

Crust–core coupling and r-mode damping in neutron stars: a toy model

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ABSTRACT

r-modes in neutron stars with crusts are damped by viscous friction at the crust–core boundary. The magnitude of this damping, evaluated by Bildsten & Ushomirsky (BU) under the assumption of a perfectly rigid crust, sets the maximum spin frequency for neutron stars spun up by accretion in low-mass X-ray binaries (LMXBs). In this paper we explore the mechanical coupling between the core r-modes and the elastic crust, using a toy model of a constant-density neutron star having a crust with a constant shear modulus. We find that, at spin frequencies in excess of ≈ 50 Hz, the r-modes strongly penetrate the crust. This reduces the relative motion (slippage) between the crust and the core compared with the rigid-crust limit. We therefore revise down, by as much as a factor of 10^2 – 10^3 , the damping rate computed by BU, significantly reducing the maximal possible spin frequency of neutron stars with solid crusts. The dependence of the crust–core slippage on the spin frequency is complicated, and is very sensitive to the physical thickness of the crust. If the crust is sufficiently thick, the curve of the critical spin frequency for the onset of the r-mode instability becomes multivalued for some temperatures; this is related to avoided crossings between the r-mode and higher-order torsional modes in the crust. The critical frequencies are comparable to the observed spins of neutron stars in LMXBs and millisecond pulsars.

Key words: dense matter – gravitation – stars: neutron – stars: oscillations – stars: rotation.

1 INTRODUCTION AND BASIC ARGUMENT

Gravitational radiation offers an exciting possibility to set an upper limit on the spin frequencies of accretionally spun-up neutron stars in close binaries (Bildsten 1998). The source for gravitational radiation may be the mass quadrupole supported by static stresses within the neutron star crust (Bildsten 1998; Ushomirsky, Cutler & Bildsten 2000), or the current quadrupole from an $l = m = 2$ r-mode, which is driven by the gravitational radiation reaction (Andersson et al. 1999; Bildsten 1998; for a review of the r-mode instability see, for example, Friedman & Lockitch 1999; for the discovery paper see Andersson 1998). In this paper we concentrate on the second option, namely that accretional spin-up triggers the r-mode instability, and the resulting spin-down torque from gravitational radiation reaction balances or exceeds the accretion torque. The instability is triggered when the r-mode growth rate due to the gravitational radiation reaction (which is a sharply increasing function of the stellar spin frequency) becomes equal to the r-mode viscous damping rate.

The viscous damping time-scale for r-modes was first computed by Lindblom, Owen & Morsink (1998). They assumed that a neutron star is a fluid ball with a polytropic equation of state, and that the viscosity is set by neutron–neutron scattering in the nuclear matter. Their viscous damping implied that, if the neutron star core temperature $T \gtrsim 10^8$ K, which is the case for the rapidly accreting neutron stars in LMXBs (Brown & Bildsten 1998; Brown 2000), the r-mode instability is triggered at a stellar spin frequency just above 100 Hz. This is inconsistent with observations: most neutron stars in LMXBs are believed to be spinning with frequencies of about 300 Hz (see van der Klis 2000 for a review), and the existence of two 1.6-ms radio pulsars means that rapidly rotating neutron stars are somehow formed in spite of the r-mode instability (Andersson et al. 1999). The r-mode damping rate would have to be ~ 250 higher than that computed by Lindblom et al. (1998) to allow for such high spins.

Lindblom et al. (1998) conjectured that, for the range of temperatures relevant for LMXBs, mutual friction between the neutron and proton superfluids will provide the missing damping. However, tour-de-force calculations of Lindblom & Mendell (1999) showed that, except for ~ 2 per cent of the allowed parameter space, mutual friction gives only a modest

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increase of the r-mode damping rate, insufficient to account for the observed high spins of neutron stars.

Bildsten & Ushomirsky (2000; hereafter BU) argued that the crust plays an important role in r-mode damping. In their picture, the r-mode lives in the liquid core, and most of the dissipation occurs in a thin (a few centimetres) viscous boundary layer between the oscillating core and the motionless (in the rotating frame) solid crust. The damping that BU computed is a factor of $\sim 10^5$ higher than that of Lindblom et al. (1998); it implies that the r-modes would not become unstable unless the spin frequency is close to break-up, which is significantly higher than the observed LMXB spins. Andersson et al. (2000) and Rieutord (2000) have revised the BU estimate, down and up respectively, by factors of order unity. Both of these corrections are valid; the net result is that the BU estimate is almost unchanged. Finally, Lindblom et al. (2000) explored the dependence of the critical frequency on the nuclear equation of state and found that, for realistic equations of state, the critical frequency is ~ 25 – 50 per cent larger than the original BU estimate. However, there is still a substantial correction unaccounted for in the previous work. In this paper, we show that, for a considerable fraction of the parameter space, the boundary-layer damping rate is a factor of 10^2 – 10^3 smaller than the BU estimate, and hence the critical frequencies for the onset of the r-mode instability are correspondingly lower, ~ 200 – 600 Hz.

