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APPLICATIONS OF MODEL THEORY TO C*-DYNAMICS

EUSEBIO GARDELLA AND MARTINO LUPINI

ABSTRACT. We initiate the study of compact group actions on C*-algebras from the perspective of model theory, and present several applications to C*-dynamics. Firstly, we prove that the continuous part of the central sequence algebra of a strongly self-absorbing action is indistinguishable from the continuous part of the sequence algebra, and in fact equivariantly isomorphic under the Continuum Hypothesis. As another application, we present a unified approach to several dimensional inequalities in C*-algebras, which is done through the notion of order zero dimension for an (equivariant) *-homomorphism. Finiteness of the order zero dimension implies that the dimension of the target algebra can be bounded by the dimension of the domain. The dimension can be, among others, decomposition rank, nuclear dimension, or Rokhlin dimension. As a consequence, we obtain new inequalities for these quantities.

As a third application we obtain the following result: if a C*-algebra A absorbs a strongly self-absorbing C*-algebra D , and α is an action of a compact group G on A with finite Rokhlin dimension with commuting towers, then α absorbs any strongly self-absorbing action of G on D . This has a number of interesting consequences, already in the case of the trivial action on D . For example, we deduce that D -stability passes from A to the crossed product. Additionally, in many cases of interest, our result restricts the possible values of the Rokhlin dimension to 0, 1 and ∞ , showing a striking parallel to the behavior of the nuclear dimension for simple C*-algebras. We also show that an action of a finite group with finite Rokhlin dimension with commuting towers automatically has the Rokhlin property if the algebra is UHF-absorbing.

1. INTRODUCTION

The use of (central) sequence algebras in the theory of operator algebras has a long history, dating back to McDuff's characterization of factors which absorb the hyperfinite II_1 -factor with separable predual R , as those whose central (W*-)sequence algebra contains a unital copy of R [55]. Applications in the context of C*-algebras are both abundant and far-reaching, and they often appear in connection with classification of C*-algebras. For instance, the fact that the central (C*-)sequence algebra (with respect to a nonprincipal filter) of a Kirchberg algebra is purely infinite and simple is a major cornerstone in the work of Kirchberg and Phillips [49], which is the starting point of the classification of Kirchberg algebras; see [47] and [58].

Another major application of central sequence algebras has been to the theory of strongly self-absorbing C*-algebras [67], which have become a fundamental part of Elliott's classification programme of nuclear C*-algebras. Indeed, the tight connections that strongly self absorbing C*-algebras have with classification, have prompted a deeper study of ultrapowers and (central) sequence algebras. In this context, the use of model-theoretic methods has become predominant [16, 18, 19, 21–23]. The most prominent features of ultrapowers are model-theoretic in nature, and include what model theorists usually refer to as Los' theorem and countable saturation. Even though relative commutants do not have a satisfactory model-theoretic analog, it is shown in [20] that for a strongly self-absorbing C*-algebra, its ultrapower and its relative commutant are indistinguishable, and in fact isomorphic assuming the Continuum Hypothesis (CH).

Ultrapowers (and relative commutants) have also been a crucial tool in the study of group actions on operator algebras. They have been used in the classification of amenable group actions on the hyperfinite II_1 -factor by Connes [14], Jones [43] and Ocneanu [57]. In their proofs, a crucial step is to show that any outer action admits equivariant embeddings of matrix algebras into its relative commutant, a condition that is now known as the Rokhlin

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property. In the context of C^* -algebras, relative commutants were used in connection with the Rokhlin property for group actions in the work of Herman-Jones [37], Kishimoto [51], Izumi [42], Hirshberg-Winter [39], and the first-named author [28]. The study of Rokhlin dimension has also made extensive use of these tools, for example in [28] and [38], as well as the more recent work on strongly self-absorbing actions [64]. As is clear from these works, the use of sequence algebras in the equivariant setting becomes even more delicate when the acting group is not discrete, since a continuous action on an operator algebra may induce a discontinuous action on its relative commutant. As such, equivariant (central) sequence algebras are interesting objects whose systematic study is justified by their wide application in the literature.

The present work takes up this task. For a given compact second countable group G , we consider actions of G on C^* -algebras (G - C^* -algebras) as structures in the framework of continuous model theory. When the group G is finite, one can regard a G -action as a usual metric structure by adding a function symbol for every element of the group. This does not work for a general compact group, since the canonical action on the ultrapower that one obtains in this way is not always continuous; see Example 2.7. On the other hand, adding a sort for the group and enforcing uniform bounds on the continuity moduli of an action would not capture the notion of ultrapower of G -actions. The solution adopted in [33], suggested by the theory of compact quantum groups and their actions on C^* -algebras, consists in replacing in the language for C^* -algebras the sort for the whole C^* -algebra with several sorts for the *isotropy components* of the action, indexed by representations of G on finite-dimensional Hilbert spaces. This gives a language $\mathcal{L}_G^{C^*}$, which has function and relations symbols corresponding to the C^* -algebra operations as well as function symbols for the restriction of the $*$ -homomorphism coding the action to the isotropy components. It is shown in [33] that G - C^* -algebras form an axiomatizable class in such a language, and explicit axioms are provided.

In this paper, we will consider G - C^* -algebras as structures with respect to several other languages. Such languages are very natural, as they correspond to notions of morphisms other than $*$ -homomorphisms—such as completely positive contractive maps, or order zero completely positive contractive maps—that are of crucial importance for the recent theory of C^* -algebras. There are important model-theoretic reasons to consider such languages. Indeed, as the recent work on the model theory of C^* -algebras has shown [19], most of the properties of C^* -algebras considered in the C^* -algebra literature can be captured model-theoretically. As most maps that arise naturally in the applications are not elementary, and often are not even $*$ -homomorphisms, it is important to keep track of the exact *complexity* of formulas needed to describe C^* -algebraic properties, including which *operations* are needed to describe them. As it turns out, the multiplication symbol in many cases can be dispensed of, and replaced with other predicates that capture the ordered operator space structure, or the “order zero” structure of a given C^* -algebra. This careful analysis will make it apparent how various regularity property are automatically preserved by several C^* -algebraic and model-theoretic constructions.

We present a number of applications to C^* -dynamics in Section 4 and Section 5. We focus mainly on strongly self-absorbing actions (in the sense of [64]), actions with finite Rokhlin dimension (in the sense of [40] and [31]), and general dimensional inequalities in C^* -algebras. The main novelty in this part is that we shift the attention from the actions themselves to the study of equivariant maps between them; this is in the spirit of KK -theory and other related theories. In particular, we consider equivariant order zero maps between C^* -dynamical systems; this is inspired in the notion of weak containment for representations and measure-preserving actions of countable groups, which are fundamental in modern ergodic theory and representation theory.

The notion of strongly self-absorbing action has been recently introduced and studied by Szabó in [64, 65], where it is shown that many familiar properties of strongly self-absorbing C^* -algebras have natural analogues for strongly self-absorbing actions. Building on this work, in Section 4 we investigate the model-theoretic properties of strongly self-absorbing actions. In particular, we show that the continuous part of the central sequence algebra of a strongly self-absorbing action is indistinguishable from the continuous part of the sequence algebra, and in fact equivariantly isomorphic assuming CH, thus generalizing results from [20]. We take the occasion to remove an unnecessary assumption present in [20], and observe that all the results hold for reduced products with respect to an arbitrary countably incomplete filter, even without the assumption that the corresponding reduced product be countably saturated. We also show that the classification problem for strongly self-absorbing actions of a fixed compact second countable group on C^* -algebras is smooth in the sense of Borel complexity theory. This is no longer the case for actions with approximately inner half-flip, even if one restricts to actions on the Cuntz algebra \mathcal{O}_2 . Indeed, we observe that the relations of conjugacy and cocycle conjugacy for $\mathbb{Z}/2\mathbb{Z}$ -actions on \mathcal{O}_2 with approximately $\mathbb{Z}/2\mathbb{Z}$ -inner half-flip are complete analytic sets. Most of the results of this section admit natural generalizations to the

case of a locally compact (not necessarily compact) second countable group G . This presents additional technical difficulties, which can be overcome by considering a more general framework than the usual framework for first order logic for metric structures. For the sake of simplicity, we will only consider the case when G is compact.

Section 5 contains applications to dimensional inequalities in C*-algebras. This is done through the notion of (equivariant) order zero dimension (with and without commuting towers) for an (equivariant) homomorphism. The case of dimension zero corresponds to the notion of positive existential embedding, which has been studied in [36] and, under the name of sequentially split *-homomorphism, in [4]. As an example, if $\alpha: G \rightarrow \text{Aut}(A)$ is an action of a compact group G on a C*-algebra A , then the Rokhlin dimension of α is equal to the G -equivariant order zero dimension of the factor embedding $\theta: A \rightarrow C(G, A)$. As an application of the syntactic characterization of G -equivariant order zero dimension together with results from [19], we obtain the following result, which is new in the non-unital case.

Theorem. *Let G be a compact group, and A be a G -C*-algebra A . Then*

$$\dim_{\text{nuc}}(A^G) \leq \dim_{\text{nuc}}(A \rtimes G) \leq (\dim_{\text{Rok}}(A) + 1)(\dim_{\text{nuc}}(A) + 1) - 1$$

and

$$\text{dr}(A^G) \leq \text{dr}(A \rtimes G) \leq (\dim_{\text{Rok}}(A) + 1)(\text{dr}(A) + 1) - 1.$$

We use results from the literature to give many other examples of *-homomorphisms with finite order zero dimension which do not come from group actions. Notable examples are the unital inclusions $\mathcal{O}_\infty \rightarrow \mathcal{O}_2$ and $\mathcal{Z} \rightarrow U$, where U is any UHF-algebra of infinite type. As a consequence of our general results, we recover and extend useful inequalities relating the nuclear dimension, decomposition rank, and Rokhlin dimension of the \mathcal{Z} - and U -stabilization of an arbitrary (G) -C*-algebra. Similar statements hold for the \mathcal{O}_∞ - and \mathcal{O}_2 -stabilizations, and this allows us to recover a result from [54]: the nuclear dimension of a Kirchberg algebra is at most 3. (The actual dimension of Kirchberg algebras has been computed in [10]: it is 1.) Some of the nuclear dimensional estimates that we derive here have also been observed in [3], while the estimates involving the decomposition rank are new.

One of our main results requires that we first prove the following equivariant generalization of the main result from [15], which is interesting in its own right.

Theorem. *Let X be a compact metrizable space of finite covering dimension, let G be a compact metrizable group, let (D, δ) be a strongly self-absorbing, unitorily regular G -C*-algebra, and let (A, α) be a separable, unital G - $C(X)$ -algebra. If A_x is G -equivariantly D -absorbing, then there is a $C(X)$ -linear G -isomorphism*

$$(A, \alpha) \cong (D \otimes C(X), \delta \otimes \iota_{C(X)}).$$

Combining the theorem above with our results related to order zero dimension, we prove the following. The second assertion is a significant generalization of previous results from [40] and [30], which only considered the case $D = \mathcal{Z}$.

Theorem. *Let G be a compact group, let D be a strongly self-absorbing C*-algebra, let A be a separable D -absorbing C*-algebra, let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of G on A with finite Rokhlin dimension with commuting towers, and let $\delta: G \rightarrow \text{Aut}(D)$ be any strongly self-absorbing action (such as the trivial action). Then (A, α) is G -equivariantly isomorphic to $(D \otimes A, \alpha \otimes \delta)$. Furthermore, the fixed point algebra A^G and the crossed product $A \rtimes G$ are D -absorbing.*

Absorption of the trivial action on the Jiang-Su algebra is particularly useful, since it opens the doors of a possible classification of actions with finite Rokhlin dimension with commuting towers. Indeed, showing absorption of well-behaved objects is a common feature in most of the classification results for group actions. These aspects will be explored in subsequent work.

We also deduce new Rokhlin dimension estimates for actions with finite Rokhlin dimension with commuting towers of finite groups on \mathcal{Z} -absorbing C*-algebras, which imply that the possible values of the Rokhlin dimension in this case are 0 and 1.

Theorem. *Let G be a finite group, let A be a C*-algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ with finite Rokhlin dimension with commuting towers. Then $\alpha \otimes \text{id}_{\mathcal{Z}}$ has Rokhlin dimension at most 1. If A is \mathcal{Z} -absorbing, then α has Rokhlin dimension at most 1.*

This represents a satisfactory parallel with the $\{0, 1, \infty\}$ -type behaviour that nuclear dimension and decomposition rank tend to have in the noncommutative setting. It is also particularly satisfactory, since proving finiteness of the Rokhlin dimension with commuting towers is a far easier task than proving that the Rokhlin dimension is (at most) 1. In the particular case when A is a commutative unital C^* -algebra $C(X)$ for some compact Hausdorff space X , such a result can be seen as a dynamical version of the main result of [66], which states that $\text{dr}(C(X) \otimes \mathcal{Z}) \leq 2$.

Finally, we also prove that finite Rokhlin dimension with commuting towers implies the Rokhlin property for finite group actions on UHF-absorbing C^* -algebras.

Theorem. *Let G be a finite group, let A be an $M_{|G|^\infty}$ -absorbing C^* -algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action with finite Rokhlin dimension with commuting towers. Then α has the Rokhlin property. This in particular applies to Cuntz algebras of the form $\mathcal{O}_{n|G|}$.*

Again, this is very relevant from a computational point of view: proving directly that an action has the Rokhlin property is often challenging, and there are not many tools available. On the other hand, Rokhlin dimensional estimates are much easier to come by, particularly for finite groups. Having access to the Rokhlin property is highly valuable, since it entails classifiability of the action, and the structure of the crossed product is extremely well-understood (see, for example, [29]).

We include an appendix, containing the relevant notions and results from model theory that are used in this paper; see also the appendix of [33]. A quick introduction to logic for metric structures can be found in [6], and as it pertains to C^* -algebras in [53], while [19] is a more complete reference for the model-theoretic study of C^* -algebras.

The model-theoretic perspective is crucial to our approach, as it allows us to isolate the semantic content of properties of G - C^* -algebras and equivariant embeddings, such as the Rokhlin property, Rokhlin dimension, G -equivariant sequentially split $*$ -homomorphism (in the terminology of [4, 5]), G -equivariant order zero dimension (introduced here). In turn, this is the fundamental ingredient to effortlessly deduce preservation results from the semantic characterizations of regularity properties of C^* -algebras obtained in [19, 36], subsuming, simplifying, and generalizing many results from the literature. The realization that the “continuous part of the ultrapower” of a G - C^* -algebra is just its ultrapower as an $\mathcal{L}_G^{C^*}$ -structure allows us to clarify its properties, including saturation and Los’ theorem, which are here deduced from general model-theoretic facts. This subsumes and technically simplifies the proof of many particular instances that had previously appeared in the literature. We then crucially use the full strength and semantic content of saturation and Los’ theorem in the proof of our main results. The model-theoretic study of G - C^* -algebras, including the notion of first-order theory, is also fundamental in our study of strongly self-absorbing G - C^* -algebras. Particularly, we show that the first-order theory provides a complete invariant (up to isomorphism) for such G - C^* -algebras. This is the crucial step in our computation of the Borel complexity of the classification problem for strongly self-absorbing G - C^* -algebras.

For the rest of the paper, G will be a second countable compact group. We denote by $C(G)$ the unital C^* -algebra of continuous, complex-valued function on G . The multiplication operation on G induces a unital $*$ -homomorphism $\Delta: C(G) \rightarrow C(G \times G) \cong C(G) \otimes C(G)$ given by $\Delta(f)(s, t) = f(st)$ for $f \in C(G)$ and $s, t \in G$. A unitary representation $\pi \in \text{Rep}(G)$ on a finite-dimensional Hilbert space \mathcal{H} defines the subspace of *matrix units* for π .

$$C(G)_\pi = \{\langle \xi, \pi(f)\eta \rangle : \xi, \eta \in \mathcal{H}, f \in C(G)\}.$$

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2. LANGUAGES FOR C^* -ALGEBRAS

2.1. The ordered selfadjoint operator space language. An *ordered selfadjoint operator space*, as defined in [8, 62], is a matricially normed and matricially ordered $*$ -vector space that admits a selfadjoint completely isometric complete order embedding into a C^* -algebra. Concretely, one can define an ordered selfadjoint operator space as a selfadjoint closed subspace of $B(H)$ with the inherited matricial norms, matricial positive cones, and involution. Ordered selfadjoint operator spaces have been abstractly characterized in [62, 63, 68], and further studied

in [8, 9, 44, 45, 56, 69]. For ordered operator spaces X and Y , we denote by $\text{CPC}(X, Y)$ the set of all selfadjoint completely positive completely contractive linear maps $X \rightarrow Y$. (Observe that in an ordered selfadjoint operator space the matrix positive cones are not necessarily spanning. Therefore a completely positive linear map on an ordered operator space is not necessarily selfadjoint.)

An ordered selfadjoint operator space X can be naturally seen as a structure in the language $\mathcal{L}^{\text{osos}}$ that contains

- sorts $M_n(X)$, with $n \in \mathbb{N}$, for the matrix amplifications of the space X , with balls centered at the origin as domains of quantification;
- a sort for each finite-dimensional C*-algebra F , with balls centered at the origin as domains of quantification;
- function symbols for the vector space operations and the involution in X and F ;
- predicate symbols for the norms in $M_n(X)$ and in F ;
- predicate symbols for the distance function from the cone of positive elements in $M_n(X)$ and in F ;
- predicate symbols for the function $F^k \times X^k \rightarrow \mathbb{R}$ given by

$$(\bar{y}, \bar{z}) \mapsto \inf_{t \in \text{CPC}(F, X)} \max_{j=1, \dots, k} \|t(y_j) - z_j\|.$$

We call such a language the *ordered selfadjoint operator space language* $\mathcal{L}^{\text{osos}}$. Observe that the $\mathcal{L}^{\text{osos}}$ -terms can be seen as degree 1 matrix *-polynomials without constant terms. These are expressions of the form

$$\alpha_1^* x_1 \beta_1 + \dots + \alpha_n^* x_n \beta_n + \gamma_1^* x_1^* \delta_1 + \dots + \gamma_n^* x_n^* \delta_n$$

where n is a positive integer, $\alpha_j, \beta_j, \gamma_j, \delta_j$, for $1 \leq j \leq n$, and are scalar matrices. It is clear that a function between ordered selfadjoint operator spaces is an $\mathcal{L}^{\text{osos}}$ -morphism if and only if it is *selfadjoint, completely positive* and *completely contractive*, and it is an $\mathcal{L}^{\text{osos}}$ -embedding if and only if it is a *selfadjoint completely isometric complete order embedding*. In particular, any C*-algebra can be seen as an $\mathcal{L}^{\text{osos}}$ -structure in the obvious way, by considering its canonical matrix norms and matrix positive cones. It is observed in [36, Appendix C], [19, Section 3 and Section 5] that all the predicates above are definable in the usual language of C*-algebras as considered in [19, 22].

An operator system is a closed, selfadjoint subspace $X \subset A$ of a *unital* C*-algebra that contains its unit. The *operator system language* \mathcal{L}^{osy} is obtained from the ordered operator space language by adding a constant symbol for the unit in X . The \mathcal{L}^{osy} -terms can be seen as degree 1 matrix *-polynomials *with a constant term*.

2.2. The order zero language. If A, B are C*-algebras, we denote by $\text{OZ}(A, B)$ the space of completely positive contractive order zero maps $A \rightarrow B$. The *order zero language* \mathcal{L}^{oz} for C*-algebras is obtained from $\mathcal{L}^{\text{osos}}$ by adding, for any finite-dimensional C*-algebra F and any $k \in \mathbb{N}$, a predicate symbol to be interpreted as the function $F^k \times A^k \rightarrow \mathbb{R}$, given by

$$(\bar{y}, \bar{z}) \mapsto \inf_{t \in \text{OZ}(F, X)} \max_{j=1, \dots, k} \|t(y_j) - z_j\|.$$

It is proved in [19, Section 5.2] that such functions are definable in the usual language of C*-algebras as considered in [19, 22]. This follows from the structure theorem for completely positive contractive order zero maps [71, Corollary 4.1] and stability of the relations defining cones of finite-dimensional C*-algebras [52, Section 3.3].

A C*-algebras can be seen as an \mathcal{L}^{oz} -structure in the obvious way. Let A and B be C*-algebras and let $f: A \rightarrow B$ be a function. Then f is an \mathcal{L}^{oz} -morphism if and only if f is a completely positive contractive order zero map.

Remark 2.1. In the order-zero language one can express the fact that a pair (a_1, a_2) of elements of a C*-algebra A are (almost) orthogonal. Indeed one can consider the canonical basis elements (e_1, e_2) of $\mathbb{C} \oplus \mathbb{C}$ and the formula $\varphi(e_1, e_2, x_1, x_2)$ defined by

$$\inf_{t \in \text{OZ}(\mathbb{C} \oplus \mathbb{C}, A)} \max \{ \|t(e_1) - x_1\|, \|t(e_2) - x_2\| \}.$$

We have that if $\varphi(e_1, e_2, a_1, a_2) < \varepsilon$, then $\|a_1 a_2\| < 2\varepsilon$. Conversely, for every $\varepsilon > 0$ there exists $\delta > 0$ such that if a_1, a_2 are positive contractions such that $\|a_1 a_2\| < \delta$, then $\varphi(e_1, e_2, a_1, a_2) < \varepsilon$.

