Deformation of extremal black holes from stringy interactions

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Black holes are a powerful setting for studying general relativity and theories beyond GR. However, analytical solutions for rotating black holes in beyond-GR theories are difficult to find because of the complexity of such theories. In this paper, we solve for the deformation to the near-horizon extremal Kerr metric due to two example string-inspired beyond-GR theories: Einstein-dilaton-Gauss-Bonnet and dynamical Chern-Simons theory. We accomplish this by making use of the enhanced symmetry group of NHEK and the weak-coupling limit of EdGB and dCS. We find that the EdGB metric deformation has a curvature singularity, while the dCS metric is regular. From these solutions, we compute orbital frequencies, horizon areas, and entropies. This sets the stage for analytically understanding the microscopic origin of black hole entropy in beyond-GR theories.

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I. INTRODUCTION

General relativity (GR), despite its huge success in describing gravity on large scales [1], must be corrected at high energies to reconcile with quantum mechanics. Black holes (BHs) may hold a key to developing a quantum theory of gravity: quantum effects can become important when gravity is strong, such as close to singularities. Quantum effects can also become important at the horizon over sufficiently long times, e.g. as Hawking radiation [2] shrinks a BH, generating arbitrarily large curvatures at the horizon, close to evaporation.

In order to go beyond GR, a huge class of alternative theories of gravity have been proposed and studied. Analytical black hole solutions can be sensitive to corrections to GR, but they are rare in beyond-GR theories. In the slow-rotation limit, BH solutions [3,4] have been found for dynamical Chern-Simons theory [5]. But for many other theories or when it comes to generic spin, it is difficult to find analytic rotating solutions. In this paper, we find BH solutions in the near-horizon extremal limit for beyond-GR theories. In particular, we make use of two theories of gravity as examples, taking the weak-coupling limit, and find the corresponding deformations to near-horizon extremal Kerr (NHEK). The two theories, both inspired by string theory, are Einstein-dilaton-Gauss-Bonnet (EdGB) [6,7] and dynamical Chern-Simons theory (dCS) [5], respectively. They both contain a dynamical scalar field that couples to curvature, correcting GR with a (different) quadratic curvature term.

After taking the weak coupling limit of a beyond-GR theory, finding the vacuum rotating solutions can be naturally formulated as finding the metric deformations to solutions in Einstein gravity, i.e. deformations to Kerr black holes (alternatively, one may expand Kerr around the $a = 0$ Schwarzschild limit, and solve for deformations around the expanded spacetime, as in [8–10]). Therefore, linear metric perturbation theory is a natural tool to address the problem. However, the perturbation equations are hard to solve unless we can use separation of variables. In the Kerr spacetime, metric perturbations do not separate, but curvature perturbations do. The most common approach is to use the Newman-Penrose formalism [11] and solve the wave equations for Weyl scalars Ψ_4 or Ψ_0. This method was developed by Teukolsky [12,13], and the partial differential equation to solve is known as the Teukolsky equation. The cost of curvature perturbations, however, is a very complicated metric-reconstruction procedure (see e.g. discussion in [14]), which only works for certain source terms, in certain gauges, and does not recover all pieces of the metric. The main difficulty in the separation of the metric perturbation equations is insufficient symmetry in the Kerr spacetime. In the near-horizon extremal scaling limit of Kerr, additional symmetries arise, and we can separate variables, as the authors showed in [15]. Therefore, in NHEK, analytical deformed solutions can be found by using linear metric perturbation theory.

The NHEK spacetime is interesting to study for several other reasons. For instance, it has been shown that the horizon instability of extremal black holes [16] can be viewed as a critical phenomenon [17]. Moreover, it was shown that near-horizon quantum states can be identified with a two-dimensional conformal field theory (CFT), via the proposed Kerr/CFT correspondence [18].

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In this paper, we focus on finding metric deformations of NHEK due to dCS and EdGB interactions in the decoupling limit. Let us emphasize, though, that this formalism is not limited to these two theories, but can be applied to finding deformed NHEK solutions in many beyond-GR theories in the decoupling limit. With the metric solutions, we compute physical properties including geodesic motion of particles and their orbital frequencies, horizon areas, and entropies. We also prove that the EdGB extremal BH is indeed singular in the decoupling limit, confirming the conjecture of [19]. One of the most important results is the calculation of the macroscopic extremal black hole entropies in beyond-GR theories. Although we only consider the near-horizon limit, the entropy results agree with extremal BH solutions (i.e. without zooming into the near-horizon region). In the NHEK spacetime, the entropy can be computed by counting the microscopic states of a two-dimensional chiral CFT [18] via the Cardy formula, which leads to the Kerr/CFT conjecture. We also expect a dual CFT description of the extremal black hole entropy for beyond-GR theories in the decoupling limit. We will not address this issue here, but our work lays the ground for studying the microscopic states of deformed extremal black holes. This may provide insight into quantum theories beyond Einstein gravity.

We organize the paper as follows. In Sec. II, we review EdGB and dCS gravity, and introduce the decoupling limit to the two theories. In Sec. III, we review the near-horizon extremal geometry, the symmetry-adapted bases, and set up the metric perturbations in near-horizon extremal Kerr spacetime as induced by the two stringy interactions. In Sec. IV, we solve for the dynamical scalar fields, construct the source term to the linearized Einstein field equation, and finally solve the metric perturbations in the "attractor" gauge. In Sec. V, we derive the timelike geodesic equations for the deformed spacetimes, and calculate the corrections to horizon areas and black hole entropies due to the two stringy interactions. We conclude and discuss future work in Sec. VI.

II. EINSTEIN-DILATON-GAUSS-BONNET AND DYNAMICAL CHERN-SIMONS GRAVITY

A. Action

We work in units where $c = 1 = \hbar$, and choose the metric signature $(-,+,+,+)$. The theories which we are considering, namely dynamical Chern-Simons gravity and Einstein-dilaton-Gauss-Bonnet, can be motivated from both low-energy effective field theory (EFT) and high-energy fundamental theory. DCS can arise from gravitational anomaly cancellation in chiral theories [20–22], including Green-Schwarz cancellation in string theory [23]. The low-energy compactified theory was explicitly presented in [24] (see references therein also). EdGB, meanwhile, can be derived by expanding the low energy string action to two loops to find the dilaton-curvature interaction [6,7].