BU considered only the leading-order transverse-motion pattern of the r-modes, and assumed that these motions do not penetrate into the crust. They estimated the magnitude of the viscous coupling between these transverse motions and the crust, and found it to be small. Therefore, their picture is that of a liquid in a bucket – the amplitude of the relative motion (slippage) between the crust and the core is just equal to the amplitude of the fluid motion at the boundary. However, small radial motions of the r-mode, neglected by BU, are very effective

at coupling the crust and the core. Above a small critical frequency, r-modes do *strongly* penetrate into the crust, and the crust and the core oscillate almost in unison with very similar amplitudes. The slippage δu in this regime is only a small fraction of the total r-mode motion, u , with typical values of $\delta u/u \sim 0.05$ – 0.1 , which results in a correction of a factor of $(\delta u/u)^2$ to the BU viscous boundary-layer damping rate.

The fact that the crust is not rigid is central to our story. In fact, the crust is more like jello: the shear modulus is much smaller than the bulk modulus. The $l = m = 2$ torsional mode in a non-rotating crust, which has the same angular displacement pattern as the corresponding r-mode, has a frequency $f_{cr} \approx 50$ Hz (McDermott et al. 1988). This frequency is a few times lower than the r-mode frequency in rapidly rotating stars, indicating that the elastic restoring force is quite weak compared to the Coriolis force which is responsible for the r-mode oscillations. Therefore, at sufficiently high spin frequencies, one would expect the crust to oscillate more or less like a liquid, with elasticity only slightly modifying the mode structure. Indeed, if the shear modulus μ of the crust were exactly zero, i.e. if the crust were completely fluid, there would be no slippage δu between the crust and the core. Since μ is non-zero but small, one would expect the slippage to be proportional to the ratio of the elastic restoring force to the Coriolis force, i.e. roughly $\delta u/u \sim (f_{cr}/f_{rmode})^2$. However, our numerical work shows that the slippage is not a monotonically decreasing function of the spin frequency; it rises sharply at the frequencies of avoided crossings between the r-mode and the higher-order torsional modes of the crust. Fig. 1 shows the slippage as a function of the spin frequency computed for two neutron star models with different crust thicknesses (10 and 20 per cent of the stellar radius).

We begin in Section 2 by outlining the equations governing the pulsations of constant-density stars with fluid cores and solid crusts and presenting the numerical results for the dependence of the