2.3. The C*-algebra language. The *C*-algebra language* \mathcal{L}^{C^*} is obtained from $\mathcal{L}^{\text{osos}}$ by adding a function symbol for the multiplication operation in $M_n(A)$, for every $n \in \mathbb{N}$. Similarly, the *unital C*-algebra language* $\mathcal{L}^{1, \text{C}^*}$ is obtained from \mathcal{L}^{C^*} by adding a constant symbol for the unit. Observe that the terms in $\mathcal{L}^{1, \text{C}^*}$ (respectively, \mathcal{L}^{C^*}) can be canonically identified with matrix *-polynomials with (respectively, without) constant term. A *matrix *-polynomial* is a linear combination of expressions of the form $X_1 \cdots X_n$ where X_j , for $j = 1, \dots, n$, is either a scalar matrix, or x or y^* for some variable x, y . A function between C*-algebras is an \mathcal{L}^{C^*} -morphism ($\mathcal{L}^{1, \text{C}^*}$ -morphism) if

and only if it is a (unital) $*$ -homomorphism, and an \mathcal{L}^{C^*} -embedding (\mathcal{L}^{1,C^*} -embedding) if and only if it is a (unital) injective $*$ -homomorphism.

Remark 2.2. The following properties of C^* -algebras have been proved to be definable by a uniform family of existential positive \mathcal{L}^{C^*} -formulas—see Definition A.1—in [19, Theorem 2.5.1 and Theorem 5.7.3]: real rank zero, stable rank at most n , quasidiagonality, simplicity, being simple and purely infinite, being simple and TAF, being abelian of real rank at most n . Considering unital C^* -algebras and positive existential \mathcal{L}^{1,C^*} -formulas gives approximate divisibility.

Remark 2.3. The following properties of C^* -algebras have been proved to be definable by a uniform family of existential positive \mathcal{L}^{C^*} -formulas among *separable* C^* -algebras in [19, Theorem 2.5.1 and Theorem 5.7.3]: being UHF; being AF; being D -absorbing for a given strongly self-absorbing C^* -algebra D ; and being \mathcal{K} -absorbing—also called *stable*—where \mathcal{K} is the algebra of compact operators.

2.4. The nuclear languages. The *nuclear ordered selfadjoint operator space language* $\mathcal{L}^{\text{osos-nuc}}$ is obtained from $\mathcal{L}^{\text{osos}}$ by adding, for every $k \in \mathbb{N}$ and every finite-dimensional C^* -algebra F , a predicate symbol for the function $X^k \times F^k \rightarrow \mathbb{R}$

$$(\bar{x}, \bar{y}) \mapsto \inf_{s \in \text{CPC}(X, F)} \max_{j=1, \dots, k} \|s(x_j) - y_j\|.$$

It is proved in [19, Section 5] that such a function is definable in the language of C^* -algebras considered in [19, 22].

In the proof of Lemma 2.4, we will need the following version of the Choi-Effros lifting theorem: if A, B are C^* -algebras, A is separable, $f: A \rightarrow B$ is a nuclear completely positive contractive map, $E \subset A$ is a finite-dimensional subspace, and $\varepsilon > 0$, then there exists a completely positive contractive map $\eta: \overline{f(A)} \rightarrow A$ such that $f \circ \eta$ is the identity map on $\overline{f(A)}$, and $\|(\eta \circ f)(x) - x\| < \varepsilon \|x\|$ for all $x \in E$. When A, B , and f are unital, this is a consequence of the Choi-Effros lifting theorem for operator systems; see [13, Lemma 3.8 and Section 4.3]. The general case can be reduced to the unital one by taking unitizations; see [11, Proposition 2.2.1 and Proposition 2.2.4].

Lemma 2.4. Let A and B be C^* -algebras, and let $f: A \rightarrow B$ be a function. Consider the following assertions:

- (1) f is a nuclear completely positive contractive map;
- (2) f is an $\mathcal{L}^{\text{osos-nuc}}$ -morphism;
- (3) f is a completely positive contractive map.

Then (1) \Rightarrow (2) \Rightarrow (3), and they are all equivalent if either A or B is nuclear.

Proof. The implication (1) \Rightarrow (2) uses the Choi-Effros lifting theorem as stated above. The proof is the same as the proof that nuclearity passes to quotients and that decomposition rank and nuclear dimension are nonincreasing under quotients; see [70, §2.9], [50, Section 3], and [72, Proposition 2.3]. The implication (2) \Rightarrow (3) is obvious. Finally, if either A or B are nuclear, then any completely positive contractive map $f: A \rightarrow B$ is nuclear, which gives (3) \Rightarrow (1). \square

The *nuclear order zero language* $\mathcal{L}^{\text{oz-nuc}}$ and the *nuclear C^* -algebra language* $\mathcal{L}^{C^*\text{-nuc}}$, are defined as above starting from \mathcal{L}^{oz} and \mathcal{L}^{C^*} , respectively. It follows from Lemma 2.4 that any nuclear completely positive contractive order zero map (nuclear $*$ -homomorphism) between C^* -algebras is an $\mathcal{L}^{\text{oz-nuc}}$ -morphism ($\mathcal{L}^{C^*\text{-nuc}}$ -morphism), and the converse holds for nuclear C^* -algebras.

It is proved in [19, Section 5] that any predicate that is definable in $\mathcal{L}^{C^*\text{-nuc}}$ is also definable in \mathcal{L}^{C^*} . However, considering the larger language $\mathcal{L}^{C^*\text{-nuc}}$ gives a more generous notion of (positive) existential formula.

Remark 2.5. The following properties of C^* -algebras are definable by a uniform family of existential positive $\mathcal{L}^{C^*\text{-nuc}}$ -formulas (see [19, Section 5]): nuclearity, having nuclear dimension at most n , and having decomposition rank at most n .

2.5. Actions of groups on C^* -algebras. Denote by $\text{Aut}(A)$ denote the group of automorphisms of A , endowed with the topology of pointwise convergence. An action of G on a C^* -algebra A is a (strongly) continuous group homomorphism $\alpha: G \rightarrow \text{Aut}(A)$. An action of G on A can be regarded as an injective nondegenerate $*$ -homomorphism

$$\alpha: A \rightarrow C(G, A),$$

defined by $\alpha(a)(g) = \alpha_{g^{-1}}(a)$ for all $g \in G$ and all $a \in A$. With respect to the identification $C(G, A) \cong C(G) \otimes A$, such a map satisfies the identity

$$(\Delta \otimes \text{id}) \circ \alpha = (\text{id} \otimes \alpha) \circ \alpha.$$

This identity characterizes the injective nondegenerate $*$ -homomorphisms that arise from actions as above.

Definition 2.6. A G - C^* -algebra is a C^* -algebra endowed with a distinguished action of G .

An irreducible representation π of G on a finite-dimensional Hilbert space defines a subspace A_π of A , called π -isotypical component or π -spectral subspace, given by

$$\{a \in A : \alpha(a) \in C(G)_\pi \otimes A\}.$$

In the particular case when π is the trivial representation, one obtains the *fixed point algebra* A^G .

It is explained in [33, Section 3.4] how G - C^* -algebras can be seen as structures in the logic for metric structures with respect to the language $\mathcal{L}_G^{C^*}$, which is the language obtained from the language of C^* -algebras \mathcal{L}^{C^*} by replacing the sort for the C^* -algebra with sorts indexed by $\text{Rep}(G)$, to be interpreted as the isotypical components. Furthermore, one adds function symbols to be interpreted as the restriction of the action to the isotypical components, regarded as maps $A_\pi \rightarrow C(G)_\pi \otimes A_\pi$. (Observe that $C(G)_\pi$ is finite-dimensional.) Explicit *axioms* for G - C^* -algebras are provided in [33, Section 3.4], thus showing that G - C^* -algebras form an $\mathcal{L}_G^{C^*}$ -axiomatizable class. For each language for C^* -algebras \mathcal{L} that we considered above, one can consider the corresponding G -equivariant version \mathcal{L}_G , which can be obtained from \mathcal{L} exactly as $\mathcal{L}_G^{C^*}$ is obtained from \mathcal{L}^{C^*} .

In the following, if A is a C^* -algebra, and \mathcal{F} is a filter over a set I , then we let $\prod_{\mathcal{F}} A$ be the corresponding reduced product. When A is a G - C^* -algebra, $\prod_{\mathcal{F}} A$ is endowed with the canonical coordinate-wise action of G . We let $\prod_{\mathcal{F}}^G A$ be the subalgebra of $a \in \prod_{\mathcal{F}} A$ such that the map $g \mapsto g^{\prod_{\mathcal{F}} A} a$ is continuous. This is a G - C^* -algebra, which can be identified with the reduced product of A with respect to \mathcal{F} when regarded as a structure in the language of G - C^* -algebras $\mathcal{L}_G^{C^*}$; see [33, Proposition 3.12].

It is worth noticing that $\prod_{\mathcal{F}}^G A$ is in general different from $\prod_{\mathcal{F}} A$ for a G - C^* -algebra A , even in the case when \mathcal{F} is an ultrafilter over \mathbb{N} , as the next example shows.

Example 2.7. The canonical inclusion of $C(G)$ inside $\prod_{\mathcal{U}}^G C(G)$ is surjective for any ultrafilter \mathcal{U} . However, the inclusion of $C(G)$ in the (nonequivariant) C^* -algebra ultrapower $\prod_{\mathcal{U}} C(G)$ is in general strict. For example, if $G = \mathbb{T}$ and $u \in C(\mathbb{T})$ is the canonical unitary generator, then the element $[u^n]$ of $\prod_{\mathcal{U}} C(\mathbb{T})$ with representative sequence $(u^n)_{n \in \mathbb{N}}$ does not belong to $C(\mathbb{T})$, since the canonical action of \mathbb{T} on $\prod_{\mathcal{U}} C(\mathbb{T})$ is not continuous at $[u^n]_{n \in \mathbb{N}}$. It follows that $C(\mathbb{T}) = \prod_{\mathcal{U}}^{\mathbb{T}} C(\mathbb{T})$ is properly contained in $\prod_{\mathcal{U}} C(\mathbb{T})$.

2.6. Languages for A -bimodules. Let A and B be C^* -algebras. Then B is an A -bimodule if it is endowed with linear maps $b \mapsto a \cdot b$ and $b \mapsto b \cdot a$ for $a \in A$, satisfying $\max\{\|a \cdot b\|, \|b \cdot a\|\} \leq \max\{\|a\|, \|b\|\}$ as well as the natural associativity requirements. When A, B are G - C^* -algebras, then we say that B is a G -equivariant A -bimodule if it is an A -bimodule satisfying $(g^A a) \cdot (g^B b) = g^B(a \cdot b)$, and $(g^B b) \cdot (g^A a) = g^B(b \cdot a)$ for all $a \in A$, $b \in B$ and $g \in G$. If $f: A \rightarrow B$ is a G -equivariant $*$ -homomorphism, then it induces a canonical G -equivariant A -bimodule structure on B , defined by $a \cdot b := f(a)b$ and $b \cdot a := bf(a)$ for $a \in A$ and $b \in B$.

We let $\mathcal{L}^{C^*, A-A}$ be the language obtained from \mathcal{L}^{C^*} by adding symbols for the A -bimodule structure. Similar definitions apply to the other languages for C^* -algebras considered above. The interpretation of an $\mathcal{L}^{C^*, A-A}$ -formula in a A -bimodule is defined in the obvious way.

2.7. The Kirchberg language. Fix a C^* -algebra A . In this subsection, we define the Kirchberg language $\mathcal{L}^K(A)$, which fits into the more flexible setting described in Subsection A.4. This language is obtained from \mathcal{L}^{C^*} by replacing the symbols for the matrix norms with pseudometric symbols d_F for every finite set F in the unit ball of A . The distinguished collection $t_A^c(x)$ of positive quantifier-free conditions that is part of the language $\mathcal{L}^K(A)$ consists of the conditions $\max_{a \in F} \|ax - xa\| = 0$ for every finite subset F of the unit ball of A .

One can regard A as an $\mathcal{L}^K(A)$ -structure by interpreting d_F on $M_n(A)$ as the pseudometric

$$(x, y) \mapsto \max_{a \in M_n(F)} \|a(x - y)\|.$$

Suppose that \mathcal{U} is an ultrafilter. Then the reduced power of A as an $\mathcal{L}^K(A)$ -structure is equal to the Kirchberg invariant $F_{\mathcal{U}}(A)$ as introduced by Kirchberg in [48]; see also [1]. Considering reduced powers instead of ultrapowers

yields the generalization of the Kirchberg invariant to arbitrary filters considered in [4, 64]. In the following, we denote by $t_A^c(x_1, \dots, x_n)$ the type $t_A^c(x_1) \cup \dots \cup t_A^c(x_n)$. If A is *unital*, then $F_{\mathcal{F}}(A)$ is equal to $A' \cap \prod_{\mathcal{F}} A$.

Let κ be an infinite cardinal that is larger than the density character of A , and let \mathcal{F} be a countably incomplete κ -good filter. (When A is separable, one can take any countably incomplete ultrafilter.) Considering an approximate unit for A shows that $F_{\mathcal{F}}(A)$ is unital. Let $t(x_1, \dots, x_n)$ be a positive primitive quantifier free \mathcal{L}^{1, C^*} -type. The corresponding *multiplier* $\mathcal{L}^K(A)$ -type $t_A^m(x_1, \dots, x_n)$ is defined as follows. Any condition in $t(\bar{x})$ should be replaced with all the conditions obtained by substituting every occurrence of a basic formula of the form $\|\mathfrak{p}(\bar{x})\|$, for some $*$ -polynomial \mathfrak{p} with constant term, with the basic formula $\|b \mathfrak{p}(\bar{x})\|$, where b is some element of the unit ball of A .

Remark 2.8. It follows from Remark A.10 that the following statements are equivalent:

- (1) $t(\bar{x})$ is realized in $F_{\mathcal{F}}(A)$,
- (2) $t^m(\bar{x})$ is realized in $F_{\mathcal{F}}(A)$,
- (3) $t^m(\bar{x})$ is approximately realized in $F_{\mathcal{F}}(A)$,
- (4) $t^m(\bar{x}) \cup t_A^c(\bar{x})$ is approximately realized in A .

Furthermore $F_{\mathcal{F}}(A)$ is positively quantifier-free \mathcal{L}^{1, C^*} - κ -saturated. When \mathcal{U} is an ultrafilter, $F_{\mathcal{U}}(A)$ is quantifier-free \mathcal{L}^{1, C^*} - κ -saturated.

Various results from [48] can be seen as consequences of Remark 2.8.

Suppose now that A is a G - C^* -algebra. Then one can consider A as an $\mathcal{L}_G^K(A)$ -structure. In this case, the reduced power of A as an $\mathcal{L}_G^K(A)$ -structure with respect to a filter \mathcal{F} —which we denote by $F_{\mathcal{F}}^G(A)$ —recovers the equivariant version of the Kirchberg invariant considered in [4, 64]. Again, the following proposition follows from the general remarks of Subsection A.4. If $t(\bar{x})$ is a positive primitive quantifier free \mathcal{L}_G^{1, C^*} -type, then the corresponding multiplier $\mathcal{L}_G^K(A)$ -type $t_A^m(\bar{x})$ can be defined as above.

Proposition 2.9. *Suppose that A is a G - C^* -algebra, κ is a cardinal larger than the density character of A , \mathcal{F} is a countably incomplete κ -good filter, and $t(\bar{x})$ is a positive primitive quantifier-free \mathcal{L}_G^{1, C^*} -type. Then $F_{\mathcal{F}}^G(A)$ is a unital G - C^* -algebra, and the following statements are equivalent:*

- (1) $t(\bar{x})$ is realized in $F_{\mathcal{F}}^G(A)$,
- (2) $t^m(\bar{x})$ is realized in $F_{\mathcal{F}}^G(A)$,
- (3) $t^m(\bar{x})$ is approximately realized in $F_{\mathcal{F}}^G(A)$,
- (4) $t(\bar{x}) \cup t_A^c(\bar{x})$ is approximately realized in A .

Furthermore $F_{\mathcal{F}}(A)$ is positively quantifier-free \mathcal{L}_G^{1, C^*} - κ -saturated.

Similar conclusions hold if one replaces filters with ultrafilters, and positive primitive quantifier free types with arbitrary quantifier free types.

3. STRONGLY SELF-ABSORBING G - C^* -ALGEBRAS

In this section, we exhibit some applications of model theory to strongly self-absorbing actions on C^* -algebras, as introduced and studied in [64, 65]. We regard G - C^* -algebras as structures in the language of G - C^* -algebras $\mathcal{L}_G^{C^*}$. An $\mathcal{L}_G^{C^*}$ -morphism between G - C^* -algebras is a G -equivariant $*$ -homomorphism, and an $\mathcal{L}_G^{C^*}$ -embedding is an injective G -equivariant $*$ -homomorphism. If A and B are G - C^* -algebras, then we denote by $A \otimes B$ the *minimal* tensor product of A and B endowed with the continuous G -action defined by $g^{A \otimes B}(a \otimes b) = (g^A a) \otimes (g^B b)$.

3.1. Positively $\mathcal{L}_G^{C^*}$ -existential injective $*$ -homomorphisms. An injective $*$ -homomorphism $\theta: A \rightarrow M$ between *separable* G - C^* -algebras is G -equivariantly sequentially split, in the sense of [4, Definition 3.3], if and only if it is positively $\mathcal{L}_G^{C^*}$ -existential, as defined in Subsection A.2. For arbitrary G - C^* -algebras, the notion of positively $\mathcal{L}_G^{C^*}$ -existential injective $*$ -homomorphism is more generous than being G -equivariantly sequentially split.

In the case of a compact group G , Theorem A.7 applied to G - C^* -algebras recovers [4, Lemma 3.6 and Corollary 3.7]. In the case of compact G , Lemma 2.3, Corollary 2.4, Proposition 2.5, Proposition 2.9, Proposition 3.8 and Corollary 3.17 of [4] are then an immediate consequence of the definition of positively $\mathcal{L}_G^{C^*}$ -existential injective $*$ -homomorphism; see Proposition A.4. Proposition 3.11 of [4] is a particular instance of [33, Proposition A.33], since the fixed point algebra A^G of a G - C^* -algebra is positively existentially definable. By appropriately choosing

the functor, one can also see that Proposition A.8 has as particular instances the following results from [4]: (I), (II), (IV) of Theorem 2.10, Proposition 3.9, Proposition 3.12, Corollary 3.14, Corollary 3.15, Proposition 3.16.

It follows from Proposition A.3 that if A, B are C*-algebras and $f: A \rightarrow B$ is a positively \mathcal{L}^{C^*} -existential injective *-homomorphism, then A has any of the properties listed in Remark 2.2 or Remark 2.3, whenever B does. The same assertion holds for any of the properties listed in Remark 2.5 when B is nuclear. In particular, this observation recovers (1), (2), (3), (4), (5), (7), (11), the first part of (12), the first half of (14), and (16) of [4, Theorem 2.11]. Other preservation results have been obtained in [26, 27]. We present here an additional preservation result does not seem to follow from the results mentioned above. Recall the definition of real rank from [7, Definition V.3.2.1].

Proposition 3.1. *Suppose that A, B are C*-algebras and $f: A \rightarrow B$ is a positively \mathcal{L}^{C^*} -existential injective *-homomorphism. If B has real rank at most n , then A has real rank at most n .*

Proof. Without loss of generality, we can assume that A, B are separable. After unitizing, we can assume that A, B are unital, f is unital, and f is a positively $\mathcal{L}^{C^*,1}$ -existential injective *-homomorphism. We identify A with its image under f . Fix selfadjoint elements $a_0, \dots, a_n \in A$ and $\varepsilon > 0$. Since B has real rank at most n , there exist selfadjoint elements $b_0, \dots, b_n \in B$ such that $b_0^2 + \dots + b_n^2$ is invertible, and $\|a_i - b_i\| < \varepsilon$ for every $i = 0, 1, \dots, n$. Since the inclusion $A \subset B$ is a positively $\mathcal{L}^{C^*,1}$ -existential *-homomorphism, we can conclude that there exist selfadjoint elements $c_0, \dots, c_n \in A$ such that $c_0^2 + \dots + c_n^2$ is invertible, and $\|c_i - a_i\| < \varepsilon$ for every $i = 0, 1, \dots, n$. \square

3.2. Commutant existential theories. The notion of weak containment and weak equivalence are introduced in the general setting of logic for metric structures in Subsection A.1. In this section we consider, in the case of G -C*-algebras, the natural *commutant* analogs of such notions. Suppose that A, B are G -C*-algebras. We say that A is *commutant positively weakly contained* in B if for some (equivalently, any) cardinal κ larger than the density character of A and B and for some (equivalently, any) countably incomplete κ -good filter \mathcal{F} on A has that every \mathcal{L}_G^{1,C^*} -type t that is realized in $F_{\mathcal{F}}^G(A)$ is also realized in $F_{\mathcal{F}}^G(B)$. Equivalently, for any unital G -C*-subalgebra C of $F_{\mathcal{F}}^G(A)$ of density character less than κ there exists a G -equivariant injective unital *-homomorphism from C to $F_{\mathcal{F}}^G(B)$. If A is unital, then A is commutant positively weakly contained in B if and only if there exists a unital *-homomorphism from A to $F_{\mathcal{F}}^G(B)$ for any filter \mathcal{F} as above. A syntactic characterization of commutant positively weak containment can be obtained using Proposition 2.9.

Suppose that A is a G -C*-algebra, and let $t_A^c(\bar{x})$ be the collection of conditions $\max_{j=1, \dots, n} \|x_j a - a x_j\| \leq 0$, for $a \in A$. Recall that if $t(\bar{x})$ is a positive (primitive) quantifier-free \mathcal{L}_G^{1,C^*} -type, then $t_A^m(\bar{x})$ denotes the positive (primitive) quantifier-free $\mathcal{L}_G^K(A)$ -type obtained from $t(\bar{x})$ by replacing every occurrence of $\|\mathfrak{p}(\bar{x})\|$ for some G -*-polynomial \mathfrak{p} with $\|b\mathfrak{p}(\bar{x})\|$ where b is an arbitrary element of A of norm at most 1. We then have that A is commutant positively weakly contained in B if and only if, for any positive primitive quantifier-free \mathcal{L}_G^{1,C^*} -type $t(\bar{x})$, $t_A^m(\bar{x}) \cup t_A^c(\bar{x})$ is approximately satisfied in A if and only if $t_B^m(\bar{x}) \cup t_B^c(\bar{x})$ is approximately satisfied in B .