The actions of dCS and EdGB both include the Einstein-Hilbert term and a scalar field that nonminimally couples to curvature. The Einstein-Hilbert action leads to standard GR. In dCS, the scalar field is an axion, while in EdGB it is a dilaton. In our discussions, there is no need to distinguish between the two scalar fields. We treat them equally as the scalar field ϑ. For both theories, we then take as our action

$$I = \int d^4x\sqrt{-g}[\mathcal{L}_{\rm EH} + \mathcal{L}_\vartheta + \mathcal{L}_{\rm int}], \tag{1}$$

with

$$\mathcal{L}_{\rm EH} = \frac{1}{2}m_{\rm pl}^2 R, \qquad \mathcal{L}_\vartheta = -\frac{1}{2}(\partial^a\vartheta)(\partial_a\vartheta), \tag{2}$$

and nonminimal scalar-curvature interaction terms for dCS and EdGB, respectively, [5–7]

$$\mathcal{L}_{\rm int}^{\rm CS} = -\frac{m_{\rm pl}}{8}\ell_{\rm CS}^2\vartheta\, {}^*RR, \qquad \mathcal{L}_{\rm int}^{\rm GB} = -\frac{m_{\rm pl}}{8}\ell_{\rm GB}^2\vartheta\, {}^*R^*R. \tag{3}$$

Here R is the Ricci scalar of the metric g_{ab}, and g is the metric determinant. The reduced Planck mass is defined through $m_{\rm pl} \equiv (8\pi G)^{-1/2}$. The scalar field ϑ has been canonically normalized such that $[\vartheta] = [M]$. In the interaction terms, we define two coupling constant $\ell_{\rm CS}$ and $\ell_{\rm GB}$ for dCS and EdGB, respectively. The two variables are dimensionful, specifically $[\ell_{\rm CS}] = [\ell_{\rm GB}] = [M]^{-1}$. That is, each of them gives the length scale of the corresponding theory, which in principle can be constrained observationally. In dCS, we encounter the Pontryagin-Chern density

$${}^*RR = {}^*R^{abcd}R_{abcd}, \tag{4}$$

while in EdGB we see minus the Euler (or Gauss-Bonnet) density

$${}^*R^*R = {}^*R^*_{abcd}R^{abcd} = -R^2 + 4R_{ab}R^{ab} - R_{abcd}R^{abcd}. \tag{5}$$

Here we have used the single- and double-dualized Riemann tensors,

$${}^*R_{abcd} \equiv \frac{1}{2}\epsilon_{ab}{}^{ef}R_{efcd}, \qquad {}^*R^*_{abcd} \equiv \frac{1}{2}{}^*R_{abef}\epsilon^{ef}{}_{cd}, \tag{6}$$

where we dualize with the completely antisymmetric Levi-Civita tensor ϵ^{abcd}.

B. Equation of motion

Variation of the action in Eq. (1) with respect to the scalar field ϑ leads to the scalar equation of motion for dCS and EdGB, respectively,

$$\square\vartheta = \frac{m_{\rm pl}}{8}\begin{cases}\ell_{\rm CS}^2\,{}^*RR, & \text{dCS}\\ \ell_{\rm GB}^2\,{}^*R^*R, & \text{EdGB}\end{cases} \tag{7}$$

where $\square = \nabla^a\nabla_a$ and ∇_a is the covariant derivative compatible with the metric. Variation of the action in Eq. (1) with respect to g^{ab} leads to the metric equation of motion,

$$m_{\rm pl}^2 G_{ab} = T_{ab}[\vartheta,\vartheta] - m_{\rm pl}\begin{cases}\ell_{\rm CS}^2 C_{ab}[\vartheta], & \text{dCS}\\ \ell_{\rm GB}^2 H_{ab}[\vartheta], & \text{EdGB}\end{cases}. \tag{8}$$

Here $T_{ab}[\vartheta,\vartheta]$ is the canonical stress-energy tensor for the scalar field ϑ,

$$T_{ab}[\vartheta,\vartheta] = \nabla_a\vartheta\nabla_b\vartheta - \frac{1}{2}g_{ab}\nabla^c\vartheta\nabla_c\vartheta. \tag{9}$$

We also define the C-tensor for dCS,

$$C_{ab}[\vartheta] = \nabla^c\nabla^d[{}^*R_{d(ab)c}\vartheta], \tag{10}$$

and introduce the H-tensor for EdGB via

$$H_{ab}[\vartheta] = \nabla^c\nabla^d[{}^*R^*_{dabc}\vartheta], \tag{11}$$

where parentheses around n indices means symmetrizing with a factor of $1/n!$.

C. Decoupling limit

We now introduce two distinct theories as the decoupling limit of dCS and EdGB, respectively, namely decoupled dynamical Chern-Simons (D^{2}CS) and decoupled dynamical Gauss-Bonnet (D^{2}GB) [25]. We will briefly review the formalism of taking the decoupling limit in dCS (see [26] for detailed discussions). The extension of this formalism to EdGB is straightforward.

We assume the corrections to GR due to the interaction terms are small, so that in the limit $\ell \to 0$, we recover standard GR. This allows us to perform a perturbative expansion of all the fields in terms of powers of $\ell_{\rm CS}$. To make the perturbation theory simpler, we introduce a formal dimensionless order-counting parameter ε. We then consider a one-parameter family of theories defined by the action I_ε, where in I_ε, we have multiplied $\mathcal{L}_{\rm int}$ by ε. This parameter can be set to 1 later.

Now we expand all fields and equations of motion in a series expansion in powers of ε. Specifically, we take $\vartheta = \vartheta^{(0)} + \varepsilon\vartheta^{(1)} + \mathcal{O}(\varepsilon^2)$, and similarly $g_{ab} = g_{ab}^{(0)} + \varepsilon h_{ab}^{(1)} + \varepsilon^2 h_{ab}^{(2)} + \mathcal{O}(\varepsilon^3)$.

