_k$ and $\partial_t u_k$ are bounded in $L^{4/3}(0,T;L^{12/11}(\Omega;\mathbb{R}^3))$. By possibly extracting again we find a not relabeled subsequence such that

$$u_k \cdot \nabla u_k \rightharpoonup u \cdot \nabla u \ \text{ and } \ \partial_t u_k \to \partial_t u \ \text{ in } \ L^{4/3}(0,T;L^{12/11}(\Omega;\mathbb{R}^3))$$

where we also used the strong convergence $u_k \to u$ in $L^2(0,T;L^6(\Omega;\mathbb{R}^3))$. This proves that

$$v_k \rightharpoonup v = \partial_t u + u\cdot\nabla u \quad\text{in}\quad L^2(0,T;L^2(\Omega;\mathbb{R}^3)).$$

We now use convexity in order to check that

$$\begin{aligned}
&\int_0^T \mathrm{e}^{-t/\varepsilon}\Big\{\frac{1}{2}\|\partial_t u + u\cdot\nabla u\|^2 + \frac{\sigma}{2}\|u\cdot\nabla u\|^2 + \frac{\nu}{2\varepsilon}\|\nabla u\|^2\Big\}dt\\
&\quad\le \liminf_{k\to\infty}\int_0^T \mathrm{e}^{-t/\varepsilon}\Big\{\frac{1}{2}\|\partial_t u_k + u_k\cdot\nabla u_k\|^2 + \frac{\sigma}{2}\|u_k\cdot\nabla u_k\|^2 + \frac{\nu}{2\varepsilon}\|\nabla u_k\|^2\Big\}dt\\
&\quad\le \liminf_{k\to\infty} I^\varepsilon(u_k) = \inf I^\varepsilon.
\end{aligned}$$

It now suffices to take the limit as $T\to\infty$ on the left-hand side above to conclude that $I^\varepsilon(u) = \min I^\varepsilon$. □

For later use we remark that Poincaré's inequality ensures that for $u\in U^\varepsilon$ with $I^\varepsilon(u)<\infty$ it holds

$$\begin{aligned}
&\mathrm{e}^{-t/2\varepsilon}(\partial_t u + u\cdot\nabla u),\ \sigma\,\mathrm{e}^{-t/2\varepsilon}u\cdot\nabla u \in L^2(0,\infty;L^2(\Omega;\mathbb{R}^3))\\
&\quad\text{and}\ \ \mathrm{e}^{-t/2\varepsilon}\nabla u\in L^2(0,\infty;L^2(\Omega;\mathbb{R}^{3\times3})).
\end{aligned}\tag{4.1}$$

5. Euler-Lagrange equations

Having established the existence of minimizers, we will now derive the corresponding Euler-Lagrange equations. The stabilization parameter $\sigma>0$ is kept fixed throughout the section.

Lemma 5.1 (Euler-Lagrange equations)**.** *Let $\sigma>0$ and u^ε be a minimizer of I^ε. For any $\varphi\in L^2_{\mathrm{loc}}(0,\infty;V)$ with $\partial_t\varphi\in L^2_{\mathrm{loc}}(0,\infty;H)$, $\varphi(0)=0$ and such that*

$$\begin{aligned}
&\mathrm{e}^{-t/2\varepsilon}\partial_t\varphi, \mathrm{e}^{-t/2\varepsilon}\varphi\cdot\nabla\varphi,\ \mathrm{e}^{-t/2\varepsilon}\varphi\cdot\nabla u^\varepsilon,\ \mathrm{e}^{-t/2\varepsilon}u^\varepsilon\cdot\nabla\varphi \in L^2(0,\infty;L^2(\Omega;\mathbb{R}^3))\\
&\textit{and}\ \ \mathrm{e}^{-t/2\varepsilon}\nabla\varphi\in L^2(0,\infty;L^2(\Omega;\mathbb{R}^{3\times3}))
\end{aligned}$$

we have that

$$\begin{aligned}
&\int_0^\infty \mathrm{e}^{-t/\varepsilon}\int_\Omega \big(\partial_t u^\varepsilon + u^\varepsilon\cdot\nabla u^\varepsilon\big)\cdot\big(\partial_t\varphi + u^\varepsilon\cdot\nabla\varphi + \varphi\cdot\nabla u^\varepsilon\big)\\
&\qquad + \sigma(u^\varepsilon\cdot\nabla u^\varepsilon)\cdot\big(u^\varepsilon\cdot\nabla\varphi + \varphi\cdot\nabla u^\varepsilon\big) + \frac{\nu}{\varepsilon}\nabla u^\varepsilon\cdot\nabla\varphi\,dx\,dt = 0.
\end{aligned}\tag{5.1}$$