Two C*-algebras are *commutant positively weakly equivalent* if they are each commutant positively weakly contained in the other. For unital nuclear G -C*-algebras, the following characterization of commutant weak \mathcal{L}_G^{1,C^*} -containment follows from the Choi-Effros lifting theorem.

Proposition 3.2. *Suppose that A is a unital nuclear G -C*-algebra, and B is a G -C*-algebra. Then A is commutant positively weakly contained in B if and only if for any separable nuclear G -invariant unital C*-subalgebra $A_0 \subset A$ and separable G -C*-subalgebra $B_0 \subset B$, there exists a sequence $(\phi_n)_{n \in \mathbb{N}}$ of completely positive contractive maps $\phi_n: A_0 \rightarrow B$ such that, for every compact subset $K \subset G$, for every $x, y \in A_0$, and for every $b \in B_0$, we have that*

$$\begin{aligned} \lim_{n \rightarrow +\infty} \|b(\phi_n(x)\phi_n(y) - \phi_n(xy))\| &= 0, & \lim_{n \rightarrow +\infty} \|\phi_n(x)b - b\phi_n(x)\| &= 0 \\ \lim_{n \rightarrow +\infty} \|b\phi_n(1) - b\| &= 0, & \text{and } \lim_{n \rightarrow +\infty} \max_{g \in K} \|b(\phi_n(g^A x) - g^B \phi_n(x))\| &= 0. \end{aligned}$$

The notions of *commutant weak containment* and commutant existential theory are defined analogously, considering arbitrary (not necessarily positive primitive) quantifier-free \mathcal{L}_G^{1,C^*} -types.

3.3. Space of separable nuclear G -C*-algebras and smooth classification. We now observe that there exists a natural standard Borel space of separable nuclear G -C*-algebras. Indeed, by Kirchberg's nuclear embedding theorem [60, Theorem 6.3.12], any separable nuclear C*-algebra is *-isomorphic to the range of a conditional expectation on \mathcal{O}_2 . Given such a conditional expectation E , set $A = E(\mathcal{O}_2)$. Write $\text{CPC}(\mathcal{O}_2)$ for the semigroup of

all completely positive contractive maps of \mathcal{O}_2 into itself, endowed with the topology of pointwise convergence in norm. Given an action $\alpha: G \rightarrow \text{Aut}(A)$, one can define a continuous function $\rho_\alpha: G \rightarrow \text{CPC}(\mathcal{O}_2)$ by $\rho_\alpha(g) := \alpha_g \circ E$. Then

- (1) $\rho_\alpha(gh) = \rho_\alpha(g) \circ \rho_\alpha(h)$ for every $g, h \in G$, and
- (2) $\rho_\alpha(1) = E$.

Conversely, any pair (E, ρ) , consisting of a conditional expectation $E: \mathcal{O}_2 \rightarrow \mathcal{O}_2$ and a continuous function $\rho: G \rightarrow \text{CPC}(\mathcal{O}_2)$ satisfying (1) and (2), arises from a continuous action of G on the range of E , as described above.

Observe that $\text{CPC}(\mathcal{O}_2)$ is a Polish space when endowed with the topology of pointwise convergence. Similarly, the space $\text{Exp}(\mathcal{O}_2)$ of conditional expectations defined on \mathcal{O}_2 is a Polish space when endowed with the topology of pointwise convergence. (This can be seen for instance by observing that conditional expectations onto a given C^* -subalgebra are precisely the idempotent maps of norm 1 mapping onto that C^* -subalgebra [7, Theorem II.6.10.2].)

The space $G\text{-}C^*\text{ALG}$ of pairs (E, ρ) arising from a continuous action of G on the image of E , is a G_δ subset of the space $\text{Exp}(\mathcal{O}_2) \times \text{CPC}(\mathcal{O}_2)$, hence a Polish space with the induced topology; see [46, Theorem 3.11]. We will regard $G\text{-}C^*\text{ALG}$ as the Polish space of separable nuclear $G\text{-}C^*$ -algebras. For an element $(E, \rho) \in G\text{-}C^*\text{ALG}$, we write $C^*(E, \rho)$ for the associated $G\text{-}C^*$ -algebra.

It is easy to see, by induction on the complexity, that any $\mathcal{L}_G^{C^*}$ -formula $\varphi(x_1, \dots, x_n, \gamma_1, \dots, \gamma_m)$ induces a Borel map $\hat{\varphi}: G\text{-}C^*\text{ALG} \times \mathcal{O}_2^n \times G^m \rightarrow \mathbb{R}$ given by

$$((E, \rho), (a_1, \dots, a_n), (g_1, \dots, g_m)) \mapsto \varphi^{C^*(E, \rho)}(E(a_1), \dots, E(a_n), g_1, \dots, g_m).$$

In other words, the $\mathcal{L}_G^{C^*}$ -theory of a separable nuclear $G\text{-}C^*$ -algebra can be computed in a Borel fashion in the parameterization $G\text{-}C^*\text{ALG}$ of $G\text{-}C^*$ -algebras. This allows one to conclude the following.

Theorem 3.3. *Separable nuclear $G\text{-}C^*$ -algebras are smoothly classifiable, in the sense of Borel complexity, up to weak $\mathcal{L}_G^{C^*}$ -equivalence and positive weak $\mathcal{L}_G^{C^*}$ -equivalence.*

An introduction to the theory of Borel complexity of equivalence relations can be found in [25]. Similar conclusions hold for unital C^* -algebras and (positive) weak \mathcal{L}_G^{1, C^*} -equivalence.

3.4. Strongly self-absorbing $G\text{-}C^*$ -algebras. We continue to fix a compact group G . Let A and B be $G\text{-}C^*$ -algebras and let $\eta_1, \eta_2: A \rightarrow B$ be unital G -*-homomorphisms. By $\mathcal{M}(B)$ we denote the multiplier algebra of B , which is endowed with a canonical *strictly continuous* G -action. Then η_1 and η_2 are said to be *G -unitarily equivalent* if there exists a unitary element u in $\mathcal{M}(B)^G$ such that $\text{Ad}(u) \circ \eta_1 = \eta_2$. Similarly, we say that η_1 and η_2 are *approximately G -unitarily equivalent* if there exists a net $(u_i)_{i \in I}$ of unitaries in $\mathcal{M}(B)^G$ such that $(\text{Ad}(u_i) \circ \eta_1)_{i \in I}$ converges pointwise to η_2 .

Definition 3.4. The $G\text{-}C^*$ -algebras (A, α) and (B, β) are said to be:

- (1) *conjugate (or G -isomorphic)*, if there exists an isomorphism $\eta: A \rightarrow B$ satisfying $\eta \circ \alpha_g = \beta_g \circ \eta$ for every $g \in G$; *cocycle conjugate* if there exists a strictly continuous map $v: G \rightarrow U(\mathcal{M}(A))$ satisfying $v_{gh} = v_g v_h$ such that the action $g \mapsto \text{Ad}(v_g) \circ \beta_g$ is conjugate to α .

We say that A is *G -equivariantly B -absorbing* if $A \otimes B$ is cocycle conjugate to B .

Definition 3.5. A $G\text{-}C^*$ -algebra D is said to have *approximately G -inner half-flip*, if it is unital and the canonical G -equivariant injective unital *-homomorphisms $\text{id}_D \otimes 1_D, 1_D \otimes \text{id}_D: D \rightarrow D \otimes D$ are approximately G -unitarily equivalent. A $G\text{-}C^*$ -algebra D is said to be a *strongly self absorbing $G\text{-}C^*$ -algebra* if it is unital and $\text{id}_D \otimes 1_D$ is approximately G -unitarily equivalent to a G -equivariant *-isomorphism.

Observe that if D has approximately G -inner half-flip, then it has approximately inner half-flip as a C^* -algebra. Similarly, if D is a strongly self-absorbing $G\text{-}C^*$ -algebra, then D is strongly self-absorbing as a C^* -algebra. Recall that any unital C^* -algebra D with approximately inner half-flip is automatically simple, nuclear, and has at most one trace; see [17].

The following remark will be used repeatedly and without further reference.

Remark 3.6. Suppose D is a strongly self-absorbing $G\text{-}C^*$ -algebra, and let A be a separable $G\text{-}C^*$ -algebra. Then A is G -equivariantly D -absorbing if and only if $A \otimes D$ is *conjugate* to A ; see [64, Theorem 4.7 and Proposition 4.8].

Thus, when working with strongly self-absorbing actions, we will mostly use conjugacy as the relevant equivalence relation, keeping in mind that it is equivalent to cocycle conjugacy.

The proof of the following theorem follows closely arguments from [20].

Theorem 3.7. *Suppose that D is a separable G -C*-algebra with approximately G -inner half-flip, and that C is a countably positively quantifier-free \mathcal{L}_G^{1,C^*} -saturated unital G -C*-algebra. Suppose that D is commutant weakly contained in C . Fix a G -equivariant unital *-homomorphism $\theta: D \rightarrow C$. The following statements hold:*

- (1) *Any two G -equivariant unital *-homomorphisms $D \rightarrow C$ are G -unitarily equivalent;*
- (2) *The inclusion $\theta(D)' \cap C \hookrightarrow C$ is an \mathcal{L}_G^{1,C^*} -existential G -equivariant unital *-homomorphism;*
- (3) *$\theta(D)' \cap C$ is an elementary \mathcal{L}_G^{1,C^*} -substructure of C ;*
- (4) *If C has density character \aleph_1 , then the inclusion $\theta(D)' \cap C \hookrightarrow C$ is approximately G -unitarily equivalent to a G -isomorphism.*

Proof. Let $\theta_1: D \rightarrow C$ be a G -equivariant unital *-homomorphism. Assume first that the ranges of θ and θ_1 commute. Choose a sequence $(u_n)_{n \in \mathbb{N}}$ of unitaries in $D \otimes D$ witnessing the fact that $\text{id}_D \otimes 1_D, 1_D \otimes \text{id}_D: D \rightarrow D \otimes D$ are approximately G -unitarily equivalent. Let $\Theta: D \otimes D \rightarrow C$ be the G -equivariant unital *-homomorphism given by $d_1 \otimes d_2 \mapsto \theta(d_1)\theta_1(d_2)$. Considering the unitaries $\Theta(u_n)$, for $n \in \mathbb{N}$, and applying the fact that A is countably positively quantifier-free \mathcal{L}_G^{1,C^*} -saturated, we obtain a unitary $u \in C^G$ satisfying $\text{Ad}(u) \circ \theta = \theta_1$. In the general case, when the ranges of θ and θ_1 do not necessarily commute, we may find a unital G -equivariant *-homomorphism $\theta_2: D \rightarrow C$ whose range commutes with those of θ and θ_1 . By the argument above, it follows that θ_2 is G -unitarily equivalent to both θ and θ_1 , so (1) follows.

We prove (2) and (3) simultaneously. Let us identify D with its image under θ . Suppose that \bar{a} is a tuple in $D' \cap C$, \bar{b} is a tuple in C , and $\varphi(\bar{x}, \bar{y})$ is an \mathcal{L}_G^{1,C^*} -formula. Let B be the G -C*-algebra generated by $D \cup \{\bar{a}, \bar{b}\}$ inside C . Observe that $B' \cap C$ satisfies the same assumptions as C . Particularly, by (1) there exists a unitary $u \in B' \cap C$ such that $g^C u = u$ for every $g \in G$ and $u^* \bar{b} u \in D' \cap C$. Hence we have $\varphi^C(\bar{a}, \bar{b}) = \varphi^{D' \cap C}(\bar{a}, u^* \bar{b} u)$, as desired.

The argument above shows that for any separable G -C*-subalgebra B of C and finite tuple \bar{b} in C there exists a unitary u in the fixed point algebra of C such that $u \in B' \cap C$ and $u^* \bar{b} u \in D' \cap C$. One can then apply the intertwining argument of [20, Theorem 2.11] to get (4). \square

Corollary 3.8. *Suppose that D is a separable G -C*-algebra with approximately G -inner half-flip, and \mathcal{F} is a countably incomplete filter. Let A be a separable unital G -C*-algebra, and let $\theta: D \rightarrow \prod_{\mathcal{F}}^G A$ be a G -equivariant unital *-homomorphism. Then:*

- (1) *Any G -equivariant unital *-homomorphism $D \rightarrow \prod_{\mathcal{F}}^G A$ is G -unitarily equivalent to θ ;*
- (2) *The inclusion $\theta(D)' \cap \prod_{\mathcal{F}}^G A \hookrightarrow \prod_{\mathcal{F}}^G A$ is an \mathcal{L}_G^{1,C^*} -existential G -equivariant unital *-homomorphism;*
- (3) *$\theta(D)' \cap \prod_{\mathcal{F}}^G A$ is an elementary \mathcal{L}_G^{1,C^*} -substructure of $\prod_{\mathcal{F}}^G A$;*
- (4) *If \mathcal{F} is a filter over \mathbb{N} and the Continuum Hypothesis holds, then $\theta(D)' \cap \prod_{\mathcal{F}}^G A$ is G -equivariantly *-isomorphic to $\prod_{\mathcal{F}}^G A$.*

Remark 3.9. Theorem 3.7 and Corollary 3.8 generalize [20, Theorem 1, Theorem 2, Corollary 2.12] in two ways: they extend them to the G -equivariant setting, and they remove the unnecessary assumption on the filter \mathcal{F} that the corresponding reduced product be countably saturated. An example of a countable incomplete filter over \mathbb{N} that does not satisfy such an assumption is provided in [20, Example 3.2].

Suppose now that D is a strongly self-absorbing G -C*-algebra. Observe that for any separable G -C*-algebra A and any countably incomplete filter \mathcal{F} , the following assertions are equivalent:

- (1) D is commutant weakly contained in A
- (2) D is positively commutant weakly contained in A ,
- (3) D embeds equivariantly into $F_{\mathcal{F}}^G(A)$.
- (4) A and $A \otimes D$ are (cocycle) conjugate;
- (5) A is G -equivariantly D -absorbing;
- (6) D is weakly \mathcal{L}_G^{1,C^*} -contained in A ;
- (7) D is positively weakly \mathcal{L}_G^{1,C^*} -contained in A .

We deduce the following rigidity result for strongly self-absorbing G - C^* -algebras.

Proposition 3.10. *Let D and E be strongly self-absorbing G - C^* -algebras. The following assertions are equivalent:*

- (1) D and E are (cocycle) conjugate;
- (2) D and E are weakly \mathcal{L}_G^{1,C^*} -equivalent;
- (3) D and E are isomorphic as C^* -algebras to the same strongly self-absorbing C^* -algebra B , and the $\text{Aut}(B)$ -orbits of D and E inside the Polish space $\text{Act}_G(B)$ of continuous actions of G on B have the same closure.

In particular, the classification of strongly self-absorbing G - C^ -algebras up to (cocycle) conjugacy is smooth.*

The equivalence of (2) and (3) in Proposition 3.10 is due to the fact that if D is a strongly self-absorbing C^* -algebra, then any injective $*$ -homomorphism $\eta: D \rightarrow \prod_{\mathcal{U}} D$, where \mathcal{U} is an ultrafilter over \mathbb{N} , admits a lift $(\eta_n)_{n \in \mathbb{N}}$ consisting of automorphisms of D .

Proposition 3.10 can be seen as the equivariant analogue of [20, Theorem 2.16, Corollary 2.17]. We would like to remark, however, that Proposition 3.10 is in principle somewhat more surprising than its nonequivariant counterpart. Indeed, while there are only very few known strongly self-absorbing C^* -algebras (and it is indeed currently known to be complete under additional regularity assumptions on the algebra), there seem to exist a greater variety of strongly self-absorbing actions on C^* -algebras. For instance, for a fixed compact group G and a fixed strongly self-absorbing C^* -algebra D , there may exist multiple (non cocycle equivalent) strongly self-absorbing actions on D . In fact, a complete list of all strongly self-absorbing actions is at the moment far out of reach.

The following consequence of Corollary 3.8 seems worth isolating.

Corollary 3.11. *Let D be a strongly self-absorbing G - C^* -algebra, let A be a separable unital G -equivariantly D -absorbing G - C^* -algebra, let \mathcal{F} be a countably incomplete filter, let $\theta: D \rightarrow \prod_{\mathcal{F}}^G A$ be a \mathcal{L}_G^{1,C^*} -embedding. Then:*

- (1) Any two G -equivariant unital $*$ -homomorphisms of D into $\prod_{\mathcal{F}}^G A$ are G -unitarily equivalent;
- (2) The inclusion $\theta(D)' \cap \prod_{\mathcal{F}}^G A \hookrightarrow \prod_{\mathcal{F}}^G A$ is an \mathcal{L}_G^{1,C^*} -existential G -equivariant unital $*$ -homomorphism;
- (3) $\theta(D)' \cap \prod_{\mathcal{F}}^G A$ is a G -elementary substructure of $\prod_{\mathcal{F}}^G A$;
- (4) If \mathcal{F} is a filter over \mathbb{N} and the Continuum Hypothesis holds, then $\theta(D)' \cap \prod_{\mathcal{F}}^G A$ is G -equivariantly $*$ -isomorphic to $\prod_{\mathcal{F}}^G A$.

Using the results above, one can provide the following model-theoretic characterization of strongly self-absorbing G - C^* -algebras, which in the nonequivariant setting is [20, Theorem 2.14]. (Recall that when G is compact, the notions of strongly self-absorbing G - C^* -algebra and strongly self-absorbing C^* -algebra coincide.)

Theorem 3.12. *Let D be a separable unital G - C^* -algebra, and let \mathcal{F} be a countably incomplete filter. Then D is a strongly self-absorbing G - C^* -algebra if and only if D is weakly \mathcal{L}_G^{1,C^*} -equivalent to $D \otimes D$, and all the G -equivariant unital $*$ -homomorphisms $D \rightarrow \prod_{\mathcal{F}}^G D$ are G -unitarily equivalent.*

Proof. The “only if” implication is a consequence of the fact that D is G -strongly cocycle conjugate to $D \otimes D$, and part (1) of Theorem 3.7. We prove the converse. Since D is weakly \mathcal{L}_G^{1,C^*} -equivalent to $D \otimes D$, we deduce that $D \otimes D$ is a G -elementary substructure of $\prod_{\mathcal{F}}^G D$, say via an embedding ρ . In particular, the G -equivariant unital $*$ -homomorphisms $\rho_1, \rho_2: D \rightarrow \prod_{\mathcal{F}}^G D$, given by $\rho_1(d) = \rho(d \otimes 1_D)$ and $\rho_2(d) = \rho(1_D \otimes d)$, for $d \in D$, are G -unitarily equivalent. It follows that D has approximately G -inner half-flip. The conclusion now follows from the implication (ii) \Rightarrow (i) in [64, Theorem 4.6]. \square

3.5. Limiting examples. We have shown in Proposition 3.10 that, for any second countable locally compact group G , the classification problem for strongly self-absorbing G -actions on C^* -algebras is smooth in the sense of Borel complexity theory. In this subsection, we observe that the same is not true for the broader class of G -actions with approximately G -inner half-flip, even if one only considers actions of $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ on the C^* -algebra \mathcal{O}_2 . The notion of complete analytic set can be found in [46, Section 22.9].

Proposition 3.13. *The relations of conjugacy and cocycle conjugacy for approximately representable \mathbb{Z}_2 -actions on \mathcal{O}_2 with Rokhlin dimension 1 and approximately \mathbb{Z}_2 -inner half-flip are complete analytic sets. Furthermore, the classification problem for such actions, up to conjugacy or cocycle conjugacy, is strictly more complicated than the classification problem for any class of countable structures with Borel isomorphism relation.*

Proof. Recall that in [42] Izumi constructed an action ν of \mathbb{Z}_2 on \mathcal{O}_2 whose crossed product $D = \mathcal{O}_2 \rtimes_{\nu} \mathbb{Z}_2$ is a Kirchberg algebra satisfying the Universal Coefficient Theorem, with trivial K_1 -group, K_0 -group isomorphic to $\mathbb{Z}[\frac{1}{2}]$, and zero element of $K_0(D)$ corresponding to the unit of D ; see [42, Lemma 4.7].

Such an action was used in [34] to prove that the relations of conjugacy and cocycle conjugacy of \mathbb{Z}_2 -actions on \mathcal{O}_2 are complete analytic sets, when regarded as subsets of $\text{Act}_{\mathbb{Z}_2}(\mathcal{O}_2) \times \text{Act}_{\mathbb{Z}_2}(\mathcal{O}_2)$. Precisely, it is proved in [34], relying on a construction of Rørdam from [59], that there exists a Borel map assigning to each uniquely 2-divisible torsion-free countable abelian group Γ a Kirchberg algebra A_{Γ} satisfying the Universal Coefficient Theorem, with trivial K_1 -group, K_0 -group isomorphic to Γ , and zero element of $K_0(A_{\Gamma})$ corresponding to the unit of A_{Γ} . Denote by $\iota_{A_{\Gamma}}$ the trivial \mathbb{Z}_2 -action on A_{Γ} . Then the function $\Gamma \mapsto \alpha_{\Gamma} := \nu \otimes \iota_{A_{\Gamma}}$ provides a Borel reduction from the relation E of isomorphism of uniquely 2-divisible torsion-free countable abelian groups to the relations of conjugacy and cocycle conjugacy of \mathbb{Z}_2 -actions on \mathcal{O}_2 . It is furthermore shown in [34], modifying an argument of Hjorth from [41], that E is a complete analytic set. Furthermore, if F is the relation of isomorphism within a class of countable structures, and if F is Borel, then F is Borel reducible to E (but not vice versa). It was furthermore observed in [34] that, for any uniquely 2-divisible torsion-free countable abelian group Γ , the action α_{Γ} has Rokhlin dimension 1, and is approximately representable.

