

GROUP AMENABILITY AND ACTIONS ON \mathcal{Z} -STABLE C^* -ALGEBRAS

EUSEBIO GARDELLA AND MARTINO LUPINI

ABSTRACT. We study strongly outer actions of discrete groups on C^* -algebras in relation to (non)amenability. In contrast to related results for amenable groups, where uniqueness of strongly outer actions on the Jiang-Su algebra is expected, we show that uniqueness fails for all nonamenable groups, and that the failure is drastic. Our main result implies that if G contains a copy of the free group, then there exist uncountable many, non-cocycle conjugate strongly outer actions of G on any Jiang-Su stable tracial C^* -algebra. Similar conclusions apply for outer actions on McDuff tracial von Neumann algebras. We moreover show that G is amenable if and only if the Bernoulli shift on a finite strongly self-absorbing C^* -algebra absorbs the trivial action on the Jiang-Su algebra. Our methods consist in a careful study of weak containments of the Koopman representations of different Bernoulli-type actions.

INTRODUCTION

Amenability for discrete groups was first introduced by von Neumann in the context of the Banach-Tarski paradox. One of the main early results in the theory, proved by Tarski, asserts that a group is amenable if and only if it admits no paradoxical decompositions. The fact that the Banach-Tarski paradox only makes use of free groups led Day to conjecture that a discrete group is nonamenable if and only if it contains the free group \mathbb{F}_2 as a subgroup. This conjecture, known as *the von Neumann problem*, was open for about 40 years, until it was disproved by Ol'shanskii.

Amenability admits a surprisingly large number of equivalent formulations. Here, we are concerned with those characterizations that are phrased in terms of actions of the group. These usually come in the form of a dichotomy: roughly speaking, they assert that there is an object in the relevant category, on which every amenable group acts in an essentially unique way, while every nonamenable group admits a continuum of non-equivalent actions. The following is an illustrative example:

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Theorem. Let G be a discrete group, and let (X, μ) be a standard atomless probability space.

- (1) If G is amenable, then all free, measure preserving, ergodic actions of G on (X, μ) are orbit equivalent.
- (2) If G is not amenable, then there exist uncountably many non-orbit equivalent free, measure preserving, ergodic actions of G on (X, μ) .

Part (1) is a combination of classical results of Dye and Ornstein-Weiss. In reference to (2), the first result in this direction is a theorem of Connes-Weiss, asserting that every nonamenable group without property (T) admits two such actions. Much more recently, Ioana proved part (2) for groups containing a copy of \mathbb{F}_2 ([12]), using the corresponding result for \mathbb{F}_2 due to Gaboriau-Popa ([8]), and finally Epstein extended the result to all nonamenable groups ([5]). In recent work ([11]), the authors improved the conclusion in part (2) above: the relation of orbit equivalence of actions of nonamenable groups is not Borel.

In the context of von Neumann algebras, and specifically for the hyperfinite II_1 -factor \mathcal{R} , amenability can also be characterized in terms of actions:

Theorem. Let G be a discrete group, and let \mathcal{R} be the hyperfinite II_1 -factor.

- (1) If G is amenable, then all outer actions of G on \mathcal{R} are cocycle conjugate.
- (2) If G is not amenable, then there exist uncountably many non-cocycle conjugate outer actions of G on \mathcal{R} .

Part (1) is due to Ocneanu ([16]), although particular cases were proved by Connes for cyclic groups ([4]), and by Jones for finite groups ([13]). Part (2) is a recent result due to Brothier-Vaes (Theorem B in [2]), generalizing previous results of Popa ([17]) and Jones ([14]).

In both theorems recalled above, the amenable case was resolved relatively early. On the other hand, the nonamenable case took much longer, and it required the invention of new and powerful tools such as Popa's celebrated deformation/rigidity theory. Indeed, it was realized that certain nonamenable groups (or certain nonamenable II_1 -factors) exhibit striking rigidity phenomena, which is best seen in the presence of property (T). The richness of the nonamenable world drove researchers in both Ergodic Theory and in von Neumann algebras to study actions of nonamenable groups on the standard atomless probability space as well as on \mathcal{R} , particularly in what relates to the complexity of their classification.

This work revolves around analogs of the above results in the context of C^* -algebras, the central theme being the case of nonamenable groups. Strongly self-absorbing C^* -algebras can be seen as the C^* -analog of the hyperfinite II_1 -factor. (Recall that a unital C^* -algebra \mathcal{D} is said to be *strongly self-absorbing* if it is infinite dimensional and there is an isomorphism $\varphi: \mathcal{D} \rightarrow \mathcal{D} \otimes_{\min} \mathcal{D}$ which is approximately unitarily equivalent to the first tensor factor embedding; see [21].) Examples of such algebras are the UHF-algebras of infinite type, the Jiang-Su algebra \mathcal{Z} , the Cuntz algebras \mathcal{O}_2 and \mathcal{O}_∞ , and their tensor products. (It is believed that this list is complete.) By a result of Winter ([22]), any strongly self-absorbing C^* -algebra absorbs \mathcal{Z} tensorially.

C^* -analogs of part (1) in the theorem above were explored in the early 1990s by Bratteli, Evans and Kishimoto ([1]), who studied a concrete family of outer actions of \mathbb{Z} on a specific UHF-algebra. Their results show that outerness in (finite) C^* -algebras is too weak a condition for an analog of Ocneanu's result to hold. They

also provided evidence for the fact that a uniqueness result may hold if one assumes outerness not only for the action, but also for its extension to the weak closure in the GNS representation. This notion is now called *strong outerness*.

Motivated by a recent breakthrough of Szabó [20], the following conjecture has been proposed in [10], the first part of which had already been suggested in [20]. We refer the reader to the introductions of [20] and [10] for motivation and relevant references (in particular, for the reason why torsion groups must be excluded).

Conjecture A. Let G be a torsion-free countable group, and let \mathcal{D} be a strongly self-absorbing C^* -algebra.

