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NUMERICAL ANALYSIS OF PARABOLIC p-LAPLACIAN: APPROXIMATION OF TRAJECTORIES*

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Abstract. The long time numerical approximation of the parabolic p-Laplacian problem with a time-independent forcing term and sufficiently smooth initial data is studied. Convergence and stability results which are *uniform* for $t \in [0, \infty)$ are established in the L^2, $W^{1,p}$ norms for the backward Euler and the Crank–Nicholson schemes with the finite element method (FEM). This result extends the existing uniform convergence results for *exponentially* contractive semigroups generated by some *semilinear* systems to *nonexponentially* contractive semigroups generated by some *quasi-linear* systems.

Key words. finite element methods, parabolic p-Laplacian, long-time dynamics, quasi-linear equations, uniform convergence

AMS subject classifications. 35K55, 37L99, 65M12, 65M15, 65M60, 65P40

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1. Introduction.

The parabolic p-Laplacian problem is a mathematical model possessing some important features shared by many practical problems, such as the non-Newtonian fluid flows (see, e.g., [22, 23, 26]) and the Smagorinsky type meteorology model (see, e.g., [27]). The dynamics of these problems, considered as dissipative dynamical systems, is important. For a general presentation on the dissipative dynamical systems, see, for instance, [13, 33]. For the numerical aspects, see [31, 32].

For *semilinear* systems, when solutions contract towards each other *exponentially* with respect to time, several classical numerical solutions approximate the corresponding true solutions uniformly well for $t \in [0, \infty)$. This has been confirmed for ordinary differential equations [30], reaction-diffusion equations [17, 24, 29], and for the Navier–Stokes equations [16]. For *semilinear* systems, there are many interesting results concerning the long-time numerical approximations. See, among many others references, [7, 8, 9, 12, 14, 15, 16, 21, 25, 31, 32] and the references cited therein. However, to the best of our knowledge, no such result is yet available for a nonsemilinear system. The purpose of this article is to extend the uniform-in-time convergence result to a quasi-linear problem, where the rate of contraction for the solutions is only *algebraic*. See [10]. This model problem covers a class of problems of monotone type. Notice that in [16], the assumption of exponential contraction is used explicitly for the analysis of the Navier–Stokes equations. Also, assuming the (one-sided) Lipschitz condition may imply exponential contraction.

New difficulties come from the strong nonlinearity. Some regularity results for the solution used in the previous analysis of the semilinear problems are not available. Also, the semigroup theory is not as easily applied as before. The p-Laplacian operator is not self-adjoint nor commutable with many other operators. For these and other reasons, it is not surprising that the error estimates obtained for such kind of problems may not be *optimal*.

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Previous results, as in [2, 35], on the finite element method (FEM) for the parabolic p-Laplacian problem where f, the right hand side of the equation, is Lipschitz continuous in time, are valid only for time $t \in [0, T]$, where $T > 0$. Obtaining an optimal convergence rate is addressed in [2], which is based on extra regularity assumptions on the solution. These assumptions are not easily verified. The regularity issue is also discussed in detail in [2].

The goal of this article is to study the convergence and stability properties of the numerical solutions valid *uniformly* for $t \in [0, \infty)$. We obtain convergence in the $L^2(\Omega)$ and $W^{1,p}(\Omega)$ norms for temporal and spatial semidiscretizations and for full discretizations. Convergence in $L^\infty(\Omega)$ norm is thus a corollary when $p > d$, where d is the dimension of the space domain of the problem. In addition, new error estimates are obtained that were not available before. We obtain these results without any extra regularity assumption, though the orders of error estimates are far from *optimal.* We apply our analysis to the backward Euler and the Crank–Nicholson schemes and, however, show that the backward Euler scheme is even asymptotically stable. Our results suggest that the backward Euler scheme has some advantage over the Crank–Nicholson scheme. But confirming this would necessitate further computing tests and a deeper analysis. The key difference between our analysis and those of [2, 35] is our exploration of the dynamical feature of the original system using some nonlinear Gronwall inequalities and the use of the monotone compactness argument of [26].

These results are extended to more general cases in [19, 20]. In this article, and in [19, 20], we discuss only the p-Laplacian problem. As part of an ongoing project, this study will be extended to more general classes of problems, especially models of non-Newtonian fluid flows and the Smagorinsky model in meteorology.

The rest of this article is organized as follows. In section 2, we recall some notations and preliminary results. In section 3, the temporal discretization is studied and the orders of uniform in time convergence are derived. In section 4, the FEM semidiscretization is studied and the uniform-in-time convergence results are established. In section 5, we discuss the full discretizations, where the results in sections 3 and 4 are naturally combined. In section 6, we give the proofs of some Gronwall type lemmas stated in section 2.

2. Notations and preliminaries.

2.1. Basic concepts. Suppose that $\Omega \subset R^d$ ($d \geq 1$) has a Lipschitz continuous boundary $\partial\Omega$ in the case $d > 1$. Assume that $p \in (2, \infty)$ or $p \in (1, 2)$. When no ambiguity occurs, we use $u(t)$ or simply u to denote the function $u(x,t) : \Omega \times R_+ \mapsto R^1$, where $R_+ = [0, \infty)$.

For the functional setting, we review some facts about the Sobolev spaces. Let H denote the space $L^2(\Omega)$, endowed with the usual inner product $\langle \cdot, \cdot \rangle$, and the usual norm

$$\|u\|_2 = \left(\int_\Omega |u|^2 dx \right)^{1/2}.$$

Denote $W^{1,p}(\Omega) = \{v | v \in L^p(\Omega), D_i v \in L^p(\Omega), i = 1, ..., d\}$, with the norm

$$\|v\|_{W^{1,p}(\Omega)} = \|v\|_{L^p(\Omega)} + \sum_{i=1}^d \|D_i v\|_{L^p(\Omega)}.$$

Denote $W_0^{1,p}(\Omega) = \{v \in W^{1,p}(\Omega) | \ \ v|_{\partial\Omega} = 0\}$ and $W^{-1,p'}(\Omega) = (W_0^{1,p}(\Omega))^*$, the dual space of $W_0^{1,p}(\Omega)$, where $1/p + 1/p' = 1$. Denote $(\cdot, \cdot)$ as the duality product for a pair in $(W_0^{1,p}(\Omega))^* \times W_0^{1,p}(\Omega)$. Denote $\|\cdot\|$ as the seminorm of $W^{1,p}(\Omega)$ and the norm of $W_0^{1,p}(\Omega)$, and $\|\cdot\|^*$ as the norm of $(W_0^{1,p}(\Omega))^*$. Let $V = W_0^{1,p}(\Omega)$ and $V' = W^{-1,p'}(\Omega)$.

Consider the following p-Laplacian problem:

$$\frac{\partial u}{\partial t} = \nabla \cdot (|\nabla u|^{p-2}\nabla u) + f, \qquad x \in \Omega, t \in (0,\infty), \tag{1}$$

$$u(x,t) = 0, \qquad x \in \partial\Omega, t \in (0,\infty), \tag{2}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{3}$$

where $u_0 \in V$ and $f \in V'$ are given. For simplicity, we suppose here that f is time-independent. In [20], the discussions are extended to the case where f is time-dependent. This problem occurs in many mathematical models of physical processes: for example, nonlinear diffusion and filtration [28] and non-Newtonian flows [1].

Define $A : V \to V'$ as follows: for every $v \in V$,

$$(Au, v) := \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx. \tag{4}$$

DEFINITION 2.1. *We say that u is a* weak solution *of* (1)–(3) *on* $[0,T]$ *if* $u \in L^p(0,T;V)$ *solves the following weak problem (WP):*

$$\frac{d}{dt}\langle u, v\rangle + (Au, v) = \langle f, v\rangle, \tag{5}$$

$$\langle u(0), v\rangle = \langle u_0, v\rangle \tag{6}$$

for all $v \in V$, in the sense of distribution on $[0,T]$*. See* [24] *and* [31] *for further details.*

2.2. Existence, uniqueness, and regularity. We recall below some classic results.

LEMMA 2.1. *Suppose $p \in (1,\infty)$, $\Omega \subset R^d$, where $d \geq 1$ and Ω is bounded. Then $W^{1,p}(\Omega) \overset{c}{\subset} L^2(\Omega)$ for $p \in (2d/(d+2),\infty)$. Moreover, for every $p \in [2d/(d+2),\infty)$, there is a $C_0 = C_0(\Omega,d,p)$, a positive constant, such that for all $u \in V$,*

$$\|u\| \geq C_0\|u\|_2.$$

The following lemma is proved in [11] for $d = 2$. It is valid for all $d \geq 1$.

LEMMA 2.2. *There exist positive constants $\alpha = \alpha(p,d)$, $\beta = \beta(p,d)$, such that the following inequalities hold:*

If $p \in (1,2)$, then for all $u, v \in V$,

$$\alpha\|u - v\|^2 \leq (Au - Av, u - v)(\|u\| + \|v\|)^{2-p},$$
$$\|Au - Av\|^* \leq \beta\|u - v\|^{p-1}.$$

If $p \in (2,\infty)$, then for all $u, v \in V$,

$$\alpha\|u - v\|^p \leq (Au - Av, u - v),$$
$$\|Au - Av\|^* \leq \beta(\|u\| + \|v\|)^{p-2}\|u - v\|.$$

Using the standard Browder–Minty theory, with Lemmas 2.1 and 2.2, it is easy to show the following result.

LEMMA 2.3. *For all $f \in V'$ and $\mu \geq 0$, there exists a unique $u \in V$, such that, for all $v \in V$,*

$$\mu\langle u, v\rangle + (Au, v) = \langle f, v\rangle.$$

Moreover,

$$\|u\|_2 + \|u\| \leq C(\Omega)(\|f\|^*)^{\frac{1}{p-1}}.$$

We collect the classical existence and regularity results from [4, 26, 33] in the following theorem.