We claim that α_{Γ} has approximately \mathbb{Z}_2 -inner half-flip. To see this, it is enough to observe that the \mathbb{Z}_2 -action ν on \mathcal{O}_2 (corresponding to the case when Γ is trivial) is strongly self-absorbing. This follows from the fact that ν is constructed as the infinite tensor product $\bigotimes_{n \in \mathbb{N}} \text{Ad}(u)$, where u is a unitary element of $\mathcal{O}_{\infty}^{\text{st}}$, using the identification $\mathcal{O}_2 \cong \bigotimes_{n \in \mathbb{N}} \mathcal{O}_{\infty}^{\text{st}}$; see [42, Section 4]. Since $\mathcal{O}_{\infty}^{\text{st}}$ is a C*-algebra with approximately inner half-flip, one can deduce from [65, Proposition 5.3] that ν is strongly self-absorbing. Since α_{Γ} is the tensor product of a strongly self-absorbing action (namely, ν) with an action with approximately \mathbb{Z}_2 -inner half-flip (namely, $\iota_{A_{\Gamma}}$), it follows that α_{Γ} has approximately \mathbb{Z}_2 -inner half-flip. This proves the claim.

Using these observations, and considering the fact that the set of \mathbb{Z}_2 -actions on \mathcal{O}_2 with approximately \mathbb{Z}_2 -inner half-flip is analytic, the result follows. \square

Clearly, similar conclusions hold for G -actions on \mathcal{O}_2 for any countable discrete group G with a quotient of order 2, such as the group of integers. This can be seen by regarding a \mathbb{Z}_2 -action as a G -action in the canonical way.

4. ORDER ZERO DIMENSION AND ROKHLIN DIMENSION

4.1. Order zero dimension. The notion of positive weak \mathcal{L} -containment between \mathcal{L} -morphisms can be defined in the general setting of logic for metric structures; see Subsection A.2. For G -C*-algebras, one has the following: a G -equivariant *-homomorphism $\theta: A \rightarrow B$ is positively weakly $\mathcal{L}_G^{C^*}$ -contained in another G -equivariant *-homomorphism $f: A \rightarrow C$ if for any separable subalgebras $A_0 \subset A$ and $B_0 \subset B$ such that $\theta(A_0) \subset B_0$, and for some (equivalently, any) countably incomplete filter \mathcal{F} , there exists a G -equivariant *-homomorphism $\gamma: B_0 \rightarrow \prod_{\mathcal{F}}^G C$ such that $(\gamma \circ \theta)|_{A_0} = (\Delta_C \circ f)|_{A_0}$, where $\Delta_C: C \rightarrow \prod_{\mathcal{F}}^G C$ is the diagonal *-homomorphism. Various equivalent formulations of this notion can be found in Subsection A.2.

We now present natural generalizations of positive weak $\mathcal{L}_G^{C^*}$ -containment where instead of a single *-homomorphism one considers a tuple of completely positive contractive order zero maps. Whenever $f: A \rightarrow B$ is a G -equivariant *-homomorphism, we will regard B as a G -equivariant A -bimodule, as defined in Subsection 2.6.

Definition 4.1. Let A , B , and C be G -C*-algebras, and let $\theta: A \rightarrow B$ and $f: A \rightarrow C$ be G -equivariant *-homomorphisms. We say that θ is G -equivariantly d -contained in f if for any separable G -C*-subalgebras $A_0 \subset A$ and $B_0 \subset B$ such that $\theta(A_0) \subset B_0$, and for some (equivalently, any) countably incomplete filter \mathcal{F} , there exist G -equivariant completely positive contractive order zero A -bimodule maps $\psi_0, \dots, \psi_d: B_0 \rightarrow \prod_{\mathcal{F}}^G C$ whose sum $\psi = \psi_0 + \dots + \psi_d$ is contractive and such that $(\psi \circ \theta)|_{A_0} = (\Delta_C \circ f)|_{A_0}$.

The notion of G -equivariant d -containment from Definition 4.1 admits a natural syntactic reformulation: $\theta: A \rightarrow B$ is G -equivariantly d -contained in $f: A \rightarrow C$ if and only if for any tuples \bar{a} in A , \bar{b} in B , and for any tuple \bar{w} of elements of a finite dimensional C*-algebra, for any positive quantifier-free $\mathcal{L}_G^{\text{oz}, A-A}$ -formulas $\varphi(\bar{z}, \bar{y})$, for any positive quantifier-free $\mathcal{L}_G^{\text{os}, A-A}$ -formulas $\psi(\bar{x}, \bar{z}, \bar{y})$, where the variables \bar{z} have finite-dimensional C*-algebras as sorts, and for any $\varepsilon > 0$, there exist tuples $\bar{c}_0, \dots, \bar{c}_d$ in C such that the following are satisfied for $j = 0, \dots, d$:

$$\psi(f(\bar{a}), \bar{w}, \bar{c}_0 + \dots + \bar{c}_d) \leq \psi(\theta(\bar{a}), \bar{w}, \bar{b}) + \varepsilon \quad \text{and} \quad \varphi(\bar{w}, \bar{c}_j) \leq \varphi(\bar{w}, \bar{b}) + \varepsilon.$$

Remark 4.2. When B is nuclear, in the syntactic characterization of d -containment, one can replace $\mathcal{L}_G^{\text{osos}, A-A}$ -formulas with $\mathcal{L}_G^{\text{osos-nuc}, A-A}$ -formulas, and $\mathcal{L}_G^{\text{oz}, A-A}$ -formulas with $\mathcal{L}_G^{\text{oz-nuc}, A-A}$ -formulas. This follows from the characterization of $\mathcal{L}_G^{\text{osos-nuc}, A-A}$ -morphisms from Lemma 2.4.

Definition 4.3. The G -equivariant order zero dimension $\dim_{\text{oz}}^G(\theta)$ of a G -equivariant $*$ -homomorphism $\theta: A \rightarrow B$ is the smallest integer $d \geq 0$ such that θ is G -equivariantly d -contained in the identity map $\text{id}_A: A \rightarrow A$. If no such d exists, we set $\dim_{\text{oz}}^G(\theta) = \infty$.

The proof of the following is an easy consequence of the syntactic characterization of G -equivariant d -containment.

Proposition 4.4. *Let Λ be a directed set.*

(1) *Let $\theta_0: A \rightarrow B$ and $\theta_1: B \rightarrow C$ be G -equivariant $*$ -homomorphisms between G - C^* -algebras. Then*

$$\dim_{\text{oz}}^G(\theta_1 \circ \theta_0) + 1 \leq (\dim_{\text{oz}}^G(\theta_1) + 1)(\dim_{\text{oz}}^G(\theta_0) + 1);$$

(2) *Let $\theta: A \rightarrow B$ be a G -equivariant $*$ -homomorphism, and let C be a G - C^* -algebra. Then*

$$\dim_{\text{oz}}^G(\theta \otimes \text{id}_C) \leq \dim_{\text{oz}}^G(\theta);$$

(3) *Let $(\{A_\lambda\}_{\lambda \in \Lambda}, \{\theta_{\lambda, \mu}\}_{\lambda, \mu \in \Lambda, \lambda < \mu})$ be a direct system of G - C^* -algebras (with G -equivariant $*$ -homomorphisms). For $\lambda \in \Lambda$, denote by*

$$\theta_{\lambda, \infty}: A_\lambda \rightarrow \varinjlim A_\mu$$

denote the canonical equivariant $$ -homomorphism. Then*

$$\dim_{\text{oz}}^G(\theta_{\lambda, \infty}) \leq \limsup_{\mu \in \Lambda} \dim_{\text{oz}}^G(\theta_{\lambda, \mu}).$$

(4) *For $j = 0, 1$, let $(\{A_\lambda^{(j)}\}_{\lambda \in \Lambda}, \{\theta_{\lambda, \mu}^{(j)}\}_{\lambda, \mu \in \Lambda, \lambda < \mu})$ be a direct system of G - C^* -algebras. Let $\{\eta_\lambda: A_\lambda^{(0)} \rightarrow A_\lambda^{(1)}\}_{\lambda \in \Lambda}$ be a family of G -equivariant $*$ -homomorphisms. Then*

$$\dim_{\text{oz}}^G(\varinjlim_{\lambda \in \Lambda} \eta_\lambda) \leq \limsup_{\mu \in \Lambda} \dim_{\text{oz}}^G(\eta_\mu).$$

Let $\theta: A \rightarrow B$ is a G -equivariant completely positive contractive order zero map. We recall that by [30, Proposition 2.3], there is a naturally induced completely positive contractive order zero map $A \rtimes G \rightarrow B \rtimes G$ between the crossed products, which we will denote in the following by $\widehat{\theta}$. If θ is a $*$ -homomorphism, then $\widehat{\theta}$ is a $*$ -homomorphism as well. If θ is an A - A -bimodule morphism, then $\widehat{\theta}$ is an A - A -bimodule morphism as well.

Lemma 4.5. Let A and B be G - C^* -algebras and let $\theta: A \rightarrow B$ be a G -equivariant $*$ -homomorphism. Then $\dim_{\text{oz}}^G(\widehat{\theta}) \leq \dim_{\text{oz}}^G(\theta)$ and $\dim_{\text{oz}}^G(\theta|_{A^G}) \leq \dim_{\text{oz}}^G(\theta)$.

Proof. Observe that if A is a G - C^* -algebra and \mathcal{F} is a countably incomplete filter, then there exists a canonical $*$ -homomorphism $(\prod_{\mathcal{F}}^G A) \rtimes G \rightarrow \prod_{\mathcal{F}}^G (A \rtimes G)$ in view of the universal property of the full crossed product; see [28]. This, together with the remarks above, proves the first assertion. The second assertion can be proved similarly observing that A^G is positively quantifier-free $\mathcal{L}_G^{C^*}$ -definable. \square

We also consider the following strengthening of the notion of d -containment.

Definition 4.6. Let A , B , and C be G - C^* -algebras of density character less than κ , and let $\theta: A \rightarrow B$ and $f: A \rightarrow C$ be G -equivariant $*$ -homomorphisms. We say that θ is G -equivariantly d -contained in f with commuting towers if for some (equivalently, any) κ -good filter \mathcal{F} , there exist G -equivariant completely positive contractive order zero A -bimodule maps $\psi_0, \dots, \psi_d: B \rightarrow \prod_{\mathcal{F}}^G C$ whose sum $\psi = \psi_0 + \dots + \psi_d$ is contractive, such that $\psi \circ \theta = \Delta_C \circ f$ and such that, for every $0 \leq i < j \leq d$, the images of $\theta(A)' \cap B$ under ψ_i and ψ_j commute.

Observe that, in Definition 4.6, since the ψ_i 's are assumed to be A -bimodule maps, the image of $A' \cap B$ under ψ_i is contained in $f(A)' \cap \prod_{\mathcal{F}}^G C$. Similarly as d -containment, the notion of G -equivariant d -containment with commuting towers from Definition 4.6 admits a natural syntactic reformulation: $\theta: A \rightarrow B$ is G -equivariantly d -contained in $f: A \rightarrow C$ if and only if for any tuples \bar{a} in A , \bar{b} in B , and $\bar{b}' \in \theta(A)' \cap B$, and for any tuple \bar{w} of elements of a finite dimensional C^* -algebra, for any positive quantifier-free $\mathcal{L}_G^{\text{oz}, A-A}$ -formulas $\varphi(\bar{z}, \bar{y}, \bar{y}')$, for any positive quantifier-free

$\mathcal{L}_G^{\text{osos}, A-A}$ -formulas $\psi(\bar{x}, \bar{z}, \bar{y})$, where the variables \bar{z} have finite-dimensional C*-algebras as sorts, and for any $\varepsilon > 0$, there exist tuples $\bar{c}_0, \dots, \bar{c}_d, \bar{c}'_0, \dots, \bar{c}'_d$ in C such that $[\bar{c}'_i, \bar{c}'_j] = 0$ for $0 \leq i < j \leq d$,

$$\psi(f(\bar{a}), \bar{w}, \bar{c}_0 + \dots + \bar{c}_d, \bar{c}'_0 + \dots + \bar{c}'_d) \leq \psi(\theta(\bar{a}), \bar{w}, \bar{b}, \bar{b}') + \varepsilon$$

and

$$\varphi(\bar{w}, \bar{c}_j) \leq \varphi(\bar{w}, \bar{b}, \bar{b}') + \varepsilon \text{ for } j = 0, \dots, d.$$

Definition 4.7. The G -equivariant order zero dimension *with commuting towers* $\dim_{\text{oz}}^{c, G}(\theta)$ of a G -equivariant *-homomorphism $\theta: A \rightarrow B$ is the smallest integer $d \geq 0$ such that θ is G -equivariantly d -contained with commuting towers in the identity map $\text{id}_A: A \rightarrow A$. If no such d exists, we set $\dim_{\text{oz}}^G(\theta) = \infty$.

4.2. Commutant d -containment. The notion of commutant positive existential $\mathcal{L}_G^{C^*}$ -theory of a G -C*-algebra has been introduced in Subsection 3.2. In this section, we consider d -dimensional generalizations of such a notion. We will regard (not necessarily unital) C*-algebras as structures in the Kirchberg language introduced in Subsection A.4. This will allow us to formulate a definition applicable in both the unital and the nonunital settings.

Definition 4.8. Let $d \in \mathbb{N}$, and let A and B be G -C*-algebras. Fix a cardinal κ larger than the density character of A and B . We say that A is *G -equivariantly commutant d -contained* in B , and write $A \lesssim_d B$, if for some (equivalently, any) countably incomplete κ -good filter \mathcal{F} , and for any separable unital G -C*-subalgebra C of $F_{\mathcal{F}}^G(A)$, there exist G -equivariant completely positive contractive order zero maps $\eta_0, \dots, \eta_d: C \rightarrow F_{\mathcal{F}}^G(B)$ with unital sum.

We say that A is *G -equivariantly commutant d -contained in B with commuting towers*, and write $A \lesssim_d^c B$, if one choose the maps $\eta_0, \dots, \eta_d: C \rightarrow F_{\mathcal{F}}^G(B)$ as above to also have pairwise commuting ranges.

Using Proposition 2.9 one can give a syntactic reformulation of Definition 4.8, which in particular shows that the choice of the countably incomplete κ -good filter \mathcal{F} is irrelevant. When A, B are separable, one can take any countably incomplete filter. It is not difficult to see that if $A \lesssim_{d-1} B$ and $B \lesssim_{k-1} C$ then $A \lesssim_{dk-1} C$.

Remark 4.9. Suppose that a separable unital G -C*-algebra A admits a G -equivariant unital *-homomorphism into $A' \cap \prod_{\mathcal{F}}^G A$ for some (equivalently, any) countably incomplete filter. Then A is G -equivariantly commutant d -contained (with commuting towers) in B if and only if there exist G -equivariant completely positive contractive order zero maps $\eta_0, \dots, \eta_d: A \rightarrow F_{\mathcal{F}}^G(B)$ (with commuting ranges) such that $\eta_0 + \dots + \eta_d$ is unital. In particular, this applies when A is commutative, or when A is strongly self-absorbing; see [64, Theorem 4.6].

Let D be a strongly self-absorbing G -C*-algebra. By [64, Theorem 4.7], a separable G -C*-algebra B is G -equivariantly D -absorbing if and only if D is commutant 0-contained in B . We will prove in Theorem 4.35 that this is in turn equivalent to D being commutant d -contained with commuting towers in B for any $d \in \mathbb{N}$.

Remark 4.10. Let A and B be separable G -C*-algebras with A G -equivariantly commutant d -contained (with commuting towers) in B . Let C be a separable subalgebra of $F_{\mathcal{F}}^G(B)$. It is a consequence of Proposition 2.9 that there exist completely positive contractive order zero maps $\eta_0, \dots, \eta_d: A \rightarrow C' \cap F_{\mathcal{F}}^G(B)$ (with commuting ranges) such that $\eta_0 + \dots + \eta_d$ is unital.

4.3. Relationship between order zero dimension and d -containment. The notion of *Rokhlin dimension* (with commuting towers) for a G -C*-algebra—see [40, Definition 1.1], [31, Definition 3.2]—can be naturally presented in terms of d -containment. Precisely, a G -C*-algebra A has Rokhlin dimension (with commuting towers) at most d if and only if the G -C*-algebra $C(G)$ endowed with the canonical left translation action Lt is G -equivariantly commutant d -contained (with commuting towers) in A . We will denote by $\dim_{\text{Rok}}(A)$ the Rokhlin dimension of a G -C*-algebra A , and by $\dim_{\text{Rok}}^c(A)$ the Rokhlin dimension with commuting towers of A . We point out that Rokhlin dimension has recently been generalized to \mathbb{R} -actions (flows) in [38].

In Proposition 4.14, we will observe that there exists a relationship between the notion of G -equivariant order zero dimension of a G -equivariant *-homomorphism introduced in Subsection 4.1, and the notion of G -equivariant commutant d -containment introduced in Subsection 4.2. Precisely, if $\theta: A \rightarrow B$ has G -equivariant order zero dimension (with commuting towers) at most d , then B is commutant d -contained (with commuting towers) in A .

Lemma 4.11. Let C be a unital G -C*-algebra, let A and B be G -C*-algebras, let κ be a cardinal larger than the density character of A and C , and let \mathcal{F} be a countably incomplete κ -good filter. Suppose that $\theta: A \rightarrow B$ is a G -equivariant *-homomorphism, and let $1_C \otimes \theta: A \rightarrow C \otimes_{\max} B$ be the map $a \mapsto 1_C \otimes \theta(a)$. If $\dim_{\text{oz}}^G(1_C \otimes \theta) \leq d < +\infty$,

then there exist G -equivariant completely positive contractive order zero maps $\eta_0, \dots, \eta_d: C \rightarrow F_{\mathcal{F}}^G(A)$ such that $\sum_{j=0}^d \eta_j$ is unital. The converse holds if $A = B$ and $\theta: A \rightarrow A$ is the identity.

Proof. Observe that θ is necessarily injective. We can therefore identify A with its image under θ inside B . Let $\psi_0, \dots, \psi_d: C \otimes_{\max} B \rightarrow \prod_{\mathcal{F}}^G A$ be G -equivariant completely positive contractive order zero A -bimodule maps witnessing the fact that $\dim_{\text{oz}}^G(1_C \otimes \theta) \leq d$. Fix $c_0 = 1, c_1, \dots, c_n \in C$. Let $t(\bar{x})$ be a positive quantifier-free $\mathcal{L}_G^{\text{oz}}$ -type that is realized by (c_0, \dots, c_n) in C . Consider the corresponding multiplier $\mathcal{L}_G^K(A)$ -type t_A^m defined as in Subsection 2.7. Let $t_A^c(\bar{x})$ be the commutant type associated with A , and consider the $\mathcal{L}_G^K(A)$ -type $q_A(\bar{y}_0, \dots, \bar{y}_d)$ consisting of conditions $\varphi(\bar{y}_j) \leq r$ for any condition $\varphi(\bar{x}) \leq r$ in $t_A^m(\bar{x}) \cup t_A^c(\bar{x})$ and $j = 0, 1, \dots, d$, and $\|a(y_{0,0} + \dots + y_{0,d}) - a\| = 0$ for every $a \in A$. Fix an approximate unit $(a_\lambda)_{\lambda \in \Lambda}$ for A . Considering the tuple $\bar{b} := (c_0 \otimes a_\lambda, \dots, c_n \otimes a_\lambda)$ in $C \otimes_{\max} B$, for large enough λ , we conclude that the type $t_A^m(\bar{x}) \cup t_A^c(\bar{x})$ is approximately realized in $C \otimes_{\max} B$. Recall that, by definition of G -equivariant order zero dimension, ψ_0, \dots, ψ_d are completely contractive order zero A -bimodule maps with contractive sums such that $(\psi_0 + \dots + \psi_d)|_A$ is the canonical diagonal G -equivariant $*$ -homomorphism $A \rightarrow \prod_{\mathcal{F}}^G A$. Therefore, by Los' theorem, considering the elements $\psi_0(\bar{b}), \dots, \psi_d(\bar{b})$ shows that the type $q_A(\bar{x})$ is approximately realized in A . The conclusion follows from quantifier-free positive $\mathcal{L}_G^K(A)$ -saturation of $F_{\mathcal{F}}^G(A)$; see Proposition 2.9.

Conversely, suppose that $A = B$, that $\theta: A \rightarrow A$ is the identity map id_A of A , and that there exist G -equivariant completely positive contractive order zero maps $\eta_0, \dots, \eta_d: C \rightarrow F_{\mathcal{F}}^G(A)$ such that $\eta_0 + \dots + \eta_d$ is unital. Then the function $F_{\mathcal{F}}^G(A) \times A \rightarrow \prod_{\mathcal{F}}^G A$, given by $([a_i]_{i \in I}, b) \mapsto [a_i b]_{i \in I}$, induces a G -equivariant $*$ -homomorphism $\Psi: F_{\mathcal{F}}^G(A) \otimes_{\max} A \rightarrow \prod_{\mathcal{F}}^G A$, by the universal property of the maximal tensor product. One can then define $\psi_j = \Psi \circ (\eta_j \otimes \text{id}_A): C \otimes_{\max} A \rightarrow \prod_{\mathcal{F}}^G A$, for $j = 0, \dots, d$. These are well defined G -equivariant completely positive contractive order zero A -bimodule maps, which witness that $\dim_{\text{oz}}^G(\theta) \leq d$. \square

Recall that, if A is a C^* -algebra and C is a unital C^* -algebra, then the relative commutant of $1_C \otimes A$ inside $C \otimes_{\max} A$ is equal to $C \otimes Z(A)$, where $Z(A)$ is the center of A ; see [2, Theorem 4]. Using this fact, the same proof as Lemma 4.11 shows the following.