In particular, any $\varphi\in C^\infty_{\mathrm{c}}(0,\infty;V_s)$, $s>5/2$, satisfies

$$\begin{aligned}
&\int_0^\infty\int_\Omega (\partial_t u^\varepsilon + u^\varepsilon\cdot\nabla u^\varepsilon)\cdot\big[\varphi + \varepsilon\partial_t\varphi + \varepsilon\varphi\cdot\nabla u^\varepsilon + \varepsilon u^\varepsilon\cdot\nabla\varphi\big]\\
&\qquad + \sigma\varepsilon(u^\varepsilon\cdot\nabla u^\varepsilon)\cdot\big[\varphi\cdot\nabla u^\varepsilon + u^\varepsilon\cdot\nabla\varphi\big] + \nu\nabla u^\varepsilon : \nabla\varphi\,dx\,dt = 0.
\end{aligned}\tag{5.2}$$

Proof. The assumptions guarantee that $u^\varepsilon + s\varphi\in U^\varepsilon$ and

$$\begin{aligned}
I^\varepsilon(u^\varepsilon + s\varphi) = \int_0^\infty \mathrm{e}^{-t/\varepsilon}\Big\{&\frac{1}{2}\|\partial_t(u^\varepsilon + s\varphi) + (u^\varepsilon + s\varphi)\cdot\nabla(u^\varepsilon + s\varphi)\|^2\\
&+ \frac{\sigma}{2}\|(u^\varepsilon + s\varphi)\cdot\nabla(u^\varepsilon + s\varphi)\|^2 + \frac{\nu}{2\varepsilon}\|\nabla(u^\varepsilon + s\varphi)\|^2\Big\}\,dt
\end{aligned}$$

is finite for any $s \in \mathbb{R}$ and is minimized at $s = 0$. Indeed, by letting $\zeta_1 = \partial_t \varphi + u^\varepsilon \cdot \nabla \varphi + \varphi \cdot \nabla u^\varepsilon$, $\zeta_2 = \varphi \cdot \nabla \varphi$, and $\xi_1 = u^\varepsilon \cdot \nabla \varphi + \varphi \cdot \nabla u^\varepsilon$, $\xi_2 = \varphi \cdot \nabla \varphi$ one can write

$$\begin{aligned} I^\varepsilon(u^\varepsilon + s\varphi) = \int_0^\infty \mathrm{e}^{-t/\varepsilon} \bigg\{ & \frac{1}{2} \| \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon + s\zeta_1 + s^2 \zeta_2 \|^2 \\ & + \frac{\sigma}{2} \| u^\varepsilon \cdot \nabla u^\varepsilon + s\xi_1 + s^2 \xi_2 \|^2 + \frac{\nu}{2\varepsilon} \| \nabla u^\varepsilon + s \nabla \varphi \|^2 \bigg\} \, dt \end{aligned}$$

By assumption we have $\mathrm{e}^{-t/2\varepsilon}\zeta_1, \mathrm{e}^{-t/2\varepsilon}\zeta_2, \mathrm{e}^{-t/2\varepsilon}\xi_1, \mathrm{e}^{-t/2\varepsilon}\xi_2 \in L^2(0,\infty; L^2(\Omega;\mathbb{R}^3))$. In combination with (4.1) this shows that $I^\varepsilon(u^\varepsilon + s\varphi)$ is differentiable with

$$\begin{aligned} 0 &= \frac{d}{ds}\Big|_{s=0} I^\varepsilon(u^\varepsilon + s\varphi) \\ &= \int_0^\infty \mathrm{e}^{-t/\varepsilon} \int_\Omega \big(\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon \big) \cdot \zeta_1 + \sigma (u^\varepsilon \cdot \nabla u^\varepsilon) \cdot \xi_1 + \frac{\nu}{\varepsilon} \nabla u^\varepsilon : \nabla \varphi \, dx \, dt \end{aligned}$$

as claimed.