In order to recover GR in the limit $\varepsilon \to 0$, at order ε^0, we have $\vartheta^{(0)} = 0$. At order ε^1, $h_{ab}^{(1)}$ has vanishing source term and thus can be set to zero as well. It is then easy to show that the EOM for the leading-order scalar field $\vartheta^{(1)}$ is at ε^1, given by

$$\square^{(0)}\vartheta^{(1)} = \frac{m_{\rm pl}}{8}\ell_{\rm CS}^2[{}^*RR]^{(0)}, \tag{12}$$

and the leading-order metric deformation enters at ε^2, which satisfies

$$m_{\rm pl}^2 G_{ab}^{(1)}[h^{(2)}] + m_{\rm pl}\ell_{\rm CS}^2 C_{ab}[\vartheta^{(1)}] = T_{ab}[\vartheta^{(1)},\vartheta^{(1)}]. \tag{13}$$

Here $G_{ab}^{(1)}[h^{(2)}]$ is the linearized Einstein operator acting on the metric deformation $h_{cd}^{(2)}$.

We now redefine our field variables in powers of $\ell_{\rm CS}$, but to do so we need another length scale against which to compare. This additional length scale is given by the typical curvature radius of the background solution, e.g. $L \sim |R_{abcd}|^{-1/2}$. For a black hole solution, this length scale will be $L \equiv GM$. We can then also pull out the scaling with powers of L from spatial derivatives and curvature tensors, by defining $\hat{\nabla} = L\nabla$ and $\hat{R}_{abcd} = L^2 R_{abcd}$. We define $\hat{h}_{ab}$ and $\hat{\vartheta}$ via

$$\vartheta^{(1)} = m_{\rm pl}\left(\frac{\ell_{\rm CS}}{GM}\right)^2\hat{\vartheta}, \qquad h_{ab}^{(2)} = \left(\frac{\ell_{\rm CS}}{GM}\right)^4\hat{h}_{ab}. \tag{14}$$

Now our hatted variables satisfy the dimensionless field equations

$$\hat{\square}^{(0)}\hat{\vartheta} = \frac{1}{8}[{}^*\hat{R}\hat{R}]^{(0)}, \qquad G_{ab}^{(1)}[\hat{h}] = S_{ab}, \tag{15}$$

with the source term $S_{ab} = T_{ab}[\hat{\vartheta},\hat{\vartheta}] - C_{ab}[\hat{\vartheta}]$.

The equations of motion in the decoupling limit of EdGB, i.e. D^{2}GB, are almost the same as Eq. (15). The only difference is that, for EdGB, we substitute ${}^*\hat{R}^*\hat{R}$ for ${}^*\hat{R}\hat{R}$, and the C-tensor in the source term should be replaced by the H-tensor.

III. NHEK AND SEPARABLE METRIC PERTURBATIONS

The metric of a generic near-horizon extremal geometry (NHEG) that makes $SL(2,\mathbb{R})\times U(1)$ symmetry manifest takes the form [27]

$$\mathrm{d}s^2 = (GM)^2\Big[v_1(\theta)\Big(-r^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2} + \beta^2\mathrm{d}\theta^2\Big) + \beta^2 v_2(\theta)(\mathrm{d}\phi - \alpha r\mathrm{d}t)^2\Big], \tag{16}$$

where v_1 and v_2 are positive functions of the polar angle θ, and α and β are constants. The spacetime has four Killing vector fields. In these Poincaré coordinates, they are given by

$$
\begin{aligned}
H_0 &= t\partial_t - r\partial_r, \\
H_+ &= \partial_t, \\
H_- &= \left(t^2 + \frac{1}{r^2}\right)\partial_t - 2tr\partial_r + \frac{2\alpha}{r}\partial_\phi, \\
Q_0 &= \partial_\phi. \qquad (17)
\end{aligned}
$$

The four generators form a representation of the Lie algebra $\mathfrak{g} \equiv \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{u}(1)$,

$$
\begin{aligned}
[H_0, H_\pm] &= \mp H_\pm, \\
[H_+, H_-] &= 2H_0, \\
[H_s, Q_0] &= 0. \quad (s = 0, \pm) \qquad (18)
\end{aligned}
$$

A crucial algebra element we will need is the Casimir element of $\mathfrak{sl}(2,\mathbb{R})$. The Casimir Ω acts on a tensor $\mathbf{t}$ via

$$\Omega \cdot \mathbf{t} = [\mathcal{L}_{H_0}(\mathcal{L}_{H_0} - \mathrm{id}) - \mathcal{L}_{H_-}\mathcal{L}_{H_+}]\mathbf{t}, \qquad (19)$$

where $\mathcal{L}_X$ is the Lie derivative along the vector field X.

The generic metric in Eq. (16) has an Einstein gravity solution, which is found with

$$
\begin{aligned}
v_1(u) &= 1 + u^2, & \alpha &= -1, \\
v_2(u) &= \frac{4(1-u^2)}{1+u^2}, & \beta &= +1, \qquad (20)
\end{aligned}
$$

where we have defined a new coordinate $u = \cos\theta$. This spacetime is called near-horizon extremal Kerr, which was first obtained by taking the near-horizon limit of extremal Kerr black holes [28].

The enhanced symmetry due to the near-horizon extremal limit enables us to separate variables in the linearized Einstein equation (LEE) in NHEK spacetime [15]. This is achieved by expanding the metric perturbations in terms of some basis functions adapted to that symmetry. For the noncompact group $SL(2,\mathbb{R})$, one can construct a *highest-weight module*, which is a unitary irreducible representation of the group. In NHEK, that is, we simultaneously diagonalize $\{\mathcal{L}_{Q_0}, \Omega, \mathcal{L}_{H_0}\}$ and label the eigenfunctions ξ by m, h, k, respectively. Here m labels the azimuthal direction, h labels the representation ("weight"), and k labels "descendants" within the same representation. We impose the highest-weight condition $\mathcal{L}_{H_+}\xi = 0$, and solve for the basis functions. Expanding the metric perturbations in terms of these bases leads to separation of variables for the LEE in NHEK spacetime. As a result, the system of partial differential equations in the LEE automatically turns into one of ordinary differential equations.