Lemma 4.12. Let C be a unital G - C^* -algebra, let A and B be G - C^* -algebras, let κ be a cardinal larger than the density character of A and C , and let \mathcal{F} be a countably incomplete κ -good filter. Suppose that $\theta: A \rightarrow B$ is a G -equivariant $*$ -homomorphism, and let $1_C \otimes \theta: A \rightarrow C \otimes_{\max} B$ be the map $a \mapsto 1_C \otimes \theta(a)$. If $\dim_{\text{oz}}^{c,G}(1_C \otimes \theta) \leq d < +\infty$, then there exist G -equivariant completely positive contractive order zero maps $\eta_0, \dots, \eta_d: C \rightarrow F_{\mathcal{F}}^G(A)$ with commuting ranges such that $\eta_0 + \dots + \eta_d$ is unital. The converse holds if $A = B$ and $\theta: A \rightarrow A$ is the identity map.

When $C = C(G)$, we deduce the following:

Lemma 4.13. Let A be a G - C^* -algebra. Denote by $\theta: A \rightarrow C(G) \otimes A$ the second factor embedding. Then $\dim_{\text{Rok}}(A) = \dim_{\text{oz}}^G(\theta)$ and $\dim_{\text{Rok}}^c(A) = \dim_{\text{oz}}^{c,G}(\theta)$.

Proposition 4.14. Suppose that $\theta: A \rightarrow B$ is a G -equivariant $*$ -homomorphism. If $\dim_{\text{oz}}^G(\theta) \leq d$, then $B \lesssim_d A$. If $\dim_{\text{oz}}^{c,G}(\theta) \leq d$, then $B \lesssim_d^c A$.

Proof. Fix a countably incomplete filter \mathcal{F} . Suppose that $\dim_{\text{oz}}^G(\theta) \leq d$. Fix a separable unital G - C^* -subalgebra C of $F_{\mathcal{F}}(B)$. Then by Lemma 4.11 the second factor embedding $1_C \otimes \text{id}_B: B \rightarrow C \otimes_{\max} B$ has order zero dimension equal to zero. By Proposition 4.4(2), the G -equivariant $*$ -homomorphism $(1_C \otimes \text{id}_B) \circ \theta: A \rightarrow C \otimes_{\max} B$ has order zero dimension at most d . Therefore by Lemma 4.11 again there exist G -equivariant completely positive contractive order zero maps $\eta_0, \dots, \eta_d: C \rightarrow F_{\mathcal{F}}^G(A)$ such that $\eta_0 + \dots + \eta_d$ is unital. By Lemma 4.11, this concludes the proof.

The second assertion can be proved in the same way, by replacing Lemma 4.11 with Lemma 4.12. \square

4.4. Dimension functions. By a *dimension function* for (nuclear) G - C^* -algebras we mean a function from the class of (nuclear) C^* -algebras to $\{0, 1, 2, \dots, \infty\}$.

Definition 4.15. A dimension function dim for G - C^* -algebras is said to be *positively $\forall\exists$ -axiomatizable* if there exists a collection \mathcal{F} of formulas $\xi(\bar{x}, \bar{z}_0, \dots, \bar{z}_d, \bar{y}_0, \dots, \bar{y}_d)$ of the form

$$\max \{ \eta(\bar{x}, \bar{z}_0, \dots, \bar{z}_d), \varphi_0(\bar{z}_0, \bar{y}_0), \dots, \varphi_d(\bar{z}_d, \bar{y}_d), \psi(\bar{x}, \bar{z}_0, \dots, \bar{z}_d, \bar{y}_0 + \dots + \bar{y}_d) \},$$

where

- (1) $\bar{z}_0, \dots, \bar{z}_d$ have finite-dimensional C*-algebras as sorts,
- (2) η is a positive quantifier-free $\mathcal{L}_G^{C^*}$ -formula,
- (3) φ is a positive quantifier-free \mathcal{L}_G^{oz} -formula,
- (4) ψ is a positive quantifier-free \mathcal{L}_G^{osos} -formula,

such that the following holds: for a G -C*-algebra A , $\dim(A) \leq d$ if and only if

$$A \models \sup_{\bar{x}} \inf_{\bar{z}_0} \dots \inf_{\bar{z}_d} \inf_{\bar{y}_0} \dots \inf_{\bar{y}_d} \xi(\bar{x}, \bar{z}_0, \dots, \bar{z}_d, \bar{y}_0, \dots, \bar{y}_d) = 0.$$

Definition 4.16. A dimension function for *nuclear* G -C*-algebras is said to be *nuclearly* positively $\forall\exists$ -axiomatizable if in Definition 4.15 we can simultaneously choose φ and ψ to be positive quantifier-free formulas in \mathcal{L}_G^{oz-nuc} and $\mathcal{L}_G^{C^*-nuc}$, respectively.

Example 4.17. The following are positively $\forall\exists$ -axiomatizable dimension functions for nuclear C*-algebras:

- (1) Nuclear dimension. Indeed, one can consider variables $(\bar{z}_0, \dots, \bar{z}_d)$ with sorts finite-dimensional C*-algebras F_0, \dots, F_d and then the formulas
 - $\eta(\bar{x}, \bar{z}_0, \dots, \bar{z}_d) \equiv \max_{j=0, \dots, d} \inf_{s \in \text{CPC}(A, F_j)} \max_k \|s(x_k) - z_{j,k}\|$;
 - $\varphi_j(\bar{y}_j, \bar{z}_j) \equiv \inf_{t \in \text{CPC}(F_j, A)} \max_k \|t(z_{j,k}) - y_{j,k}\|$, for a fixed $j = 0, \dots, d$;
 - $\psi(\bar{x}, \bar{z}_0, \dots, \bar{z}_d, \bar{y}) \equiv \max_k \|x_k - y_k\|$.
- (2) Decomposition rank. In fact, one may just consider the same formulas η and ψ as in (1), together with

$$\varphi_j(\bar{z}_j, \bar{y}_j) \equiv \inf_{t \in \text{CPC}(F_j, A)} \max_k \|t(z_{j,k}) - y_{j,k}\|, \text{ for } j = 0, \dots, d.$$

If \bar{x}, \bar{y} are n -tuples of variables, we write $\delta(\bar{x}, \bar{y})$ for the formula

$$\max_{1 \leq j, k \leq n} \|x_j y_k - y_k x_j\|.$$

Definition 4.18. A dimension function \dim for (separable) G -C*-algebras is said to be *commutant positively existentially axiomatizable* if there exists a collection \mathcal{F} of formulas $\xi(\bar{x}, \bar{y}_0, \dots, \bar{y}_d)$ of the form

$$\max_{\substack{0 \leq j < k \leq d \\ 1 \leq \ell \leq n}} \{\delta(\bar{x}, \bar{y}_j), \varphi(\bar{y}_j), \|x_\ell(y_{0,j} + \dots + y_{d,j}) - x_\ell\|\},$$

where φ is a quantifier-free positive \mathcal{L}_G^{oz} -formula with parameters from finite-dimensional C*-algebras, such that the following holds: for a (separable) G -C*-algebra A , one has $\dim(A) \leq d$ if and only if

$$A \models \sup_{\bar{x}} \inf_{\bar{y}_0} \dots \inf_{\bar{y}_d} \xi(\bar{x}, \bar{y}_0, \dots, \bar{y}_d) = 0.$$

Definition 4.19. Suppose that \dim is a dimension function for (separable) G -C*-algebras. We say that \dim is *commutant positively existentially axiomatizable with commuting towers* if there exists a collection \mathcal{F} of formulas $\xi(\bar{x}, \bar{y}_0, \dots, \bar{y}_d)$ of the form

$$\max_{\substack{0 \leq j < k \leq d \\ 1 \leq \ell \leq n}} \{\delta(\bar{x}, \bar{y}_j), \delta(\bar{y}_j, \bar{y}_k), \varphi(\bar{z}_j, \bar{y}_j), \|x_\ell(y_{0,j} + \dots + y_{d,j}) - x_\ell\|\}$$

where φ is a positive quantifier-free \mathcal{L}_G^{oz} -formula with parameters from finite-dimensional C*-algebras, such that the following holds: for a (separable) G -C*-algebra A , one has $\dim(A) \leq d$ if and only if

$$A \models \sup_{\bar{x}} \inf_{\bar{y}_0} \dots \inf_{\bar{y}_d} \xi(\bar{x}, \bar{z}_0, \dots, \bar{z}_d, \bar{y}_0, \dots, \bar{y}_d) = 0.$$

Example 4.20. Suppose that C is a fixed G -C*-algebra. Set $\dim(A) \leq d$ if and only if C is commutant d -contained in C (with commuting towers). Then \dim is a dimension function for G -C*-algebras that is commutant positively existentially axiomatizable (with commuting towers).

In the particular case when C is the G -C*-algebra $C(G)$ endowed with the canonical shift action of G , this says that Rokhlin dimension (with commuting towers) is a commutant positively existentially axiomatizable (with commuting towers) dimension function.

The following is a consequence of Definition 4.19 and the syntactic characterization of commutant d -containment.

Proposition 4.21. *Let \dim be a dimension function for separable G - C^* -algebras that is positively existentially axiomatizable (with commuting towers). Let A and B be separable G - C^* -algebras such that A is commutant d -contained (with commuting towers) in B . Then*

$$\dim(B) + 1 \leq (d + 1)(\dim(A) + 1).$$

Similarly, the following fact is a consequence of the syntactic characterization of G -equivariant d -containment, Remark 4.2, and Proposition 4.14.

Proposition 4.22. *Let A and B be G - C^* -algebras, and let $\theta: A \rightarrow B$ be a G -equivariant $*$ -homomorphism. Suppose that \dim is a dimension function for C^* -algebras. If \dim is positively $\forall\exists$ -axiomatizable dimension or commutant positively existentially axiomatizable, then*

$$\dim(A) + 1 \leq (\dim_{\text{oz}}^G(\theta) + 1)(\dim(B) + 1).$$

Moreover, if B is nuclear and \dim is nuclearly $\forall\exists$ -axiomatizable, then again

$$\dim(A) + 1 \leq (\dim_{\text{oz}}^G(\theta) + 1)(\dim(B) + 1).$$

In particular, Proposition 4.22 applies when \dim is either nuclear dimension \dim_{nuc} , decomposition rank dr , or Rokhlin dimension \dim_{Rok} ; see Example 4.17 and Example 4.20. More generally, one can define the nuclear dimension and decomposition rank of a $*$ -homomorphism $f: A \rightarrow B$, and then show that if $\theta: A \rightarrow B$ is d -contained in f , then

$$\dim_{\text{nuc}}(\theta) + 1 \leq (d + 1)(\dim_{\text{nuc}}(f) + 1) \quad \text{and} \quad \text{dr}(\theta) + 1 \leq (d + 1)(\text{dr}(f) + 1).$$

The following result relates the order zero dimension of the canonical inclusions $A^G \rightarrow A$ and $A \rtimes G \rightarrow A \otimes \mathcal{K}(L^2(G))$ to the Rokhlin dimension of a G - C^* -algebra A .

Proposition 4.23. *Let A be a G - C^* -algebra A , and denote by $\iota: A^G \hookrightarrow A$ and $\sigma: A \rtimes G \rightarrow A \otimes \mathcal{K}(L^2(G))$ the canonical inclusion maps. Then $\dim_{\text{oz}}(\iota) \leq \dim_{\text{Rok}}(A)$ and $\dim_{\text{oz}}(\sigma) \leq \dim_{\text{Rok}}(A)$.*

Proof. Denote by $\theta: A \rightarrow C(G) \otimes A$ the second factor embedding. Let Lt denote the action of G on $C(G)$ by left translation, and denote by α the given action on A . Endow $C(G) \otimes A$ with the tensor product action $\gamma = \text{Lt} \otimes \alpha$. Then θ is G -equivariant, and hence it induces a $*$ -homomorphism $A \rtimes G \rightarrow (C(G) \otimes A) \rtimes G$. Observe that $(C(G) \otimes A, \gamma)$ is canonically G -equivariantly isomorphic to $(C(G) \otimes A, \text{Lt} \otimes \iota_A)$ by [29, Proposition 2.3]. Then the crossed product $(C(G) \otimes A) \rtimes_{\gamma} G$ is canonically isomorphic to $A \otimes \mathcal{K}(L^2(G))$, and the fixed point algebra $(C(G) \otimes A)^{\gamma}$ is canonically isomorphic to A . It follows that the map $\hat{\theta}$ —defined right before Lemma 4.5—is canonically conjugate to σ , and $\theta|_{A^G}$ is canonically conjugate to ι .

Using Lemma 4.13 at the first step, Lemma 4.5 at the second, and the above observations at the third, we get

$$\dim_{\text{Rok}}(A) = \dim_{\text{oz}}^G(\theta) \geq \dim_{\text{oz}}(\hat{\theta}) = \dim_{\text{oz}}(\hat{\sigma}).$$

Similarly, we have $\dim_{\text{Rok}}(A) \geq \dim_{\text{oz}}(\iota)$, as desired. \square

Corollary 4.24. *Let A be a G - C^* -algebra A , and let \dim be a positively $\forall\exists$ -axiomatizable dimension function for C^* -algebras. Then*

$$\dim(A^G) + 1 \leq (\dim_{\text{Rok}}(A) + 1)(\dim(A) + 1),$$

and

$$\dim(A \rtimes G) \leq (\dim_{\text{Rok}}(A) + 1)(\dim(A) + 1).$$

For separable unital A , the following first appeared as [30, Theorem 3.3]. The particular case of commuting towers has also been independently obtained in [32], using completely different methods.

Corollary 4.25. *Let A be a G - C^* -algebra A . Then*

$$\dim_{\text{nuc}}(A^G) + 1 \leq \dim_{\text{nuc}}(A \rtimes G) + 1 \leq (\dim_{\text{Rok}}(A) + 1)(\dim_{\text{nuc}}(A) + 1)$$

and

$$\text{dr}(A^G) + 1 \leq \text{dr}(A \rtimes G) + 1 \leq (\dim_{\text{Rok}}(A) + 1)(\text{dr}(A) + 1).$$

Proof. The first inequalities in Corollary 4.25 are due to the fact that the fixed point algebra A^G of a G -C*-algebra is a corner of the crossed product $A \rtimes G$ —see [61, Theorem]—and the fact that decomposition rank and nuclear dimension are nonincreasing when passing to hereditary subalgebras; see [50, Proposition 3.8] and [72, Proposition 2.5]. The second inequalities are an immediate consequence of Corollary 4.24 and Example 4.17. \square

4.5. Bundles. In this subsection, we generalize the main result of [15] to equivariant bundles; see Theorem 4.28. This result will be crucial in our applications to actions with finite Rokhlin dimension in Subsections 6.4 and 6.5.

We will need the following equivariant version of the Choi-Effros lifting theorem for compact groups.

Proposition 4.26. *Let (A, α) and (B, β) be G -C*-algebras, and let $\varphi: A \rightarrow B$ be a surjective, G -equivariant, nuclear *-homomorphism. Then there exists a G -equivariant completely positive contractive lift $\sigma: B \rightarrow A$. If φ is unital, then we can also choose σ to be unital.*

Proof. Use Choi-Effros to find a completely positive contractive lift $\rho: B \rightarrow A$ (which may be chosen to be unital if φ is). If μ denotes the normalized Haar measure on G , then it is easy to check that the map $\sigma: B \rightarrow A$ given by $\sigma(b) = \int_G \alpha_g(\rho(\beta_{g^{-1}}(b))) \, d\mu$, for all $b \in B$, is as in the statement. \square

Suppose that X is a compact metrizable space. The definition of $C(X)$ -algebra can be found in [15, Definition 2.1]. We consider here the natural equivariant analog of a $C(X)$ -algebra:

Definition 4.27. Let A be a $C(X)$ -algebra. For $x \in X$, denote by U_x the open subset $X \setminus \{x\}$ of X , and denote by $A(U_x)$ the corresponding ideal of A . We say that A is a G - $C(X)$ -algebra, if A is endowed with an action $\alpha: G \rightarrow \text{Aut}(A)$ satisfying $\alpha_g(A(U_x)) \subset A(U_x)$ for all $x \in X$ and all $g \in G$.

In the context of the above definition, given $x \in X$, denote by A_x the quotient $A/A(U_x)$ and by $\pi_x: A \rightarrow A_x$ the canonical quotient map. Then α induces actions $\alpha^{(x)}: G \rightarrow \text{Aut}(A_x)$, that make each π_x equivariant.

The definition of *unitarily regular action* is given in [65, Definition 1.18]. Observe that the trivial action on a strongly self-adsorbing C*-algebra is unitarily regular. More generally, this applies to any strongly self-absorbing G -C*-algebra that G -equivariantly absorbs the trivial action on the Jiang-Su algebra; see [65, Proposition 1.20]. The main theorem of this subsection is the following:

Theorem 4.28. *Let X be a compact metrizable space of finite covering dimension. Let (D, δ) be a strongly self-absorbing, unitarily regular G -C*-algebra, and let (A, α) be a separable, unital G - $C(X)$ -algebra such that A_x is G -equivariantly isomorphic to D , for all $x \in X$. Then there is a G -equivariant $C(X)$ -linear isomorphism*

$$(A, \alpha) \cong (C(X) \otimes D, \iota_{C(X)} \otimes \delta).$$

Our proof follows the lines of Dadarlat-Winter's proof of the nonequivariant version of Theorem 4.28 from [15, Section 4]. In fact, for the sake of succinctness, we only mention what changes are needed in said proof, and leave the smaller details to the reader. Similar results for general locally compact groups are explored in [24].

Throughout the rest of the subsection, we fix a compact metrizable space X , a strongly self-absorbing G -C*-algebra (D, δ) , and a separable unital G - $C(X)$ -algebra A .

Definition 4.29. Let (B, β) and (C, γ) be G -C*-algebras, let $\varepsilon > 0$ and let $F \subset B$ be a compact set. We say that a linear map $\varphi: B \rightarrow C$ is ε -multiplicative (respectively, ε -equivariant) on F , if $\|\varphi(b_1 b_2) - \varphi(b_1)\varphi(b_2)\| < \varepsilon$ for all $b_1, b_2 \in F$ (respectively, $\|\gamma_g(\varphi(b)) - \varphi(\beta_g(b))\| < \varepsilon$ for all $g \in G$ and for all $b \in F$).

The following is the analog of Proposition 4.1 of [15].

Proposition 4.30. *Denote by $\mu: C(X) \rightarrow A$ the structure map. Suppose that for any $\varepsilon > 0$ and for any compact subsets $F \subset A$, $H_1 \subset C(X)$ and $H \subset D$, there are completely positive contractive maps $\psi: A \rightarrow C(X) \otimes D$ and $\varphi: C(X) \otimes D \rightarrow A$ satisfying*

- (1) $\|(\varphi \circ \psi)(a) - a\| < \varepsilon$ for all $a \in F$;
- (2) $\|\varphi(f \otimes 1_D) - \mu(f)\| < \varepsilon$ for all $f \in H_1$;
- (3) $\|(\psi \circ \mu)(f) - f \otimes 1_D\| < \varepsilon$ for all $f \in H_1$;
- (4) φ is ε -multiplicative and ε -equivariant on $(1_{C(X)} \otimes \text{id}_D)(H_2)$;
- (5) ψ is ε -multiplicative and ε -equivariant on F .

Then there is a G -equivariant $C(X)$ -linear isomorphism $(A, \alpha) \cong (C(X) \otimes D, \iota_{C(X)} \otimes \delta)$.

Proof. The only thing that needs to be checked is that the isomorphisms $\overline{\varphi}$ and $\overline{\psi}$ constructed in [15], are equivariant, which is a routine computation. \square

We need an equivariant version of [15, Proposition 3.5], in order to prove the analog of [15, Lemma 4.5]. We note here that when $\varepsilon > 0$ is small enough, then any unitary in A_ε^G can be perturbed to a nearby unitary in A^G . Moreover, if the original unitary can be connected to the unit within A_ε^G , then its perturbation can be connected to the unit by a path in A^G .

Proposition 4.31. *Suppose that D is unitarily regular. Then for any finite set $F \subset D$ and every $\varepsilon > 0$, there exist a finite set $H \subset D$ and $\delta > 0$ with the following property: for any unital D -absorbing G - C^* -algebra A , and any unital completely positive maps $\varphi, \psi: D \rightarrow A$ that are δ -multiplicative and δ -equivariant on H , there is a unitary $u \in \mathcal{U}_0(A^G)$ such that $\|\varphi(d) - u\psi(d)u^*\| < \varepsilon$ for all $d \in F$.*

Proof. The proof in [15] applies almost verbatim, with the following changes: the maps Φ and Ψ are also equivariant. Instead of [67, Corollary 1.12], use [64, Proposition 3.4(iii)]; the obtained unitary V can be chosen to belong to $(B \otimes D)^G$, and similarly with V_n . The equivariant analog of [67, Proposition 1.9] is straightforward to show for compact groups (choosing unitaries in the fixed point algebra). The unital homomorphisms θ_n can then be chosen to be equivariant, and the maps γ_n are also equivariant. Finally, one must use [65, Theorem 2.15] instead of [15, Theorem 3.1] (this is where unitary regularity of the action on D is used). \square

Lemma 4.2 in [15] goes through with only minor changes:

Lemma 4.32. *Adopt the notation from [15, Lemma 4.2]. Assume furthermore that D is unitarily regular and that the maps σ_1 and σ_2 are $\delta(F, \gamma)$ -equivariant on $\mathcal{E}(F, \gamma)$. Then there is a continuous path $(u_t)_{t \in [0,1]}$ of unitaries in $(C(K) \otimes D)^G$ satisfying $u_0 = 1_{C(K)} \otimes 1_D$ and $\|u_1\sigma_1(d)u_1^* - \sigma_2(d)\| < \gamma$ for all $d \in F \cdot F$.*

Proof. Replace every application of [15, Proposition 3.5] with an application of Proposition 4.31. \square

We need an equivariant analog of a local approximate trivialization; see [15, Definition 4.3]. Since our notation differs slightly from the one used in said paper, we reproduce the definition entirely.