- (1) If G is amenable, then any two strongly outer actions of G on \mathcal{D} are cocycle conjugate.
- (2) If G is not amenable, then there exist uncountably many non-cocycle conjugate strongly outer actions of G on \mathcal{D} .

The main result of [20] asserts that part (1) holds when \mathcal{D} is either a UHF-algebra of the Jiang-Su algebra and G is abelian, while [10] asserts that part (2) holds when \mathcal{D} is a UHF-algebra and for groups containing a subgroup with relative property (T).

In this work, we continue the study of strongly outer actions on C^* -algebras, and specifically on strongly self-absorbing C^* -algebras. In particular, we are interested in constructing many non-cocycle conjugate actions for nonamenable groups. We focus on a specific and very rich class of actions, which we call *generalized (non-commutative) Bernoulli shifts*. These are constructed as follows: given a strongly self-absorbing C^* -algebra \mathcal{D} and an action $G \curvearrowright^\sigma X$ of a discrete group G on a countable set X , we consider the action of G on $\otimes_{x \in X} \mathcal{D} \cong \mathcal{D}$ given by permuting the tensor factors according to σ .

For an arbitrary group G , it seems difficult to produce actions of this form other than the usual Bernoulli shift $G \curvearrowright \otimes_{g \in G} \mathcal{D}$ and the trivial action of G on \mathcal{D} . However, considering these actions leads to a new characterization of amenability:

Theorem B. Let G be a countable discrete group, and let \mathcal{D} be a finite strongly self-absorbing C^* -algebra. Then G is amenable if and only if the Bernoulli shift $G \curvearrowright \bigotimes_{g \in G} \mathcal{D}$ absorbs tensorially (up to cocycle conjugacy) the trivial action $\text{id}_{\mathcal{Z}}$ on \mathcal{Z} .

This result implies, in particular, a weak form of part (2) of Conjecture A: every nonamenable group admits two non-cocycle conjugate strongly outer actions on \mathcal{D} , namely, the Bernoulli shift and its stabilization with $\text{id}_{\mathcal{Z}}$.

We obtain stronger results for groups having sufficiently many finite subquotients. A particular instance of our main result (Theorem 4.5) confirms part (2) of Conjecture A for groups containing \mathbb{F}_2 :

Theorem C. Let G be a discrete group containing a copy of \mathbb{F}_2 , and let A be a \mathcal{Z} -absorbing tracial C^* -algebra. (For example, a strongly self-absorbing C^* -algebra.) Then there exist uncountably many non-cocycle conjugate, strongly outer actions of G on A , acting via asymptotically inner automorphisms of A .

The fact that the actions we construct are pointwise asymptotically inner implies that these actions are not distinguishable by any kind of K - or KK -theoretical

invariant, or even the Cuntz semigroup. The way in which we distinguish them is via the weak equivalence class of the associated Koopman representation.

Theorem C is the C*-version of Ioana's result on non-orbit equivalent actions from [12]. Epstein later combined Ioana's result with Gaboriau-Lyon's measurable solution to the von Neumann problem [7] to generalize Ioana's work to *all* nonamenable groups. A very interesting and promising problem is to find an analog of the main result in [7] in the context of strongly outer actions on C*-algebras, or at least for outer actions on \mathcal{R} . A satisfactory solution would allow us to prove Theorem C (and Theorem D below) for all nonamenable groups.

Finally, our methods allow us to replace \mathcal{Z} with the hyperfinite II₁-factor \mathcal{R} , thus obtaining the following generalization of a recent result of Brothier-Vaes:

Theorem D. Let G be a discrete group containing a copy of \mathbb{F}_2 , and let M be a McDuff tracial von Neumann algebra. Then there exist uncountably many non-cocycle conjugate, strongly outer actions of G on M , acting via asymptotically inner automorphisms of M .

Even in the case when $M = \mathcal{R}$, our proof is more elementary than the proof of Theorem B in [2], and avoids the use of Popa's advanced techniques from [18] and [17]. It should be mentioned that Theorem B in [2] applies not just to groups containing \mathbb{F}_2 , but to arbitrary nonamenable groups.

The rest of the paper is organized as follows. In Section 1, we establish a number of basic facts about subgroups with finite index that will be important in the later sections. In Section 2, we study the generalized Bernoulli shift associated with an action $G \curvearrowright^\sigma X$ on a discrete set X , and relate its Koopman representation to the canonical unitary representation of G on $\ell^2(X)$. In Section 3, we specialize to a particular family of generalized Bernoulli shifts, obtained from finite subquotients of G . Finally, Section 4 contains the proofs of our main results (Theorem 4.3 and Theorem 4.5), from which Theorems B, C and D follow.

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1. QUASIREGULAR REPRESENTATIONS

Given a C*-algebra D , we will denote by $\mathcal{S}(D)$ the space of states over D . We begin by recalling the notion of quasiregular representation.

Definition 1.1. Let G be a discrete group, and let H be subgroup. We denote by $\lambda_{G/H}: G \rightarrow \mathcal{U}(\ell^2(G/H))$ the unitary representation induced by the canonical left translation action of G on G/H . We call $\lambda_{G/H}$ the *quasiregular representation* associated with H .

Let G be a discrete group and let $\mu: G \rightarrow \mathcal{U}(\mathcal{H}_\mu)$ and $\nu: G \rightarrow \mathcal{U}(\mathcal{H}_\nu)$ be unitary representations. Recall that μ is said to be (*unitarily*) *contained* in ν , written $\mu \subseteq \nu$, if there exists an isometry $\varphi: \mathcal{H}_\mu \rightarrow \mathcal{H}_\nu$ satisfying $\varphi \circ \mu_g = \nu_g \circ \varphi$ for all $g \in G$. When φ can be chosen to be a unitary, we say that μ and ν are *unitarily equivalent*, and write $\mu \cong \nu$.

Lemma 1.2. Let G be a discrete group, and let H_1, \dots, H_n be subgroups of G whose indices in G are finite. Set $H = H_1 \cap \dots \cap H_n$. Then:

- (1) H has finite index in G , and $[G : H] \leq [G : H_1] \cdots [G : H_n]$. If the indices of H_1, \dots, H_n in G are relatively prime, then equality holds.