If the LEE system has a source term, and that source term is a linear combination of a finite number of representations, then the metric perturbations can also be expanded as a sum of those same representations. As we will see, for both EdGB and dCS gravity in the decoupling limit, the source term on the rhs of Eq. (15) will have the same $SL(2,\mathbb{R}) \times U(1)$ symmetry as the background spacetime. This enables us to solve for the linear metric deformations analytically.

IV. SOLVING FOR THE METRIC DEFORMATIONS

In this section, we find solutions of the leading-order scalar fields, construct the source terms on the rhs of Eq. (15) for D^{2}CS and D^{2}GB, respectively, and finally solve for the metric deformations.

A. Solutions for scalars and construction of source

In a Ricci-flat spacetime (like Kerr), the I curvature invariant [29] agrees with $I = \frac{1}{16}(-{}^*R^*R + i{}^*RR)$. In NHEK, this takes the value $\hat{I} = 3/(1-iu)^6$. The imaginary and (minus) real parts of $\hat{I}$ thus give compact ways of expressing the source terms for the scalar equations of motion of, respectively, D^{2}CS and D^{2}GB.

In D^{2}CS, the leading-order scalar equation of motion admits an axion solution which is regular everywhere. This scalar field is given by

$$\hat{\vartheta}^{(1)} = \frac{1}{4}\left[\frac{u(u^4 + 2u^2 - 7)}{(u^2+1)^3} + 2\arctan u\right] + \text{const.} \qquad (21)$$

This also agrees with the solution presented in [30]. Because the theory is shift-symmetric, we are free to set the constant term to zero. We then construct the source $S_{ab}[\hat{\vartheta}^{(1)}, \hat{\vartheta}^{(1)}]$ in Eq. (15) for D^{2}CS.

In D^{2}GB, we find the leading-order scalar solution is

$$
\begin{aligned}
\hat{\vartheta}^{(1)} = d_2 &+ \frac{\log(u^2+1)}{4} - \frac{u^4 + 4u^2 - 1}{2(u^2+1)^3} \\
&+ \left(-\frac{d_1}{2} - \frac{1}{4}\right)\log(1-u) \\
&+ \left(\frac{d_1}{2} - \frac{1}{4}\right)\log(1+u), \qquad (22)
\end{aligned}
$$

where d_1 and d_2 are constants. Unlike the D^{2}CS case, it is not possible to remove both logarithmic divergences at $u = \pm 1$ by choosing specific values of d_1 and d_2. It is possible to cancel the divergence at one pole or the other, but not both. We set $d_1 = 0$ so that the scalar field retains the reflection symmetry, $u \to -u$, of the background spacetime. Again by shift symmetry, we are free to set the additive constant $d_2 = 0$, and then construct the source term S_{ab} accordingly. The source S_{ab} remains irregular at the two poles $u = \pm 1$.

Let us remark on an important common feature of the two source terms. For either theory,

$$\mathcal{L}_X S_{ab} = 0, \tag{23}$$

where $X \in \{H_0, H_\pm, Q_0\}$. That is, if we decompose the source term using the symmetry-adapted scalar, vector and tensor bases, the source term only contains the $m = h = k = 0$ component. Therefore, on the lhs of the LEE, the metric perturbations only have stationary axisymmetric basis components, either for D^{2}CS or D^{2}GB. These components live in both the highest-weight and lowest-weight representations of NHEK's isometry group.

B. dCS-deformed NHEK

We now seek the solutions to the linearized metric perturbation equations of NHEK sourced by the two stringy interactions. Expansions of the metric perturbations into the basis functions turn the systems of partial differential equations in LEE into ten coupled ordinary differential equations (ODEs) in u, which we solve in this subsection.

So far we haven't chosen any gauge condition. Since the linear metric perturbations have the same $SL(2,\mathbb{R}) \times U(1)$ symmetry as the background NHEK spacetime, we can fix the gauge by requiring an "attractor form" [27] of the deformed solutions as in Eq. (16). That is, we only consider the following shifts in the metric parameters. Recalling that the metric is corrected at order ε^2, we have

$$\begin{aligned} v_1(u) &\to v_1(u) + \varepsilon^2 \delta v_1(u), \qquad \alpha \to \alpha + \varepsilon^2 \delta\alpha, \\ v_2(u) &\to v_2(u) + \varepsilon^2 \delta v_2(u), \qquad \beta \to \beta + \varepsilon^2 \delta\beta. \end{aligned} \tag{24}$$

We call this gauge choice the attractor gauge. This ansatz is, by construction, in the $m = h = k = 0$ representation of NHEK's isometry group. Therefore, it always makes the $SL(2,\mathbb{R}) \times U(1)$ symmetry manifest.

For D^{2}CS, the linear metric deformations are found to be the following complicated expressions, which we also plot in Fig. 1:

$$\delta v_1(u) = f_1(u) + \frac{1}{53760(u^2+1)^5}\mathcal{P}_1^{\mathrm{D^2CS}}[u], \tag{25}$$

$$\delta v_2(u) = f_2(u) - \frac{(u^2-1)}{6720(u^2+1)^7}\mathcal{P}_2^{\mathrm{D^2CS}}[u], \tag{26}$$

where

$$\begin{aligned} f_1(u) =& \frac{1}{3}c_1(-u^2+4u-1) + \frac{1}{3}c_2(2u^2-5u+2) \\ & - \frac{1}{3}c_3 u\sqrt{1-u^2} - \frac{4}{3}\delta\beta(u^2+1) + 2\delta\beta u\sqrt{1-u^2}\sin^{-1}u \\ & + \frac{975u\sqrt{1-u^2}\tan^{-1}(\frac{\sqrt{2}u}{\sqrt{1-u^2}})}{512\sqrt{2}} - \frac{3}{16}u\tan^{-1}u, \end{aligned} \tag{27}$$