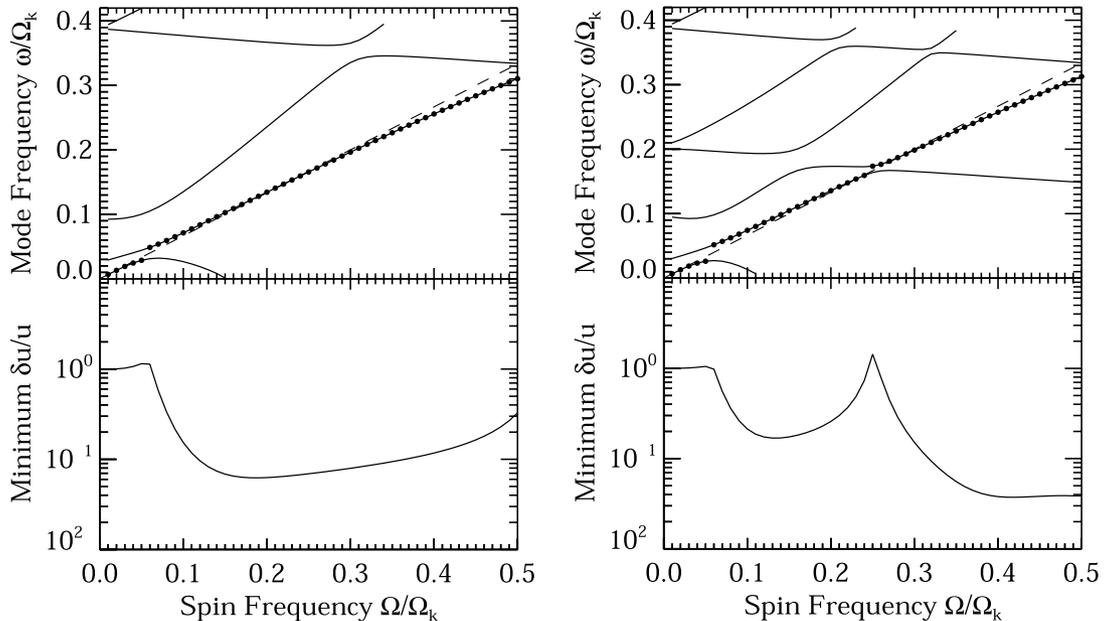


Figure 1. Top left: Mode angular frequency ω (in the rotating frame) as a function of the angular spin frequency Ω for the five lowest-frequency modes of the thin-crust model. The dashed line shows the ‘classical’ r-mode dispersion relation, $\omega = 2\Omega/3$, and the dots mark the mode closest to this relation, which has the smallest slippage. Bottom left: absolute value of the slippage of the mode marked with thick dots in the top panel (i.e., the minimum slippage) as a function of Ω . Right: same as left, but for the thick-crust model.

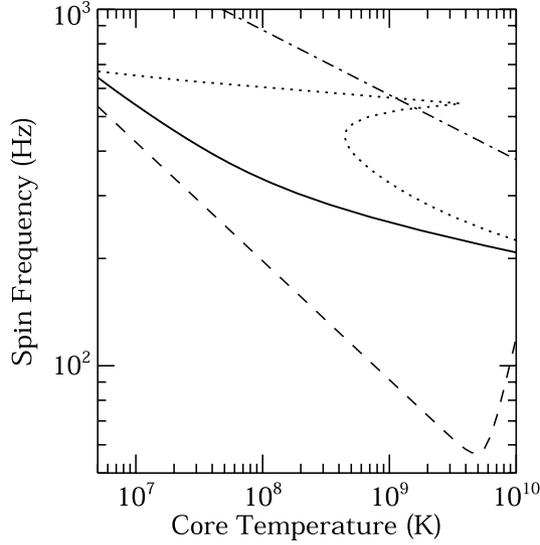


Figure 2. Critical frequencies as functions of the core temperature for the modes in the thin-crust model (solid line) and the thick-crust model (dotted line). For comparison, we also display the critical frequencies for the $n = 1$ polytrope fluid model (dashed line), and for a model with a crust and $\delta u/u = 1$ independent of the spin (dot-dashed line).

slippage $\delta u/u$ on the spin frequency. In Section 3 we use the computed slippages to construct the critical r -mode stability curves (see Fig. 2). The resulting curves are quite sensitive to the thickness of the neutron star crust. In particular, for a sufficiently thick crust, the stability curve has a peculiar multivalued structure, which is a consequence of the non-monotonic dependence of the relative slippage on the spin frequency. The stability curve determines the maximum spin of a neutron star as a function of temperature; we find that it generally goes through the region of the observed spin frequencies of neutron stars in LMXBs and in millisecond radio pulsars. Broad implications of our results are discussed in Section 4.

2 CORE–CRUST BOUNDARY SLIPPAGE

2.1 Basic formalism

Oscillation modes of a rotating neutron star with an elastic crust were analysed by Lee & Strohmayer (1996; hereafter LS). Their analysis was performed for the case of slow rotation, which was treated as a perturbation to other restoring forces, and the oblateness of the equilibrium configuration was neglected. We broadly utilize their general formalism; however, we derive equations specialized to the constant-density case. For r -modes, the correction to the eigenfrequency due to the non-sphericity of the equilibrium configuration is of the same order as the correction due to the radial motion (Provost et al. 1981). However, we expect that the oblateness of the star will not change the *qualitative* picture for the crust–core slippage: at high enough spin frequencies, the core and the crust will move almost in unison. The *quantitative* effects of non-sphericity are not treated in this paper.

For our toy model, we assume that the outer part, $r_{\text{cr}} < r < R$, of the neutron star of radius R possesses a small shear modulus $\mu(r)$, such that $\mu(r) = \text{constant}$. The neutron star fluid is assumed to be incompressible, has a constant density ρ , and is rotating with angular frequency Ω .