- (2) We have $\lambda_{G/H} \subseteq \lambda_{G/H_1} \oplus \cdots \oplus \lambda_{G/H_n}$, and this containment is an equivalence whenever the indices of H_1, \dots, H_n in G are relatively prime.

Proof. It is enough to prove both parts for $n = 2$. We begin with some notation. Consider the diagonal action on $\ell^2(G/H_1) \otimes \ell^2(G/H_2) \cong \ell^2(G/H_1 \times G/H_2)$ via $\lambda_{G/H_1} \otimes \lambda_{G/H_2}$. Set $x = (H_1, H_2) \in G/H_1 \times G/H_2$. Then the stabilizer of x is precisely $H_1 \cap H_2$. Define a map $\psi: G \rightarrow G/H_1 \times G/H_2$ by $\psi(g) = g \cdot x$ for all $g \in G$.

- (1). Since ψ is the orbit map associated with x , we have

$$[G : H_1 \cap H_2] = |\psi(x)| \leq [G : H_1][G : H_2].$$

When $[G : H_1]$ and $[G : H_2]$ are coprime, one checks that ψ is surjective, so we get equality.

(2). Consider the induced map $\widehat{\psi}: G/(H_1 \cap H_2) \rightarrow G/H_1 \times G/H_2$, which is given by $\widehat{\psi}(g(H_1 \cap H_2)) = (gH_1, gH_2)$ for all $g \in G$. Define $\varphi: \ell^2(G/(H_1 \cap H_2)) \rightarrow \ell^2(G/H_1 \times G/H_2)$ on the canonical orthonormal basis by setting $\varphi(\delta_{g(H_1 \cap H_2)}) = \delta_{\widehat{\psi}(g)}$ for all $g \in G$. It is clear that φ is an isometry, and that it is a unitary if $[G : H_1]$ and $[G : H_2]$ are coprime. It remains to show that φ intertwines $\lambda_{G/(H_1 \cap H_2)}$ and $\lambda_{G/H_1} \oplus \lambda_{G/H_2}$. Given $g, k \in G$, we have

$$\begin{aligned} (\lambda_{G/H_1}(k) \oplus \lambda_{G/H_2}(k)) (\varphi(\delta_{g(H_1 \cap H_2)})) &= (\lambda_{G/H_1}(k) \oplus \lambda_{G/H_2}(k)) (\delta_{(gH_1, gH_2)}) \\ &= \delta_{kgH_1, kgH_2} \\ &= \varphi(\delta_{kg(H_1 \cap H_2)}) \\ &= \varphi(\lambda_{G/(H_1 \cap H_2)}(k)(\delta_{g(H_1 \cap H_2)})). \end{aligned}$$

This finishes the proof. \square

Our final lemma is well-known, so we only sketch the proof; see also [15].

Lemma 1.3. Let G be a discrete group, let S be a subgroup of G , and let H be a subgroup of S with $[S : H] < \infty$. Then $\lambda_{G/S} \subseteq \lambda_{G/H}$.

Proof. Let $\pi: G/H \rightarrow G/S$ be the canonical quotient map. Then $\varphi: \ell^2(G/S) \rightarrow \ell^2(G/H)$ given by $\varphi(\xi) = \frac{1}{\sqrt{[S:H]}} \xi \circ \pi$ for all $\xi \in \ell^2(G/S)$, is an equivariant isometry. \square

In particular, if S is a *finite* subgroup of G , then $\lambda_{G/S} \subseteq \lambda_G$.

2. GENERALIZED BERNOULLI SHIFTS

In this section, we study a class of group actions on C^* -algebras which are obtained from permutation actions of G on (discrete) sets. First, we need to discuss how the GNS construction behaves with respect to infinite tensor products.

2.1. Infinite tensor products and the GNS construction. We briefly review the GNS construction.

Definition 2.1. Let D be a C^* -algebra, and let $\phi: D \rightarrow \mathbb{C}$ be a state on D . Define $\langle \cdot, \cdot \rangle_\phi: D \times D \rightarrow \mathbb{C}$ by $\langle a, b \rangle_\phi = \phi(a^*b)$ for all $a, b \in D$. Let \mathcal{H}_ϕ^D denote the Hilbert space obtained as the Hausdorff completion of D in the seminorm induced by $\langle \cdot, \cdot \rangle_\phi$. We denote by $\iota_\phi^D: D \rightarrow \mathcal{H}_\phi^D$ the canonical map with dense image. When D is clear from the context, we will simply write \mathcal{H}_ϕ and ι_ϕ .

There is a canonical representation of D on \mathcal{H}_ϕ by left multiplication, and we denote by \overline{D}^ϕ the weak closure of the image of D in this representation. Then ϕ extends to a faithful normal state on \overline{D}^ϕ , which we usually denote again by ϕ .

When D is a von Neumann algebra and ϕ is a normal, faithful state on it, then $\overline{D}^\phi = D$ and the extension of ϕ is just ϕ again.

We turn to infinite tensor products of Hilbert spaces.

Definition 2.2. Let \mathcal{H} be a Hilbert space, let $\eta \in \mathcal{H}$ be a unit vector, and let X be a discrete set. We define the tensor product of \mathcal{H} over X (along η) to be the completion of

$$\text{span} \left\{ \bigotimes_{x \in X} \xi_x : \xi_x \in \mathcal{H}, \text{ and } \xi_x = \eta \text{ for all but finitely many } x \in X \right\},$$

in the norm induced by the pre-inner product given by

$$\left\langle \bigotimes_{x \in X} \xi_x, \bigotimes_{x \in X} \zeta_x \right\rangle = \prod_{x \in X} \langle \xi_x, \zeta_x \rangle.$$

(Observe that all but finitely many of the multiplicative factors above are equal to 1, so that the product is indeed finite.)

It will be convenient to have a description of an orthonormal basis of an infinite tensor product of Hilbert spaces.