$$\begin{aligned} f_2(u) =& \frac{8c_3u\sqrt{1-u^2}}{3(u^2+1)^2} + \frac{4c_1(u^4+4u^3-4u-1)}{3(u^2+1)^2} \\ & - \frac{4c_2(2u^4+5u^3-5u-2)}{3(u^2+1)^2} + \frac{40\delta\beta(u^2-1)}{3(u^2+1)} \\ & - \frac{16\delta\beta u\sqrt{1-u^2}\sin^{-1}u}{(u^2+1)^2} + \frac{\delta\alpha(8-8u^2)}{u^2+1} \\ & - \frac{975u\sqrt{1-u^2}\tan^{-1}(\frac{\sqrt{2}u}{\sqrt{1-u^2}})}{64\sqrt{2}(u^2+1)^2} - \frac{3u(u^2-1)\tan^{-1}u}{4(u^2+1)^2}, \end{aligned} \tag{28}$$

and the polynomials $\mathcal{P}_1^{\mathrm{D^2CS}}[u]$ and $\mathcal{P}_2^{\mathrm{D^2CS}}[u]$ are given by

$$\begin{aligned} \mathcal{P}_1^{\mathrm{D^2CS}}[u] =& -58501u^{12} - 222147u^{10} - 255058u^8 \\ & + 11754u^6 + 323735u^4 - 149799u^2 + 4416, \end{aligned} \tag{29}$$

$$\begin{aligned} \mathcal{P}_2^{\mathrm{D^2CS}}[u] =& 280u^{12} - 52341u^{10} - 252928u^8 \\ & - 472090u^6 - 536680u^4 + 26583u^2 - 18792. \end{aligned} \tag{30}$$

Here c_1, c_2 and c_3 are integration constants. It is straightforward to see these three constants, together with $\delta\alpha$ and $\delta\beta$, correspond to different homogeneous solutions to the LEE. These solutions are finite on the domain $u \in [-1, +1]$, but would have infinite derivative at the poles $u = \pm 1$ without an appropriate choice of $\delta\beta$. By demanding regularity at the two poles and reflection symmetry of the deformed metric, we set

$$\delta\beta = -\frac{975}{1024\sqrt{2}}, \qquad c_3 = 0, \qquad c_2 = \frac{4c_1}{5}. \tag{31}$$

Note that $\delta\alpha$ will shift the Killing vector H_-. By demanding that the perturbed spacetime has the same Killing vectors as NHEK, we also set $\delta\alpha = 0$. After inserting the solutions from (31) back into the metric, we only need to fix c_1. Collecting the terms proportional to c_1, one immediately finds that

$$(\text{coefficient of } c_1) \propto \frac{\partial g_{ab}^{(0)}}{\partial M}. \tag{32}$$

This means the homogeneous solution associated with c_1 shifts the mass of the black hole. Since we don't want the mass shift, we fix $c_1 = 0$. With these parameter choices, we obtain the regular solution to the LEE sourced by the dCS interaction in the decoupling limit. We call the newly found spacetime "dCS-deformed NHEK."

C. EdGB-deformed NHEK

For D^{2}GB, in the attractor gauge, the linear metric deformations are found to be

$$\delta v_1(u) = f_1(u) + \frac{1}{8}(-u^2+u-1)\log(1-u) + \frac{1}{8}(-u^2-u-1)\log(1+u) + \frac{1}{8}(u^2+1)\log(u^2+1) - \frac{3u\sqrt{1-u^2}\tan^{-1}\left(\frac{\sqrt{2}u}{\sqrt{1-u^2}}\right)}{256\sqrt{2}} + \frac{1}{53760(u^2+1)^5}\mathcal{P}_1^{\mathrm{D^2GB}}[u], \tag{33}$$

$$\delta v_2 = f_2(u) + \frac{(u^4-u^3+u-1)\log(1+u)}{2(u^2+1)^2} + \frac{(u^4+u^3-u-1)\log(1-u)}{2(u^2+1)^2} + \frac{(1-u^2)\log(u^2+1)}{2u^2+2} + \frac{3u\sqrt{1-u^2}\tan^{-1}(\frac{\sqrt{2}u}{\sqrt{1-u^2}})}{32\sqrt{2}(u^2+1)^2} + \frac{(u^2-1)}{6720(u^2+1)^7}\mathcal{P}_2^{\mathrm{D^2GB}}[u], \tag{34}$$

where the functions $f_1(u)$, $f_2(u)$ are identical to the D^{2}CS case and given in Eqs. (27) and (28), and where the polynomials $\mathcal{P}_1^{\mathrm{D^2GB}}[u]$ and $\mathcal{P}_2^{\mathrm{D^2GB}}[u]$ are given by

$$\mathcal{P}_1^{\mathrm{D^2GB}}[u] = -27459u^{12} - 82773u^{10} - 42302u^8 + 81766u^6 - 18815u^4 + 298479u^2 + 11264, \tag{35}$$

$$\mathcal{P}_2^{\mathrm{D^2GB}}[u] = 35859u^{10} + 152792u^8 + 226230u^6 + 10160u^4 + 205503u^2 - 5632. \tag{36}$$

As in the D^{2}CS case, the constant $\delta\beta$ can be chosen so as to cancel a square-root behavior at the poles which would have infinite derivative. However, the important difference from D^{2}CS is the appearance of log terms in Eqs. (33) and (34). There are no integration constants which can cancel these logarithmic divergences.

Still, canceling the square-root behavior and assuming reflection symmetry in u, we find

$$\delta\beta = -\frac{969}{1024\sqrt{2}}, \qquad c_3 = 0, \qquad c_2 = \frac{4c_1}{5}. \tag{37}$$

We also fix $\delta\alpha = 0$ to preserve the Killing vector fields of NHEK, and set $c_1 = 0$ to avoid a mass shift. After fixing all constants, these functions are plotted in Fig. 1. We call the corresponding spacetime "EdGB-deformed NHEK." This metric deformation has a true curvature singularity at the poles, $u = \pm 1$, which we discuss further in Sec. VI.

