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Equivalence of Hawking and Unruh Temperatures and Entropies Through Flat Space Embeddings

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Abstract

We present a unified description of temperature and entropy in spaces with either "true" or "accelerated observer" horizons: In their (higher dimensional) global embedding Minkowski geometries, the relevant detectors have constant accelerations a_G; associated with their Rindler horizons are temperature $a_G/2\pi$ and entropy equal to 1/4 the horizon area. Both quantities agree with those calculated in the original curved spaces. As one example of this equivalence, we obtain the temperature and entropy of Schwarzschild geometry from its flat D=6 embedding.

The relation between Hawking and Unruh effects has been extensively studied, and the emergence of temperature and entropy due to their respective "real" or "accelerated" horizons is well-understood. Recently, it was shown that the temperatures measured by accelerated detectors in de Sitter (dS) and Anti de Sitter (AdS) geometries can be obtained [1] from their corresponding constant (Rindler) accelerations in the appropriate global embedding Minkowski spacetimes (GEMS). Our purpose here is to generalize this method to provide a unified kinematical treatment of both effects in terms of accelerated motions in the GEMS; any Einstein geometry can be so embedded [2]. As a first illustration, we obtain the entropy of dS in this way. We will then derive, as our main example, the equivalence between temperature and entropy measured by the usual static detector in Schwarzschild geometry and their values as calculated from its Rindler-like motion in the (D=6) GEMS. A more complete discussion of this equivalence, and of its validity for other important geometries such as BTZ, Schwarzschild-AdS, Schwarzschild-dS and Riessner-Nordstrom, will be presented elsewhere.

Recall first that in flat space, observers with constant acceleration of magnitude a, who follow a timelike Killing vector field ξ that encounters an event horizon, will thereby measure a temperature, $2\pi T = a$. It is also well-known that the connection between surface gravity k_H and temperature,

$$k_H = g_{00}^{1/2}\, 2\pi T \tag{1}$$

holds both in black hole spaces and for Rindler motions [3]; x^0 is the timelike Killing vector of rest detectors and k_H is defined as the horizon value of $[-\frac{1}{2}(D_\mu \xi_\nu)^2]^{1/2}$. In Rindler coordinates, the longitudinal interval is $ds^2 = L^2 e^{2\zeta}(d\tau^2 - d\zeta^2)$ and ζ=const detectors have acceleration $a = L^{-1}e^{-\zeta}$; they see an event horizon at $\zeta = -\infty$ ($x^2 - t^2 = 0$), where the Killing vector ∂_τ is null. There, $k_H = 1$, hence (1) immediately reproduces $2\pi T = a$. The entropy of these observers is also known [4]: it is $A/4$, where A is the (transverse) horizon area; for "unrestricted" Rindler motion it is in general infinite, but not (as we shall see) in the GEMS context when real horizons are being represented.

Next, for orientation, we summarize the GEMS approach in its original dS/AdS setting. These spaces are hyperboloids

$$\eta_{AB} z^A z^B = \mp R^2 \tag{2}$$

in flat, $ds^2 = \eta_{AB} dz^A dz^B$, GEMS. Here $A, B = 0..D$, $\eta_{AB} = diag(1, -1.. -1, \mp 1)$ and upper/lower signs always refer to dS/AdS respectively. Now consider $z^2 = ... = z^{D-1} = 0$, $z^D = Z$ trajectories; these obey $(z^1)^2 - (z^0)^2 = \pm(R^2 - Z^2) \equiv a_G^{-2}$, and describe constant acceleration Rindler motions of the form "$x^2 - t^2 = a^{-2}$ " for Z=const. Such detectors therefore measure $2\pi T = a_G$, which we rewrite in terms of the original dS/AdS accelerations a_D as

$$2\pi T = a_G = (\pm R^{-2} + a_D^2)^{1/2} \ . \tag{3}$$

This relation is a particular case of the familiar Gauss–Codazzi–Ricci equation: detectors following a timelike Killing vector ξ in the physical space have acceleration [5] $a_D = D_\xi \xi / |\xi|^2$ with