Lemma 2.3. Let \mathcal{H} be a Hilbert space, let $\eta \in \mathcal{H}$ be a unit vector, and let X be a discrete set. Denote by κ the dimension of \mathcal{H} . Let $\{\eta_n : n \in \kappa\}$ be an orthonormal basis for \mathcal{H} with $\eta_0 = \eta$. Then the set

$$\mathcal{F} = \{\xi : X \rightarrow \kappa : \xi(x) = 0 \text{ for all but finitely many } x \in X\}$$

can be canonically identified with an orthonormal basis for $\bigotimes_{x \in X} \mathcal{H}$.

We will need infinite (minimal) tensor products of *unital* C^* -algebras and infinite (spatial) tensor products of von Neumann algebras (along states).

Let D be a unital C^* -algebra, and let X be a countable set. Write $\mathbb{P}_f(X)$ for the set of all finite subsets of X , ordered by inclusion. We define the tensor product $\bigotimes_{x \in X} D$ to be the direct limit of the minimal tensor products $\bigotimes_{x \in S} D$, for $S \in \mathbb{P}_f(X)$, with the canonical connecting maps $\iota_{S,T} : \bigotimes_{x \in S} D \rightarrow \bigotimes_{x \in T} D$ given by $\iota_{S,T}(d) = d \otimes 1_{T \setminus S}$ for $d \in \bigotimes_{x \in S} D$, whenever $S, T \in \mathbb{P}_f(X)$ satisfy $S \subseteq T$. If ϕ is a state on D , then the direct limit of the states $\bigotimes_{x \in S} \phi$, for $S \in \mathbb{P}_f(X)$, defines a state on $\bigotimes_{x \in X} D$, which we denote by $\bigotimes_{x \in X} \phi$. If ϕ is a trace, then so is $\bigotimes_{x \in X} \phi$.

Remark 2.4. The tensor product $\bigotimes_{x \in X} D$ is canonically isomorphic to the C^* -algebra of operators on $\bigotimes_{x \in X} \mathcal{H}$ generated by the operators of the form $\bigotimes_{x \in X} a_x$, where $a_x \in D$ and $a_x = 1_D$ for all but finitely many $x \in X$.

If M is a von Neumann algebra and ϕ is a normal state on it, the (spatial) tensor product of M over X with respect to ϕ is defined similarly: by considering M as a unital C^* -algebra, we construct its infinite C^* -algebraic tensor product $\bigotimes_{x \in X} M$ as before. Then $\bigotimes_{x \in X} \phi$ is a state on it, and we define the infinite spatial tensor product $\overline{\bigotimes_{x \in X} M}$ to be the weak- $*$ closure of $\bigotimes_{x \in X} M$ in the GNS representation of $\bigotimes_{x \in X} \phi$.

For notational convenience, we will not make a distinction between the minimal tensor product of C^* -algebras and the spatial tensor product of von Neumann algebras. Here, we will denote both of them by \otimes .

Next, we show that GNS constructions commute with infinite tensor products.

Theorem 2.5. Let D be either a unital C^* -algebra or a von Neumann algebra, let X be a discrete set, and let $\phi: D \rightarrow \mathbb{C}$ be a (normal) state. Set $\eta = \iota_\phi(1_D) \in \mathcal{H}_\phi$. We will abbreviate $\phi^X = \bigotimes_{x \in X} \phi$. Then there is a canonical unitary

$$u: \bigotimes_{x \in X} \mathcal{H}_\phi \rightarrow \mathcal{H}_{\phi^X}$$

determined on a dense subset by

$$u \left(\bigotimes_{x \in X} \iota_\phi(a_x) \right) = \iota_{\phi^X} \left(\bigotimes_{x \in X} a_x \right),$$

where $a_x \in D$ for all $x \in X$, and $a_x = 1_D$ for all but finitely many $x \in X$. (The tensor product $\bigotimes_{x \in X} \mathcal{H}_\phi$ is taken along η .)

Proof. We only outline the proof in the case that D is a C^* -algebra, since the case of a von Neumann algebra is essentially identical. Let $x \in X$ and write $\psi_x: D \rightarrow \bigotimes_{x \in X} D$ for the x -th tensor factor embedding. Since $\phi = \phi^X \circ \psi_x$, it follows that ψ_x induces a Hilbert space isometry $u_x: \mathcal{H}_\phi \rightarrow \mathcal{H}_{\phi^X}$. Then u_x satisfies $u_x \circ \iota_\phi = \iota_{\phi^X} \circ \psi_x$.

Denote by $\theta_x: \mathcal{H}_\phi \rightarrow \bigotimes_{x \in X} \mathcal{H}_\phi$ the canonical isometry as the x -th tensor factor.

By the universal property of the tensor product, there exists a bounded linear map $u: \bigotimes_{x \in X} \mathcal{H}_\phi \rightarrow \mathcal{H}_{\phi^X}$ satisfying $u_x = u \circ \theta_x$ for all $x \in X$. It is then easy to check that u is a unitary, and that it satisfies the identity in the statement. We omit the details. \square

2.2. Generalized Bernoulli shifts. We begin by introducing some useful notation and terminology. Let G be a discrete group, and let X be a discrete set. By a (set) action of G on X we mean a homomorphism $\sigma: G \rightarrow \text{Perm}(X)$ from G to the group $\text{Perm}(X)$ of permutations of X . We usually abbreviate this to $G \curvearrowright^\sigma X$. Also, for $g \in G$ and $x \in X$, we write $g \cdot x$ for $\sigma(g)(x)$.

Definition 2.6. Let G be a countable group, let X be a countable set, and let $G \curvearrowright^\sigma X$ be an action. Endow X with the counting measure, and let D be a unital C^* -algebra or a tracial von Neumann algebra.

- (1) The *unitary representation associated with σ* is the unitary representation $u_\sigma: G \rightarrow \mathcal{U}(\ell^2(X))$ given by $u_\sigma(g)(\delta_x) = \delta_{g^{-1} \cdot x}$ for all $g \in G$ and all $x \in X$.

