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Error correction and convex programming

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This article discusses a recently proposed error correction method involving convex optimization [1]. From an encoded and corrupted real-valued message, a receiver would like to determine the original message. A few entries of the encoded message are corrupted arbitrarily (which we call gross errors) and all the entries of the encoded message are corrupted slightly. We show that it is possible to recover the message with nearly the same accuracy as in the setting where no gross errors occur.

1 Introduction

We are concerned with the following coding problem: a real-valued signal $x \in \mathbb{R}^n$ is encoded using a linear code $A \in \mathbb{R}^{m\times n}$ assumed to have full rank with $m > n$. The encoded message is then corrupted by $e \in \mathbb{R}^m$ and received as $y = Ax + e$. We would like to recover x knowing only A and y.

In [2] it is shown that if e is sufficiently sparse (meaning that the number of entries of Ax that are corrupted is sufficiently small) and A satisfies certain conditions, e will be the solution to a certain linear program and so x can be recovered exactly. Note that we have assumed nothing about e other than its sparsity, and in particular have made no restriction on the ℓ_2 norm of e.

This result is in part interesting because linear programs, and more generally convex programs, are computationally tractable, meaning that fairly large convex programs can be quickly solved using interior point solvers or other methods.

2 Gross Errors and Small Errors

While the above result is interesting, it may be a bit unrealistic to expect that all but a few entries of Ax are received with infinite precision. A more realistic model is that a fraction of the entries of Ax are corrupted arbitrarily and that all the entries are corrupted by small errors. These small errors can be thought of as quantization error, channel noise, etc. Thus we are in the setup $y = Ax + e + z$ where e is a sparse vector and $z \in \mathbb{R}^m$ represents the small errors.

As we are now corrupting all the entries of Ax, we can no longer hope for exact recovery of x. However, one might expect that it is possible to recover x accurately. If there were no gross errors, a reasonable method to reconstruct x is via least-squares

$$x^{Ideal} = (A^*A)^{-1}A^*y,$$

which has the recovery error

$$\|x^{Ideal} - x\|_{\ell_2} = \|(A^*A)^{-1}A^*z\|_{\ell_2}.$$

The question we then ask in our new setup is whether it is possible to do almost as well as this ideal error without knowing e in advance and in a computationally tractable manner. This is indeed the case, and two different reconstruction methods, both of which involve solving a convex optimization problem, can be used. For simplicity, we will assume from now on that the columns of A are orthonormal.

2.1 Decoding by second-order cone programming

Consider the following convex program, which can be recast as a second-order cone program.

$$(SOCP) \quad \min \|y - A\tilde{x} - \tilde{z}\|_{\ell_1} \quad \text{such that} \quad \|\tilde{z}\|_{\ell_2} \le \epsilon,\ A^*\tilde{z} = 0.$$

In [1] the following theorem is proved:

Theorem 2.1 *If A obeys certain conditions (which depend on the sparsity of e) and $\|P_{V^\perp}z\|_{\ell_2} \le \epsilon$, then the solution $\hat{x}$ to (SOCP) obeys*

$$\|\hat{x} - x\|_{\ell_2} \le C \cdot \frac{\epsilon}{\sqrt{1 - n/m}} + \|x^{Ideal} - x\|_{\ell_2}.$$