THEOREM 2.1. *If $f \in V'$ and $u_0 \in H$, then the problem (5)–(6) has a unique solution $u \in L^p(0,T;V)$ for $T > 0$. Thus $u \in C[R_+, H]$, u' and $Au \in L^{p'}(0,T;V')$, for $T > 0$, and $u_0 \mapsto u(t)$ is continuous in H. Furthermore for $p \in (2,\infty)$, if $u_0 \in V\bigcap D(A)$, i.e., $Au_0 \in H$, and $f \in H$, then for $T > 0$, $u \in L^\infty(0,T;V)$ and u' and $Au \in L^\infty(0,T;H)$. Moreover, $u \in L^\infty(R_+;V)$.*

Remark 2.1. For any $p \in (1,\infty)$, if $f \in H$ and $u_0 \in V$ then $u \in L^\infty(R_+;V)$. Further, when $u_0 \in V\bigcap D(A)$, we have $u \in C(R_+, V)$ and u' and $Au \in L^\infty(R_+, H)$. These results can be proved by using the stability results obtained in section 3 and the classic monotone compactness argument of [26]. For details, see [18]. These facts are frequently used later. □

2.3. Gronwall lemmas. We recall several versions of the Gronwall lemmas, which are useful later. We prove Lemma 2.4, Lemma 2.7, and Lemma 2.8 in section 6. Lemmas 2.5 and 2.9 are the discrete versions of Lemma 2.4 and Lemma 2.8 and can be proved without much difficulty by following the ideas of the proofs of Lemmas 2.4 and 2.8.

LEMMA 2.4. *Suppose $q > 0$, $\gamma > 0$, $\delta \geq 0$ and $y(t)$ is nonnegative and absolutely continuous such that for $t \in (0,\infty)$*

$$\frac{dy}{dt} + \gamma y^q(t) \leq \delta.$$

Then

$$y(t) \leq \max\left\{y(0), \left(\frac{\delta}{\gamma}\right)^{1/q}\right\}.$$

LEMMA 2.5. *Suppose q,γ,$\Delta t > 0$, $\delta \geq 0$ and y_n's are nonnegative such that for $n = 1, 2, \ldots,$*

$$\frac{y_n - y_{n-1}}{\Delta t} + \gamma y_n^q \leq \delta.$$

Then

$$y_n \leq \max\left\{y_0, \left(\frac{\delta}{\gamma}\right)^{1/q}\right\}.$$

The following two lemmas are improved versions of Lemmas 2.4 and 2.5 for the case where $q \in (1, \infty)$. Lemma 2.6 is a slightly improved restatement of Ghidaglia's Gronwall lemma. See section III.5.2 of [33] for the proof. The inequalities derived below are actually sharp.

LEMMA 2.6. *Suppose $q > 1$, $\gamma > 0$, $\delta \geq 0$ and y is nonnegative and absolutely continuous on $(0, \infty)$ and satisfying*

$$\frac{dy}{dt} + \gamma y^q \leq \delta.$$

Then, for $t \geq 0$,

$$y(t) \leq \begin{cases} (\frac{\delta}{\gamma})^{1/q} & \text{if } y^q(0) \leq \frac{\delta}{\gamma}, \\ (\frac{\delta}{\gamma})^{1/q} + \{[y(0) - (\frac{\delta}{\gamma})^{1/q}]^{1-q} + \gamma(q-1)t\}^{1/(1-q)} & \text{otherwise.} \end{cases}$$

LEMMA 2.7. *Suppose $q > 1$, $\gamma > 0$, $\Delta t > 0$, $\delta \geq 0$ and $\{y_n\}_1^\infty$ is a nonnegative sequence such that*

$$\frac{y_n - y_{n-1}}{\Delta t} + \gamma y_n^q \leq \delta.$$

Then, for Δt small enough and $n = 1, 2, ...,$ it holds that

$$y_n \leq (\frac{\delta}{\gamma})^{1/q}$$

if $y_0^q \leq \frac{\delta}{\gamma}$. Otherwise

$$y_n \leq (\frac{\delta}{\gamma})^{1/q} + \left\{ \left[y_0 - \left(\frac{\delta}{\gamma}\right)^{1/q} \right]^{1-q} + \gamma(q-1)n\Delta t \left[1 - q\gamma \left(y_0 - \left(\frac{\delta}{\gamma}\right)^{q-1} \right) \Delta t \right] \right\}^{1/(1-q)}.$$

The following two lemmas are improved versions of Lemmas 2.4 and 2.5 for the case where $q \in (0, 1)$. The inequalities given below are not necessarily sharp.

LEMMA 2.8. *Suppose $q \in (0, 1)$, γ,$\delta > 0$ and y is nonnegative and absolutely continuous such that for $t > 0$*

$$\frac{dy}{dt} + \gamma y^q(t) \leq \delta.$$

Then, for $t > 0$,

$$y(t) \leq \begin{cases} (\frac{\delta}{\gamma})^{1/q} & \text{if } y^q(0) \leq \frac{\delta}{\gamma}, \\ (\frac{\delta}{\gamma}) y^{1-q}(0) + [y(0) - \frac{\delta}{\gamma} y^{1-q}(0)] exp(-\frac{\gamma t}{y^{1-q}(0)}) & \text{otherwise.} \end{cases}$$

LEMMA 2.9. *Suppose $q \in (0, 1)$, $\gamma > 0$, $\delta \geq 0$, $\Delta t > 0$ and $\{y_n\}_0^\infty$ is a nonnegative sequence such that for $n = 1, 2, \ldots,$*

$$\frac{y_n - y_{n-1}}{\Delta t} + \gamma y_n^q \leq \delta.$$

Then, for $n = 0, 1, \ldots,$

$$y_n \leq \begin{cases} (\frac{\delta}{\gamma})^{1/q} & \text{if } y_0^q \leq \frac{\delta}{\gamma}, \\ (\frac{\delta}{\gamma}) y_0^{1-q} + (y_0 - \frac{\delta}{\gamma} y_0^{1-q})(1 + \gamma y_0^{q-1} \Delta t)^{-n} & \text{otherwise.} \end{cases}$$

Finally, we recall the following uniform Gronwall lemma presented in section III.1.1.3 of [33], which is a powerful tool for time uniform *a priori* estimates.

LEMMA 2.10 (uniform Gronwall lemma). *Let g, h, y be locally integrable functions on (t_0, ∞), such that y' is locally integrable on (t_0, ∞) and for $t \in (t_0, \infty)$*

$$\frac{dy}{dt} \leq gy + h,$$

for $t \in (t_0, \infty)$

$$\int_t^{t+r} g(s)ds \leq a_1, \int_t^{t+r} h(s)ds \leq a_2, \int_t^{t+r} y(s)ds \leq a_3,$$

where r, a_1, a_2, a_3 are constants. Then, for $t \in (t_0, \infty)$,

$$y(t+r) \leq exp(a_1)\left(a_2 + \frac{a_3}{r}\right).$$

3. The time discretizations. Let $\{t_i\}_{i=0}^{\infty}$ be a uniform partition of R_+ with $t_i = i\Delta t$ for time step $\Delta t > 0$.

3.1. The backward Euler scheme. Consider the following recursive nonlinear elliptic problem:

Given $f \in V'$, $u_0 \in H$, find $u_i \in V$, such that for all $v \in V$,

$$\left\langle \frac{u_i - u_{i-1}}{\Delta t}, v \right\rangle + (Au_i, v) = \langle f, v \rangle, \tag{7}$$

$$u_0 = u_0(x), \tag{8}$$

where $u_i = u_i(x), i = 1, 2, \ldots$.

By Lemma 2.1, $V \subset H \subset V'$. By Lemma 2.3, there is a unique sequence $\{u_i\}_1^{\infty} \subset V$, defined by (7).

First, we give some uniform stability results in Lemma 3.1–3.4 that are crucial in getting the uniform convergence results.

LEMMA 3.1. *If $f \in H$, $u_0 \in H$, then $\{\|u_i\|_2\}_0^{\infty}$ is uniformly bounded by a constant $C(\|u_0\|_2, \|f\|_2)$.*

Proof. Choosing $v = u_i$, (7) becomes

$$\|u_i\|_2^2 + \Delta t \|u_i\|^p = \Delta t \langle f, u_i \rangle + \langle u_{i-1}, u_i \rangle.$$

Thus Lemma 2.1 gives

$$\|u_i\|_2 + \Delta t C_0^p \|u_i\|_2^{p-1} \leq \|u_{i-1}\|_2 + \Delta t \|f\|_2.$$

So, by Lemma 2.5, $\|u_i\|_2$ is uniformly bounded with respect to i and Δt. We can get better estimates using Lemmas 2.7 and 2.9. ∎

If $f \in V'$, the result similar to Lemma 3.1 holds.

LEMMA 3.2. *If $f \in H$ and $u_0 \in H$, then $\{\|(u_i - u_{i-1})/\Delta t\|_2\}_0^{\infty}$ is monotonically decreasing. If in addition, $u_0 \in V \bigcap D(A)$, then $\{\|(u_i - u_{i-1})/\Delta t\|_2\}_0^{\infty}$ and $\{\|u_i\|\}_0^{\infty}$ are uniformly bounded with respect to i and Δt. Moreover, the following estimates hold:*

If $p \in (2, \infty)$, then for Δt small enough,

$$\left\|\frac{u_i - u_{i-1}}{\Delta t}\right\|_2 \leq L\left(i, \left\|\frac{u_1 - u_0}{\Delta t}\right\|_2\right),$$

where $L(i,z) = \{z^{2-p} + \alpha C_p(p-2)(\Delta t)^{p-2}(i-1)\Delta t[1-(p-1)\alpha C_p(\Delta t)^{p-1}z]\}^{\frac{1}{2-p}}$.