Definition 4.33. For $n \in \mathbb{N}$, we write $p: [0, 1]^n \rightarrow [0, 1]$ for the first coordinate projection. Given a compact subset $X \subset [0, 1]^n$, set $Y = p(X)$. If $C \subset Y$ is a closed subset, we write $X_C = p^{-1}(C)$. Let A be a unital G - $C(X)$ -algebra A . We abbreviate A_{X_C} to A_C , and $A_{X_{\{s\}}}$ to A_s , for $s \in Y$, while the fiber maps are denoted π_C and π_s , respectively. (We will not distinguish, as far as the notation is concerned, between fiber maps of different $C(X)$ -algebras associated to the same closed subset of X .)

Suppose that D is a strongly self-absorbing G - C^* -algebra, that each fiber of A is G -equivariantly isomorphic to D , and that for each $s \in Y$, there is a G - $C(X_s)$ -algebra isomorphism $A_s \cong C(X_s) \otimes D$. Let $\eta > 0$, let $t \in Y$, and let $\theta: A_t \rightarrow C(X_t) \otimes D$ be a G - $C(X_t)$ -algebra isomorphism. Fix compact subsets $F \subset A$ containing 1_A , $H \subset C(X) \otimes D$, and $\widehat{H} \subset C(X_t) \otimes D$.

Let $Y^{(t)}$ be a closed neighborhood of t in Y . A G -equivariant $(\theta, F, H, \widehat{H}, \eta)$ -trivialization of A over $Y^{(t)}$ is a family of diagrams, indexed over $s \in Y^{(t)}$, as follows:

$$\begin{array}{ccccc}
 & & A & & \\
 & & \downarrow \pi_{Y^{(t)}} & & \\
 & & A_{Y^{(t)}} & \xrightarrow{\pi_s} & A_s \\
 & \nearrow \sigma^{(t)} & \downarrow \theta^{(t)} & & \downarrow \theta_s^{(t)} \\
 C(X) \otimes D & \xrightarrow{\pi_{Y^{(t)}}} & C(Y^{(t)}) \otimes D & \xrightarrow{\iota^{(t)}} & \prod_{r \in Y^{(t)}} C(X_r) \otimes D & \xrightarrow{\pi_s} & C(X_s) \otimes D \\
 & \searrow \zeta^{(t)} & & \downarrow \pi_t & & \\
 & & & C(X_t) \otimes D, & &
 \end{array}$$

where all C^* -algebras are G - $C(X)$ -algebras in the obvious way; each map is G -equivariant, unital and completely positive; and conditions (i) through (xii) in [15, Definition 4.3] are satisfied.

Existence of equivariant local approximate trivialisations, in the sense of the definition above, is established similarly as in the nonequivariant case:

Lemma 4.34. Adopt the notation and assumptions of the first two paragraphs of Definition 4.33, and assume moreover that D is unitarily regular. Then there exist a closed neighborhood $Y^{(t)} \subset Y$ of t and a G -equivariant $(\theta, F, H, \widehat{H}, \eta)$ -trivialisaton of A over $Y^{(t)}$.

Proof. Again, the proof given in [15] requires only minor changes: the isomorphisms $\widetilde{\theta}_s^{(t)} : A_s \rightarrow C(X_s) \otimes D$ are chosen to be G -equivariant. Also, the G -equivariant, unital completely positive lifts $\widetilde{\zeta}^{(t)} : C(X_t) \rightarrow C(X_{\widetilde{Y}^{(t)}})$ and $\widetilde{\sigma}^{(t)} : D \rightarrow A_{\widetilde{Y}^{(t)}}$ are obtained using Proposition 4.26. The applications of [15, Lemma 4.2] are replaced by applications of Lemma 4.32. In particular, $\widetilde{u}^{(s)}$ can be chosen to be G -invariant. It follows that $\theta_s^{(t)}$ is equivariant, since so are $\widetilde{\pi}^{(s)}$ and $\widetilde{\theta}_s^{(t)}$. The verification of (xi) and (xii) in Definition 4.33 is routine, and we omit it. \square

Finally, we come to the proof of the main result of this section:

Proof of Theorem 4.28. Use Proposition 4.30 instead of [15, Proposition 4.1]. The basis of induction must also assume that $\theta_t : A_t \rightarrow C(X_t) \otimes D$ is G -equivariant. Apply Lemma 4.34 in place of [15, Lemma 4.5]. The unital completely positive maps $\lambda^{(i)}, \varrho^{(i)} : C(X_{y_i}) \otimes D \rightarrow C(X_{t_i}) \otimes D$ are G -equivariant because so are $\zeta^{(y_i)}, \sigma^{(y_i)}, \pi_{t_i}, \theta_{t_i}^{(y_i)}$, and $\theta_{t_i}^{(y_{i+1})}$. The unitaries $u_t^{(i)}$, for $t \in [0, 1]$ and $i \in I$, can be chosen to be G -invariant by Lemma 4.32; in other words, the path $t \mapsto u_t^{(i)}$ determines a G -invariant unitary in $C([0, 1]) \otimes C(X_{t_1}) \otimes D$, where $C([0, 1])$ carries the trivial G -action. The unitaries defined in (31) and (32) are automatically G -invariant. Finally, the maps $\psi : A \rightarrow C(X) \otimes D$ and $\varphi : C(X) \otimes D \rightarrow A$ are readily checked to be equivariant (observe that the structure map of a G - $C(X)$ -algebra is equivariant when $C(X)$ is endowed with the trivial G -action). This finishes the proof. \square

4.6. G -equivariant D -absorption. We start by providing a new characterization of G -equivariant D -absorption. The nonequivariant case (when the group is trivial), has recently been observed in [38].

Theorem 4.35. Let A be a separable G - C^* -algebra, let \mathcal{F} be a countably incomplete filter, and let D be a strongly self-absorbing, unitarily regular G - C^* -algebra. Fix $d \in \mathbb{N}$. Then A is G -equivariantly D -absorbing if and only if there exist G -equivariant completely positive contractive order zero maps $\psi_0, \dots, \psi_d : D \rightarrow F_{\mathcal{F}}^G(A)$ with commuting ranges such that $\psi_0 + \dots + \psi_d$ is unital.

Proof. By [64, Theorem 3.7], being D -absorbing is equivalent to the condition in Theorem 4.35 with $d = 0$. We now prove the converse implication. We let $C(D) = C_0((0, 1]) \otimes D$ denote the cone of D , and $C(D)^\dagger$ denote its minimal unitization, endowed with the canonical G -action. The tensor product $C(D)^\dagger \otimes \dots \otimes C(D)^\dagger$ of $d+1$ copies of $C(D)^\dagger$ has a canonical G -equivariant character. We let E be its kernel, which is a G -invariant ideal. Observe that if B is a unital C^* -algebra, then $(d+1)$ -tuples of G -equivariant completely positive contractive order zero maps $D \rightarrow B$ with commuting ranges and unital sum, are into one-to-one correspondence with unital G -equivariant *-homomorphisms $E \rightarrow B$. This follows from the structure theorem for completely positive contractive order zero maps [71, Corollary 4.1]—or, more precisely, its equivariant counterpart [31, Corollary 2.10]—and the universal properties of unitization and tensor products. Therefore, in order to conclude the proof, it is enough to show that E is G -equivariantly D -absorbing.

Denote by X the spectrum of the center of E , which is a subspace of the $(d+1)$ -dimensional cube $[0, 1]^{d+1}$. Thus, X is a compact metrizable space. Moreover, the G - C^* -algebra E is easily seen to be a G - $C(X)$ -algebra with fibers isomorphic to D . By Theorem 4.28, we conclude that E is G -equivariantly isomorphic to $C(X) \otimes D$, and in particular is G -equivariantly D -absorbing. This finishes the proof. \square

In view of Remark 4.9, one can reformulate Theorem 4.35 by asserting that A is G -equivariantly D -absorbing if and only if it is commutant d -contained in D with commuting towers for some $d \in \mathbb{N}$.

Corollary 4.36. Let A be a separable G - C^* -algebra, let \mathcal{F} be a countably incomplete filter, and let D be a strongly self-absorbing, unitarily regular G - C^* -algebra. Then A is D -absorbing if and only if there exist $d \in \mathbb{N}$ and completely positive contractive order zero maps $\psi_0, \dots, \psi_d : D \rightarrow F_{\mathcal{F}}^G(A)$ with commuting ranges such that $\psi_0 + \dots + \psi_d$ is unital.

Suppose that D is a strongly self-absorbing G - C^* -algebra. Consider the $\{0, \infty\}$ -valued dimension function for separable G - C^* -algebras obtained by setting $\dim_D(A) = 0$ if and only if A is G -equivariantly D -absorbing. The following proposition is an immediate consequence of Theorem 4.35; see also Example 4.20.

Proposition 4.37. *Let D be a strongly self-absorbing, unitarily regular G - C^* -algebra. Then \dim_D , as defined above, is commutant positively existentially axiomatizable with commuting towers.*

The following is the main result of this subsection. The conclusion is new even in the nonequivariant setting. Recall that \mathcal{Z} -absorbing strongly self-absorbing actions are automatically unitarily regular.

Corollary 4.38. *Let A and B be separable G - C^* -algebras, and let D be a unitarily regular, strongly self-absorbing G - C^* -algebra. If A is G -equivariantly D -absorbing and $A \lesssim_d^c B$ for some $d \in \mathbb{N}$, then B is G -equivariantly D -absorbing.*

Proof. If A is G -equivariantly D -absorbing, then $D \lesssim_0^c A$. If furthermore $A \lesssim_d^c B$, then we have $D \lesssim_d^c B$. Therefore B is G -equivariantly D -absorbing by Theorem 4.35 and Remark 4.9. \square

4.7. Examples and applications to dimensional inequalities. In this section, we exhibit some examples of embeddings with finite order zero dimension, and use them to deduce some dimensional inequalities, particularly for nuclear dimension and decomposition rank. We need to extract a technical fact from Section 5 of [54]. If a, b are elements of a C^* -algebra A , we write $a \approx_\varepsilon b$ to denote that $\|a - b\| < \varepsilon$.

Lemma 4.39. *Let $n \in \mathbb{N}$, and let $\varepsilon > 0$. Then there exist completely positive contractive maps $\lambda_0, \lambda_1: M_n \rightarrow \mathcal{Z}$ such that $\lambda_0(1_{M_n}) + \lambda_1(1_{M_n}) \approx_\varepsilon 1_{\mathcal{Z}}$.*

Proof. See the first part of proof of Theorem 1.1 in Section 5 of [54]. \square

Theorem 4.40. *Let U be a UHF-algebra, and let $\theta: \mathcal{Z} \rightarrow U$ be any unital embedding. Then $\dim_{\text{oz}}(\theta) = 1$.*

Proof. Since any two unital embeddings of \mathcal{Z} into U are approximately unitarily equivalent, and $\mathcal{Z} \otimes U$ is isomorphic to U , we may assume, without loss of generality, that θ is the first tensor factor embedding $\mathcal{Z} \rightarrow \mathcal{Z} \otimes U$. Let \mathcal{F} be the filter of cofinite subsets of \mathbb{N} . Write U as an increasing union $U = \overline{\bigcup_{n \in \mathbb{N}} M_{k_n}}$ of matrix algebras M_{k_n} . Using injectivity of M_{k_n} , choose a conditional expectation $E_n: U \rightarrow M_{k_n}$. For $n, m \in \mathbb{N}$, let $\lambda_0^{(n,m)}, \lambda_1^{(n,m)}: M_{k_n} \rightarrow \mathcal{Z}$ denote the order zero maps obtained from Lemma 4.39 for $\varepsilon = 1/m$. For $j = 0, 1$, set

$$\lambda_j^{(n)} = (\lambda_j^{(n,m)})_{m \in \mathbb{N}}: M_{k_n} \rightarrow \prod_{\mathcal{F}} \mathcal{Z},$$

which is an order zero map. Note that $\lambda_0^{(n)}(1_{M_{k_n}}) + \lambda_1^{(n)}(1_{M_{k_n}})$ is equal to the identity of $\prod_{\mathcal{F}} \mathcal{Z}$. For $j = 0, 1$, let $\psi_j: U \rightarrow \prod_{\mathcal{F}}(\prod_{\mathcal{F}} \mathcal{Z})$ be given by $\psi_j(x) = (\lambda_j(E_n(x)))_{n \in \mathbb{N}}$ for all $x \in U$. Then ψ_j is order zero, and $\psi_0(1_U) + \psi_1(1_U)$ is equal to the identity of $\prod_{\mathcal{F}}(\prod_{\mathcal{F}} \mathcal{Z})$. We obtain a commutative diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\quad} & U \\ \downarrow & & \downarrow \psi_j \\ \prod_{\mathcal{F}} \mathcal{Z} & \xrightarrow{\quad} & \prod_{\mathcal{F}}(\prod_{\mathcal{F}} \mathcal{Z}), \end{array}$$

where the maps from \mathbb{C} are the canonical unital homomorphisms, and the lower horizontal map is the canonical diagonal $*$ -homomorphism $\Delta_{\prod_{\mathcal{F}} \mathcal{Z}}: \prod_{\mathcal{F}} \mathcal{Z} \rightarrow \prod_{\mathcal{F}}(\prod_{\mathcal{F}} \mathcal{Z})$. We claim that there are completely positive contractive order zero maps $\varphi_0, \varphi_1: U \rightarrow \prod_{\mathcal{F}} \mathcal{Z}$ such that $\psi_j = \Delta_{\prod_{\mathcal{F}} \mathcal{Z}} \circ \varphi_j$ for $j = 0, 1$. This (and in fact, a more general statement) can be proved along the lines of [29, Lemma 4.18], replacing condition (2) in its proof with the following:

$$\left\| (\psi_j)_m^{(nr)}(b^*) - (\psi_j)_m^{(nr)}(b)^* \right\| < \frac{1}{r} \quad \text{and} \quad \left\| (\psi_j)_m^{(nr)}(cc') \right\| < \frac{1}{r}$$

whenever $b, c, c' \in G_r$ satisfy $cc'^*c' = c'^* = c'c = 0$. We omit the details.

The fact that $\dim_{\text{oz}}(\theta) \leq 1$ now follows from Lemma 4.11. It remains to show that $\dim_{\text{oz}}(\theta) > 0$. Since $\dim_{\text{nuc}}(\mathcal{Z}) = 1$ and $\dim_{\text{nuc}}(U) = 0$, the claim follows from Proposition 4.22 for $\dim = \dim_{\text{nuc}}$. \square

In the proof of the next theorem, given C^* -algebras A and B , given $\varepsilon > 0$ and given a finite subset $F \subset A$, we say that a linear map $\varphi: A \rightarrow B$ is ε -order zero on F , if $\|\varphi(ab)\| < \varepsilon$ for all $a, b \in F$ satisfying $ab = a^*b = ab^* = a^*b^* = 0$.

Theorem 4.41. *Let A be a unital Kirchberg algebra, and let $\theta: A \rightarrow \mathcal{O}_2$ be any unital embedding. Then $\dim_{\text{oz}}(\theta) \leq 1$. Moreover, $\dim_{\text{oz}}(\theta) = 1$ unless $A = \mathcal{O}_2$.*

Proof. Assume first that $A = \mathcal{O}_\infty$. As in the proof of Theorem 4.40, we may assume, without loss of generality, that θ is the first tensor factor embedding $\mathcal{O}_\infty \rightarrow \mathcal{O}_\infty \otimes \mathcal{O}_2$. We will verify the finitary version of order zero dimension. To that effect, let $\varepsilon > 0$, and let $F \subset \mathcal{O}_\infty$ and $H \subset \mathcal{O}_2$ be finite subsets consisting of positive contractions.

Use [31, Lemma 4.17]—see also the first part of the proof of Theorem 3.3 in [3]—to find *-homomorphisms $\varphi_0, \varphi_1: \mathcal{O}_2 \rightarrow \mathcal{O}_\infty$ and positive contractions $k_0, k_1 \in \mathcal{O}_2$ such that $\|\varphi_0(k_0) + \varphi_1(k_1) - 1_{\mathcal{O}_\infty}\| < \varepsilon/5$. Since \mathcal{O}_∞ is isomorphic to its infinite tensor product, we may choose φ_0 and φ_1 to satisfy $\|\varphi_j(y)a - a\varphi_j(y)\| < \|y\|\varepsilon/5$ for $j = 0, 1$, for all $y \in \mathcal{O}_2$ and for all $a \in F$. (For instance, find $m \in \mathbb{N}$ and a finite subset $F' \subset \otimes_{j=1}^m \mathcal{O}_\infty \subset \otimes_{j=1}^\infty \mathcal{O}_\infty$, such that for every $a \in F$ there exists $a' \in F'$ with $\|a - a'\| < \varepsilon/5$. With $\iota_{m+1}: \mathcal{O}_\infty \rightarrow \otimes_{j=1}^\infty \mathcal{O}_\infty$ denoting the $(m+1)$ -st tensor factor embedding, the maps $\varphi_j \circ \iota_{m+1}$, for $j = 0, 1$, will satisfy the condition above.) Likewise, since \mathcal{O}_2 is isomorphic to its infinite tensor product, we may also assume that $\|k_j b - b k_j\| < \varepsilon/5$ for $j = 0, 1$ and for all $b \in H$. Define completely positive contractive maps $\gamma_0, \gamma_1: \mathcal{O}_\infty \otimes \mathcal{O}_2 \rightarrow \mathcal{O}_\infty$ on simple tensors as follows: for $x \in \mathcal{O}_\infty$ and for positive $y \in \mathcal{O}_2$, set

$$\gamma_j(x \otimes y) = \varphi_j(k_j)^{1/2} \varphi_j(y)^{1/2} x \varphi_j(y)^{1/2} \varphi_j(k_j)^{1/2}, \quad \text{for } j = 0, 1.$$

We claim that γ_0 and γ_1 are ε -order zero on $F \otimes H$, that $((\gamma_0 + \gamma_1) \circ \theta)(a) \approx_\varepsilon a$, and that $\gamma_j(a) \approx_\varepsilon \gamma_j(1)a$ for all $j = 0, 1$ and all $a \in F$. To show the first part of the claim, it is enough to observe that when $x \in F$ and $y \in H$, we have $\gamma_j(x \otimes y) \approx_{4\varepsilon/5} \varphi_j(k_j y)x$ for $j = 0, 1$. For the second one, a similar reasoning applies, since for $a \in F$ we have $\gamma_j(a \otimes 1_{\mathcal{O}_2}) \approx_{2\varepsilon/5} \varphi_j(k_j)a$, and hence

$$(\gamma_0 + \gamma_1)(a \otimes 1_{\mathcal{O}_2}) \approx_{4\varepsilon/5} (\varphi_0(k_0) + \varphi_1(k_1))a \approx_{\varepsilon/5} a,$$

as desired. The third part of the claim also follows, since we have $\gamma_j(a) \approx_{2\varepsilon/5} \varphi_j(k_j)a = \gamma_j(1)a$ for $j = 0, 1$ and for $a \in F$. This proves the result for $A = \mathcal{O}_\infty$.

When A is an arbitrary Kirchberg algebra, the claim follows from the first part of the proof and part (2) of Proposition 4.4, together with Kirchberg's absorption theorems $A \otimes \mathcal{O}_\infty \cong A$ and $A \otimes \mathcal{O}_\infty \cong \mathcal{O}_2$.

When $A = \mathcal{O}_2$, then any inclusion into \mathcal{O}_2 is approximately unitarily equivalent to the identity, which clearly has order zero dimension zero. Since having a positively existential embedding into \mathcal{O}_2 implies absorbing \mathcal{O}_2 , it follows that $\dim_{\text{oz}}(\theta) = 1$ whenever A is not \mathcal{O}_2 . \square

In particular, we recover from Theorem 4.41 the following dimensional estimate from [54, Theorem 7.1]. The actual nuclear dimension of Kirchberg algebras has recently been computed in [10, Theorem G]: it is always 1. We nevertheless present this consequence to illustrate the applicability of our techniques.

Corollary 4.42. *Let A be a Kirchberg algebra. Then $\dim_{\text{nuc}}(A) \leq 3$.*

Proof. This follows immediately from Theorem 4.41, Proposition 4.22, and the fact that $\dim_{\text{nuc}}(\mathcal{O}_2) = 1$. \square

In the next result, we endow $\mathcal{Z}, \mathcal{O}_2, \mathcal{O}_\infty$ and the UHF-algebra with the trivial G -action, and we endow all tensor products with the diagonal action.

Theorem 4.43. *Let A be a G -C*-algebra, and let \dim be a positively $\forall\exists$ -axiomatizable dimension function for G -C*-algebra. Let U be a UHF-algebra of infinite type. Then*

$$\dim(A \otimes \mathcal{Z}) \leq 2 \dim(A \otimes U) + 1 \quad \text{and} \quad \dim(A \otimes \mathcal{O}_\infty) \leq 2 \dim(A \otimes \mathcal{O}_2) + 1.$$

Proof. This is a consequence of Theorem 4.40, Theorem 4.41, part (2) of Proposition 4.4, and Proposition 4.22. \square

We want to highlight two important consequences of Theorem 4.43. One of them is obtained by letting \dim be the Rokhlin dimension. In this case, and again endowing $\mathcal{Z}, \mathcal{O}_2, \mathcal{O}_\infty$ and the UHF-algebra with the trivial G -action, and all tensor products with the diagonal action, we deduce the following dimensional inequalities (compare with Section 4 of [31]).

Corollary 4.44. *Let A be a G -C*-algebra, and let U be a UHF-algebra of infinite type. Then*

$$\dim_{\text{Rok}}(A \otimes \mathcal{Z}) \leq 2 \dim_{\text{Rok}}(A \otimes U) + 1, \quad \text{and} \quad \dim_{\text{Rok}}(A \otimes \mathcal{O}_\infty) \leq 2 \dim_{\text{Rok}}(A \otimes \mathcal{O}_2) + 1.$$

The other consequence is obtained by letting \dim be either the nuclear dimension or the decomposition rank. The estimates involving nuclear dimension have previously been observed in [3, Section 3], while the estimates for the decomposition rank are new.