Consider now $\varphi \in C_c^\infty(0,\infty, V_s)$, $s > 5/2$, and let $\psi = \mathrm{e}^{t/\varepsilon}\varphi$ so that $\partial_t \psi = \mathrm{e}^{t/\varepsilon}\partial_t\varphi + \varepsilon^{-1}\mathrm{e}^{t/\varepsilon}\varphi \in L^2(0,\infty; L^2(\Omega;\mathbb{R}^3))$ by Poincaré's inequality. Since $V_s \hookrightarrow W^{1,\infty}(\Omega;\mathbb{R}^3)$ and φ has compact support in t we infer from (4.1) that ψ satisfies the assumptions of the first part of the Lemma. Equation (5.2) is now a direct consequence of (5.1) applied to ψ. □

We may equivalently write equality (5.2) as

$$\begin{aligned} \int_0^\infty \int_\Omega & \varepsilon \big[\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon \big] \cdot \partial_t \varphi + \big[\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon \big] \cdot \varphi \\ & + \varepsilon (\nabla u^\varepsilon)^\top \big[\partial_t u^\varepsilon + (1+\sigma) u^\varepsilon \cdot \nabla u^\varepsilon \big] \cdot \varphi \\ & + \big[\varepsilon \big(\partial_t u^\varepsilon + (1+\sigma) u^\varepsilon \cdot \nabla u^\varepsilon \big) \otimes u^\varepsilon + \nu \nabla u^\varepsilon \big] : \nabla \varphi \, dx \, dt = 0 \end{aligned} \tag{5.3}$$

which corresponds indeed the weak formulation of (1.3). By using the quadratic function B introduced in (3.3) we furthermore set

$$v^\varepsilon = \partial_t u^\varepsilon + B(u^\varepsilon) = \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon \tag{5.4}$$

where the latter equality follows from the fact that $u^\varepsilon \cdot \nabla u^\varepsilon \in L^2(\Omega;\mathbb{R}^3)$. (More precisely, $B(u^\varepsilon) \in H \subset V'$ is the L^2-orthogonal projection of $u^\varepsilon \cdot \nabla u^\varepsilon$ onto H.) Since $u^\varepsilon, \partial_t u^\varepsilon, u^\varepsilon \cdot \nabla u^\varepsilon \in L^2_{\mathrm{loc}}(0,\infty; L^2(\Omega;\mathbb{R}^3))$, $\nabla u^\varepsilon \in L^2_{\mathrm{loc}}(0,\infty; L^2(\Omega;\mathbb{R}^{3\times 3}))$, and $V_s \hookrightarrow W^{1,\infty}(\Omega;\mathbb{R}^3)$ for $s > 5/2$, we have that f^ε and g^ε, defined by

$$\langle f^\varepsilon(t), \psi \rangle = \int_\Omega \varepsilon (\nabla u^\varepsilon)^\top \big[\partial_t u^\varepsilon + (1+\sigma) u^\varepsilon \cdot \nabla u^\varepsilon \big] \cdot \psi \, dx, \tag{5.5}$$

$$\langle g^\varepsilon(t), \psi \rangle = \int_\Omega \varepsilon \big[\partial_t u^\varepsilon + (1+\sigma) u^\varepsilon \cdot \nabla u^\varepsilon \big] \otimes u^\varepsilon : \nabla \psi \, dx \tag{5.6}$$

for a.e. t for all $\psi \in V_s$ are elements of $L^1_{\mathrm{loc}}(0,\infty; V_s')$ with

$$\| f^\varepsilon \|_{L^1(0,T;V_s')} \le C\varepsilon \| \nabla u^\varepsilon \|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{3\times 3}))} \| \partial_t u^\varepsilon + (1+\sigma) u^\varepsilon \cdot \nabla u^\varepsilon \|_{L^2(0,T;L^2(\Omega;\mathbb{R}^n))}, \tag{5.7}$$

$$\| g^\varepsilon \|_{L^1(0,T;V_s')} \le C\varepsilon \| \partial_t u^\varepsilon + (1+\sigma) u^\varepsilon \cdot \nabla u^\varepsilon \|_{L^2(0,T;L^2(\Omega;\mathbb{R}^n))} \| u^\varepsilon \|_{L^2(0,T;L^2(\Omega;\mathbb{R}^n))} \tag{5.8}$$

for any $T > 0$. By recalling the definition (3.1) of the Stokes operator $A : V \to V' \hookrightarrow V_s'$, we obtain from Lemma 5.1 the following.