The general displacement \mathbf{u} of a fluid element, in a frame rotating with the neutron star, is given by (Zahn 1966; LS):

$$\mathbf{u} = \sum_l [u_l(r)Y_{lm}\hat{\mathbf{r}} + v_l(r)\nabla Y_{lm} + w_l(r)\hat{\mathbf{r}} \times \nabla Y_{lm}] e^{i\omega t}, \quad (1)$$

where ω is the mode frequency in the rotating frame, m is the fixed azimuthal number, and ∇Y_{lm} is the gradient of the spherical harmonic evaluated on a unit sphere. We are interested in the $l = m$ r -mode. For a purely fluid star, to $\mathcal{O}(\Omega/\Omega_K)^2$, where $\Omega_K = (GM/R^3)^{1/2}$ is the Keplerian angular velocity, the displacement for this mode is given by

$$\mathbf{u} = (w_l\hat{\mathbf{r}} \times \nabla Y_{ll} + v_{l+1}\nabla Y_{l+1, l} + u_{l+1}Y_{l+1, l}\hat{\mathbf{r}}) e^{i\omega t}; \quad (2)$$

that is, v_{l+1}/w_l and u_{l+1}/w_l are of order $(\Omega/\Omega_K)^2$. In the solid crust, the displacement vector need not satisfy the ordering assumed in equation (2). However, at least for modes with ω close to $2\Omega/3$, the Coriolis force is the dominant restoring force, and we expect v_{l+1}/w_l and u_{l+1}/w_l to be small; this ordering is confirmed by our numerical results (see Section 3).

We now define dimensionless frequencies $\tilde{\Omega} = \Omega/\Omega_K$ and $\tilde{\omega} = \omega/\Omega_K$, as well as the dependent variables similar to those utilized by LS:

$$z_1 = \frac{u_{l+1}}{r}, \quad (3a)$$

$$z_2 = 2 \frac{du_{l+1}}{dr}, \quad (3b)$$

$$z_3 = \frac{v_{l+1}}{r}, \quad (3c)$$

$$z_4 = \frac{dv_{l+1}}{dr} - \frac{v_{l+1}}{r} + \frac{u_{l+1}}{r}, \quad (3d)$$

$$z_5 = i \frac{w_l}{r}, \quad (3e)$$

$$z_6 = i \left(\frac{dw_l}{dr} - \frac{w_l}{r} \right). \quad (3f)$$

In the constant-density case, z_2 is not a dynamical variable, as it can be eliminated using the continuity equation, $\nabla \cdot \mathbf{u} = 0$, or

$$\frac{1}{2}z_2 + 2z_1 - (l+1)(l+2)z_3 = 0. \quad (4)$$

However, even in the $\rho = \text{constant}$ case, the Eulerian pressure perturbation is non-zero, and, to leading order, is given by $\delta p = \delta p_{l+1}(r)Y_{l+1, l}$. We define the dimensionless pressure perturbation, $\tilde{p} = \delta p_{l+1}/\rho g r$, where $g = GM(r)/r^2$ is the local gravitational acceleration. Finally, we define the dimensionless shear modulus, $\tilde{\mu} = \mu/\rho g r$. Throughout this paper, we use $\tilde{\mu} = 1.5 \times 10^{-4}(R/r)^2$, chosen so that the frequency of the $l = m = 2$ fundamental torsional crustal mode is close to the ≈ 50 Hz value computed by McDermott et al. (1988) for a non-rotating neutron star; the shear-wave speed corresponding to this choice is $c_t = (\mu/\rho)^{1/2} = 1.7 \times 10^8 \text{ cm s}^{-1}$.

These variables allow us to write the equations of motion in the

crust as a system of first-order differential equations:

$$r \frac{dz_1}{dr} = (l+1)(l+2)z_3 - 3z_1, \quad (5a)$$

$$r \frac{d\tilde{p}}{dr} = -2\tilde{p} + \tilde{\omega}^2 z_1 - 2\tilde{\omega}\tilde{\Omega}l z_3 - 2\tilde{\omega}\tilde{\Omega}J'_{l+1} z_5 \\ + \tilde{\mu}(l+1)(l+2)(z_4 + 2z_3 - 2z_1), \quad (5b)$$