- (2) The *generalized Bernoulli shift associated with σ* is the action $\beta_{\sigma,D}: G \rightarrow \text{Aut}(\otimes_{x \in X} D)$ given by permuting the tensor factors according to $G \curvearrowright^\sigma X$.

Notation 2.7. Let G be a discrete group. We will denote by σ_G the action of left translation $G \curvearrowright G$, so that u_{σ_G} is the left regular representation $\lambda_G: G \rightarrow \mathcal{U}(\ell^2(G))$. Similarly, if H is a subgroup of G , we will denote by $\sigma_{G/H}$ the canonical action $G \curvearrowright G/H$ by left translation of left cosets, so that $u_{\sigma_{G/H}}$ is the quasiregular representation $\lambda_{G/H}: G \rightarrow \mathcal{U}(\ell^2(G/H))$ from Definition 1.1.

We will also need the Koopman construction, which is a way of obtaining unitary representations from group actions.

Definition 2.8. Let G be a countable group, let (D, ϕ) be a unital C^* -algebra with a state ϕ , and let $\alpha: G \rightarrow \text{Aut}(D)$ be a group action satisfying $\phi \circ \alpha_g = \phi$ for all $g \in G$.

- The *Koopman representation of α (with respect to ϕ)* is the unitary representation $\kappa_\phi(\alpha): G \rightarrow \mathcal{U}(\mathcal{H}_\phi)$ determined by $\kappa(\alpha)_g(\iota_\phi(a)) = \iota_\phi(\alpha_g(a))$ for all $g \in G$ and all $a \in D$.
- The *reduced Koopman representation of α (with respect to ϕ)*, denoted by $\kappa_\phi(\alpha)^{(0)}$, is the restriction of $\kappa(\alpha)$ to the orthogonal complement of $\iota_\phi(1_D)$.

Since the state ϕ will usually be fixed, we will often omit it in the notation of the Koopman representation. (The only exception to this will occur in Proposition 2.24.)

We denote by 1_G the trivial unitary representation of G on \mathbb{C} .

Remark 2.9. In the notation of the definition above, $\kappa(\alpha)$ is unitarily equivalent to $\kappa(\alpha)^{(0)} \oplus 1_G$.

We proceed to collect some elementary lemmas that will be needed later. If $\alpha: G \rightarrow \text{Aut}(D, \phi)$ is a state-preserving action of a discrete group G on a C^* -algebra D with a distinguished state ϕ , we write $\bar{\alpha}^\phi: G \rightarrow \text{Aut}(\bar{D}^\phi, \phi)$ for its weak extension. It is clear that the (reduced) Koopman representations of α and of $\bar{\alpha}^\phi$ (with respect to ϕ) agree in a natural way.

The following notation will be useful throughout.

Notation 2.10. Let G be a countable group, let $G \curvearrowright^\sigma X$ be an action of G on a countable set X , and let \mathcal{H} be a Hilbert space with a distinguished unit vector η . We denote by $\kappa_\sigma^\mathcal{H}: G \rightarrow \mathcal{U}(\otimes_{x \in X} \mathcal{H})$ the unitary representation given by permuting the tensor factors according to $G \curvearrowright^\sigma X$. When \mathcal{H} is clear from the context (as it usually will), we simply write κ_σ . We define $\kappa_\sigma^{(0)}$ to be the restriction of κ_σ to the orthogonal complement of η in $\bigotimes_{x \in X} \mathcal{H}$.

The notation for κ_σ and $\kappa_\sigma^{(0)}$ is justified by parts (2) and (3) below.

Lemma 2.11. Let G be a countable group, let (D, ϕ) be a unital C^* -algebra endowed with a state ϕ , and let $G \curvearrowright^\sigma X$ be an action of G on a countable set X .

- (1) The actions $\bar{\beta}_{\sigma,D}^\phi$ and $\beta_{\sigma,\bar{D}^\phi}$ are both canonically conjugate.
- (2) The Koopman representations of $\beta_{\sigma,D}$, of $\bar{\beta}_{\sigma,D}^\phi$, and of $\beta_{\sigma,\bar{D}^\phi}$ are all canonically unitarily equivalent to κ_σ .
- (3) The reduced Koopman representations of $\beta_{\sigma,D}$, of $\bar{\beta}_{\sigma,D}^\phi$ and of $\beta_{\sigma,\bar{D}^\phi}$ are all canonically unitarily equivalent to $\kappa_\sigma^{(0)}$.

Proof. Observe that ϕ extends to a (normal, faithful) state on \overline{D}^ϕ , which we denote again by ϕ . It is easy to check that the GNS constructions of D and \overline{D}^ϕ (with respect to ϕ) agree in a natural way. Using this observation together with Theorem 2.5, the proof of the lemma is straightforward, and we omit the details. \square

Lemma 2.12. For $n \in \mathbb{N}$, let $G \curvearrowright^{\sigma_n} X_n$ be actions, and let $G \curvearrowright^\sigma X$ denote the disjoint union of them (with $X = \bigsqcup_{n \in \mathbb{N}} X_n$). Let (D, ϕ) be a unital C^* -algebra endowed with a state ϕ .

- (1) The representations u_σ and $\bigoplus_{n \in \mathbb{N}} u_{\sigma_n}$ are canonically unitarily equivalent.
- (2) The actions $\beta_{\sigma, D}$ and $\bigotimes_{n \in \mathbb{N}} \beta_{\sigma_n, D}$ are canonically conjugate.
- (3) The representations κ_σ and $\bigotimes_{n \in \mathbb{N}} \kappa_{\sigma_n}$ are canonically unitarily equivalent.
- (4) The representations $\kappa_\sigma^{(0)}$ and $\bigotimes_{n \in \mathbb{N}} \kappa_{\sigma_n}^{(0)}$ are canonically unitarily equivalent.