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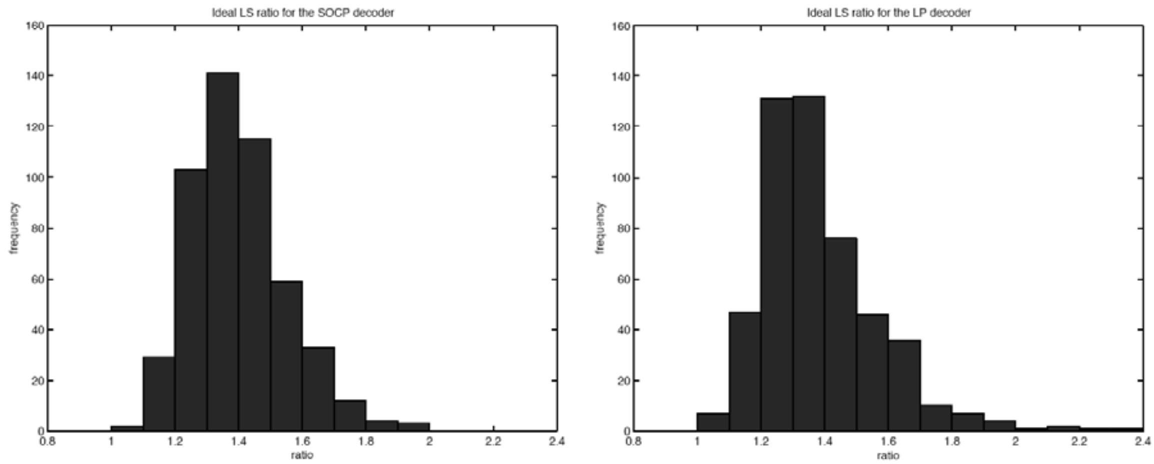

Fig. 1 Histograms of $\|\hat{x} - x\|/\|x^{Ideal} - x\|$ for 500 repeated numerical experiments where A, z have Gaussian entries and e has 10% nonzero entries. In the left figure, $\hat{x}$ is obtained by solving (SOCP) and in the right by solving (LP). See [1] for more details.

While we have omitted the precise statement of the conditions on A in the interest of limited space, we note that it is also shown in [1] that if the (orthonormal) columns of A are selected uniformly at random, then the above estimate holds with overwhelming probability if e is sufficiently sparse.

If z is taken to be Gaussian and the ratio m/n is fixed, we obtain the following corollary.

Corollary 2.2 *If $z \sim N(0, I\sigma^2)$ and $m = 2n$, we have that $\|\hat{x} - x\|_{\ell_2}^2 \leq C \cdot m\sigma^2$ with overwhelming probability.*

Because $\mathbb{E}\|x^{Ideal} - x\|_{\ell_2}^2 = n\sigma^2 = m\sigma^2/2$ (recall that the columns of A are orthonormal), we get that the SOCP reconstruction error is within a constant factor $2C$ of the ideal MSE. Numerical experiments suggest that the constant is small. See Figure 1 for the results of 500 experiments which show that, in the set up of the experiment, $2C$ almost never exceeded 2.

2.2 Decoding by linear programming

One could also attack the coding problem via linear programming. Consider the following program which can be reformulated as a linear program.

$$(LP) \quad \min \|y - A\tilde{x} - \tilde{z}\|_{\ell_1} \quad \text{such that} \quad \|\tilde{z}\|_{\ell_\infty} \leq \lambda,\ A^*\tilde{z} = 0.$$

Theorem 2.3 *If A obeys certain conditions (which depend on the sparsity k of e), and $\|P_{V^\perp} z\|_{\ell_\infty} \leq \lambda$, the solution $\hat{x}$ to (LP) obeys*

$$\|\hat{x} - x\|_{\ell_2} \leq C \cdot \sqrt{k} \cdot \frac{\lambda}{1 - n/m} + \|x^{Ideal} - x\|_{\ell_2}.$$

There are several differences to note between the two methods. One is that the reconstruction error scales differently, so that in certain setups one method may be preferable to the other. Also, it is interesting to note that in the (LP) decoder, there is an explicit dependence on the sparsity k of e which is not present in the (SOCP) decoder. It is possible that for very sparse gross errors the (LP) decoder is more accurate than its (SOCP) counterpart. In our numerical experiments both methods perform very similarly.

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References

[1] E. Candès and P. Randall, Highly Robust Error Correction by Convex Programming. Submitted. Available at `http://arxiv.org/abs/cs/0612124`.

[2] E. Candès and T. Tao, Decoding by Linear Programming. IEEE Trans. Inform. Theory, **51**(12), 4203-4215 (2005).