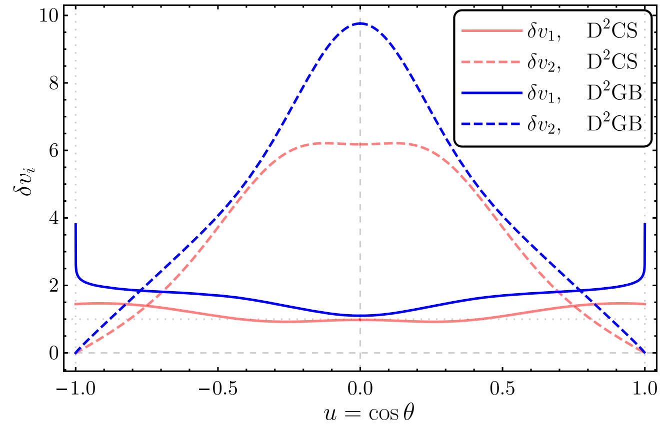

FIG. 1. The metric deformation functions δv_1 (solid) and δv_2 (dashed) as functions of u, for both dCS-deformed (red) and EdGB-deformed (blue) NHEK. Note that in D^{2}GB, δv_1 blows up at the two poles $u = \pm 1$.

V. PROPERTIES OF SOLUTIONS

A. Orbits

In this subsection, we derive the geodesic equations for a particle in the deformed NHEK spacetime. Since the NHEK background and the deformed solutions have the same isometry group, we consider the spacetime with the general metric in Eq. (16). The relativistic Hamiltonian for geodesic motion of a particle can be defined as

$$H(x^a, p_b) = \frac{1}{2}g^{ab}p_a p_b, \tag{38}$$

where p_a are the conjugate momenta of the particle. By drawing analogy to geodesic motion in Kerr spacetime, we can similarly find three constants of motion: energy $E \equiv -p_t$, z angular momentum $L_z \equiv p_\phi$, and Carter's constant $\mathcal{C}$. The Carter constant comes from separating the radial and polar motions. Note, however, that because our Killing vector field ∂_t is different from the asymptotically timelike KVF (with norm -1 at infinity), our energy is different from the usual Kerr orbital energy [31]. Following the Hamilton-Jacobi approach [32], we define the characteristic function W via

$$W = -\frac{1}{2}\kappa\lambda - Et + \int\frac{\sqrt{R(r)}}{\beta^2 r^2}dr + \int\sqrt{\Theta(\theta)}d\theta + L_z\phi, \tag{39}$$

where λ is the affine parameter and $\frac{1}{2}\kappa$ is the value of the Hamiltonian evaluated along the world line of the particle. $R(r)$ and $\Theta(\theta)$ are given by

$$R(r) = \beta^4(E - \alpha L_z r)^2 - \beta^2 \mathcal{C} r^2,$$
$$\Theta(\theta) = \mathcal{C} - \frac{v_1(\theta)}{v_2(\theta)} L_z^2 + M^2\beta^2 v_1(\theta)\kappa. \tag{40}$$

Since $p_a = \frac{\partial W}{\partial x^a}$, we obtain the following geodesic equations of motion,

$$\Sigma \frac{dt}{d\lambda} = \frac{\beta^2}{r^2}(E - \alpha L_z r),$$
$$\Sigma \frac{dr}{d\lambda} = \pm\sqrt{R(r)},$$
$$\Sigma \frac{d\theta}{d\lambda} = \pm\sqrt{\Theta(\theta)},$$
$$\Sigma \frac{d\phi}{d\lambda} = \frac{\alpha\beta^2}{r}(E - \alpha L_z r) + \frac{v_1(\theta)}{v_2(\theta)} L_z, \tag{41}$$

where $\Sigma = M^2\beta^2 v_1(\theta)$. These integrals can be directly performed after defining the "Mino time" τ, where $d\tau = d\lambda/\Sigma$ (this again differs from the usual Mino time in the asymptotic region of Kerr, because our time coordinate differs).

In particular, let us consider circular equatorial motion, i.e. $\theta = \pi/2 = \theta_0$. For such motion we only need E and L_z to determine the orbit. For a time-like orbit with four-velocity u^a, $g_{ab}u^a u^b = -1$, we have that

$$\left(\frac{dr}{d\lambda}\right)^2 = V(r), \tag{42}$$

where the effective potential $V(r)$ is given by

$$V(r) = \frac{(E - \alpha L_z r)^2}{M^4 v_1^2(\theta_0)} - \frac{r^2}{M^2 v_1(\theta_0)} - \frac{L_z^2 r^2}{M^4\beta^2 v_1(\theta_0) v_2(\theta_0)}. \tag{43}$$

Solving for the conditions of circular motion, we obtain

$$E = 0, \qquad L_z = \pm\frac{M\beta\sqrt{v_1(\theta_0)v_2(\theta_0)}}{\sqrt{-v_1(\theta_0) + \alpha^2\beta^2 v_2(\theta_0)}}. \tag{44}$$

The corresponding circular orbits $r = r_0$ are all marginally stable, i.e. $V''(r)|_{r=r_0} = 0$. After integrating out the azimuthal motion we also obtain that $\phi = \phi_0 + \omega_\phi t$, where the angular frequency ω_ϕ is given by

$$\omega_\phi = \left(\alpha - \frac{v_1(\theta_0)}{\alpha\beta^2 v_2(\theta_0)}\right) r_0. \tag{45}$$

The fact that all circular equatorial orbits are essentially the same, with a different angular frequency, is due to the dilation symmetry of the spacetime. That is, the metric is invariant under $r \to cr$ and $t \to t/c$ for any constant $c \in (0, +\infty)$. As a result, in Eq. (45), the radius-frequency relationship has to be compatible with the dilation symmetry.

Plugging in the D^{2}CS solutions, we find the angular frequency of the equatorial circular orbits to be

$$\omega_\phi^{\mathrm{D^2CS}} = \left[-\frac{3}{4} + \frac{25}{128}\left(\frac{\ell_{\mathrm{CS}}}{GM}\right)^4 + \mathcal{O}(\varepsilon^3)\right] r_0. \tag{46}$$

Similarly for the D^{2}GB solutions, the angular frequency is found to be

$$\omega_\phi^{\mathrm{D^2GB}} = \left(-\frac{3}{4} + \mathcal{O}(\varepsilon^3)\right) r_0. \tag{47}$$

Therefore, at the leading order in the metric perturbations, EdGB-type interactions do not lead to corrections to the angular frequency of circular equatorial orbits in an extremal black hole, in the near-horizon limit.