If $p \in (1,2)$, *then*

$$\left\|\frac{u_i - u_{i-1}}{\Delta t}\right\|_2 \le \left\|\frac{u_1 - u_0}{\Delta t}\right\|_2 (1 + \alpha C_p \Delta t)^{-i+1}.$$

Proof. Setting $v = u_i - u_{i-1}$, (7) becomes

$$\left\langle \frac{u_i - u_{i-1}}{\Delta t}, u_i - u_{i-1} \right\rangle + (Au_i, u_i - u_{i-1}) = \langle f, u_i - u_{i-1}\rangle.$$

Similarly,

$$\left\langle \frac{u_{i-1} - u_{i-2}}{\Delta t}, u_i - u_{i-1} \right\rangle + (Au_{i-1}, u_i - u_{i-1}) = \langle f, u_i - u_{i-1}\rangle.$$

Subtracting the later equation from the former gives

$$\left\langle \frac{u_i - u_{i-1}}{\Delta t}, u_i - u_{i-1} \right\rangle + (Au_i - Au_{i-1}, u_i - u_{i-1}) = \left\langle \frac{u_{i-1} - u_{i-2}}{\Delta t}, u_i - u_{i-1} \right\rangle. \tag{9}$$

Now it follows from Lemma 2.2 that

$$\left\|\frac{u_i - u_{i-1}}{\Delta t}\right\|_2 \le \left\|\frac{u_{i-1} - u_{i-2}}{\Delta t}\right\|_2.$$

So $\{\|(u_i - u_{i-1})/\Delta t\|_2\}_0^\infty$ is monotonically decreasing. Notice that

$$\left\langle \frac{u_1 - u_0}{\Delta t}, u_1 - u_0 \right\rangle + (Au_1 - Au_0, u_1 - u_0) = \langle f, u_1 - u_0\rangle - (Au_0, u_1 - u_0).$$

Thus

$$\left\|\frac{u_i - u_{i-1}}{\Delta t}\right\|_2 \le \left\|\frac{u_1 - u_0}{\Delta t}\right\|_2 \le \|f - Au_0\|_2.$$

Moreover

$$\|u_i\|^p = (Au_i, u_i) = \left\langle f - \frac{u_i - u_{i-1}}{\Delta t}, u_i \right\rangle \le (\|f\|_2 + \|f - Au_0\|_2)\|u_i\|_2.$$

Thus $\{\|u_n\|\}_0^\infty$ is bounded uniformly with respect to n and Δt by Lemma 3.1.

If $p \in (2,\infty)$, then by Lemma 2.1, Lemma 2.2, and (9), there exists $C_p = C_p(C_0, p) > 0$, such that

$$\left\|\frac{u_i - u_{i-1}}{\Delta t}\right\|_2 + \alpha C_p(\Delta t)^{p-2}\Delta t \left\|\frac{u_i - u_{i-1}}{\Delta t}\right\|_2^{p-1} \le \left\|\frac{u_{i-1} - u_{i-2}}{\Delta t}\right\|_2.$$

The estimate then follows immediately by Lemma 2.7.

If $p \in (1,2)$, then by Lemma 2.2 and (9),

$$\left\|\frac{u_i - u_{i-1}}{\Delta t}\right\|_2 + \alpha C_0(\|u_i\| + \|u_{i-1}\|)^{p-2}\|u_i - u_{i-1}\| \le \left\|\frac{u_{i-1} - u_{i-2}}{\Delta t}\right\|_2.$$

Since $\{\|u_i\|\}_0^\infty$ is uniformly bounded with respect to i and Δt, by Lemma 2.1, there exists $C_p = C_p(C_0, p, \|f\|_2, \|f - Au_0\|_2) > 0$, such that

$$\left\|\frac{u_i - u_{i-1}}{\Delta t}\right\|_2 + \alpha C_p \|u_i - u_{i-1}\|_2 \le \left\|\frac{u_{i-1} - u_{i-2}}{\Delta t}\right\|_2.$$

Thus the estimation follows from induction on i. ∎

If $u_0 \in V$ and $f \in H$, then $\{\|u_i\|\}_0^\infty$ is still bounded uniformly with respect to n. See [19, 20].

Essentially following the proof of Lemma 4 of [35], we have the following lemma.

LEMMA 3.3. *If $f \in H$, $u_0 \in V \bigcap D(A)$, then $\{\|Au_i\|_2\}_0^\infty$ is uniformly bounded with respect to i and Δt.*

Let $\{t_{n,i}\}_{i=0}^\infty$ be a uniform partition of R_+. Let $t_{n,i} = i\Delta t_n$, with time step $\Delta t_n > 0$. Let $\{u_{n,i}\}_{i=0}^\infty$ be the solution set defined by (7)–(8) with Δt replaced by Δt_n. Notice that the n in the subscripts of $u_{n,i}$ is used to specify the difference between the corresponding quantities with different time steps Δt_n.

Define

$$u_n(t) := \frac{t - t_{n,i}}{\Delta t_n} u_{n,i+1} + \frac{t_{n,i+1} - t}{\Delta t_n} u_{n,i}, \qquad t_{n,i} < t \le t_{n,i+1}, \qquad i = 0, 1, \ldots,$$
$$\bar{u}_n(t) := u_{n,i+1}, \qquad t_{n,i} < t \le t_{n,i+1}, \qquad i = 0, 1, \ldots,$$

where $t_{n,i} = i\Delta t_n$. We sometimes drop n in $u_{n,i}$ and Δt_n when no ambiguity occurs. By the above definition, Lemmas 3.1 and 3.2, we immediately have the following lemma.

LEMMA 3.4. *For every $\Delta t_n > 0$, if $u_0 \in H$, $f \in H$, then the functions $\|u_n(t)\|_2$ and $\|\bar{u}_n(t)\|_2$ are bounded uniformly for $t \in R_+$ with bounds independent of n and Δt_n; if $u_0 \in V \bigcap D(A)$, $f \in H$, the functions $\|u_n(t)\|$ and $\|\bar{u}_n(t)\|$ are bounded uniformly for $t \in R_+$ with bounds independent of n and Δt_n.*

Remark 3.1. We do *not* have an estimate on $\|Au_n(t)\|_2$ which is still an unsolved problem. However, by making use of the classic monotone compactness argument, one can prove that, when $u_0 \in V \bigcap D(A)$, $A\bar{u}_n$ converges weak-star in $L^\infty(R_+; H)$ to $Au \in L^\infty(R_+; H)$, and thus $u' \in L^\infty(R_+; H)$. Alternatively, since $\|Au_0\|_2$ is bounded, $\|u_n'(t)\|_2$ is bounded uniformly for $t \in [0, \infty)$. Using the classic monotone compactness argument, u_n' converges weak-star $L^\infty(R_+; H)$ to $u' \in L^\infty(R_+; H)$, thus $Au \in L^\infty(R_+; H)$. See [18] for details. Notice that these facts will be used frequently later. ∎

Now we give the uniform convergence theorem in the space H.

THEOREM 3.1 (uniform convergence in H). *Suppose that $f \in H$, $u_0 \in V \bigcap D(A)$. Then $\{u_n\}$ is a Cauchy sequence in $C[R_+; H]$, and it converges to u in $C[R_+; H]$ uniformly with respect to t as $\Delta t \to 0$. Moreover, there exists a $C(t) \ge 0$, independent of Δt and uniformly bounded for $t \in R_+$, such that, for $p \in (1, \infty)$ and $t \in R_+$,*

$$\|u(t) - u_n(t)\|_2 \le C(t)(\Delta t)^{1/r},$$

where $r = \max\{2, p\}$. Further, if $p \in (1, 2)$ then $C(t) \to 0$ exponentially as $t \to +\infty$ and if $p \in (2, \infty)$ then for $\Delta t > 0$ fixed and sufficiently small, $C(t) = O(t^{1/p(2-p)})$, as $t \to \infty$.

Proof. By definition of $u_n(t)$,

$$\left\langle \frac{du_n}{dt}, v \right\rangle + (Au_{n,i+1}, v) = \langle f, v \rangle, \qquad t_{n,i} < t \le t_{n,i+1}.$$

By Theorem 2.1,

$$\left\langle \frac{du}{dt}, v \right\rangle + (Au, v) = \langle f, v\rangle, \qquad t_{n,i} < t \le t_{n,i+1}.$$

Let $v = u - u_{n,i+1}$ in the above equations, and subtracting the latter from the former, we have for $t \in (t_{n,i}, t_{n,i+1}]$, using Theorem 2.1 and Remark 2.1,

$$\frac{1}{2}\frac{d}{dt}\|u - u_n\|_2^2 + (Au - Au_{n,i+1}, u - u_{n,i+1}) = (u' - u_n', u_{n,i+1} - u_n).$$

By Lemma 2.2, Lemma 3.2, and Remark 3.1, there are positive constants c_1, c_2 independent of t, Δt and n such that

$$\frac{d}{dt}\|u - u_n\|_2^2 + c_1\|u - u_{n,i+1}\|^r \le c_2 \left\|\frac{u_{n,i+1} - u_{n,i}}{\Delta t}\right\|_2 \Delta t.$$

For $r \ge 2$, there exists a positive constant, $C_r > 0$, depending only on r such that

$$\|u - u_n\|^r = \|u - u_{n,i+1} + u_{n,i+1} - u_n\|^r \le C_r(\|u - u_{n,i+1}\|^r + \|u_{n,i+1} - u_n\|^r).$$

Notice also that by Lemma 2.2, Lemma 3.2, and Lemma 3.3 there exists a positive constant, $C_r' > 0$, independent of t, n and Δt such that

$$\|u_{n,i+1} - u_{n,i}\|^r \le C_r'\|u_{n,i+1} - u_{n,i}\|_2 \le C_r' \left\|\frac{u_{n,i+1} - u_{n,i}}{\Delta t}\right\|_2 \Delta t.$$

Thus there are positive constants c_3, c_4 independent of t, Δt and n such that

$$\frac{d}{dt}\|u - u_n\|_2^2 + c_3\|u - u_n\|^r \le c_4 \left\|\frac{u_{n,i+1} - u_{n,i}}{\Delta t}\right\|_2 \Delta t.$$

The error estimate in the H norm is obtained by an application of Lemma 2.1, Lemma 2.4, and Lemma 3.2. $C(t)$ is obviously uniformly bounded since $\|(u_{n,i+1} - u_{n,i})/\Delta t\|_2$ is by Lemma 3.2. Better estimation of $C(t)$ is as follows.