Corollary 4.45. Let A be a C^* -algebra, and let U be any UHF-algebra of infinite type. Then

$$\dim_{\text{nuc}}(A \otimes \mathcal{Z}) \leq 2 \dim_{\text{nuc}}(A \otimes U) + 1, \quad \text{and} \quad \dim_{\text{nuc}}(A \otimes \mathcal{O}_\infty) \leq 2 \dim_{\text{nuc}}(A \otimes \mathcal{O}_2) + 1.$$

Furthermore,

$$\text{dr}(A \otimes \mathcal{Z}) \leq 2 \text{dr}(A \otimes U) + 1.$$

4.8. Rokhlin dimension and strongly self-absorbing G - C^* -algebras. In this subsection, and since we consider different actions on the same C^* -algebra, we denote by $\dim_{\text{Rok}}^c(A, \alpha)$ the Rokhlin dimension with commuting towers of the G - C^* -algebra (A, α) . The following is one of our main technical results.

Theorem 4.46. Let α be a continuous action of G on a C^* -algebra A . If $\dim_{\text{Rok}}(A, \alpha) \leq d$, then $(A, \iota_A) \lesssim_d (A, \alpha)$. If $\dim_{\text{Rok}}^c(A, \alpha) \leq d$, then $(A, \iota_A) \lesssim_d^c (A, \alpha)$.

Proof. We prove the first assertion. The proof of the second assertion is analogous. Fix a nonprincipal ultrafilter \mathcal{U} over \mathbb{N} . We denote by $F_{\mathcal{U}}(A)$ the Kirchberg invariant of (A, ι_A) (endowed with the trivial action), and by $F_{\mathcal{U}}^G(A)$ the Kirchberg invariant of (A, α) (endowed with the canonical G -action obtained from α). Since $\dim_{\text{Rok}}^c(\alpha) \leq d$, it follows from the reformulation of Rokhlin dimension in terms of commutant d -containment and Proposition 2.9 that for any separable C^* -subalgebra C of $F_{\mathcal{U}}(A)$ there exist G -equivariant completely positive contractive order zero maps $\psi_0, \dots, \psi_d: C(G) \rightarrow C' \cap F_{\mathcal{U}}^G(A)$ such that $\psi_0 + \dots + \psi_d$ is unital.

Fix a separable C^* -subalgebra C of $F_{\mathcal{U}}^G(A)$ containing A . When G is finite, the maps witnessing that $(A, \alpha) \lesssim_d (A, \iota_A)$ can be constructed explicitly, so we outline this first. For $g \in G$, let $\delta_g \in C(G)$ be the characteristic function of $\{g\}$. Define maps $\eta_j: C \rightarrow F_{\mathcal{U}}^G(A)$, for $j = 0, \dots, d$, by $\eta_j(x) = \sum_{g \in G} \psi_j(\delta_g) \alpha_g(x)$. Then these maps witness the fact that (A, ι_A) is G -equivariantly commutant d -contained in (A, α) .

Suppose now that G is an arbitrary compact second countable group. Below, if a and b are elements of a C^* -algebra and $\varepsilon > 0$, we write $a \approx_\varepsilon b$ to mean that $\|a - b\| < \varepsilon$. Let ρ be a left invariant metric on G . Fix a finite subset F of positive elements of C and $\varepsilon > 0$. The argument in [29, Proposition 2.11] shows that there exist $\delta > 0$, a finite subset K of G , and a partition of unity $(f_g)_{g \in K}$ of G satisfying:

- (1) $f_g \in C(G)$ is a positive contraction for all $g \in G$;
- (2) f_g and f_h are orthogonal whenever $g, h \in G$ satisfy $\rho(g, h) > \delta$;
- (3) for every $a \in F$, we have $\alpha_g(a) \approx_\varepsilon \alpha_h(a)$ whenever $g, h \in G$ satisfy $\rho(g, h) < \delta$; and
- (4) $\alpha_h(\sum_{g \in K} \psi_j(f_g)a) \approx_\varepsilon \sum_{g \in K} \psi_j(f_g)a$ for all $h \in G$ and all $a \in F$.

Define now $\eta_j: C \rightarrow F_{\mathcal{U}}^G(A)$ by $\eta_j(x) = \sum_{g \in K} \psi_j(f_g) \alpha_g(x)$ for $j = 0, \dots, d$. Observe that η_0, \dots, η_d are completely positive contractive maps. Furthermore, for every $0 \leq j \leq d$, if $a, b \in F$ satisfy $ab \approx_\varepsilon 0$, then (1) and (2) imply that

$$\eta_j(a) \eta_j(b) = \sum_{g, h \in K} \psi_j(f_g) \psi_j(f_h) \alpha_g(a) \alpha_h(b) \approx_\varepsilon \sum_{g, h \in K} \psi_j(f_g) \psi_j(f_h) \alpha_g(ab) \approx_\varepsilon 0.$$

By (3), we have $\alpha_g(\eta_j(a)) \approx_\varepsilon \eta_j(a)$ for every $g \in G$, every $j = 0, \dots, d$, and every $a \in F$. Since $\varepsilon > 0$ and $F \subset C_+$ are arbitrary, it follows from Proposition 2.9 that there exist G -equivariant completely positive contractive order zero maps $\eta_0, \dots, \eta_d: C \rightarrow F_{\mathcal{F}}^G(A)$ with unital sum. Since this is true for every separable C^* -subalgebra C of $F_{\mathcal{F}}(A)$, we conclude that $(A, \iota_A) \lesssim_d (A, \alpha)$, as desired. \square

The following corollary is then a consequence of Theorem 4.46 and Proposition 4.21.

Corollary 4.47. Let \dim be a dimension function for G - C^* -algebras, A is a C^* -algebra, α is a continuous action of a G on A , and ι_A is the trivial G -action on A . If \dim is positively existentially axiomatizable, then

$$\dim(A, \alpha) + 1 \leq (\dim_{\text{Rok}}(A, \alpha) + 1)(\dim(A, \iota_A) + 1).$$

If \dim is commutant positively existentially axiomatizable, then

$$\dim(A, \alpha) + 1 \leq (\dim_{\text{Rok}}^c(A, \alpha) + 1)(\dim(A, \iota_A) + 1).$$

We now arrive at one of the main results of this section. It asserts that, for a strongly self-absorbing C*-algebra, G -actions with finite Rokhlin dimension with commuting towers, on D -absorbing C*-algebras, automatically absorb the trivial action on D .

Theorem 4.48. *Let D be a strongly self-absorbing C*-algebra, let A be a separable D -absorbing C*-algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of G with $\dim_{\text{Rok}}^c(A, \alpha) < \infty$. Then (A, α) is G -equivariantly (D, ι_D) -absorbing.*

Proof. Let \dim_D be the $\{0, \infty\}$ -valued dimension function for G -C*-algebras which is finite if and only if the given G -C*-algebra is (D, ι_D) -absorbing. The action (D, ι_D) is unitarily regular since it absorbs $(\mathcal{Z}, \iota_{\mathcal{Z}})$ tensorially; see [65, Proposition 1.20]. It follows from this and Proposition 4.37 that \dim_D is commutant positively existentially axiomatizable with commuting towers. The result now follows from Corollary 4.47. \square

Corollary 4.49. *Let D be a strongly self-absorbing C*-algebra, and let (A, α) be a separable G -C*-algebra with $\dim_{\text{Rok}}^c(A, \alpha) < \infty$. If A is (nonequivariantly) D -absorbing, then so are A^G and $A \rtimes_{\alpha} G$.*

Proof. By Theorem 4.48, there is a G -equivariant isomorphism between (A, α) and $(A \otimes D, \alpha \otimes \iota_D)$. Upon taking crossed products, we deduce that

$$A \rtimes_{\alpha} G \cong (A \otimes D) \rtimes_{\alpha \otimes \iota_D} G \cong (A \rtimes_{\alpha} G) \otimes D,$$

so $A \rtimes_{\alpha} G$ is D -absorbing. The same applies to the fixed point algebra, since we have $A^{\alpha} \cong (A \otimes D)^{\alpha \otimes \iota_D} = A^{\alpha} \otimes D$. \square

Corollary 4.49 is a significant generalization of previously known results concerning Jiang-Su absorption: for finite groups this was shown by Hirshberg-Winter-Zacharias in [40, Theorem 5.9], and in [28, Theorem 5.4.4] by the first-named author for compact groups. Similar results have been independently obtained with different methods in [32].

Observe that in the next result we do not require the strongly self-absorbing action to be unitarily regular, unlike in Theorem 4.28 or 4.38.

Theorem 4.50. *Let A be a separable C*-algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action. Suppose that A absorbs a strongly self-absorbing C*-algebra D , and let $\delta: G \rightarrow \text{Aut}(D)$ be any strongly self-absorbing action.*

- (1) *If $\dim_{\text{Rok}}(A, \alpha) = d < \infty$, then $(D, \delta) \lesssim_d (A, \alpha)$.*
- (2) *If $\dim_{\text{Rok}}^c(A, \alpha) = d < \infty$, then $(D, \delta) \lesssim_d^c (A, \alpha)$. Moreover, (A, α) is G -equivariantly (D, δ) -absorbing.*

Proof. (1). Suppose that $\dim_{\text{Rok}}(A, \alpha) = d < +\infty$. By Lemma 4.13, the second-factor embedding $\theta: (A, \alpha) \rightarrow (C(G) \otimes A, \text{Lt} \otimes \alpha)$ has G -equivariant order zero dimension at most d . Since $A \otimes D \cong D$, it follows from Proposition 2.3 in [29] that $(C(G) \otimes A, \text{Lt} \otimes \alpha)$ is conjugate to $(C(G) \otimes A \otimes D, \text{Lt} \otimes \alpha \otimes \delta)$. In other words, $(C(G) \otimes A, \text{Lt} \otimes \alpha)$ is G -equivariantly (D, δ) -absorbing. By the implication (ii) \Rightarrow (iv) in [4, Theorem 4.29] the first-factor embedding $\eta: (C(G) \otimes A, \text{Lt} \otimes \alpha) \rightarrow (C(G) \otimes A \otimes D, \text{Lt} \otimes \alpha \otimes \delta)$ has G -equivariant order zero dimension zero. By item (1) in Proposition 4.4, the composition

$$\eta \circ \theta: (A, \alpha) \rightarrow (C(G) \otimes A \otimes D, \text{Lt} \otimes \alpha \otimes \delta)$$

has dimension at most d . Observe that $(\eta \circ \theta)(A) = 1_{C(G)} \otimes a \otimes 1_D$ for all $a \in A$. It follows that the first-factor embedding $(A, \alpha) \rightarrow (A \otimes D, \alpha \otimes \delta)$ has G -equivariant dimension at most d (this embedding is really just $\eta \circ \theta$, once its codomain is truncated). We conclude from Proposition 4.14 that $(A \otimes D, \alpha \otimes \delta) \lesssim_d (A, \alpha)$, and in particular $(D, \delta) \lesssim_d (A, \alpha)$.

(2). Fix a nonprincipal ultrafilter \mathcal{U} over \mathbb{N} . Suppose that $\dim_{\text{Rok}}^c(A, \alpha) \leq d$. We want to show that $(D, \delta) \lesssim_d^c (A, \alpha)$. In view of Remark 4.9, it is enough to show that there exist G -equivariant unital completely positive contractive maps $\eta_0, \dots, \eta_d: (D, \delta) \rightarrow (F_{\mathcal{U}}^G(A), \alpha)$ with commuting ranges such that $\sum_{j=0}^d \eta_j$ is unital. In order to illustrate the ideas of the proof, we begin by considering the unital case, since the argument is easier to follow in this case.

Assume that A is unital, so that $F_{\mathcal{U}}^G(A)$ is equal to $A' \cap \prod_{\mathcal{U}}^G A$. By assumption, there exist G -equivariant completely positive contractive order zero A -bimodule maps $\psi_0, \dots, \psi_d: (C(G) \otimes A, \text{Lt} \otimes \alpha) \rightarrow (\prod_{\mathcal{U}}^G A, \alpha)$, such that $(\psi_0 + \dots + \psi_d) \circ (1 \otimes \text{id}_A)$ is the diagonal inclusion of A into $\prod_{\mathcal{U}}^G A$, and $\psi_j(C(G) \otimes 1)$ and $\psi_k(C(G) \otimes 1)$ commute for $0 \leq j < k \leq d$. Define C to be the separable C*-subalgebra of $\prod_{\mathcal{U}}^G A$ generated by A together with

the ranges of ψ_0, \dots, ψ_d . By Theorem 4.48, (A, α) is (D, ι_D) -absorbing. Use Theorem 4.35 to find a G -equivariant *-homomorphism $\theta: D \rightarrow C' \cap \prod_{\mathcal{U}}^G A$. For $j = 0, \dots, d$, define a G -equivariant completely positive contractive order zero A -bimodules map

$$\phi_j: (C(G) \otimes A \otimes D, \text{Lt} \otimes \alpha \otimes \iota_D) \rightarrow \left(\prod_{\mathcal{U}}^G A, \alpha \right)$$

by setting $\phi_j(f \otimes a \otimes d) = \psi_j(f \otimes a)\theta(d)$ for all $f \in C(G)$, all $a \in A$, and all $d \in D$. Observe that $\phi_j(C(G) \otimes 1_A \otimes D)$ commutes with $\phi_k(C(G) \otimes 1_A \otimes D)$ whenever $0 \leq j < k \leq d$.

Fix a G -equivariant isomorphism $(C(G) \otimes D, \text{Lt} \otimes \iota_D) \rightarrow (C(G) \otimes D, \text{Lt} \otimes \delta)$, and tensor it with the identity on A to obtain a G -equivariant *-isomorphism

$$\pi: (C(G) \otimes A \otimes D, \text{Lt} \otimes \alpha \otimes \delta) \rightarrow (C(G) \otimes A \otimes D, \text{Lt} \otimes \alpha \otimes \iota_D)$$

satisfying $\pi(1_{C(G)} \otimes a \otimes 1_D) = 1_{C(G)} \otimes a \otimes 1_D$ for every $a \in A$. For $j = 0, \dots, d$, let

$$\eta_j: (D, \delta) \rightarrow \left(A' \cap \prod_{\mathcal{U}}^G A, \alpha \right)$$

be the G -equivariant completely positive order zero map given by $\eta_j(d) = (\phi_j \circ \pi)(1_{C(G)} \otimes 1_A \otimes d)$ for all $d \in D$. Then $\eta_j(D)$ commutes with $\eta_k(D)$ whenever $0 \leq j < k \leq d$, and $\sum_{j=0}^d \eta_j(1) = 1$. We conclude that $(D, \delta) \lesssim_d^c (A, \alpha)$.

We consider now the general case when A is not necessarily unital. Fix maps $\psi_0, \dots, \psi_d: (C(G) \otimes A, \text{Lt} \otimes \alpha) \rightarrow \left(\prod_{\mathcal{U}}^G A, \alpha \right)$ as before. Let C the separable C^* -subalgebra of $\prod_{\mathcal{U}}^G A$ generated by the ranges of ψ_0, \dots, ψ_d . As above, find a G -equivariant *-homomorphism

$$\theta: (D, \iota_D) \rightarrow \frac{C' \cap \prod_{\mathcal{U}}^G A}{\text{Ann}(A, C' \cap \prod_{\mathcal{U}}^G A)}.$$

Let π be as before, and consider the canonical G -equivariant *-homomorphism

$$\Psi: C \otimes_{\max} \frac{C' \cap \prod_{\mathcal{U}}^G A}{\text{Ann}(A, C' \cap \prod_{\mathcal{U}}^G A)} \rightarrow \prod_{\mathcal{U}}^G A$$

defined as in the proof of Lemma 4.11. For $j = 0, \dots, d$, set

$$\phi_j = \Psi \circ (\psi_j \otimes \theta): C(G) \otimes A \otimes D \rightarrow \prod_{\mathcal{U}}^G A.$$

Then ϕ_j is a G -equivariant order zero A -bimodule map.

Let $(u_\lambda)_{\lambda \in \Lambda}$ be an increasing approximate identity for A . For every $\lambda \in \Lambda$ and $j = 0, \dots, d$, define

$$\eta_{j,\lambda}: (D, \delta) \rightarrow \left(\prod_{\mathcal{U}}^G A, \alpha \right)$$

by $\eta_{j,\lambda}(d) = (\phi_j \circ \pi)(1_{C(G)} \otimes u_\lambda \otimes d)$. These maps satisfy are completely positive contractive order zero, have commuting ranges, and

$$\|[\eta_{j,\lambda}(d), a]\| \rightarrow 0 \quad \text{and} \quad \|a(\eta_{0,\lambda} + \dots + \eta_{d,\lambda})(1) - a\| \rightarrow 0,$$

for every $d \in D$ and $a \in A$. By countable saturation of $\left(\prod_{\mathcal{U}}^G A, \alpha \right)$, there exist G -equivariant completely positive order zero maps

$$\tilde{\eta}_j: (D, \delta) \rightarrow \left(A' \cap \prod_{\mathcal{U}}^G A, \alpha \right)$$

satisfying $a [(\tilde{\eta}_{0,\lambda} + \dots + \tilde{\eta}_{d,\lambda})(1)] = a$ for every $a \in A$. Composing such maps with the canonical quotient mapping

$$A' \cap \prod_{\mathcal{U}}^G A \rightarrow \frac{A' \cap \prod_{\mathcal{U}}^G A}{\text{Ann}(A, A' \cap \prod_{\mathcal{U}}^G A)} = F_{\mathcal{U}}^G(GA)$$

gives G -equivariant completely positive order zero maps with commuting ranges $\eta_0, \dots, \eta_d: (D, \delta) \rightarrow (F_{\mathcal{U}}^G(A), \alpha)$, which witness the fact that $(D, \delta) \lesssim_d^c (A, \alpha)$.

We now justify the second claim. When δ is unitarily regular, the fact that $(D, \delta) \lesssim_d^c (A, \alpha)$ implies that (A, α) is G -equivariantly (D, δ) -absorbing is a consequence of Corollary 4.38. Now suppose that δ is an arbitrary strongly self-absorbing action, and consider $\delta^{\mathbb{Z}} = \delta \otimes \text{id}_{\mathbb{Z}}$, regarded as an action of G on $D \otimes \mathbb{Z} \cong D$. Then $\delta^{\mathbb{Z}}$ is unitarily

regular, and hence α absorbs $\delta^{\mathcal{Z}}$ by the above paragraph. Since $\text{id}_{\mathcal{Z}}$ is unitarily regular and A is \mathcal{Z} -absorbing (because it is D -absorbing), we also deduce that α absorbs $\text{id}_{\mathcal{Z}}$. Putting these things together, we deduce that

$$\alpha \cong \alpha \otimes \delta^{\mathcal{Z}} = \alpha \otimes \text{id}_{\mathcal{Z}} \otimes \delta \cong \alpha \otimes \delta.$$

In other words, α absorbs δ , and the proof is finished. \square

Theorem 4.50 has a number of new and strong consequences, as already the case of the trivial action on D is new. For example, we derive now some dimension reduction type results. Roughly speaking, these statements say that, in some contexts, the Rokhlin property is *equivalent* to finite Rokhlin dimension with commuting towers. (By definition, a compact group action has the Rokhlin property if it has Rokhlin dimension zero.)

These are useful results, since proving directly that an action has the Rokhlin property is often challenging, and there are not many tools available. On the other hand, Rokhlin dimensional estimates are much easier to come by, particularly for finite groups. It follows from our results that, in some cases, knowing that the Rokhlin dimension is finite is enough to deduce the Rokhlin property. Having access to the Rokhlin property is extremely valuable, since it entails classifiability (of the action), and the structure of the crossed product is extremely well-understood; see [29].

Corollary 4.51. Let A be an \mathcal{O}_2 -absorbing C*-algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action with $\dim_{\text{Rok}}^c(\alpha) < \infty$. Then α has the Rokhlin property.

Proof. Let $\delta: G \rightarrow \text{Aut}(\mathcal{O}_2)$ be any action with the Rokhlin property; one such action is constructed in [28]. By the classification theorem in [35], the action δ is strongly self-absorbing. It follows from Theorem 4.50 that α absorbs δ , so α has the Rokhlin property. \square

Corollary 4.52. Let G be a finite group, let A be an $M_{|G|^\infty}$ -absorbing C*-algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action with $\dim_{\text{Rok}}^c(\alpha) < \infty$. Then α has the Rokhlin property. This in particular applies to Cuntz algebras of the form $\mathcal{O}_{n|G|}$.

Proof. Let $\delta: G \rightarrow \text{Aut}(M_{|G|^\infty})$ be the infinite tensor product of conjugation by the left regular representation of G on $\ell^2(G)$. This action is well-known to have the Rokhlin property, since $C(G)$ embeds equivariantly into $B(\ell^2(G)) \cong M_{|G|}$ as multiplication operators. It is elementary to show that such an action is strongly self-absorbing. It follows from Theorem 4.50 that α absorbs δ , so α has the Rokhlin property. \square

Finally, we obtain some new Rokhlin-dimensional estimates. The following one represents a satisfactory parallel with the $\{0, 1, \infty\}$ -type behaviour that nuclear dimension and decomposition rank tend to have in the noncommutative setting. It is also particularly satisfactory, since proving finiteness of the Rokhlin dimension is a far easier task than proving that it is (at most) 1.

Corollary 4.53. Let G be a finite group, let A be a C*-algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ satisfy $\dim_{\text{Rok}}^c(\alpha) < \infty$. Then

$$\dim_{\text{Rok}}(\alpha \otimes \text{id}_{\mathcal{Z}}) \leq 1.$$

If A is \mathcal{Z} -absorbing, then

$$\dim_{\text{Rok}}(\alpha) \leq 1.$$

Proof. We apply Corollary 4.44 with $U = M_{|G|^\infty}$, so that $\dim_{\text{Rok}}(\alpha \otimes \text{id}_U) = 0$ by Corollary 4.52. The second assertion follows from the first one and Theorem 4.50. \square

The next result is a dynamical version of the main result of [66], which states that $\text{dr}(C(X) \otimes \mathcal{Z}) \leq 2$.