Corollary 5.2 (Euler-Lagrange equations, strong form)**.** *Let u^ε be a minimizer of I^ε on U^ε and $v^\varepsilon, f^\varepsilon, g^\varepsilon$ be defined in* (5.4), (5.5), (5.6), *respectively. Then*

$$\int_0^\infty \left(\varepsilon v^\varepsilon, \partial_t \varphi\right)_H + \langle v^\varepsilon + f^\varepsilon + g^\varepsilon + \nu A u^\varepsilon, \varphi\rangle \, dt = 0 \tag{5.9}$$

for all $\varphi \in C_c^\infty(0,\infty;V_s)$, $s > 5/2$. In fact, u^ε is a solution to the elliptic problem

$$-\varepsilon \partial_t v^\varepsilon + v^\varepsilon + f^\varepsilon + g^\varepsilon + \nu A u^\varepsilon = 0 \tag{5.10}$$

in $L^1_{\rm loc}(0,\infty;V_s')$.

Proof. First note that relation (5.9) is an immediate consequence of the former (5.3). Hence, equation (5.10) holds in the sense of distributions $\mathcal{D}'(0,\infty;V_s')$ on $(0,\infty)$ with values in V_s'. In particular, as $v^\varepsilon, f^\varepsilon, g^\varepsilon, Au^\varepsilon \in L^1_{\rm loc}(0,\infty;V_s')$ we have that $\partial_t v^\varepsilon \in L^1_{\rm loc}(0,\infty;V_s')$ as well. □

6. A-priori estimates

In order to pass to the limit $\varepsilon \to 0$ in the Euler-Lagrange equations and prove Theorem 3.2 we now provide estimates on the minimizers u^ε. This requires $\sigma > 1/8$, which is assumed throughout this section. At first, moving from Lemma 5.1 we prove in Proposition 6.1 an energy estimate which formally consists in testing the Euler-Lagrange equations by u^ε. Then, Proposition 6.2 exploits the strong form of the Euler-Lagrange equations from Corollary 5.2 in order to deduce bounds in dual spaces. Their derivation with the help of the convolution kernel K (see (6.5)) is inspired by a similar reasoning in [30].

Proposition 6.1 (Energy estimate)**.** *Let $\sigma > 1/8$. All minimizers $u^\varepsilon \in U^\varepsilon$ of I^ε satisfy $\partial_t u^\varepsilon$, $u^\varepsilon \cdot \nabla u^\varepsilon \in L^2(0,\infty;L^2(\Omega;\mathbb{R}^3))$, $u^\varepsilon \in L^2(0,\infty;H^1(\Omega;\mathbb{R}^3)) \cap L^\infty(0,\infty;L^2(\Omega;\mathbb{R}^3))$, and*

$$\|u^\varepsilon\|_{L^2(0,\infty;H^1(\Omega;\mathbb{R}^3))} + \|u^\varepsilon\|_{L^\infty(0,\infty;L^2(\Omega;\mathbb{R}^3))} \le C$$

as well as

$$\sqrt{\varepsilon}\|\partial_t u^\varepsilon\|_{L^2(0,\infty;L^2(\Omega;\mathbb{R}^3))} + \sqrt{\varepsilon}\|u^\varepsilon \cdot \nabla u^\varepsilon\|_{L^2(0,\infty;L^2(\Omega;\mathbb{R}^3))} \le C.$$

More explicitly, the following inequality holds

$$\|u^\varepsilon(T)\|^2 + 2\nu \int_0^T (1 - {\rm e}^{-t/\varepsilon})\|\nabla u^\varepsilon(t)\|^2 \, dt \le \|u_0^\varepsilon\|^2 \tag{6.1}$$

for all $T > 0$.

Proof. Let us start by estimating u^ε on the initial short-time interval $(0,\varepsilon)$. By letting $c = \min\{1,\sigma,\nu\}/(2e) > 0$ we have

$$\begin{aligned}
&\frac{c}{\varepsilon}\int_0^\varepsilon \varepsilon\|\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon\|^2 + \varepsilon\|u^\varepsilon \cdot \nabla u^\varepsilon\|^2 + \|\nabla u^\varepsilon\|^2 \, dt \\
&\le \int_0^\infty {\rm e}^{-t/\varepsilon}\Big\{\frac{1}{2}\|\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon\|^2 + \frac{\sigma}{2}\|u^\varepsilon \cdot \nabla u^\varepsilon\|^2 + \frac{\nu}{2\varepsilon}\|\nabla u^\varepsilon\|^2\Big\} \, dt \\
&= I(u^\varepsilon) \le C\varepsilon^{-1}
\end{aligned}$$

by Proposition 4.1, and thus

$$\int_0^\varepsilon \varepsilon\|\partial_t u^\varepsilon\|^2 + \varepsilon\|u^\varepsilon \cdot \nabla u^\varepsilon\|^2 + \|\nabla u^\varepsilon\|^2 \, dt \le C. \tag{6.2}$$