For $p \in (1, 2)$, notice that by Lemma 3.2 there are constants $c, c' > 0$ such that

$$\left\|\frac{u_{n,i+1} - u_{n,i}}{\Delta t}\right\|_2 \le c\exp(-c'\alpha C_p t).$$

So $C(t)$ goes to 0 exponentially as $t \to +\infty$.

For $p \in (2, \infty)$, when $\Delta t > 0$ is *fixed*, notice that there is a constant $c > 0$ such that

$$\left\|\frac{u_{n,i+1} - u_{n,i}}{\Delta t}\right\|_2 \le ct^{1/(2-p)}. \qquad \square$$

In fact, for $p \in (2, \infty)$ we can still get control of $C(t)$ *without* fixing Δt. Notice the fact that for a, $b > 0$ and $\theta \in (0, 1]$,

$$a + b \ge a^\theta b^{1-\theta}.$$

Then, by Lemma 3.2, there is a constant $c > 0$ such that

$$\left\|\frac{u_{n,i+1} - u_{n,i}}{\Delta t}\right\|_2 \le ct^{\theta/(2-p)}(\Delta t)^{-\theta}.$$

Thus, by Lemma 2.4, there is a constant $c > 0$ such that

$$\|u(t) - u_n(t)\|_2 \le ct^{\theta/p(2-p)}(\Delta t)^{(1-\theta)/p}.$$

This comment can be used in the following theorem to relax Δt.

Next we give a uniform convergence result in the space of V.

THEOREM 3.2 (uniform convergence in V). *Suppose that $f \in H$ and $u_0 \in V \bigcap D(A)$. Then $\{u_n\}$ is a Cauchy sequence in $C[R_+; V]$ and it converges to $u \in C[R_+; V]$ uniformly with respect to n as $\Delta t \to 0$. Moreover, there exists a $C(t) \ge 0$ independent of Δt and uniformly bounded for $t \in R_+$ such that*

$$\|u_n(t) - u(t)\| \le C(t)(\Delta t)^{1/r^2},$$

where $r = \max\{2, p\}$. Further, if $p \in (1, 2)$ then $C(t)$ goes to 0 exponentially as $t \to \infty$ and if $p \in (2, \infty)$ then for fixed $\Delta t > 0$, $C(t) = O(t^{1/p^2(2-p)})$ as $t \to \infty$.

Proof. By definition of $u_n(t)$, for all $t \in (t_i, t_{i+1})$,

$$\begin{aligned}\|u_n(t) - u(t)\| &= \left\| \frac{t - t_i}{\Delta t} u_i + \frac{t_{i+1} - t}{\Delta t} u_{i+1} - u(t) \right\| \\ &\le \frac{t - t_i}{\Delta t} \|u_i - u(i\Delta t)\| + \frac{t_{i+1} - t}{\Delta t} \|u_{i+1} - u((i+1)\Delta t)\| \\ &\quad + \frac{t - t_i}{\Delta t} \|u(t) - u(i\Delta t)\| + \frac{t_{i+1} - t}{\Delta t} \|u(t) - u((i+1)\Delta t)\|.\end{aligned}$$

If $p \in (2, \infty)$ then by Lemma 2.2,

$$\begin{aligned}\alpha \|u_i - u(i\Delta t)\|^p &\le (Au_i - Au(i\Delta t), u_i - u(i\Delta t)) \\ &\le (\|Au_i\|_2 + \|Au(i\Delta t)\|_2)\|u_i - u(i\Delta t)\|_2\end{aligned}$$

and

$$\begin{aligned}\alpha \|u(t) - u(i\Delta t)\|^p &\le (Au(t) - Au(i\Delta t), u(t) - u(i\Delta t)) \\ &\le (\|Au(t)\|_2 + \|Au(i\Delta t)\|_2)\|u(t) - u(i\Delta t)\|_2 \\ &\le (\|Au(t)\|_2 + \|Au(i\Delta t)\|_2)\|u'(t')\|_2 \Delta t,\end{aligned}$$

where $t' \in [t_i, t_{i+1}]$.

The case $p \in (1, 2)$ can be treated similarly. We omit it for brevity. Using the above estimates, Lemma 3.3, Theorem 3.1, and Remark 3.1, the theorem is proved. ∎

Notice the comment following the proof of Theorem 3.1 dealing with the situation when Δt is not fixed.

The above two theorems show that even if Δt does not go to zero as $n \to \infty$, we still get convergence in H and V. This is interesting from the point of view of dynamics.

The above theorems give uniform convergence results for the backward Euler scheme. The proofs apply as well to the time discretization of semidiscrete methods, such as FEM, to be discussed in section 4.

Remark 3.2. 1. From the above theorems, we see that $u \in C(R_+, V)$. In [35], a local result similar to Theorem 3.1 is obtained for $p \in (2, \infty)$ and f Lipschitz continuous with respect to t in H, which is crucial in getting the final local convergence result of [35]. The proof is different from ours. The key point in [35] is proving the local result $u \in C([0,T],V)$ for $T > 0$. We get this result in the global version as a by-product of Theorems 3.1 and 3.2. Our proof here is much simpler since we are able to use Theorem 2.1 and also Remark 3.1. The case of f being Lipschitz continuous in H can be treated similarly. See [19, 20] for details.

2. If $u_0 \in H$, we can still use the classic monotone compact argument presented in Chapter II of [26] to show that $\{u_n\}_0^\infty$ still converges to u in some sense, but then, u is only the unique solution of (5)–(6) in the sense of distribution on $(0,T)$ only, for any $T > 0$, rather than pointwise. See [20] for details. □

The proof of the following stability results is omitted for brevity. Using Lemma 2.2, it can be obtained by Lemma 2.7 for $p \in (2,\infty)$ and by induction for $p \in (1,2)$.

THEOREM 3.3 (stability). *Let $\{u_n\}_0^\infty$, $\{v_n\}_0^\infty$ be two different sequences solving (7) with different initial $u_0, v_0 \in V \bigcap D(A)$. Let $w_n := u_n - v_n$. Then there exists a $C = C(\alpha,p,f)$ and a $C' = C'(\alpha,p,f,u_0,v_0) > 0$, such that for all $n = 0,1,\ldots,$ the following hold:*

If $p \in (1,2)$, then

$$\|w_n\|_2 \le \|w_0\|_2 e^{-Cn\Delta t},$$
$$\|w_n\| \le C'\|w_n\|_2^{1/2}.$$

If $p \in (2,\infty)$ and $\Delta t > 0$ is sufficiently small, then

$$\|w_n\|_2 \le \|w_0\|_2\{1 + \|w_0\|_2^{p-2}\alpha C_0^p(p-2)n\Delta t(1-(p-2)\alpha C_0^p\|w_0\|^{p-2}\Delta t)\}^{1/(2-p)},$$
$$\|w_n\| \le C'\|w_n\|_2^{1/p}.$$

For the above H estimates to hold, we require only that $u_0, v_0 \in H$.

This theorem shows that the backward Euler scheme is not only unconditionally stable, i.e., there is no restriction on the ratio of temporal step size to spatial mesh size, but is also asymptotically stable. In fact, for $p \in (1,2)$, it is even exponentially stable.

3.2. The Crank–Nicholson scheme. We summarize some of the convergence and stability results for the Crank–Nicholson scheme. They are also valid for the more general θ-methods. For brevity, the proofs are omitted. See [18] for details.

Consider the following recursive nonlinear elliptic problems:
Given $u_{i-1} \in V$, find $u_i \in V$), such that

$$\left\langle \frac{u_i - u_{i-1}}{\Delta t}, v \right\rangle + \left(A\frac{u_i + u_{i-1}}{2}, v \right) = \langle f, v\rangle, \tag{10}$$
$$u_0 = u_0(x), \tag{11}$$

where $u_i = u_i(x), i = 1,2,\ldots.$

We can define the solution sequence $\{u_n\}$ of (10)–(11) as before and prove the following results following the ideas above.

THEOREM 3.4 (convergence). *Suppose that $f \in H$, $u_0 \in V \bigcap D(A)$. Then $\{u_n\}$ is a Cauchy sequence in $C[R_+;V]$ and it converges to $u \in C[R_+;V]$ uniformly with respect to n as $\Delta t \to 0$. Moreover, there exists a $C = C(u_0,f) \ge 0$ independent of Δt_n and n such that the following estimates hold:*

If $p \in (2,\infty)$, then

$$\|u_n(t) - u(t)\|_2 \le C(u_0,f)(\Delta t_n)^{p'/p^2},$$
$$\|u_n(t) - u(t)\| \le C(u_0,f)(\Delta t_n)^{p'/p^3}.$$

If $p \in (1,2)$, *then*

$$\|u_n(t) - u(t)\|_2 \le C(u_0, f)(\Delta t_n)^{(p-1)/2},$$
$$\|u_n(t) - u(t)\| \le C(u_0, f)(\Delta t_n)^{(p-1)/4}.$$

THEOREM 3.5 (stability). *Let* $\{u_n\}_0^\infty$, $\{v_n\}_0^\infty$ *be two different sequences solving* (11) *with different initial data* $u_0, v_0 \in V \bigcap D(A)$. *Let* $w_n := u_n - v_n$. *Then, for* $n = 0, 1, \dots,$

$$\|w_n\|_2 \le \|w_0\|_2.$$

Moreover, there exists a $C > 0$ *independent of* n *and* Δt, *such that for* $n = 0, 1, \dots,$

$$\|w_n\| \le C\|w_n\|_2^{1/p} \qquad \text{if } p \in (2, \infty),$$
$$\|w_n\| \le C\|w_n\|_2^{1/2} \qquad \text{if } p \in (1, 2).$$

Notice that we use $u_0 \in V$ for the existence result. Even though we still have the unconditional stability here, we could not get the asymptotic stability. The convergence and stability results for this scheme suggest no advantage over backward Euler scheme, especially when high regularity is unavailable. However, confirming this point needs further computing tests and deeper analyses.