Proof. All parts follow from the facts that $\ell^2(\bigsqcup_{n \in \mathbb{N}} X_n)$ is canonically isometrically isomorphic to $\bigoplus_{n \in \mathbb{N}} \ell^2(X_n)$, as witnessed by a unitary intertwining the representations σ and $\bigoplus_{n \in \mathbb{N}} \sigma_n$; and that $\bigotimes_{x \in X} D$ is canonically isomorphic to $\bigotimes_{n \in \mathbb{N}} \bigotimes_{x \in X_n} D$, as witnessed by an isomorphism that intertwines $\beta_{\sigma, D}$ and $\bigotimes_{n \in \mathbb{N}} \beta_{\sigma_n, D}$. \square

We need to recall the following definition.

Definition 2.13. Let D be a C^* -algebra, and let $\phi \in \mathcal{S}(D)$ be a state on it. Given $a \in D$, we set $\|a\|_{2, \phi} = \phi(a^*a)^{1/2}$.

Remark 2.14. If D is a C^* -algebra and $\phi \in \mathcal{S}(D)$, then $\|ab\|_{2, \phi} \leq \|a\| \|b\|_{2, \phi}$ for all $a, b \in D$.

By definition, \mathcal{H}_ϕ is the (Hausdorff) completion of D with respect to $\|\cdot\|_{2, \phi}$. On the other hand, with D_1 denoting the norm-unit ball of D , Kaplansky's density theorem implies that the weak closure \overline{D}^ϕ of D in the GNS construction for ϕ can be identified with $\overline{\text{span}} \left(\overline{D_1}^{\|\cdot\|_{2, \phi}} \right) \subseteq \mathcal{B}(\mathcal{H}_\phi)$.

The following Rokhlin-type property will allow us to relate the unitary representations u_σ from Definition 2.6 and κ_σ from Notation 2.10; see Proposition 2.22.

Definition 2.15. Let G be a countable group, let D be a C^* -algebra, let $\phi \in \mathcal{S}(D)$, and let $G \curvearrowright^\sigma X$ be an action of G on a countable set X . An action $\alpha: G \rightarrow \text{Aut}(D)$ preserving ϕ is said to have the σ -Rokhlin property (with respect to ϕ) if there exist projections $p_x^{(n)} \in \overline{D}^\phi$, for $x \in X$ and $n \in \mathbb{N}$, satisfying the following conditions:

- (1) $\lim_{n \rightarrow \infty} \|\overline{\alpha}_g^\phi(p_x^{(n)}) - p_{g \cdot x}^{(n)}\|_{2, \phi} \rightarrow 0$ for every $g \in G$ and $x \in X$.
- (2) $\lim_{n \rightarrow \infty} \|ap_x^{(n)} - p_x^{(n)}a\|_{2, \phi} \rightarrow 0$ for every $a \in D$ and $x \in X$.
- (3) $\phi(p_x^{(n)} p_y^{(n)}) = \phi(p_x^{(n)}) \phi(p_y^{(n)})$ for all $x, y \in X$ with $x \neq y$ and for all $n \in \mathbb{N}$.
- (4) $\phi(p_x^{(n)})$ does not depend on x or on n , and belongs to $\mathbb{Q} \cap (0, 1)$.

Condition (3) asserts that the different projections are ‘‘independent’’, while Condition (4) is a non-triviality condition, since any action satisfies conditions (1), (2) and (3) above with $p_x^{(n)} = 0$ for all $x \in X$ and all $n \in \mathbb{N}$.

Having a projection whose value on ϕ is rational and nontrivial is a mild condition, although it is not automatic, even if ϕ is a trace. For example, $\mathbb{C} \oplus \mathbb{C}$ with

trace τ determined by $\tau(1, 0) = \sqrt{2}/2$, has no projections of rational trace other than 0 and 1.

In contrast to the example above, our next result, which is of independent interest, shows that every infinite-dimensional tracial von Neumann algebra always has projections of arbitrarily small (rational) trace.

Theorem 2.16. Let M be an infinite dimensional von Neumann algebra, let $\phi \in \mathcal{S}(M)$ be a normal faithful state, and let $\mathcal{P}(M)$ be the set of its projections. Then there exists $0 < \varepsilon \leq 1$ such that $[0, \varepsilon] \subseteq \phi(\mathcal{P}(M))$.

Proof. Without loss of generality, we will assume that M has separable predual. We divide the proof into three cases.

Case 1: the center of M is infinite dimensional. Then there are a standard probability space (Y, μ) whose measure is not concentrated on finitely many points, and an isomorphism $(Z(M), \phi) \cong L^\infty(X, \mu)$. It suffices to show that for every sufficiently small $r \in [0, 1]$, there exists a measurable subset $E \subseteq X$ with $\mu(E) = r$. (This implies $\phi(\mathcal{P}(M)) = [0, 1]$.) Again we consider two cases.

- *Case 1.1: Y is purely atomic.* Let $r \in [0, 1]$. Find $y_1 \in A$ such that $\mu(\{y_1\}) \leq r$, and set $Y_1 = \{y_1\}$. If $\mu(Y_1) < r$, find $y_2 \in Y$ such that $\mu(\{y_1, y_2\}) \leq r$, and set $Y_2 = Y_1 \cup \{y_2\}$. This process terminates if there are $y_1, \dots, y_n \in Y$ such that $\mu(\{y_1, \dots, y_n\}) = r$. Otherwise there is a sequence $(Y_n)_{n \in \mathbb{N}}$ of subsets of Y such that $(\chi_{Y_n})_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^1(Y, \mu)$. Then its limit must be the characteristic function of a set $E \subseteq Y$ of measure r .
- *Case 1.2: Y is not purely atomic.* In this case, after replacing Y with a Borel subset, and then renormalizing the restriction of the measure, we can assume that (Y, μ) is atomless, and hence isomorphic to $[0, 1]$ with the Lebesgue measure. In this case, the conclusion is obvious.

Case 2: M is a factor. Since M is infinite dimensional, it contains the hyperfinite II_1 -factor \mathcal{R} as a (unital) subfactor. Upon restricting ϕ to any infinite dimensional abelian subalgebra of \mathcal{R} , it follows that $\phi(\mathcal{P}(M)) = [0, 1]$.