Corollary 4.54. Let G be a finite group, let X be a compact Hausdorff space, and let $\alpha: G \rightarrow \text{Aut}(C(X))$ be induced by a free action of G on X . Then

$$\dim_{\text{Rok}}(\alpha \otimes \text{id}_{\mathcal{Z}}) \leq 1.$$

Proof. This is an immediate consequence of Corollary 4.53, since α has finite Rokhlin dimension (with commuting towers) by Theorem 4.2 in [31]. \square

APPENDIX

Here we recall some basic notions from first order logic for metric structures. We will consider here and in the following a *multi-sorted language* \mathcal{L} . This is endowed with a collection of *sorts*. A collection of *domains*, as well as a collection of *pseudometric symbols*, are associated with each given sort. We refer the reader to [33, Appendix] for details concerning the semantic and syntax in this setting, which is slightly more general than the usual setting of logic for metric structures as considered in [6] or [19, 22]. In particular, the notions of (positive/positive primitive) \mathcal{L} -formula and \mathcal{L} -type, (positive quantifier-free) κ -saturated structure, \mathcal{L} -definable set, reduced product, and ultraproduct, can be found in [33, Appendix].

An \mathcal{L} -embedding $\theta : M \rightarrow N$ between \mathcal{L} -structures is a collection of functions $\theta_S : \mathcal{S}^M \rightarrow \mathcal{S}^N$ such that $\theta_S(D^M) \subseteq D^N$ for every domain D associated with \mathcal{S} , and $\varphi(\theta(\bar{a})) = \varphi(\bar{a})$ for any quantifier-free \mathcal{L} -formula $\varphi(\bar{x})$ and tuple \bar{a} in M . Similarly, an \mathcal{L} -morphism θ between \mathcal{L} -structures is a collection of functions $\theta_S : \mathcal{S}^M \rightarrow \mathcal{S}^N$ such that $\theta_S(D^M) \subseteq D^N$ for every domain associated with \mathcal{S} , such that $\varphi(\theta(\bar{a})) \leq \varphi(\bar{a})$ for any atomic \mathcal{L} -formula $\varphi(\bar{x})$ and any tuple \bar{a} in M . If \mathcal{F} is a filter over a set I and $(M_i)_{i \in I}$ is an I -sequence of \mathcal{L} -structures, we let $\prod_{\mathcal{F}} M_i$ be the corresponding reduced product. In the case when $(M_i)_{i \in I}$ is constantly equal to a fixed \mathcal{L} -structure M , one obtains the reduced \mathcal{L} -power $\prod_{\mathcal{F}} M$ of M with respect to \mathcal{F} .

A.1. Existential theories. Suppose that M is an \mathcal{L} -structure. The *existential \mathcal{L} -theory* $\text{Th}_{\exists}^{\mathcal{L}}(M)$ of M is the function $\varphi \mapsto \varphi^M$ that assigns to an existential \mathcal{L} -sentence φ its value φ^M in M . We say that M is *weakly \mathcal{L} -contained* in N if $\text{Th}_{\exists}^{\mathcal{L}}(M) \leq \text{Th}_{\exists}^{\mathcal{L}}(N)$, and *weakly \mathcal{L} -equivalent* to N if M and N have the same existential \mathcal{L} -theory. We will identify the existential \mathcal{L} -theory of an \mathcal{L} -structure with its weak \mathcal{L} -equivalence class. It follows from saturation of ultrapowers and Los' theorem that M is weakly \mathcal{L} -contained in N if and only if for some (equivalently, any) countably incomplete ultrafilter \mathcal{U} , every separable substructure of M admits an \mathcal{L} -embedding into $\prod_{\mathcal{U}} N$. This is equivalent to the assertion that if a quantifier-free \mathcal{L} -type is approximately realized in M , then it is approximately realized in N .

A class \mathfrak{C} of structures is said to be *existentially \mathcal{L} -axiomatizable* if there is a collection (φ_i) of existential \mathcal{L} -sentences such that, for every \mathcal{L} -structure M , $M \in \mathfrak{C}$ if and only if $\varphi_i^M \leq 0$ for every $i \in I$. More generally, we consider the following notion, which has been introduced in [19, Definition 5.7.1].

Definition A.1. A class \mathfrak{C} of (separable) structures is said to be *definable by a uniform family of existential \mathcal{L} -formulas* if, for every $k \in \mathbb{N}$, there exist $n_k \in \mathbb{N}$ and an uniformly equicontinuous collections $\mathcal{F}_k(x_1, \dots, x_{n_k})$ of existential \mathcal{L} -formulas, such that a (separable) \mathcal{L} -structure M belongs to \mathfrak{C} if and only if for every $k \in \mathbb{N}$ and every $\bar{a} \in M^{n_k}$ there exists $\varphi \in \mathcal{F}_k$, such that $M \models \varphi(\bar{a}) \leq 1/k$.

Observe that if \mathfrak{C} is a class of (separable) structures definable by a uniform family of existential \mathcal{L} -formulas, then \mathfrak{C} is closed under (countable) direct limits. We say that a property is definable by a uniform family of existential \mathcal{L} -formulas if the class of \mathcal{L} -structures satisfying that property is.

The notions of existential positive \mathcal{L} -theory, positive weak \mathcal{L} -containment, positive weak \mathcal{L} -equivalence, positively existentially \mathcal{L} -axiomatizable class, and class definable by a uniform family of existential positive primitive \mathcal{L} -formulas are defined as above, by only considering existential positive \mathcal{L} -formulas. It follows from Los' Theorem and Proposition A.5 below that M is positively weakly \mathcal{L} -contained in N if and only if for some (equivalently, any) countably incomplete filter \mathcal{F} , every separable substructure of M admits an \mathcal{L} -morphism to $\prod_{\mathcal{F}} N$.

A.2. Existential theories of embeddings. Let A, M be \mathcal{L} -structures and $\theta : A \rightarrow M$ an \mathcal{L} -embedding. We can regard (M, θ_M) as a structure in the language $\mathcal{L}(A)$ obtained by adding a constant symbol c_a for any element $a \in A$. The interpretation of c_a in (M, θ) is the image $\theta(a)$ of a under θ . One can then define the notions of quantifier-free $\mathcal{L}(A)$ -formula and quantifier-free $\mathcal{L}(A)$ -type. The same definition as in Subsubsection A.1 gives the notion of weak \mathcal{L} -containment, weak \mathcal{L} -equivalence, and existential \mathcal{L} -theory for embeddings $\theta_M : A \rightarrow M$ and $\theta_N : A \rightarrow N$. As in the case of \mathcal{L} -structures, one can say that θ_M is weakly \mathcal{L} -contained in θ_N if and only if for any separable substructures $A_0 \subset A$ and $M_0 \subset M$ such that $\theta_M(A_0) \subset M_0$, and for some (equivalently, any) countably incomplete ultrafilter \mathcal{U} , there exists an \mathcal{L} -embedding $\eta : M_0 \rightarrow \prod_{\mathcal{U}} N$ such that $\Delta_N \circ \theta_N|_{A_0} = \eta \circ \theta_M|_{A_0}$.

Definition A.2. An \mathcal{L} -embedding $\theta_M : A \rightarrow M$ is said to be *\mathcal{L} -existential* if for any quantifier-free \mathcal{L} -formula $\varphi(\bar{x}, \bar{y})$ and any tuple $\bar{a} \in A$, the value of $\inf_{\bar{y}} \varphi(\bar{a}, \bar{y})$ in A is the same as the value of $\inf_{\bar{y}} \varphi(\theta_M(\bar{a}), \bar{y})$ in M .

It is easy to see that $\theta_M: A \rightarrow M$ is \mathcal{L} -existential if and only if θ_M is weakly \mathcal{L} -contained in the identity embedding $\text{id}_A: A \rightarrow A$.

Similarly, one can define the notion of positively \mathcal{L} -existential \mathcal{L} -embedding $\theta_M: A \rightarrow M$, by only considering existential positive \mathcal{L} -formulas. The following fact follows easily from the definitions.

Proposition A.3. *Suppose that \mathfrak{C} is a class of structures that is definable by a uniform family of existential positive \mathcal{L} -formulas. If $\theta_M: A \rightarrow M$ is a positively \mathcal{L} -existential \mathcal{L} -embedding and $M \in \mathfrak{C}$, then $A \in \mathfrak{C}$.*

Proposition A.4. *Let Λ be a directed set. The following properties follow easily from the definition.*

- (1) *The composition of positive \mathcal{L} -existential \mathcal{L} -embeddings is a positively \mathcal{L} -existential \mathcal{L} -embedding.*
- (2) *Let $(\{M_\lambda\}_{\lambda \in \Lambda}, \{\theta_{\lambda, \mu}\}_{\lambda, \mu \in \Lambda, \lambda < \mu})$ be a direct system of \mathcal{L} -structures with positively \mathcal{L} -existential \mathcal{L} -embeddings $\theta_{\lambda, \mu}: M_\lambda \rightarrow M_\mu$ for $\lambda < \mu$. If M is the corresponding direct limit, then the canonical \mathcal{L} -embedding of M_λ into M , for $\lambda \in \Lambda$, is positively \mathcal{L} -existential.*
- (3) *For $j = 0, 1$, let $(\{M_\lambda^{(j)}\}_{\lambda \in \Lambda}, \{\theta_{\lambda, \mu}^{(j)}\}_{\lambda, \mu \in \Lambda, \lambda < \mu})$ be a direct system of \mathcal{L} -structures. Let $\{\eta_\lambda: M_\lambda^{(0)} \rightarrow M_\lambda^{(1)}\}_{\lambda \in \Lambda}$ be a family of intertwining positively \mathcal{L} -existential \mathcal{L} -embeddings. Then*

$$\varinjlim \eta_\lambda: \varinjlim M_\lambda^{(0)} \rightarrow \varinjlim M_\lambda^{(1)}$$

is a positively \mathcal{L} -existential \mathcal{L} -embedding.

The analogue of Remark A.4 holds for \mathcal{L} -existential \mathcal{L} -embeddings as well.

A.3. Saturation of ultrapowers. The notion of κ -good ultrafilter has been introduced in the case of model theory for discrete structures in the classical monograph [12, Section 6.1]. The notion of κ -good filter can be defined exactly as for ultrafilters. Theorem 6.1.4 of [12] shows that countably incomplete κ -good ultrafilters exist for any cardinal κ . Every countably incomplete ultrafilter is \aleph_1 -good; see [12, Exercise 6.1.2]. In particular, every nonprincipal ultrafilter over a countable set is \aleph_1 -good. The same proof as [12, Theorem 6.1.8] shows the following.

Proposition A.5. *Suppose that κ is a cardinal larger than the density character of \mathcal{L} . Suppose that M is an \mathcal{L} -structure and \mathcal{U} is a countably incomplete κ -good ultrafilter. Then $\prod_{\mathcal{U}} M$ is \mathcal{L} - κ -saturated.*

If \mathcal{F} is a countably incomplete κ -good filter, then $\prod_{\mathcal{F}} M$ is positively quantifier-free \mathcal{L} - κ -saturated.

Using Proposition A.5 one can easily deduce the following characterization of \mathcal{L} -existential \mathcal{L} -embeddings.

Theorem A.6. *Let A and M be \mathcal{L} -structures, and let $\theta: A \rightarrow M$ be an \mathcal{L} -embedding. Let κ be a cardinal greater than the density character of M and the density character of \mathcal{L} . The following assertions are equivalent:*

- (1) *θ is an \mathcal{L} -existential \mathcal{L} -embedding;*
- (2) *there exist an \mathcal{L} -structure N and an \mathcal{L} -embedding $\eta: M \rightarrow N$ such that $\eta \circ \theta: A \rightarrow N$ is an \mathcal{L} -existential \mathcal{L} -embedding;*
- (3) *if N is a quantifier-free \mathcal{L} - κ -saturated \mathcal{L} -structure, and $\theta_N: A \rightarrow N$ is an \mathcal{L} -embedding, then there exists an \mathcal{L} -embedding $\eta: M \rightarrow N$ such that $\eta \circ \theta = \theta_N$;*
- (4) *for some (equivalently, any) countably incomplete ultrafilter \mathcal{U} , and for every separable $A_0 \subset A$ and $M_0 \subset M$ such that $\theta_M(A_0) \subset M_0$, there exists an \mathcal{L} -embedding $\eta: M_0 \rightarrow \prod_{\mathcal{U}} A$ such that $\eta \circ \theta_M|_{A_0} = \Delta_A|_{A_0}$.*

A similar characterization can be given for positively \mathcal{L} -existential \mathcal{L} -embeddings.

Theorem A.7. *Let A and M be \mathcal{L} -structures, and let $\theta: A \rightarrow M$ be an \mathcal{L} -embedding. Let κ be a cardinal larger than the density character of M and the density character of \mathcal{L} . The following assertions are equivalent:*

- (1) *θ is a positively \mathcal{L} -existential \mathcal{L} -embedding;*
- (2) *there exist an \mathcal{L} -structure N and an \mathcal{L} -morphism $\eta: M \rightarrow N$ such that $\eta \circ \theta: A \rightarrow N$ is an \mathcal{L} -existential \mathcal{L} -embedding;*
- (3) *if N is a quantifier-free positively quantifier-free \mathcal{L} - κ -saturated \mathcal{L} -structure, and $\theta_N: A \rightarrow N$ is an \mathcal{L} -embedding, then there exists an \mathcal{L} -morphism $\eta: M \rightarrow N$ such that $\eta \circ \theta = \theta_N$;*
- (4) *for some (equivalently, any) countably incomplete ultrafilter \mathcal{F} , and for every separable $A_0 \subset A$ and $M_0 \subset M$ such that $\theta_M(A_0) \subset M_0$, there exists an \mathcal{L} -morphism $\eta: M_0 \rightarrow \prod_{\mathcal{F}} A$ such that $\eta \circ \theta_M|_{A_0} = \Delta_A|_{A_0}$.*

We isolate the following fact, which is an immediate consequence of the semantic characterization of positive \mathcal{L} -existential \mathcal{L} -embedding. If F is a functor between two categories, we denote by $F(\theta)$ the image of a morphism θ under F . We regard the class of \mathcal{L} -structures as a category with \mathcal{L} -morphisms as morphisms.

Proposition A.8. *Suppose that $\mathcal{L}^{(0)}$ and $\mathcal{L}^{(1)}$ are languages. Let F be a functor from the category of $\mathcal{L}^{(0)}$ -structures to the category of $\mathcal{L}^{(1)}$ -structures. Assume that F preserves direct limits and that for any separable $\mathcal{L}^{(0)}$ -structure M and nonprincipal ultrafilter \mathcal{U} over \mathbb{N} , there exists an $\mathcal{L}^{(1)}$ -morphism $\rho_M: F(\prod_{\mathcal{U}} M) \rightarrow \prod_{\mathcal{U}} F(M)$ such that $\rho_M \circ F(\Delta_M) = \Delta_{F(M)}$. If A and M are $\mathcal{L}^{(0)}$ -structures in \mathfrak{C} and $\theta_M: A \rightarrow M$ is a positive $\mathcal{L}^{(0)}$ -existential $\mathcal{L}^{(0)}$ -embedding, then $F(\theta_M)$ is a positive $\mathcal{L}^{(1)}$ -existential $\mathcal{L}^{(1)}$ -embedding.*

Proof. Since F preserves direct limits, it is enough to consider the case when M is separable. In this case, the conclusion is the consequence of the first assumption on the functor F and Condition (3) of Theorem A.6. \square

In particular, Proposition A.8 applies when the functor F preserves both direct limits and ultraproducts.

A.4. A more general framework. In this subsection, we consider a more general framework, that will later allow us to deal with not necessarily unital C^* -algebras. For simplicity, we restrict to the single-sorted case. Consider the language \mathcal{L} that contains

- a collection of function symbols,
- a collection of relation symbols,
- a directed collection \mathcal{D} of pseudometric symbols, and
- a distinguished collection $p(x)$ of quantifier-free positive primitive \mathcal{L} -conditions.

The pseudometric symbols are to be interpreted as pseudometrics in a given \mathcal{L} -structure. We denote by $p^+(x)$ denotes the collection of conditions $\varphi(x) \leq r + \varepsilon$ whenever $\varphi(x) \leq r$ is a condition in p . We let $\wp_{\text{fin}}(p^+)$ be the collection of finite subsets of p^+ . We will assume that if $d_0(t_0(\bar{x}), t_1(\bar{x})) \leq r$ is a condition in p for some $d_0 \in \mathcal{D}$ and \mathcal{L} -terms t_0, t_1 , then the condition $d(t_0(\bar{x}), t_1(\bar{x})) \leq r$ also belongs to p for every $d \in \mathcal{D}$. Furthermore, we assume that for any relation symbol B in \mathcal{L} and function symbol f in \mathcal{L} , the language \mathcal{L} contains functions $\varpi_B: \mathbb{R} \rightarrow \mathbb{R} \times \mathcal{D} \times \wp_{\text{fin}}(p^+)$ and $\varpi_f: \mathbb{R} \times \mathcal{D} \rightarrow \mathbb{R} \times \mathcal{D} \times \wp_{\text{fin}}(p^+)$.

Definition A.9. An \mathcal{L} -structure is a set M endowed with an interpretation B^M of any function or relation symbol in \mathcal{L} such that:

- (1) the pseudometric symbols in \mathcal{D} are interpreted as pseudometrics on M ;
- (2) for any n -ary relation symbol B and any $\varepsilon_0 > 0$, if $\varpi_B(\varepsilon_0) = (\varepsilon_1, d_1, q_1)$, then for any realizations \bar{a}, \bar{b} of q_1 in M with $\max_i d_1^M(a_i, b_i) \leq \varepsilon_1$, one has $|B(\bar{a}) - B(\bar{b})| \leq \varepsilon_0$;
- (3) for any n -ary function symbol f , any $\varepsilon > 0$, and any $d_0 \in \mathcal{D}$, if $\varpi_f(\varepsilon, d_0, q_0) = (\varepsilon_1, d_1, q_1)$, then for any realizations \bar{a}, \bar{b} of q_1 in M with $\max_i d_1^M(a_i, b_i) \leq \varepsilon_1$, then $f(\bar{a})$ is a realization of q_0 and $d_0^M(f(\bar{a}), f(\bar{b})) \leq \varepsilon_0$.

The notions of \mathcal{L} -formulas and \mathcal{L} -types in this setting are defined in the usual way.

Suppose that $(M_i)_{i \in I}$ is a collection of \mathcal{L} -structures, and \mathcal{F} is a filter over I . We let $M = \prod_{i \in I} M_i$ be the cartesian product. For every $i \in I$ and $d \in \mathcal{D}$, define a pseudometric d^M on M by

$$d^M((a_i)_{i \in I}, (b_i)_{i \in I}) = \limsup_{i \rightarrow \mathcal{F}} d^{M_i}(a_i, b_i).$$

Let now $M_{\mathcal{F}}$ be the quotient of M by the equivalence relation $(a_i)_{i \in I} \sim (b_i)_{i \in I}$ if and only if $d^M((a_i)_{i \in I}, (b_i)_{i \in I}) = 0$ for every $d \in \mathcal{D}$. As before, we denote by \mathbf{a} the equivalence class of the collection $(a_i)_{i \in I}$. Set $\prod_{\mathcal{F}} M_i$ to be the set of $\mathbf{a} \in M_{\mathcal{F}}$ such that for every $q \in \wp_{\text{fin}}(p^+)$ the set $\{i \in I: a_i \text{ is a realization of } q\}$ belongs to \mathcal{F} .

The interpretation in $\prod_{\mathcal{F}} M_i$ of function and relation symbols from \mathcal{L} is also defined in the usual way. For instance, if B is an n -ary relation symbol from \mathcal{L} and $\mathbf{a}_1, \dots, \mathbf{a}_n \in \prod_{\mathcal{F}} M_i$, then we let

$$B^{\prod_{\mathcal{F}} M_i}(\mathbf{a}_1, \dots, \mathbf{a}_n) = \limsup_{i \rightarrow \mathcal{F}} B(a_{1,i}, \dots, a_{n,i}).$$

Similarly, if f is an n -ary function symbol from \mathcal{L} and $\mathbf{a}_1, \dots, \mathbf{a}_n \in \prod_{\mathcal{F}} M_i$, we let $f^{\prod_{\mathcal{F}} M_i}(\mathbf{a}_1, \dots, \mathbf{a}_n)$ be the element with representative sequence

$$(f(a_{1,i}, \dots, a_{n,i}))_{i \in I}.$$

The definition of \mathcal{L} -structure guarantee that these definitions do not depend on the representatives, and define an \mathcal{L} -structure $\prod_{\mathcal{F}} M_i$, which we call the *reduced product*.

Remark A.10. Let M be an \mathcal{L} -structure, let $t(\bar{x})$ be a positive primitive quantifier free type. Let κ be a cardinal larger than the density character of M and the density character of \mathcal{L} , and let \mathcal{F} be a countably incomplete κ -good filter. It follows from Proposition A.5 that the following statements are equivalent:

- (1) $t(\bar{x})$ is realized in $\prod_{\mathcal{F}} M$
- (2) $t(\bar{x})$ is approximately realized in $\prod_{\mathcal{F}} M$
- (3) $t(\bar{x}) \cup p(x_1) \cup \dots \cup p(x_n)$ is approximately realized in M .

By expanding the language to include constants to name elements of $\prod_{\mathcal{F}} M$, one can deduce that $\prod_{\mathcal{F}} M$ is positively quantifier-free \mathcal{L} - κ -saturated. One may also consider countably incomplete κ -good *ultrafilters* (rather than filters) and arbitrary quantifier-free types.

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