$$r \frac{dz_3}{dr} = z_4 - z_1, \quad (5c)$$

$$r \frac{dz_4}{dr} = -2z_1 + 2[l(l+3) + 1]z_3 - 3z_4 \\ + \frac{1}{\tilde{\mu}} \left[\tilde{p} + \frac{2lJ'_{l+1}\tilde{\omega}\tilde{\Omega}}{l+1} z_5 + \frac{2l\tilde{\omega}\tilde{\Omega}}{(l+1)(l+2)} (z_1 + z_3) - \tilde{\omega}^2 z_3 \right], \quad (5d)$$

$$r \frac{dz_5}{dr} = z_6, \quad (5e)$$

$$r \frac{dz_6}{dr} = -3z_6 + \{l(l+1) - 2\}z_5 \\ + \frac{1}{\tilde{\mu}} \left[\left(\frac{2\tilde{\omega}\tilde{\Omega}}{l+1} - \tilde{\omega}^2 \right) z_5 + \frac{2J'_{l+1}\tilde{\omega}\tilde{\Omega}}{l+1} (z_1 + (l+2)z_3) \right], \quad (5f)$$

where $J'_{l+1} = (2l+3)^{-1/2}$. The derivation of these equations is analogous to that of equations (9)–(21) of LS. In the liquid core, the only non-trivial equations are

$$r \frac{d\tilde{p}}{dr} = (l-1)\tilde{p}, \quad (6a)$$

$$r \frac{dz_1}{dr} = -(l+4)z_1 - \left(\tilde{\omega} - \frac{2\tilde{\Omega}}{l+1} \right) \frac{(l+1)^4(2l+3)}{8l\tilde{\Omega}^3} \tilde{p}, \quad (6b)$$

and

$$z_5 = -\frac{(l+1)\tilde{p}}{2lJ'_{l+1}\tilde{\omega}\tilde{\Omega}} \quad (7)$$

(see, for example, Kokkotas & Stergioulas 1999 or LS).

Both at the crust–core boundary, and at the top of the crust, we require the tangential traction to be zero; that is,

$$z_4 = z_6 = 0. \quad (8)$$

We hence neglect the viscous coupling between the transverse r-mode motions and the crust. This is the coupling considered and found negligible by BU. At the crust–core boundary the radial displacement z_1 is continuous, and so is the radial component of the traction, $\Delta p - 2\mu du_{l+1}/dr$; or

$$z_1(r_c^-) = z_1(r_c^+), \quad (9a)$$

$$\tilde{p}(r_c^+) - \tilde{\mu}[2(l+1)(l+2)z_3(r_c^+) - 4z_1] = \tilde{p}(r_c^-). \quad (9b)$$

At the top of the crust, the radial component of the traction is zero:

$$\tilde{p}(R) - z_1(R) - \tilde{\mu}[2(l+1)(l+2)z_3(R) - 4z_1(R)] = 0 \quad (10)$$

Equations (8)–(10) provide the boundary conditions for the equations of motion. We normalize our solutions to have unit transverse displacement just inside the liquid core:

$$z_5(r_c^-) = 1. \quad (11)$$

With this normalization, the slippage is

$$\delta u/u = \{[z_5(r_c^-) - z_5(r_c^+)]^2 + [z_3(r_c^-) - z_3(r_c^+)]^2\}^{1/2}$$

(where we used our assumption, $z_3 \ll z_5$).

2.2 Numerical results

The Keplerian spin frequency Ω_K , the shear modulus μ , and the ratio of the crust thickness to the stellar radius r_c/R are the only parameters necessary to describe a constant-density star (we neglect the thin surface ocean and the modes that are present in it, as they are unlikely to influence the crust–core slippage). The crust thickness is roughly proportional to the scaleheight at the crust–core boundary, and is hence $\propto R^2/M$. In order to survey the range of parameters, we carry out our calculations for two models: a thin-crust model with $r_c/R = 0.9$, and a thick-crust model with $r_c/R = 0.8$. Because of the differences in the normal-mode spectrum between these two cases, we shall see that the r-mode instability curves for the two models have qualitatively different behaviour in the temperature range of interest for accreting neutron stars, namely $T = 10^8$ – 10^9 K.