Case 3: the center of M is finite dimensional. In this case, M is a direct sum of factors $M \cong N_1 \oplus \dots \oplus N_m$. Since M is infinite dimensional, there exists $k = 1, \dots, m$ such that N_k is infinite dimensional. In this case, with 1_k denoting the unit of N_k in M , it follows from Case 2 that $[0, \phi(1_k)] \subseteq \phi(\mathcal{P}(M))$. \square

Next, we construct examples of actions with the σ -Rokhlin property using generalized Bernoulli shifts.

Proposition 2.17. Let G be a countable group, let D be a unital C^* -algebra, and let $\phi \in \mathcal{S}(D)$ be a state for which \overline{D}^ϕ is not one-dimensional. Let $G \curvearrowright^\sigma X$ be an action of G on a countable set X . Then $\bigotimes_{n \in \mathbb{N}} \beta_{\sigma, D}$ has the σ -Rokhlin property.

Moreover, if $\alpha: G \rightarrow \text{Aut}(A)$ is any action on a C^* -algebra A , and ϕ_A is an α -invariant state on it, then $\alpha \otimes \bigotimes_{n \in \mathbb{N}} \beta_{\sigma, D}$ has the σ -Rokhlin property (with respect to $\phi_A \otimes \bigotimes_{n \in \mathbb{N}} \phi$).

Proof. Observe that $\bigotimes_{n \in \mathbb{N}} \beta_{\sigma, D}$ is conjugate to $\bigotimes_{n \in \mathbb{N}} \beta_{\sigma, \bigotimes_{n \in \mathbb{N}} D}$, and that the weak closure of $\bigotimes_{n \in \mathbb{N}} D$ with respect to ϕ is infinite dimensional. Thus we may assume, without loss of generality, that \overline{D}^ϕ is infinite dimensional.

Use Theorem 2.16 to fix a projection $p \in \overline{D}^\phi$ with $\phi(p) \in \mathbb{Q} \cap (0, 1)$. For $n \in \mathbb{N}$ and $x \in X$, denote by

$$\iota_x^{(n)}: \overline{D}^\phi \rightarrow \bigotimes_{n \in \mathbb{N}} \bigotimes_{x \in X} \overline{D}^\phi$$

the canonical embedding into the (n, x) -th tensor factor, and set $p_x^{(n)} = \iota_x^{(n)}(p)$. Since $(\overline{\beta}_{\sigma, D}^\phi)_g \circ \iota_x^{(n)} = \iota_{g \cdot x}^{(n)}$ for all $g \in G$, all $x \in X$ and all $n \in \mathbb{N}$, condition (1) is guaranteed. The remaining conditions are easily checked.

For the last assertion, if $\{p_x^{(n)}: x \in X, n \in \mathbb{N}\}$ are projections as in Definition 2.15 for $\bigotimes_{n \in \mathbb{N}} \overline{\beta}_{\sigma, D}^\phi$, then $\{1 \otimes p_x^{(n)}: x \in X, n \in \mathbb{N}\}$ witness the fact that $\alpha \otimes \bigotimes_{n \in \mathbb{N}} \beta_{\sigma, D}$ has the σ -Rokhlin property with respect to $\phi_A \otimes \bigotimes_{n \in \mathbb{N}} \phi$. \square

In particular, if σ is conjugate to its infinite amplification $\sigma \times \text{id}_{\mathbb{N}}$, then $\beta_{\sigma, D}$ always has the σ -Rokhlin property.

We recall the definition of weak containment for representations in the sense of Zimmer; see also Definition 1.3 in [3].

Definition 2.18. Let G be a discrete group, and let $\mu: G \rightarrow \mathcal{U}(\mathcal{H}_\mu)$ and $\nu: G \rightarrow \mathcal{U}(\mathcal{H}_\nu)$ be unitary representations. We say that μ is *weakly contained in ν in the sense of Zimmer*, in symbols $\mu \prec_Z \nu$, if for any $\varepsilon > 0$, for any $\xi_1, \dots, \xi_n \in \mathcal{H}_\mu$, for any finite subset $F \subseteq G$, and for any $\varepsilon > 0$, there exist $\eta_1, \dots, \eta_n \in \mathcal{H}_\nu$ satisfying

$$|\langle \mu_g(\xi_j), \xi_k \rangle - \langle \nu_g(\eta_j), \eta_k \rangle| < \varepsilon$$

for all $g \in F$ and for all $j, k = 1, \dots, n$.

Finally, we say that μ and ν are *weakly equivalent in the sense of Zimmer*, written $\mu \sim_Z \nu$, if $\mu \prec_Z \nu$ and $\nu \prec_Z \mu$.

We will not be using the standard notion of weak containment, which is weaker. (In fact, one can show μ is weakly contained in ν in the usual sense if and only if μ is weakly contained, in the sense of Zimmer, in the infinite amplification of ν .) It is obvious that, when G is countable, $\mu \prec_Z \nu$ if and only if, for every separable subrepresentation μ' of μ , we have $\mu' \prec_Z \nu$.

Below, we present a characterization of weak containment in the sense of Zimmer that will be convenient for our purposes. We need a short discussion on ultrapowers of unitary representations first.

Let \mathcal{U} be a free ultrafilter over an index set I and let \mathcal{H} be a Hilbert space. Set

$$\mathcal{H}^\mathcal{U} = \ell^\infty(I, \mathcal{H}) / \{(\xi_j)_{j \in I} \in \ell^\infty(I, \mathcal{H}) : \lim_{j \rightarrow \mathcal{U}} \|\xi_j\| = 0\},$$

endowed with the quotient norm. The class in $\mathcal{H}^\mathcal{U}$ of a sequence $\xi \in \ell^\infty(I, \mathcal{H})$ is denoted by $[\xi]$. Then $\mathcal{H}^\mathcal{U}$ is a Hilbert space with respect to

$$\langle [\xi], [\eta] \rangle = \lim_{j \rightarrow \mathcal{U}} \langle \xi_j, \eta_j \rangle$$

for all $\xi, \eta \in \ell^\infty(I, \mathcal{H})$. If $\nu: G \rightarrow \mathcal{U}(\mathcal{H})$ is a unitary representation of a discrete group G on \mathcal{H} , then there is an induced representation $\nu^\mathcal{U}: G \rightarrow \mathcal{U}(\mathcal{H}^\mathcal{U})$ given by $\nu_g^\mathcal{U}([\xi]) = [(\nu_g(\xi_j))_{j \in I}]$ for all $g \in G$ and all $\xi \in \ell^\infty(I, \mathcal{H})$.