In order to deduce an estimate on $[\varepsilon, T]$, for $T > 0$, we let $\eta \in W^{1,\infty}(0,\infty)$ be defined as

$$\eta(t) = \begin{cases} \mathrm{e}^{t/\varepsilon} - 1 & \text{for } t \le T, \\ (\mathrm{e}^{T/\varepsilon} - 1) & \text{for } t \ge T. \end{cases}$$

and apply Lemma 5.1 to $\varphi = \eta u^\varepsilon$. We obtain

$$\begin{aligned} 0 &= \int_0^\infty \mathrm{e}^{-t/\varepsilon} \int_\Omega \eta(t) \Big\{ (\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon) \cdot \big[\partial_t u^\varepsilon + 2u^\varepsilon \cdot \nabla u^\varepsilon \big] \\ &\qquad + 2\sigma |u^\varepsilon \cdot \nabla u^\varepsilon|^2 + \frac{\nu}{\varepsilon} |\nabla u^\varepsilon|^2 \Big\} + \eta'(t)(\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon) \cdot u^\varepsilon \, dx \, dt \\ &= \int_0^\infty \mathrm{e}^{-t/\varepsilon} \eta(t) \int_\Omega |\partial_t u^\varepsilon|^2 + 3\partial_t u^\varepsilon \cdot (u^\varepsilon \cdot \nabla u^\varepsilon) + 2(1+\sigma)|u^\varepsilon \cdot \nabla u^\varepsilon|^2 + \frac{\nu}{\varepsilon} |\nabla u^\varepsilon|^2 \, dx \, dt \\ &\quad + \int_0^T \frac{1}{\varepsilon} \int_\Omega \partial_t u^\varepsilon \cdot u^\varepsilon \, dx \, dt, \end{aligned}$$

where we have also used the fact that $\int_\Omega (u^\varepsilon \cdot \nabla u^\varepsilon) \cdot u^\varepsilon \, dx = 0$. It follows that

$$\begin{aligned} 0 = \int_0^\infty \mathrm{e}^{-t/\varepsilon} \eta(t) \int_\Omega \varepsilon |\partial_t u^\varepsilon + \tfrac{3}{2} u^\varepsilon \cdot \nabla u^\varepsilon|^2 + \varepsilon(2\sigma - \tfrac{1}{4}) |u^\varepsilon \cdot \nabla u^\varepsilon|^2 + \nu |\nabla u^\varepsilon|^2 \, dx \ dt \\ + \frac{1}{2} \int_\Omega |u^\varepsilon(T)|^2 dx - \frac{1}{2} \int_\Omega |u^\varepsilon(0)|^2 \, dx. \end{aligned}$$

Since $\eta(t) \ge 0$ for all t and $t \mapsto \mathrm{e}^{-t/\varepsilon} \eta(t) = 1 - \mathrm{e}^{-t/\varepsilon}$ increases on $[0, T]$, by defining $c = \min\{1, 2\sigma - 1/4, \nu\}(1 - \mathrm{e}^{-1}) > 0$ we get

$$\begin{aligned} & c \int_\varepsilon^T \int_\Omega \varepsilon |\partial_t u^\varepsilon + \tfrac{3}{2} u^\varepsilon \cdot \nabla u^\varepsilon|^2 + \varepsilon |u^\varepsilon \cdot \nabla u^\varepsilon|^2 + |\nabla u^\varepsilon|^2 \, dx \ dt \\ & \le \int_0^T \mathrm{e}^{-t/\varepsilon} \eta(t) \int_\Omega \varepsilon |\partial_t u^\varepsilon + \tfrac{3}{2} u^\varepsilon \cdot \nabla u^\varepsilon|^2 + \varepsilon(2\sigma - \tfrac{1}{4}) |u^\varepsilon \cdot \nabla u^\varepsilon|^2 + \nu |\nabla u^\varepsilon|^2 \, dx \ dt \\ & \le \frac{1}{2} \|u_0^\varepsilon\|^2 - \frac{1}{2} \|u^\varepsilon(T)\|^2. \end{aligned} \tag{6.3}$$