4. A semi-discrete finite element method. For simplicity, we assume that Ω is convex with piecewise smooth boundary $\partial\Omega$ where $d \ge 1$. In this case, interpolation theory is immediately available. However, the discussion given here can be extended to more general cases using standard techniques.

Let $S_h(\Omega)$ be a *regular* conforming finite element space of V, where h is the maximum of the diameters of the elements of triangulation. See, e.g., [6]. Let $\Pi_h : V \to S_h(\Omega)$ be the Lagrangian interpolation operator. Define the operator $P_h : V \to S_h(\Omega)$ such that for $v \in V$,

$$(AP_h v, v_h) = (Av, v_h) \quad \text{for all } v_h \in S_h(\Omega).$$

We recall and establish some important properties about P_h. The first convergence property was proved in [11] for $d = 2$, which is also true for $d \ge 1$.

LEMMA 4.1 (convergence). *For every* $v \in V$,

$$\lim_{h\to 0} \|v - P_h v\| = 0.$$

Due to Lemma 2.2, it is easy to show the following.

LEMMA 4.2 (stability). *Suppose that* $v, \tilde{v} \in V$. *Then* $\|P_h v\| \le \|v\|$. *Moreover,*

$$\|P_h v - P_h \tilde{v}\|^{p-1} \le \frac{\beta}{\alpha}(\|v\| + \|\tilde{v}\|)^{p-2}\|v - \tilde{v}\| \qquad \text{for all } p \in (2, \infty),$$
$$\|P_h v - P_h \tilde{v}\| \le \frac{\beta}{\alpha}(\|v\| + \|\tilde{v}\|)^{2-p}\|v - \tilde{v}\|^{p-1} \qquad \text{for all } p \in (1, 2).$$

Next we establish a uniform convergence lemma.

LEMMA 4.3 (uniform convergence). *Suppose* $v \in C[R_+ \bigcup \{\infty\}, V]$. *Then*

$$\lim_{h\to 0^+} \|v(t) - P_h v(t)\| = 0,$$

where the convergence is uniform for $t \in R_+ \bigcup\{\infty\}$.

Proof. The convergence follows from Lemma 4.1. We need only to show it is uniformly true. We prove it by a contradiction. Suppose there exist $\varepsilon_0 > 0$, $\{h_n\}_{n=1}^{\infty} \subset R_+$ and $\{t_n\}_{n=1}^{\infty} \subset R_+$ such that $h_n \to 0$ as $n \to \infty$, and

$$\|v(t_n) - P_{h_n} v(t_n)\| \geq \varepsilon_0.$$

Since $\{t_n\}_{n=1}^{\infty} \subset R_+$, there is a $t^* \in R_+$ or $t^* = \infty$ such that $\{t_n\}_{n=1}^{\infty}$ has a subsequence, without loss of generality still denoted as $\{t_n\}_{n=1}^{\infty}$, such that $t_n \to t^*$, as $n \to \infty$. Since

$$\|v(t_n) - P_{h_n} v(t_n)\| \leq \|v(t_n) - v(t^*)\| + \|v(t^*) - P_{h_n} v(t^*)\| + \|P_{h_n} v(t^*) - P_{h_n} v(t_n)\|$$

and

$$\lim_{t \to t^*} \|v(t) - v(t^*)\| = 0,$$

by Lemma 4.1, Lemma 4.2,

$$\lim_{n \to \infty} \|v(t_n) - P_{h_n} v(t_n)\| = 0,$$

which is a contradiction. □

This lemma is quite general, as it requires only the projection operator P_h to have the convergence property and the uniform boundedness in h, properties often satisfied by many similar projection operators.

Using an argument of [34] (see also [5],), with the boundedness of the projection operators P_h and Π_h uniform in h, i.e., the fact that for every $v \in V$, there exists a $c \geq 0$, a constant independent of v, such that $\|P_h v\|, \|\Pi_h v\| \leq c\|v\|$, it is easy to prove the following approximation property of P_h.

LEMMA 4.4 (approximation). *For every* $v \in V$, *there exists a* $C = C(\|v\|) \geq 0$ *such that*

$$\|v - P_h v\| \leq C\|v - \Pi_h v\|^s,$$

where $s = \frac{2}{p}$ *and if* $p \in (2, \infty)$; $s = \frac{p}{2}$, *if* $p \in (1, 2)$.

Consider the following semidiscrete FEM scheme.

Find $u_h \in S_h(\Omega)$, such that for all $v_h \in S_h(\Omega)$,

$$\left\langle \frac{du_h}{dt}, v_h \right\rangle + (Au_h, v_h) = \langle f, v_h \rangle, \tag{12}$$

$$(Au_h(0), v_h) = (Au(0), v_h). \tag{13}$$

LEMMA 4.5 (existence and uniqueness). *If* $u_0 \in V$ *and* $f \in H$, *then there exists a unique* $u_h \in L^{\infty}(R_+, S_h(\Omega) \bigcap V)$ *solving* (12)–(13).

Proof. Local existence of u_h follows from Peano's theorem and Lemma 4.7. The uniqueness of $u_h(t)$ follows from the monotonicity of the operator A. The global existence of $u_h(t)$ follows from the boundedness of $u_h(t)$ uniform in time, which is shown as follows.

Setting $v_h = u_h(t)$ in (12), we have

$$\frac{1}{2} \frac{d}{dt} \|u_h(t)\|_2^2 + \alpha \|u_h(t)\|^p \leq \|f\|_2 \|u_h(t)\|_2. \tag{14}$$

By Lemma 2.1, it is easy to show that there exists a $C_1 = C_1(\alpha, p) > 0$ such that for $t \in [0, T)$,

$$\frac{d}{dt}\|u_h(t)\|_2 + C_1\|u_h\|_2^{p-1} \leq \|f\|_2.$$

By Lemma 2.4, $\|u_h(t)\|_2$ is uniformly bounded for $t \in [0, T)$ and the bound is independent of T. By (14), using Hölder's inequality, there exists $C_i = C_i(\alpha, p) > 0, i = 2, 3,$ such that

$$\frac{d}{dt}\|u_h(t)\|_2^2 + C_2\|u_h(t)\|^p \leq C_3\|f\|_2^q,$$

where $1/p + 1/q = 1$. By integration, for $t \in [0, T - r)$, and $r > 0$,

$$\|u_h(t+r)\|_2^2 + C_2\int_{s=t}^{t+r}\|u_h(s)\|^p ds \leq \|u_h(t)\|_2^2 + C_3 r\|f\|_2^q.$$

Setting $v_h = u_h'(t) := \frac{d}{dt}u_h(t)$ in (12),

$$\|u_h'(t)\|_2^2 + \frac{1}{p}\frac{d}{dt}\|u_h(t)\|^p = \langle f, u_h'(t)\rangle \leq \frac{1}{2}\|f\|_2^2 + \frac{1}{2}\|u_h'(t)\|_2^2.$$

Thus

$$\frac{d}{dt}\|u_h(t)\|^p \leq \frac{p}{2}\|f\|_2^2.$$

By integration, for $t \in [0, T - r)$, and $r > 0$,

$$\frac{p}{2}\int_{s=t}^{t+r}\|f\|_2^2 ds \leq \frac{pr}{2}\|f\|_2^2.$$

By the above estimates and the uniform Gronwall lemma, $\|u_h(t)\|$ is also uniformly bounded for all $t \in [0, T)$, with the bound independent of T. □

The discussion in section 3 can be easily adapted to show the following results for (12)–(13). Denote $\{u_{h,n}\}_0^\infty$ as the backward Euler solution sequence for (12)–(13). In particular, when u_n, $\overline{u}_n$ are replaced with $u_{h,n}$, $\overline{u}_{h,n}$, Lemma 3.4 still holds. Besides, we have the following result.

LEMMA 4.6. *Suppose that* f *and* $u_0 \in V \bigcap D(A)$. *Then* $\|u_h'(t)\|_2$ *and* $\{\|Au_h(t)\|_{2,h}^*\}$ *are uniformly bounded for* $t \in R_+$, *where*

$$\|Au_h(t)\|_{2,h}^* := \sup_{v_h \in S_h(\Omega)} \frac{|(Au_h(t), v_h)|}{\|v_h\|_2}.$$

Proof. Notice that Lemma 3.2 is still valid if we replace u_i and u_{i-1} with $u_{h,i}$ and $u_{h,i-1}$ respectively. In order to prove this, we need only to check the initial data. This is obtained easily from the following equality:

$$\begin{aligned}&\frac{1}{\Delta t}\langle u_{h,1} - u_{h,0}, u_{h,1} - u_{h,0}\rangle + (Au_{h,1} - Au_{h,0}, u_{h,1} - u_{h,0})\\ &= \langle f, u_{h,1} - u_{h,0}\rangle - (Au_0, u_{h,1} - u_{h,0}).\end{aligned}$$

Thus, by the classical monotone compact argument, $(u_{h,n} - u_{h,n-1})/\Delta t$ converges weakly to u_h' in $H \bigcap S_h(\Omega)$. Also, $\|Au_h(t)\|_{2,h}^* \leq \|f\|_2 + \|f - Au_0\|_2$. Thus the lemma is proved. □

Thanks to the Gronwall lemmas in section 2.3, the stability estimates in terms of both global boundedness and perturbation of the solutions to (5)–(6) can be obtained easily. In the following theorem, we state only the results on the perturbation for (5)–(6), since it will be used later. Using Lemma 2.2 and Remark 2.1, the proof of this theorem can be obtained by Lemma 2.6 for $p \in (2, \infty)$ and by integration for $p \in (1, 2)$. Similar results are also available for all the discrete problems discussed in this article.