Again, because our time differs from the time coordinate in the asymptotic region, these frequencies are not the asymptotically observable orbital frequencies. Such observable quantities were computed for slowly-rotating BHs in D^{2}CS in [4] and in D^{2}GB in [9,10,33–35].

B. Location and area of deformed horizons

Since NHEK is not asymptotically flat, it does not have an event horizon. However, because of what the near-horizon limit is designed to do—to zoom in on the horizon region—the scaling limit of the Kerr event horizon gives rise to the horizon of the Poincaré patch. This Poincaré horizon has the same geometric properties as in Kerr, and thus it has the same area and entropy.

We can identify the location of this Killing horizon by considering observers whose world lines are along real linear combinations $c_t\partial_t + c_\phi\partial_\phi$, with c_t, c_ϕ real constants, such that their world lines are timelike. At the horizon, these world lines are forced to be null. For any metric of the NHEG form (16), the horizon is at $r = 0$. Therefore, in attractor gauge, the coordinate location of the horizon is not deformed after including the scalar-gravity coupling in the action.

A cross section of the deformed-NHEK horizon is still homeomorphic to a two-sphere S^2, but the total area has changed. Because the horizon is Killing, we can compute the area along any spatial cross section $\mathcal{H}$ carrying coordinates x. The horizon areas of the two deformed solutions are both given by

$$A_{\mathrm{deformed}} = \oint_{\mathcal{H}} \sqrt{\gamma}\, \mathrm{d}^2 x$$
$$= A_{\mathrm{NHEK}} \times \left[1 + \eta\left(\frac{\ell}{GM}\right)^4 + \mathcal{O}(\varepsilon^3)\right], \tag{48}$$

where ℓ is $\ell_{\rm CS}$ or $\ell_{\rm GB}$ when appropriate. Here γ is the determinant of the induced metric on $\mathcal{H}$. $A_{\rm NHEK}$ is the horizon area of an extremal Kerr black hole, which is given by $A_{\rm NHEK} = 8\pi(GM)^2$. The constant η varies for the two deformed solutions. For D^{2}CS and D^{2}GB, respectively, we find

$$\eta_{\rm D^2CS} = (4875\sqrt{2} - 1380\pi - 3928)/7680 \approx -0.18, \tag{49}$$

$$\eta_{\rm D^2GB} = (1615\sqrt{2} - 300\pi - 464 - 320\log 2)/2560 \approx +0.26. \tag{50}$$

Despite the fact that EdGB-deformed NHEK has a true curvature singularity, this singularity is integrable, leading to a finite correction to the horizon area.

Note that while considering deformed NHEK, the entropy no longer equals the horizon area, since the stringy interactions also contribute microscopic degrees of freedom. The horizon areas computed here will be used in the following subsection to calculate the entropy of the two deformed solutions.

C. Thermodynamics of horizons

The macroscopic entropy of a Killing horizon is interpreted as the Noether charge associated with the Killing vector field which generates the horizon [36,37]. In any diffeomorphism invariant theory with a Lagrangian $\mathcal{L} = \mathcal{L}(\phi, \nabla_a\phi, g_{ab}, R_{abcd}, \nabla_e R_{abcd}, \ldots)$, where ϕ are matter fields, the black hole entropy can be written as an integral over a horizon cross section $\mathcal{H}$ [38]. Again, since the horizon is Killing, any spacelike cross section will do. This entropy integral is

$$S = -2\pi \oint_{\mathcal{H}} \frac{\delta\mathcal{L}}{\delta R_{abcd}} \hat{\epsilon}_{ab}\hat{\epsilon}_{cd}\bar{\epsilon}. \tag{51}$$

Here $\bar{\epsilon}$ is the induced volume form on the $D-2$-dimensional cross section, and $\hat{\epsilon}_{ab}$ is the binormal. The binormal has been normalized such that $\hat{\epsilon}_{ab}\hat{\epsilon}^{ab} = -2$.

The NHEK solution does not have an event horizon; however, we can still get the correct entropy of the extremal black hole by performing the integral over the cross section of the Poincaré horizon. The entropy of the NHEK solution can then be obtained by evaluating Eq. (51) in Einstein-Hilbert theory $\mathcal{L} = \mathcal{L}_{\rm EH}$. It is not surprising that we arrive at the Bekenstein-Hawking entropy for the extremal Kerr black hole [2,39],

$$S_{\rm NHEK} = 2\pi m_{\rm pl}^2 A_{\rm NHEK} = \frac{A_{\rm NHEK}}{4G}. \tag{52}$$

Similarly, in D^{2}CS and D^{2}GB, by computing the entropy corrections due to stringy degrees of freedom, we will be able to obtain the entropies of the deformed-NHEK solutions in the two theories. Note, however, that the entropy results agree with the extremal BH solutions, since the Poincaré horizon is the scaling limit of the extremal BH event horizon. The corrections to the entropy are due to high-energy stringy degrees of freedom becoming activated.

In either dCS or EdGB gravity, the scalar field Lagrangian $\mathcal{L}_\vartheta$ does not contribute to the entropy while the interaction term $\mathcal{L}_{\rm int}$ does. Therefore, in a full theory with action given by Eq. (1), the entropy of a stationary black hole solution with horizon cross section $\mathcal{H}$ is

$$S = 2\pi m_{\rm pl}^2 \oint_{\mathcal{H}} \bar{\epsilon} + S_{\rm int}, \tag{53}$$

where we have defined $S_{\rm int}$ via

$$S_{\rm int} = -2\pi \oint_{\mathcal{H}} \frac{\delta\mathcal{L}_{\rm int}}{\delta R_{abcd}} \hat{\epsilon}_{ab}\hat{\epsilon}_{cd}\bar{\epsilon}. \tag{54}$$