The left panel of Fig. 1 displays the results of our calculations for $l = m = 2$ for the thin-crust model, while the right panel shows the results for the thick-crust model. Let us first consider the top plots, which show the dispersion relation, $\omega(\Omega)$, for the two models. Contrary to the case of a purely fluid star, where, in the isentropic case, there is only one ‘classical’ $l = m$ r-mode, all the modes displayed in Fig. 1 have displacements of the form given by equation (2), with $w_l \propto r^l$ in the core. However, only the mode with ω closest to the ‘classical’ r-mode value, $2\Omega/3$, is of interest to us. This mode is denoted by thick dots in Fig. 1. When ω is much different from $2\Omega/3$, the displacements in the crust are much bigger than the displacements in the core, resulting in large slippage, and, hence, large damping. In essence, unless the mode frequency is close to the r-mode value, the elasticity of the crust plays the dominant role in determining the mode properties, and the mode is a crustal mode, rather than a core r-mode.

Consider the low-spin-frequency [$\Omega \ll (\mu/\rho R^2)^{1/2} \approx 0.05\Omega_K$] limit of our results. In this case, the mode with the lowest frequency in Fig. 1 is the ‘classical’ $l = m$ r-mode, which has the transverse displacement $w_2 \propto r^2$ in the core and negligible displacements in the crust. This behaviour (liquid in a bucket) is the approximation used by BU. The modes at higher frequencies are torsional crustal oscillations, which have appreciable displacements in the crust, and negligible displacements in the core. The fundamental (nodeless) torsional mode has $\omega = 0.03\Omega_K$ for both models. The lack of sensitivity of the frequency of this mode to the thickness of the crust was first noted by Hansen & Cioffi (1980). The modes at higher frequencies are overtones, and we note that the mode spectrum for the thick-crust model is denser than that of the thin-crust model. Qualitatively, the mode frequency is determined by fitting a certain number of nodes of the eigenfunction between the boundaries of the crust, and hence a thicker crust supports more modes.

As the spin frequency is increased beyond $\Omega \approx (\mu/\rho R^2)^{1/2}$, the behaviour of the modes changes. The frequency of the core r-modes rises to meet with the frequencies of the modes that originally resided in the crust, resulting in avoided crossings. At the first avoided crossing, the core r-mode is coupled with the lowest-order torsional mode of the crust; at this avoided crossing the core r-mode strongly penetrates the crust. As the spin frequency increases beyond the first crossing, the crust–core slippage decreases along the branch closest to the ‘classical’ r-mode frequency, $\omega = 2\Omega/3$ (marked by a dashed line in Fig. 1). Since the boundary-layer damping rate is proportional to $(\delta u/u)^2$, this branch is most interesting for us as it corresponds to the r-mode with the lowest damping, and, hence, the greatest instability.

The relative slippage as a function of the spin frequency is plotted in the bottom panels of Fig. 1. We always plot slippage of the r-mode from the branch with the lowest damping, marked by thick dots in Fig. 1 (i.e. the most unstable r-mode). As the spin frequency increases beyond the first crossing, the slippage decreases to ≈ 0.1 . However, the thick-crust model has another avoided crossing at $\Omega \approx 0.25\Omega_K$, and the slippage rises at this frequency because the core r-mode couples resonantly with a higher-order torsional crustal mode. In the thick-crust model, the ‘valley’ in which the second mode dominates only extends to $\Omega \approx 0.25\Omega_K$. After the avoided crossing, the third mode has the smallest slip, $4 \times 10^{-2} \leq \delta u/u \leq 1$. Note that the slip value is ≈ 1 near the avoided crossing and is ≥ 0.1 for $0.25\Omega_K \leq \Omega \leq 0.3\Omega_K$, or for spin frequencies between ~ 500 and 600 Hz. In this regime, the viscous boundary-layer damping rate approaches the high BU value, and so the critical spin frequency for the r-mode instability is high as well. As we now show, the shape of the r-mode stability curve depends quite sensitively on the thickness of the crust.

3 CRITICAL STABILITY CURVES

The dissipation rate in the viscous boundary layer is set by the amplitude of the relative motion between the core and the crust. BU assumed that the crust remains static in the rotating frame, and hence the amount of boundary slippage $\delta u/u = 1$. As shown in the previous section, however, the slippage $\delta u/u$ is generally significantly less than 1. Therefore, the BU estimate of the damping rate has to be multiplied by $(\delta u/u)^2$. The corrected r-mode damping rate is [cf. equation (4) of BU]

$$\frac{1}{\tau_{\text{vbl}}} \approx 0.01 \text{ s}^{-1} \frac{R_6^2 F^{1/2} \rho_b}{M_{1.4} T_8 \rho} \left(\frac{f}{\text{kHz}} \right)^{1/2} \left(\frac{\delta u}{u} \right)^2, \quad (12)$$