THEOREM 4.1 (stability). *Let $u(t)$ and $v(t)$ be the solutions of* (5) *with initial data $u_0, v_0 \in H$. Then, there exists a $C_1 = C_1(\alpha, p, f) > 0$, such that for $t \in R_+$,*

$$\|u(t) - v(t)\|_2 \leq \|u(0) - v(0)\|_2 \{1 + \|u(0) - v(0)\|_2^{p-2} C_1 t\}^{1/(2-p)} \quad \text{if } p \in (2, \infty),$$
$$\|u(t) - v(t)\|_2 \leq \|u(0) - v(0)\|_2 e^{-C_1 t} \quad \text{if } p \in (1, 2).$$

Further, if $u_0, v_0 \in V \bigcap D(A)$, then there exists a $C_2 = C_2(\alpha, p, f, u_0, v_0) > 0$ such that for $t \in R_+$,

$$\|u(t) - v(t)\| \leq C_2 \|u(t) - v(t)\|_2^{1/p} \quad \text{if } p \in (2, \infty),$$
$$\|u(t) - v(t)\| \leq C_2 \|u(t) - v(t)\|_2^{1/2} \quad \text{if } p \in (1, 2).$$

From the above theorem, we see that for $p \in (1, 2)$, the contraction is at least exponential. While for $p \in (2, \infty)$ and $f = 0$, it is shown to be algebraic only. See [10].

Define $\bar{u} \in V$ to be the solution of the following problem: for all $v \in V$,

$$(A\bar{u}, v) = \langle f, v \rangle.$$

By Lemma 2.3, $\bar{u}$ exists and is unique.

Define $e(t) = u(t) - \bar{u} \in V$ for $t \in R_+$. Then, for $v \in V$,

$$\left\langle \frac{de(t)}{dt}, v \right\rangle + (Au - A\bar{u}, v) = 0.$$

By Theorem 4.1, we have immediately the following theorem.

THEOREM 4.2 (stability). *There exists a $C_1 = C_1(\alpha, p, f) > 0$ such that for $t \in R_+$,*

$$\|e(t)\|_2 \leq \|e(0)\|_2 \{1 + \|e(0)\|_2^{p-2} C_1 t\}^{1/(2-p)} \quad \text{if } p \in (2, \infty),$$
$$\|e(t)\|_2 \leq \|e(0)\|_2 e^{-C_1 t} \quad \text{if } p \in (1, 2).$$

Further, if $u_0, v_0 \in V \bigcap D(A)$, then there exists a $C_2 = C_2(\alpha, p, f, u_0, v_0) > 0$ such that for $t \in R_+$,

$$\|e(t)\| \leq C_2 \|e(t)\|_2^{1/p} \quad \text{if } p \in (2, \infty),$$
$$\|e(t)\| \leq C_2 \|e(t)\|_2^{1/2} \quad \text{if } p \in (1, 2).$$

By Theorem 4.2, we have the following result from Lemma 4.3 immediately.

COROLLARY 4.1. *Suppose $f \in H$, $u_0 \in V \bigcap D(A)$ and u is the solution of* (5)–(6). *Then*

$$\lim_{h \to 0^+} \|u(t) - P_h u(t)\| = 0,$$

where the convergence is uniform for $t \in R_+$.

Suppose u is the solution of (5)–(6), u_h is the solution of (12)–(13). Then it is easy to show the following equality: i.e., for all $v_h \in S_h(\Omega)$ and for all $t > 0$,

$$\text{(15)}\quad \frac{1}{2}\frac{d}{dt}\|u-u_h\|_2^2 + (Au - Au_h, u-u_h) = \langle u' - u_h', u - v_h\rangle + (Au - Au_h, u - v_h).$$

This equality is always used in our later convergence analysis.

Now we state and prove our main uniform convergence theorem.

THEOREM 4.3 (uniform convergence). *Suppose that $f \in H$ and $u_0 \in V \bigcap D(A)$. Then $u_h \in C[R_+; S_h(\Omega)]$ and it converges to $u \in C[R_+; V]$ uniformly for $t \in R_+ \bigcup\{\infty\}$, as $h \to 0$.*

Proof. We prove the case $p \in (2,\infty)$. The case $p \in (1,2)$ can be treated similarly.

It is easy to see that, by letting $v_h = P_h u$ in (15), there exists a $C = C(\beta, p, C_0, u_0, f) \geq 0$ such that

$$\text{(16)}\qquad \frac{1}{2}\frac{d}{dt}\|u-u_h\|_2^2 + \alpha C_0^p\|u-u_h\|_2^p \leq C\|u - P_h u\|.$$

By Corollary 4.1, for every $\varepsilon > 0$, there exists a $h_\varepsilon > 0$ such that for $h \in (0, h_\varepsilon)$,

$$\frac{1}{2}\frac{d}{dt}\|u-u_h\|_2^2 + \alpha C_0^p\|u-u_h\|_2^p \leq C\varepsilon.$$

By Lemma 2.4,

$$\|u(t)-u_h(t)\|_2 \leq \max\left\{\|u_0 - P_h u_0\|_2, \left(\frac{C\varepsilon}{\alpha C_0^p}\right)^{1/p}\right\} \leq \max\left\{C_0^{-1}\varepsilon, \left(\frac{C\varepsilon}{\alpha C_0^p}\right)^{1/p}\right\}.$$

Thus, for $t \in R_+$,

$$\text{(17)}\qquad \lim_{h\to 0}\|u(t)-u_h(t)\|_2 = 0,$$

where the convergence is uniform for $t \in R_+$. Moreover, for $t \in R_+$,

$$\text{(18)}\quad \lim_{h\to 0}\|P_h u(t) - u_h(t)\|_2 \leq \lim_{h\to 0}\|P_h u(t) - u(t)\|_2 + \lim_{h\to 0}\|u(t)-u_h(t)\|_2 = 0,$$

where the convergence is uniform for $t \in R_+$.

We now prove the uniform convergence in V. Notice that for $p \in (2,\infty)$ and $t \in R_+$,

$$\begin{aligned}\alpha\|u(t)-u_h(t)\|^p &\leq (Au - Au_h, u-u_h)\\ &= (Au, u-u_h) - (Au_h, u - P_h u) - (Au_h, P_h u - u_h)\\ &\leq \|Au\|_2\|u-u_h\|_2 + \|u_h\|^{p-1}\|u - P_h u\| + \|Au_h\|_{2,h}^*\|P_h u - u_h\|_2.\end{aligned}$$

Thus, by Remark 2.1, (17), Lemma 4.5, Corollary 4.1, Lemma 4.6, and (18), the above inequality yields that for $t \in R_+$,

$$\lim_{h\to 0}\|u(t)-u_h(t)\| = 0,$$

and the convergence is uniform for $t \in R_+$. ◻

In the above theorem, we assumed only $u_0 \in V \bigcap D(A)$. Here we are not concerned much with "optimum" convergence order, but rather the convergence uniform in time, with no extra regularity requirements.

We give two theorems below that allow us to get higher convergence rates. For simplicity, we use c's to denote some generic positive constants that are independent of h, t and might be different in different occasions. Further dependence of the c's will be specified only if it is important for the discussion.

THEOREM 4.4 (error estimates in H). *Suppose that $f \in H$, $u_0 \in V \bigcap D(A)$ and $r = \max\{2, p\}$. Then there exists a constant $c > 0$ independent of h and t such that for $t \geq 0$ and for any $v_h \in S_h(\Omega)$,*

$$\|u(t) - u_h(t)\|_2^2 \leq c \max\{\|u_0 - u_h(0)\|_2^2, \|u - v_h\|_{L^\infty(R_+,V)}^{2/r}\},$$

$$\|u(t) - u_h(t)\|_2^2 + \|u(t) - u_h(t)\|_{L^r(R_+,V)}^r \leq c\{\|u_0 - u_h(0)\|_2^2 + \|u - v_h\|_{L^1(R_+,V)}\}.$$

In particular, we can choose $v_h = P_h u$ and $u_h(0) = P_h u_0$.

Proof. It is easy to see from (15), using Lemma 2.2, that there exists a positive constant, $C = C(\beta, p, C_0, u_0, f)$, independent of t and h, such that for all $v_h \in S_h(\Omega)$,

$$\frac{1}{2}\frac{d}{dt}\|u - u_h\|_2^2 + \alpha C_\alpha \|u - u_h\|^r \leq C\|u - v_h\|. \tag{19}$$

The first estimate follows from (19), using Lemma 2.4, and the second from a direct integration of (19) and the use of Lemma 2.2. □

We improve the estimates in the above theorem by making use of the following two technical lemmas from [2].

LEMMA 4.7. *For $p \in (1, \infty)$ and $\delta \geq 0$, there exist positive constants c_1, c_2, which depend only on p and d, such that for all ξ, $\eta \in R^d$ and $d \geq 1$,*

$$||\xi|^{p-2}\xi - |\eta|^{p-2}\eta| \leq c_1|\xi - \eta|^{1-\delta}(|\xi| + |\eta|)^{p-2+\delta},$$

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta) \geq c_2|\xi - \eta|^{2+\delta}(|\xi| + |\eta|)^{p-2-\delta}.$$

LEMMA 4.8. *For $p \in (1, \infty)$, there exists a $\epsilon_0 > 0$ such that for all a, σ_1 and $\sigma_2 \geq 0$ and for all $\epsilon \in (0, \epsilon_0)$,*

$$(a + \sigma_1)^{p-2}\sigma_1\sigma_2 \leq \epsilon(a + \sigma_1)^{p-2}\sigma_1^2 + C(\epsilon^{-1})(a + \sigma_2)^{p-2}\sigma_2^2.$$

The following lemma is motivated by the proof of Theorem 3.2 of [2].