$$a_G^2 = a_D^2 + \alpha^2 |\xi|^{-4} \tag{4}$$

where α is the second fundamental form [2]; this relation extends to cases such as Schwarzschild where the GEMS is more than one dimension higher. It should not be inferred from (4) that there is always a meaningful temperature (in either the original or the embedding space), since α^2 need not always be positive; for example it is $\alpha^2 |\xi|^{-4} = -R^{-2}$ for AdS. After all, the flat space Unruh description itself is only meaningful when $a_G^2 \geq 0$. We refer to [1] for details. To calculate entropy here, we must restrict the (transverse) horizon area integration according to the definitions of the embedding coordinates. For (D=4) dS the three-fold integral over $dz^2 dz^3 dz^4$ is restricted by the condition $[(z^2)^2 + (z^3)^2 + (z^4)^2]^{1/2} = R$, resulting in the same finite 2-surface value $4\pi R^2$ as calculated directly in D=4. For the corresponding AdS observers, the coordinate restriction is instead $[-(z^2)^2 - (z^3)^2 + (z^4)^2]^{1/2} = R$ and the area integral is now infinite. This is not surprising since AdS, like the Rindler wedge, has no true horizon and its D=4 entropy is also infinite.

Let us now turn to the Schwarzschild metric to exemplify spaces with "real" horizons. Here the GEMS [6], which covers the usual Kruskal [7] extension, has D=6 $ds^2 = \eta_{AB} dz^A dz^B$ $\eta_{AB} = diag(1, -1, -1, -1, -1, -1)$, with

$$z^0 = 4m(1-u)^{1/2} \sinh t/4m \quad z^1 = 4m(1-u)^{1/2} \cosh t/4m \tag{5}$$

$$z^2 = -2m \int du [u + u^2 + u^3]^{1/2} u^{-2}, \;\; (z^3, z^4, z^5) = (x, y, z), \;\; u \equiv 2m/r, \;\; r^2 \equiv x^2 + y^2 + z^2.$$

This is a global embedding, with extendability to $r < 2m$. [In incomplete embedding spaces, such as those in [8], that cover only the exterior region $r > 2m$, observers see no event horizon, hence no loss of information or temperature.] The Hawking detectors with constant

r in D=4 are here Unruh detectors; their D=6 motions are the now familiar hyperbolic trajectories

$$(z^1)^2 - (z^0)^2 = 16m^2(1-u) \equiv a_6^{-2} \ . \tag{6}$$

We therefore immediately infer the usual Schwarzschild local Hawking (T) and black hole (T_0) temperatures

$$2\pi T = a_6 = [16m^2(1-u)]^{-1/2}, \quad T_0 = g_{00}^{1/2} T = (8\pi m)^{-1} \ , \tag{7}$$

from the purely kinematical GEMS motion's Unruh temperature.

We consider next the equivalence of entropy definitions. Here we must restrict the fourfold transverse integration over $dz^2 ... dz^5$ to the underlying manifold using the embedding definition (5), including the condition $r = 2m$ implied by $(z^0)^2 - (z^1)^2 = 0$, to be enforced by introduction of the obvious delta function with r expressed as $[(z^3)^2 + (z^4)^2 + (z^5)^2]^{1/2}$. This just restricts the z^3, z^4, z^5 volume to the surface of $r = 2m$, while the z^2 integral is just equal to 1; then it is a simple matter to verify that the integration gives the transverse area $4\pi(2m)^2$, precisely that of the Schwarzschild horizon, and thereby establish the equivalence.

We have here exploited the GEMS approach to unify the definitions of temperatures and entropies whatever the origin of their event horizons. The "equivalence" that emerged is closely related to its usual Einstein meaning, even though we are dealing here with nonlocal properties. From the definitions of temperature and entropy with flat space accelerated motions, the GEMS mechanism ensures the correspondence with curved space horizons as well, at least for sufficiently simple horizon structures where detectors are mapped into Rindler ones; the quantitative equivalence is then essentially guaranteed. We hope to return to other, "non-horizon", uses of GEMS, for example those involving rotation and superradiance.

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