where $f = \Omega/2\pi$ is the neutron star spin frequency, R_6 , $M_{1.4}$ and T_8 are the radius, the mass and the core temperature of the neutron star in units of 10 km, $1.4 M_\odot$, and 10^8 K, respectively, ρ is the density at the crust–core interface, $\rho_b = 1.5 \times 10^{14} \text{ g cm}^{-3}$ is the estimated value of this density (Pethick et al. 1995), and F is the fitting parameter used by Cutler & Lindblom (1987) to parametrize the shear viscosity found by Flowers & Itoh (1979). The Ekman layer correction introduced by Rieutord (2000) for the constant-density case is of order unity and is not included in equation (12); see Lindblom et al. (2000) for a treatment applicable for more realistic, non-uniform-density stars. We also neglected the contribution of the elastic energy of the crust to the total mode energy, as well as the small deviation of the mode frequency from $\omega = 2\Omega/3$.

The r-mode growth rate due to gravitational radiation is given by

$$\frac{1}{\tau_{\text{gw}}} \approx 0.02 \text{ s}^{-1} M_{1.4} R_6^4 \left(\frac{f}{\text{kHz}} \right)^6. \quad (13)$$

Note that this expression is different from the one used by BU. Andersson et al. (2000) pointed out that BU incorrectly rescaled the τ_{gw} used by Lindblom et al. (1998) to the canonical $R = 10$ km. Andersson et al. (2000) gave their own estimate of the r-mode growth rate for a constant-density star. Equation (13) is the correct τ_{gw} for an $n = 1$ polytrope, neglecting the deviation of the transverse displacement from $w_l \propto r^l$ in the crust, as well as the small deviation of the mode frequency from $\omega = 2\Omega/3$.

While we computed the slippage using a constant-density model, we utilize τ_{vbl} (equation 12) and τ_{gw} (equation 13), which are appropriate for a more realistic $n = 1$ polytrope. Compared with more realistic models, the use of τ_{gw} and τ_{vbl} for a constant-density

model overestimates the gravitational radiation growth rate (by putting more mass at larger radii), and underestimates the strength of the boundary-layer damping (by virtue of a larger mode energy). Both of these effects result in lower critical frequencies for the onset of the r-mode instability. Slippage calculations for more realistic models are in progress.

The stability curve is obtained by equating the gravitational-radiation-induced growth rate with the viscous dissipation rate. Since we are primarily interested in temperatures below the melting temperature of the crust, $\approx 10^{10}$ K, we neglect the dissipation due to bulk viscosity. We also ignore the dissipation due to the shear in the interior of the star, as well as any potentially important dissipation mechanisms that may be operating in the crust, and presume that the viscous boundary layer is the dominant energy sink. Since we do not have an analytical expression for the slippage, it is more convenient to evaluate the critical temperature, T_8 , such that $\tau_{\text{gw}} = \tau_{\text{vbl}}$:

$$T_8 = 0.5 \frac{F^{1/2} \rho_b}{R_6^2 M_{1.4}^2 \rho} \left(\frac{f}{\text{kHz}} \right)^{-5.5} \left(\frac{\delta u}{u} \right)^2. \quad (14)$$

Fig. 2 shows the critical stability curves for models with thin (solid line) and thick (dotted line) crusts, with F , $M_{1.4}$, R_6 and ρ_b/ρ all equal to one. In other words, the proper label for the abscissa of this plot is $TR_6^2 M_{1.4}^2 F^{-1/2} \rho/\rho_b$. For comparison, we also plot the stability curve for a purely fluid $n = 1$ polytrope (Lindblom et al. 1998, 1999, dashed line), as well as the curve with $\delta u/u = 1$ independent of spin frequency (dash-dotted line; cf. BU). Neglecting the shear viscosity in the interior of the star is clearly justified.

For the thin-crust model (solid line), the slippage in the frequency range of interest does not rise or fall sharply, and so the critical frequency is a single-valued, monotonically decreasing function of T . However, for the neutron star with a thick crust, the slippage and the damping rise sharply at the spin frequency close to the second avoided crossing. Therefore, the critical temperature is not a monotonically decreasing function of the spin frequency f , and the stability curve is multivalued. Such a non-trivial stability curve should be interpreted in terms of the critical temperature, rather than the critical frequency: for temperatures larger than the critical one given by equation (14), i.e. for points in the T - f plane that lie to the right of the viscous-boundary-layer part of stability curve, the r-mode is unstable and grows. Otherwise, the r-mode is stable and decays.