LEMMA 4.9. *Suppose that $f \in H$, $u_0 \in V \bigcap D(A)$, u and u_h are the solutions to (5)–(6) and (12)–(13), respectively. Then, for all $v_h \in S_h(\Omega)$, there exist constants $c_3, c_4 > 0$, such that, for all $t \in R_+$,*

$$\frac{1}{2}\frac{d}{dt}\|u - u_h\|_2^2 + c_3 \int_\Omega (|\nabla u| + |\nabla u_h|)^{p-2}|\nabla(u - u_h)|^2 dx$$
$$\leq \langle u' - u_h', u - v_h\rangle + c_4 \int_\Omega (|\nabla u| + |\nabla v_h|)^{p-2}|\nabla(u - v_h)|^2 dx.$$

Proof. By Lemma 4.7 with $\delta = 0$ and using (15),

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u - u_h\|_2^2 + c_2 \int_\Omega (|\nabla u| + |\nabla u_h|)^{p-2}|\nabla(u - u_h)|^2 dx \\
&\leq \frac{1}{2}\frac{d}{dt}\|u - u_h\|_2^2 + (Au - Au_h, u - u_h) \\
&= \langle u' - u_h', u - v_h\rangle + (Au - Au_h, u - v_h) \\
&\leq \langle u' - u_h', u - v_h\rangle + c_1 \int_\Omega (|\nabla u| + |\nabla u_h|)^{p-2}|\nabla(u - u_h)||\nabla(u - v_h)| dx.
\end{aligned}$$

Notice that for all $\xi,\, \eta \in R^d$,

$$(|\xi| + |\eta|)/2 \le |\xi| + |\xi - \eta| \le 2(|\xi| + |\eta|), \tag{20}$$

and by Lemma 4.8,

$$\begin{aligned}
&\int_\Omega (|\nabla u| + |\nabla(u-u_h)|)^{p-2} |\nabla(u-u_h)||\nabla(u-v_h)|dx \\
&\le \epsilon \int_\Omega (|\nabla u| + |\nabla(u-u_h)|)^{p-2} |\nabla(u-u_h)|^2 dx \\
&\quad + C_\epsilon \int_\Omega (|\nabla u| + |\nabla(u-v_h)|)^{p-2} |\nabla(u-v_h)|^2 dx.
\end{aligned}$$

Thus the lemma is proved by considering the above three inequalities. ∎

Now we state and prove our theorem.

THEOREM 4.5 (error estimates in H). *Suppose that $f \in H$, $u_0 \in V \bigcap D(A)$, $r = \max\{2, p\}$, and $s = \min\{2, p\}$. Then for all $v_h \in S_h(\Omega)$, there exists a constant $c > 0$ independent of h and t, depending on $\|v_h\|$, such that for $t \ge 0$ and for any $v_h \in S_h(\Omega)$,*

$$\begin{aligned}
&\|u(t) - u_h(t)\|_2^2 \le c \max\{\|u_0 - u_h(0)\|_2^2, \|u - v_h\|_{L^\infty(R_+,H)}^{2/r} + \|u - v_h\|_{L^\infty(R_+,V)}^{2s/r}\}, \\
&\|u(t) - u_h(t)\|_2^2 + \|u(t) - u_h(t)\|_{L^r(R_+,V)}^r \\
&\qquad \le c\{\|u_0 - u_h(0)\|_2^2 + \|u - v_h\|_{L^1(R_+,H)} + \|u - v_h\|_{L^s(R_+,V)}^s\}.
\end{aligned}$$

In particular, we can choose $v_h = P_h u$ and $u_h(0) = P_h u_0$.

Proof. We show that there exist c_5, $c_6 > 0$ independent of h, t such that, for all $t > 0$,

$$\frac{1}{2}\frac{d}{dt}\|u - u_h\|_2^2 + c_5 \|u - u_h\|^r \le \langle u' - u_h', u - v_h \rangle + c_6 \|u - v_h\|^s, \tag{21}$$

from which the theorem follows using Lemmas 2.4 and 2.2 for the first estimate and integrating (21) and Lemma 2.2 for the second.

If $p \in (2, \infty)$, it is easy to see, by Lemmas 4.7 and 4.9, that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u - u_h\|_2^2 + c\|u - u_h\|^p \\
&\le \frac{1}{2}\frac{d}{dt}\|u - u_h\|_2^2 + c_3 \int_\Omega (|\nabla u| + |\nabla u_h|)^{p-2} |\nabla(u-u_h)|^2 dx \\
&\le \langle u' - u_h', u - v_h \rangle + c_4 \int_\Omega (|\nabla u| + |\nabla v_h|)^{p-2} |\nabla(u-v_h)|^2 dx \\
&\le \langle u' - u_h', u - v_h \rangle + c\|u - v_h\|^2,
\end{aligned}$$

where the Hölder's inequality is used for the last inequality.

If $p \in (1, 2)$, letting $q = p(2-p)/2$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u - u_h\|_2^2 + c\|u - u_h\|^2 \\
&= \frac{1}{2}\frac{d}{dt}\|u - u_h\|_2^2 + c\left\{\int_\Omega (|\nabla u| + |\nabla u_h|)^q (|\nabla u| + |\nabla u_h|)^{-q} |\nabla(u-u_h)|^p dx\right\}^{2/p}
\end{aligned}$$

$$\leq \frac{1}{2}\frac{d}{dt}\|u-u_h\|_2^2 + c_3\int_\Omega (|\nabla u| + |\nabla u_h|)^{p-2}|\nabla(u-u_h)|^2 dx$$
$$\leq \langle u' - u_h', u - v_h\rangle + c_4\int_\Omega (|\nabla u| + |\nabla v_h|)^{p-2}|\nabla(u-v_h)|^2 dx$$
$$\leq \langle u' - u_h', u - v_h\rangle + c\int_\Omega (|\nabla u| + |\nabla(u-v_h)|)^{p-2}|\nabla(u-v_h)|^2 dx$$
$$\leq \langle u' - u_h', u - v_h\rangle + c\|u-v_h\|^p,$$

where the Hölder's inequality is used for the first inequality, Lemma 4.9 for the second, and (20) for the third. ◻

Remark 4.1. 1. The above theorem can be considered as a global version of some local results obtained in [2]. The choice of v_h here is just for estimating errors and not for computing. But $\|v_h\|$ must be uniformly bounded with respect to time if v_h is chosen as time-dependent. However, this requirement is not explicitly needed for Theorem 4.4. Considering Lemma 4.4, $\Pi_h u$ is better than $P_h u$ in getting higher order error estimates. Even if $u \notin C(\bar{\Omega})$, a *nonlocal* interpolation operator can still be used. Details are omitted here. Comparing (19) and (21), we see that (21) can give better estimates on the rate of convergence.

2. The choice of $u_h(0)$ has more freedom, though this is indeed to be computed. A first glance of the estimates in Theorems 4.4 and 4.5 seems to suggest that if there is a large error in approximating the initial condition, then this large error will be kept forever. This is so for local in time convergence. But things are better in our case. In fact, we only applied Lemma 2.4 to get the estimates of Theorems 4.4 and 4.5. For $r = \max\{2, p\}$, if we use Lemma 2.6 for the case $r > 2$ and the direct integration for the case $r = 2$, then it is easy to see that the error in the initial condition will be damped away either algebraically or exponentially as $t \to \infty$.

In fact, under the conditions of Lemma 2.6, it is easy to see that even if $\delta \geq 0$ is time-dependent,

$$y(t) \leq \left(\frac{\delta}{\gamma}\right)^{1/q} + [\gamma(q-1)t]^{1/(1-q)}.$$

Thus, for $p \in (2,\infty)$, we have from (19) that there is a constant $c > 0$ such that

$$\|u(t) - u_h(t)\|_2^2 \leq c\|u(t) - v_h\|^{2/r} + ct^{2/(2-r)},$$

and from (21) that there is a constant $c > 0$ such that

$$\|u(t) - u_h(t)\|_2^2 \leq c(\|u(t) - v_h\|_2^{2/r} + \|u(t) - v_h\|^{2s/r}) + ct^{2/(2-r)}.$$

So, for $p \in (2,\infty)$, when t is very large, the initial error can in fact be ignored completely.

The case $p \in (1,2)$ can be treated by a direct integration and it is easy to see that the decay of the influence of the initial error is exponential as $t \to \infty$. These results are not available from the analysis of [2, 35], as dynamics was not considered there.

3. For $p \in (2,\infty)$, the condition $u \in L^2(0,T;W^{2,2}(\Omega)) \bigcap L^\infty(0,T;W^{1,\infty}(\Omega))$ was imposed in [2] to get their desired error bound. (The same error bound holds for $u \in L^p(0,T;W^{2,p}(\Omega))$; but this is possibly harder to establish for $p > 2$.) As shown above, even for global case some estimates can still be obtained without this condition. Of course, dropping this condition might slightly lower the order of the error estimates. ◻

Finally we give an error estimate in V.

THEOREM 4.6 (error estimate in V). *Suppose that $f \in H$, $u_0 \in V \bigcap D(A)$ and $r = \max\{2, p\}$. Then there exists a constant $c > 0$ independent of h and t such that for $t \geq 0$ and for any $v_h \in S_h(\Omega)$,*

$$\|u(t) - u_h(t)\|^r \leq c(\|u(t) - u_h(t)\|_2 + \|u(t) - v_h\|).$$

In particular, we can choose $v_h = \Pi_h u$.

Proof. For $p \in (2, \infty)$,

$$\begin{aligned}\alpha\|u - u_h\|^p &\leq (Au - Au_h, u - u_h)\\ &= (Au, u - u_h) - (Au_h, u - v_h) - (Au_h, v_h - u_h)\\ &\leq \|Au\|_2\|u - u_h\|_2 + \|u_h\|^{p-1}\|u - v_h\| + \|Au_h\|^*_{2,h}\|v_h - u_h\|_2\\ &\leq \|Au\|_2\|u - u_h\|_2 + \|u_h\|^{p-1}\|u - v_h\| + \|Au_h\|^*_{2,h}(\|u - v_h\|_2 + \|u - u_h\|_2).\end{aligned}$$

The case $p \in (1, 2)$ can be treated similarly. □

5. The full discretizations with unconditionally stable implicit schemes. By Theorems 3.1 and 3.2, noticing Lemmas 4.5 and 4.6, it is easy to see that the following convergence results hold.