Combining (6.2), (6.3), and (3.6) we see that

$$\|u^\varepsilon(T)\|^2 + \int_0^T \varepsilon \|\partial_t u^\varepsilon\|^2 + \varepsilon \|u^\varepsilon \cdot \nabla u^\varepsilon\|^2 + \|\nabla u^\varepsilon\|^2 \, dt \le C.$$

As $T > 0$ is arbitrary, this proves the statement. □

Proposition 6.2 (Dual estimate)**.** *Let $\sigma > 1/8$ and $s > 5/2$. All minimizers $u^\varepsilon \in U^\varepsilon$ of I^ε satisfy*

$$\|\partial_t u^\varepsilon\|_{L^2(0,\infty;V_s')} + \|B(u^\varepsilon)\|_{L^2(0,\infty;V_s')} \le C.$$

Proof. Let $0 \le \tau \le T$. By Proposition 6.1 and equations (5.7), (5.8), and (3.2) we have the estimates

$$\|f^\varepsilon + g^\varepsilon\|_{L^1(0,\infty;V_s')} \le C\sqrt{\varepsilon} \qquad \text{and} \qquad \|Au^\varepsilon\|_{L^2(0,\infty;V_s')} \le C. \tag{6.4}$$

Multiplying the Euler-Lagrange equation (5.10) by $\varepsilon^{-1} \mathrm{e}^{-t/\varepsilon}$, integrating over $[\tau, T]$, and multiplying by $\mathrm{e}^{\tau/\varepsilon}$, we find that v^ε as defined in (5.4) satisfies

$$v(\tau) = \mathrm{e}^{(\tau - T)/\varepsilon} v(T) - \int_\tau^T \frac{\mathrm{e}^{(\tau - t)/\varepsilon}}{\varepsilon} \big(f^\varepsilon(t) + g^\varepsilon(t) + \nu A u^\varepsilon(t) \big) \, dt.$$

As $v^\varepsilon \in L^2(0,\infty;H)$ by Proposition 6.1 we may send $T \to \infty$ along a sequence with $v^\varepsilon(T) \to 0$ in H $(\hookrightarrow V_s')$ and see that

$$v^\varepsilon(\tau) = -\int_\tau^\infty \frac{\mathrm{e}^{(\tau-t)/\varepsilon}}{\varepsilon}\big(f^\varepsilon(t) + g^\varepsilon(t) + \nu A u^\varepsilon(t)\big)\, dt.$$

Extending $f^\varepsilon, g^\varepsilon$ and Au^ε by 0 on $(-\infty,0)$ to all of $\mathbb{R}$, this can be written as

$$v^\varepsilon = K * (f^\varepsilon + g^\varepsilon + \nu A u^\varepsilon)$$

on $[0,\infty)$ where the kernel K is given by

$$K(t) = \begin{cases} \varepsilon^{-1}\mathrm{e}^{t/\varepsilon} & \text{for } \ t \le 0, \\ 0 & \text{for } \ t > 0. \end{cases} \tag{6.5}$$

This entails that

$$\|v^\varepsilon\|_{L^2(0,\infty;V_s')} \le C \tag{6.6}$$

since by Young's inequality and (6.4) we can estimate

$$\|K * (f^\varepsilon + g^\varepsilon)\|_{L^2(\mathbb{R};V_s')} \le \|K\|_{L^2(\mathbb{R})}\|f^\varepsilon + g^\varepsilon\|_{L^1(\mathbb{R};V_s')} \le C$$

and

$$\|\nu K * Au^\varepsilon\|_{L^2(\mathbb{R};V_s')} \le \|K\|_{L^1(\mathbb{R})}\|\nu A u^\varepsilon\|_{L^2(\mathbb{R};V_s')} \le C,$$

where we have used that $\|K\|_{L^1(\mathbb{R})} = 1$ and $\|K\|_{L^2(\mathbb{R})} = 1/\sqrt{2\varepsilon}$. As by Proposition 6.1 and bound (3.5) one also has that