4 DISCUSSION, BUT NO BETS

Observationally, the spin frequencies of neutron stars in LMXBs have been inferred from the QPO-like flux modulation during the type I X-ray bursts (the so-called burst QPOs; see van der Klis 2000 for a review). In two out of the six sources displaying the burst QPOs, the frequency is close to 300 Hz, while in the remaining four it is close to 600 Hz. In one of the latter four systems, Miller (1999) found a subharmonic at about 300 Hz, and has argued that this subharmonic corresponds to the true spin frequency. The same relation is sometimes believed to hold for the other ≈ 600 Hz bursters. In addition, ≈ 300 Hz spin is often argued on the basis of a beat-frequency interpretation of the kilohertz QPO peak separation in all six of the burst QPO systems, as well as for the neutron stars in 13 additional LMXBs that do not display burst QPOs. On the other hand, there is no compelling theoretical picture for why the

subharmonics of the ≈ 600 Hz burst QPOs should be so weak or invisible, so a bimodal spin distribution is certainly not ruled out.

The new critical stability curves computed in this paper for two toy models of neutron stars cut right through the observed range of spins both for neutron stars in LMXBs and for millisecond pulsars, for temperatures of interest for accreting neutron stars (between 10^8 and 10^9 K; Brown & Bildsten 1998; Brown 2000). An accreting neutron star can continue its spin-up so long as it is located to the left of the stability curve in Fig. 2. However, once it crosses the critical stability curve and is located to the right of it, the r-mode will grow and halt the spin-up, probably forcing the neutron star into a thermal runaway cycle (Levin 1999; Spruit 1999). The details of what happens during this runaway and how violent it is are a topic of current research. For example, the displacements induced in the crust may cause it break when the strain (which is of the order of the r-mode amplitude α , as defined in Lindblom et al. 1998) exceeds a critical value ($\approx 10^{-2}$, see Section 6.1 of Ushomirsky et al. 2000 and references therein for a summary of estimates of crustal breaking strains), or melt if the heating due to the viscous boundary layer raises its temperature above $\approx 10^{10}$ K (see Lindblom et al. 2000 for the calculation of the r-mode amplitude required to melt the crust). The evolution of the spin frequency and the temperature depends on the saturation of the r-mode growth, which is not currently understood. Despite these uncertainties, it is clear that, if the r-mode instability operates in accreting neutron stars, the non-trivial shape of the stability curve and its sensitivity to the crustal structure will be reflected in a peculiar, non-uniform spin distribution of accreting neutron stars and millisecond pulsars.

In this paper we adopted a very simple model of the neutron star crust. However, the salient feature of the problem, the fact that the shear restoring force in the crust is much smaller than the Coriolis force, is present in our simple model. Therefore, qualitative results of our calculation should persist even when the approximation of constant-density, incompressible crust is relaxed. The participation of the crust in the r-mode oscillation will significantly reduce the effectiveness of boundary-layer damping, thus lowering, by a factor of several (compared to the absolutely rigid crust values), the critical frequencies for the onset of the r-mode instability. The presence of avoided crossings will be reflected in the shape and possible multivaluedness of the stability curve. Stratified, compressible crusts have more modes in the relevant frequency range, and hence the avoided crossings will be more numerous than in the case of an incompressible constant-density crust.¹ Realistic neutron stars, therefore, might have a more complicated shape of the r-mode stability curve than shown in Fig. 2.

Much work is necessary in order to make robust comparisons with observations. R-mode calculations for more realistic neutron star models, taking into account the complicated structure of the crust, are in progress. We have not considered the possible superfluid nature of neutrons and protons at the crust–core interface. In particular, we have not considered the pinning of the

superfluid vortices to the crust. We assumed that the boundary layer is laminar, which may not be true for large r-mode amplitudes. Significant magnetic field at the crust–core boundary may also affect the crust–core coupling (Rezzolla et al. 2000; Ho & Lai 2000). Thus, our estimate of the r-mode damping may change when more complex physics is considered.

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¹ Since completing this work, we have learned of an independent and simultaneous work by Yoshida & Lee (2000). They computed the mode structure for more realistic neutron star crusts than those discussed in our paper, and also demonstrated the existence of avoided crossings. However, these authors did not discuss the reduction of the crust–core slippage at high spin frequencies and its implication for the r-mode stability curve.