Compared to Einstein gravity, dCS- and EdGB-deformed NHEK receive entropy corrections from two sources: the deformation of the horizon area, and the string interaction term $S_{\rm int}$. In dCS theory, the correction to the entropy due to the scalar-gravity interaction term is given by

$$S_{\rm int}^{\rm CS} = \frac{\pi}{2} m_{\rm pl}\ell_{\rm CS}^2 \oint_{\mathcal{H}} \vartheta {}^*R^{abcd}\hat{\epsilon}_{ab}\hat{\epsilon}_{cd}\bar{\epsilon}. \tag{55}$$

Similarly, we find the correction to entropy via the EdGB interaction is

$$S_{\rm int}^{\rm GB} = \frac{\pi}{2} m_{\rm pl}\ell_{\rm GB}^2 \oint_{\mathcal{H}} \vartheta {}^*R^{*abcd}\hat{\epsilon}_{ab}\hat{\epsilon}_{cd}\bar{\epsilon}. \tag{56}$$

Now let us explore the effect of taking the decoupling limit and compute the leading-order corrections to the entropy of extremal Kerr in D^{2}CS and D^{2}GB theories. The leading-order scalar field is already at ε^1 while the metric perturbations correct at order ε^2, thus we can evaluate Eqs. (55) and (56) using the original NHEK metric. Combining the horizon area calculations given by Eq. (48), the entropies of the two deformed NHEK solutions can both be written as

$$S_{\rm deformed} = S_{\rm NHEK}\left[1 + \xi\left(\frac{\ell}{GM}\right)^4 + \mathcal{O}(\varepsilon^3)\right], \tag{57}$$

where the constant ξ for D^{2}CS and D^{2}GB are given by

$$\xi_{\rm D^2CS} = (4875\sqrt{2} + 360\pi^2 - 868\pi - 3928)/7680 \approx +0.49, \tag{58}$$

$$\xi_{\mathrm{D^2GB}} = (360\pi^2 + 4845\sqrt{2} - 1392 - 960\log 2 - 4\pi(480\log 2 - 607))/7680 \approx +1.54. \tag{59}$$

Here as well, despite the EdGB scalar solution having a singularity at the poles, the singularity is integrable, leading to a finite correction to the entropy. Note that both entropy corrections are positive, as should be the case when adding new degrees of freedom to the underlying microscopic theory.

VI. DISCUSSION AND FUTURE WORK

We have obtained analytic solutions for the linearized metric deformations to near-horizon extremal Kerr spacetimes as induced by dCS and EdGB interactions in the decoupling limit. In this limit, the metric deformations solve linearized Einstein equations with a source term arising from the dilaton or axion field and the background metric. We decomposed the metric perturbations using basis functions adapted to the $SL(2,\mathbb{R})\times U(1)$ isometry and turn the systems of field equations into solvable ODEs. The resulting solution in D^{2}CS, dCS-deformed NHEK, is regular everywhere, while in D^{2}GB, EdGB-deformed NHEK has a true curvature singularity at the poles, discussed further below. We studied timelike orbits in these two newly found spacetimes. In particular, for circular equatorial orbits, we computed the leading-order corrections to the angular frequencies, which are observables for subextremal black holes by gravitational wave experiments. Finally, we computed the corrections to the horizon areas and the macroscopic entropies of the extremal black hole solutions in D^{2}CS and D^{2}GB. The positive entropy corrections are related to the inclusion of new degrees of freedom in the theory.

EdGB-deformed NHEK is irregular at the poles $u=\pm1$, no matter how we choose the constants of integration. This irregular behavior originates from the source term built from the dilaton field, since the dilaton has an unavoidable logarithmic singularity at the poles, as discussed in Sec. IV A. This leads to a true curvature singularity, which can be seen as follows. We can find the singularity without solving for $\hat{h}^{(2)}$ by simply tracing the equation of motion Eq. (15). Since the background Ricci scalar and the first-order metric deformation both vanish ($\hat{R}^{(0)}=0=\hat{h}^{(1)}$), the deformation $\delta\hat{R}^{(2)}$ is a gauge-invariant quantity. Now, the uu component of the source tensor, $S_{ab}^{\mathrm{D^2GB}}$, contains $(\partial_u\hat{\vartheta})^2$ and $\partial_u^2\hat{\vartheta}$, which give a pole of order two at $u=\pm1$. The inverse metric component g^{uu} only contributes a single zero at the poles. Thus the trace of the source term $g^{ab}S_{ab}^{\mathrm{D^2GB}}$ blows up with a pole of order 1 at $u=\pm1$, and we have an unavoidable curvature singularity.

This problem with extremal EdGB solutions was previously mentioned in [40] and discussed further in Appendix B of [19]. They presented numerical evidence and an analytic argument that the extremal limit does not admit regular solutions, for any values of the GB coupling parameter. Here, we have proven that there are no regular solutions, in the decoupling limit. While our analysis is restricted to the decoupling limit, based on the gauge-invariant argument above, we have proven that the extremal limit is indeed singular for EdGB.

We still lack a clear physical understanding of this curvature singularity. The simplest interpretation is that this is a sign of a breakdown of EdGB when treated as an EFT, and that this singularity is cured by the inclusion of operators at the same or higher order (such as those which were discarded in the truncation of [7]). This situation would be a counterexample to Hadar and Reall's recent claim that EFT does not break down at an extremal horizon [41].

Future work.—The near-horizon near-extremal Kerr (near-NHEK) spacetime has the same $SL(2,\mathbb{R})\times U(1)$ isometry as the NHEK spacetime. Therefore, we expect all this work can be extended to near-NHEK directly. The techniques we used here can also be used for any other beyond-GR theory which has a continuous limit to GR. Therefore, we can also solve for deformed NHEK solutions in a broad class of theories. It may be possible to use matched asymptotic expansions to combine perturbation theory about (near-)NHEK and Schwarzschild, in order to build beyond-GR metric solutions valid for all values of spin, $0\le a\le M$.

On the observational side, the angular frequencies of the near-extremal Kerr ISCO may be determined accurately in future gravitational wave experiments, providing a useful way to test general relativity.

Finally, this work may be helpful in understanding quantum theories beyond Einstein gravity. We have computed the macroscopic entropies of extremal black holes, which must be associated with corresponding microscopic entropies. This may be possible with an analog of the Kerr/CFT correspondence.

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