$$\begin{aligned}\|B(u^\varepsilon)\|_{L^2(0,\infty;V_s')} &\le C\|u^\varepsilon \cdot \nabla u^\varepsilon\|_{L^2(0,\infty;L^1(\Omega;\mathbb{R}^3))} \\ &\le C\|u^\varepsilon\|_{L^\infty(0,\infty;L^2(\Omega;\mathbb{R}^3))}\|\nabla u^\varepsilon\|_{L^2(0,\infty;L^2(\Omega;\mathbb{R}^3))} \le C,\end{aligned}$$

the claim follows from this inequality and (6.6). □

7. Convergence

This section is eventually devoted to the proof of Theorem 3.2. Let u^ε be a minimizer of I^ε on U^ε and define v^ε as in (5.4). By the a-priori estimates of Propositions 6.1 and 6.2 (recall that we assume here $\sigma > 1/8$) there exists $u \in L^2(0,\infty;V) \cap L^\infty(0,\infty;H)$ with $\partial_t u \in L^2(0,\infty;V_s')$, $s > 5/2$, such that, for a subsequence (not relabeled),

$$u^\varepsilon \rightharpoonup u \quad \text{in } L^2(0,\infty;V), \qquad u^\varepsilon \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,\infty;H) \tag{7.1}$$

as well as

$$\partial_t u^\varepsilon \rightharpoonup \partial_t u \quad \text{in } L^2(0,\infty;V_s'). \tag{7.2}$$

This, in particular, entails that

$$Au^\varepsilon \rightharpoonup Au \quad \text{in } L^2(0,\infty;V_s') \tag{7.3}$$

as well, since A is bounded, see (3.2). Moreover, (5.4), (5.7), and (5.8) in combination with Proposition 6.1 yield

$$\varepsilon v^\varepsilon \to 0 \quad \text{in } L^2(0,\infty;L^2(\Omega;\mathbb{R}^3)) \qquad \text{and} \tag{7.4}$$

$$f^\varepsilon + g^\varepsilon \to 0 \quad \text{in } L^1(0,\infty;V_s'). \tag{7.5}$$

By the Aubin-Lions Lemma we obtain $u^\varepsilon \to u$ strongly in $L^2_{\mathrm{loc}}(0,\infty;H)$ so that (7.1), Proposition 6.2, and $V_s \hookrightarrow L^\infty(\Omega)$ yield

$$B(u^\varepsilon) \rightharpoonup B(u) \quad \text{in } L^2(0,\infty;V_s'). \tag{7.6}$$

From (7.2) and (7.6) we thus get

$$v^\varepsilon \rightharpoonup \partial_t u + B(u) \quad \text{in } L^2(0,\infty;V_s'). \tag{7.7}$$

Recall now from Corollary 5.2 that u^ε is a solution to (5.9) for any $\varphi \in C_c^\infty((0,\infty);V_s)$. By using (7.4), (7.7), (7.5), and (7.3) we can now pass to the limit $\varepsilon \to 0$ in (5.9) and get

$$\int_0^\infty \langle \partial_t u + B(u) + \nu A u, \varphi \rangle \, dt = 0$$

for any $\varphi \in C_c^\infty((0,\infty);V_s)$ and so

$$\partial_t u + \ B(u) + \nu A u = 0 \tag{7.8}$$

in $L^2(0,\infty;V_s')$ and a.e. in time.

Since $W^{1,1}(0,T,V_s') \hookrightarrow C([0,T],V_s')$ for any $T > 0$, we also have $u^\varepsilon \rightharpoonup u$ in $C([0,T],V_s')$ and u satisfies the initial condition $u(0) = u_0$ because of (3.6). As $u \in L^2(0,\infty;V)$, the bounds (3.2) and (3.4) entail that $Au \in L^2(0,\infty;V')$ and $B(u) \in L^1(0,\infty;V')$, respectively. By comparison in (7.8) one has that $\partial_t u \in L^1(0,\infty;V')$ as well so that the equation holds in V' for a.a. times and u is indeed a Leray-Hopf solution.

Eventually, the energy inequality (3.9) follows by lower semicontinuity when passing to the limit $\varepsilon \to 0$ in inequality (6.1) by using the convergence $u_0^\varepsilon \to u_0$ in H from (3.6), $u^\varepsilon(T) \rightharpoonup u(T)$ in $L^2(\Omega;\mathbb{R}^3)$, and $(1 - \mathrm{e}^{-t/\varepsilon})^{1/2}\nabla u^\varepsilon \rightharpoonup \nabla u$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^{3\times 3}))$.

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