THEOREM 5.1. *Suppose $\{u_{h,n}\}$ is the sequence generated by the backward Euler scheme for the semidiscrete problem defined in the last section. Then, it is a Cauchy sequence in $C[R_+; V]$, and it converges to $u_h \in C[R_+; V]$ uniformly with respect to n, as $\Delta t \to 0$. Moreover, there exists $c(t) \geq 0$, a constant independent of $\Delta t, h$ and bounded uniformly with respect to t, such that for $p \in (1, \infty)$ and $t \in R_+$,*

$$\|u_{h,n}(t) - u_h(t)\|_2 + \|u_{h,n}(t) - u_h(t)\|^r \leq c(t)(\Delta t_n)^{1/r},$$

where $r = max\{2, p\}$.

However, still much more effort, both computational and analytical, needs to be made on how to solve the nonlinear algebraic system derived from the full discretization of (12)–(13) efficiently.

Similar treatment discussed above applies to the Crank–Nicholson scheme.

THEOREM 5.2. *Suppose that $\{u_{h,n}\}$ is the sequence generated by the Crank–Nicholson scheme for the semidiscrete problem defined in the last subsection. Then it is a Cauchy sequence in $C[R_+; V]$, and it converges to $u_h \in C[R_+; V]$ uniformly with respect to n, as $\Delta t \to 0$. Moreover, there exists $C(u_0, f) \geq 0$, a constant independent of $\Delta t, h$, such that the following estimates hold:*

1) *If $p \in (2, \infty)$, then*

$$\|u_{h,n}(t) - u_h(t)\|_2 \leq C(u_0, f)(\Delta t_n)^{p'/p^2},$$
$$\|u_{h,n}(t) - u_h(t)\| \leq C(u_0, f)(\Delta t_n)^{p'/p^3}.$$

2) *If $p \in (1, 2)$, then*

$$\|u_{h,n}(t) - u_h(t)\|_2 \leq C(u_0, f)(\Delta t_n)^{(p-1)/2},$$
$$\|u_{h,n}(t) - u_h(t)\| \leq C(u_0, f)(\Delta t_n)^{(p-1)/4}.$$

Remark 5.1. Combining the above results and those of section 4, we get all the convergence and stability results for the full discretization. Notice that the $c(t)$ in Theorem 5.1 and C in Theorem 5.2 are independent of h. This means that for the full discretization, our convergence results are independent of the order of the limit procedure as $t, h \to 0$. □

6. Appendix. In this section, we give the proofs of some of the Gronwall inequalities stated in section 2.3.

1. *Proof of Lemma* 2.4.

(i) If $y^q(0) \leq \delta/\gamma$, then $y^q(t) \leq \delta/\gamma$ for $t \in (0, \infty)$. Otherwise, there exist t_0, t_1 such that $0 \leq t_0 < t_1$, $y^q(t_0) = \delta/\gamma$ and $y^q(t) > \delta/\gamma$ for all $t \in (t_0, t_1]$. Thus there exists a $t^* \in [t_0, t_1]$ such that

$$y'(t^*) = \frac{y(t_1) - y(t_0)}{t_1 - t_0} > 0.$$

However, this is a contradiction since $y'(t) \leq \delta - \gamma y^q(t) \leq 0$ for $t \in [t_0, t_1]$.

(ii) If $y^q(0) \geq \delta/\gamma$, then $y(t) \leq y(0)$ for $t > 0$. This is obvious when $y'(t) \leq 0$ for $t > 0$. When it is not this case, since $y'(0) \leq \delta - \gamma y^q(0) < 0$, then there exists a $t_0 > 0$ such that $y'(t_0) = 0$ and $y'(t) < 0$ for $t \in (0, t_0)$. Thus $y(t) < y(0)$ for $t \in (0, t_0)$. Notice that

$$y^q(t_0) \leq \frac{\delta - y'(t_0)}{\gamma} = \frac{\delta}{\gamma} < y^q(0).$$

Thus, by (i), $y^q(t) \leq \delta/\gamma < y^q(0)$ for $t \in (t_0, \infty)$. □

2. *Proof of Lemma* 2.7. First, we prove the following claim:

Suppose that $q > 1, \gamma > 0, \Delta t > 0$ *and* $\{z_n\}_0^\infty$ *is a nonnegative sequence such that for* $n = 1, 2, \ldots,$

$$\frac{z_n - z_{n-1}}{\Delta t} + \gamma z_n^q = 0.$$

If $\Delta t \leq z_0^{q-1}/(q\gamma)$, *then for* $n = 1, 2, \ldots,$

$$z_0[1 + \gamma z_0^{q-1}(q-1)n\Delta t]^{\frac{-1}{q-1}} \leq z_n \leq z_0[1 + \gamma z_0^{q-1}(q-1)n\Delta t(1 - q\gamma z_0^{q-1}\Delta t)]^{\frac{-1}{q-1}}.$$

Proof of claim. Suppose $z_0 > 0$. Let $\gamma = \alpha^{q-1}$, where $\alpha > 0$ and for $n = 0, 1, 2, \ldots,$ $w_n = \alpha z_n$. Then

$$\frac{w_n - w_{n-1}}{\Delta t} + w_n^q = 0.$$

Since $w_0 > 0$, it is easy to see by induction that for $n = 1, 2, \ldots$, $w_n > 0$. Thus, for $n = 1, 2, \ldots$, $w_n < w_{n-1}$. A similar conclusion thus follows for $\{z_n\}_0^\infty$.

We now show that if $\Delta t \leq w_0^{1-q}/q$, then for all $n = 0, 1, 2, \ldots,$

$$w_0[1 + w_0^{q-1}(q-1)n\Delta t]^{1/(1-q)} \leq w_n \leq w_0[1 + w_0^{q-1}(q-1)n\Delta t(1 - pw_0^{q-1}\Delta t)]^{1/(1-q)},$$

from which the claim follows.

By the mean value theorem, there exists a $\theta_n \in [w_n, w_{n-1}]$ such that

$$\begin{aligned} w_n^{1-q} - w_{n-1}^{1-q} &= (1-q)\theta_n^{-q}(w_n - w_{n-1}) = (q-1)\theta_n^{-q}(w_{n-1} - w_n) \\ &= (q-1)\theta_n^{-q} w_n^q \Delta t \leq (q-1)\Delta t. \end{aligned}$$

By induction,

$$w_n \geq w_0[1 + w_0^{q-1}(q-1)n\Delta t]^{1/(1-q)}.$$

Notice also that

$$\begin{aligned} w_n^{1-q} - w_{n-1}^{1-q} &= (q-1)\theta_n^{-q} w_n^p \Delta t \\ &= (q-1)\theta_n^{-q} w_{n-1}^q \Delta t + (q-1)\theta_n^{-q}(w_n^q - w_{n-1}^q)\Delta t \\ &\geq (q-1)\Delta t + (q-1)\theta_n^{-q}(w_n^q - w_{n-1}^q)\Delta t. \end{aligned}$$

By the mean value theorem, there exists a $\gamma_n \in [w_n, w_{n-1}]$ such that

$$w_n^q - w_{n-1}^q = q\gamma_n^{q-1}(w_n - w_{n-1}) = -q\gamma_n^{q-1} w_n^q \Delta t \geq -q w_0^{q-1} w_n^q \Delta t.$$

Thus

$$\begin{aligned} w_n^{1-q} - w_{n-1}^{1-q} &\geq (q-1)\Delta t - (q-1)\theta_n^{-q} p w_0^{q-1} w_n^q (\Delta t)^2 \\ &\geq (q-1)\Delta t - q(q-1) w_0^{q-1}(\Delta t)^2 \\ &= (q-1)\Delta t(1 - q w_0^{q-1}\Delta t). \end{aligned}$$

By induction, if $\Delta t \leq 1/(q w_0^{q-1})$, then

$$w_n \leq w_0[1 + w_0^{q-1}(q-1)n\Delta t(1 - q w_0^{q-1}\Delta t)]^{1/(1-q)}.$$

Thus the claim is proved.

Now we state the following fact, the checking of which is left to the readers.

Let f, g be increasing functions of $x \in R^1$ and f be strictly increasing. Let $\{u_n\}_0^\infty, \{v_n\}_0^\infty$ be two sequences such that $u_0 \leq v_0$ and for $n = 1, 2, \ldots,$

$$f(u_n) \leq g(u_{n-1}), f(v_n) \geq g(v_{n-1}).$$

Then, for $n = 1, 2, ...,$

$$u_n \leq v_n.$$

Using the claim and the fact above, together with Lemma 2.5, we can prove the lemma easily with simple calculations. □

3. *Proof of Lemma* 2.8. (i) If $y^q(0) \leq \delta/\gamma$, then by Lemma 2.4, for $t \in [0, \infty)$, $y(t) \leq (\delta/\gamma)^{1/q}$.

(ii) If $y^q(0) > \delta/\gamma$, then there is a $t_0 \in (0, \infty]$, which is the largest real number or $+\infty$ such that for $t \in [0, t_0)$, $y(t) > (\delta/\gamma)^{1/q}$. Then, for $t \in [0, t_0)$, $y'(t) \leq \delta - \gamma y^q(t) < 0$. Thus, for $t \in [0, t_0)$, $(\delta/\gamma)^{1/q} < y(t) \leq y(0)$. Noticing $q \in (0, 1)$, we have for $t \in [0, t_0)$

$$y'(t) \leq \delta - \gamma y^q(t) \leq \delta - \gamma y^{q-1}(0) y(t).$$

Thus, by integration, for $t \in [0, t_0)$,

$$y(t) \leq y^{1-q}(0)\frac{\delta}{\gamma} + (y(0) - y^{1-q}(0)\frac{\delta}{\gamma}) exp\left(-\frac{\gamma t}{y^{1-q}(0)}\right).$$

Notice that $y^{1-q}(0)(\delta/\gamma) \geq (\delta/\gamma)^{1/q}$. Combining (i), the lemma is proved. □

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