Remark 2.19. Adopt the notation from the discussion above. If ν_1 and ν_2 are unitary representations of G on Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 , respectively, it is easy to verify that $(\nu_1 \oplus \nu_2)^\mathcal{U}$ is naturally unitarily conjugate to $\nu_1^\mathcal{U} \oplus \nu_2^\mathcal{U}$. The analogous statement for infinite direct sums is not true.

The following result is probably known to the experts. Since we were not able to find it in the literature, we provide a short proof.

Proposition 2.20. Let G be a countable discrete group, and let $\mu: G \rightarrow \mathcal{U}(\mathcal{H}_\mu)$ and $\nu: G \rightarrow \mathcal{U}(\mathcal{H}_\nu)$ be unitary representations, where \mathcal{H}_μ is separable. Let \mathcal{U} be a free ultrafilter on \mathbb{N} . Then $\mu \prec_Z \nu$ if and only if $\mu \subseteq \nu^\mathcal{U}$.

Proof. Both directions are easy, and follow, for example, by combining Los' theorem (for the "if" implication) and countable saturation of ultrapowers (for the "only if" implication). See, for example, Sections 2.3 and 4.3 in [6]. \square

In our next preparatory result, the case when μ is the trivial representation is well known.

Proposition 2.21. Let G be a countable discrete group, and let $\mu: G \rightarrow \mathcal{U}(\mathcal{H}_\mu)$ and $\nu_j: G \rightarrow \mathcal{U}(\mathcal{H}_j)$, for $j = 1, \dots, n$, be unitary representations. Assume that μ is irreducible and finite-dimensional, and that $\mu \prec_Z \nu_1 \oplus \dots \oplus \nu_n$. Then there exists $k \in \{1, \dots, n\}$ such that $\mu \prec_Z \nu_k$.

Proof. Without loss of generality we can assume that \mathcal{H}_μ is separable. Let \mathcal{U} be any free ultrafilter on \mathbb{N} . Use Proposition 2.20 to choose an equivariant isometry $\varphi: \mathcal{H}_\mu \rightarrow (\mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_n)^\mathcal{U}$ witnessing the fact that $\mu \prec_Z \nu_1 \oplus \dots \oplus \nu_n$. We identify $(\mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_n)^\mathcal{U}$ equivariantly with $\mathcal{H}_1^\mathcal{U} \oplus \dots \oplus \mathcal{H}_n^\mathcal{U}$ in a canonical way via Remark 2.19. For $j = 1, \dots, n$, we denote by $\varphi_j: \mathcal{H}_\mu \rightarrow \mathcal{H}_j^\mathcal{U}$ the composition of φ with the canonical projection onto $\mathcal{H}_j^\mathcal{U}$.

Since φ is nonzero, there exists $k \in \{1, \dots, n\}$ such that φ_k is nonzero. Since μ is irreducible and finite-dimensional, by Schur's lemma φ_k is a scalar multiple of an isometry. This concludes the proof. \square

The main use of the σ -Rokhlin property is given by the combination of the next proposition with Proposition 2.24 below.

Proposition 2.22. Let G be a countable discrete group, let D be a C^* -algebra or a von Neumann algebra, let $\phi \in \mathcal{S}(D)$ be a (normal) state on it, and let $G \curvearrowright^\sigma X$ be an action of G on a countable set X . If $\alpha: G \rightarrow \text{Aut}(D)$ is a ϕ -preserving action with the σ -Rokhlin property, then

$$u_\sigma \oplus 1_G \prec_Z \kappa(\alpha) \quad \text{and} \quad u_\sigma \prec_Z \kappa(\alpha)^{(0)}.$$

Proof. It suffices to show the second part, since $\kappa(\alpha) \cong \kappa(\alpha)^{(0)} \oplus 1_G$. Suppose that α has the σ -Rokhlin property, and let $\{p_x^{(n)}: x \in X, n \in \mathbb{N}\}$ be a family of projections in \overline{D}^ϕ as in Definition 2.15. Choose positive integers $r, t \in \mathbb{N}$ with $r < t$ satisfying $\phi(p_x^{(n)}) = r/t$ for all $x \in X$ and all $n \in \mathbb{N}$. For $x \in X$ and $n \in \mathbb{N}$, set $\xi_x^{(n)} = r - tp_x^{(n)} \in \mathcal{H}_\phi$. Since $\phi(r - tp_x^{(n)}) = 0$, it follows that $\xi_x^{(n)}$ belongs to the orthogonal complement of the unit. Moreover, for $n \in \mathbb{N}$ and $x, y \in X$ with $x \neq y$, we have

$$\begin{aligned} \langle \xi_x^{(n)}, \xi_y^{(n)} \rangle &= \phi((r - tp_x^{(n)})^*(r - tp_y^{(n)})) \\ &= r^2 - rt\phi(p_x^{(n)}) - rt\phi(p_y^{(n)}) + t^2\phi(p_x^{(n)}p_y^{(n)}) = 0. \end{aligned}$$

In particular, $\xi_x^{(n)} \perp \xi_y^{(n)}$ whenever $x \neq y$. Similarly,

$$\langle \xi_x^{(n)}, \xi_x^{(n)} \rangle = r^2 - rt\phi(p_x^{(n)}) - rt\phi(p_x^{(n)}) + t^2\phi(p_x^{(n)}) = r(t - r).$$

Additionally, it is immediate that $\lim_{n \rightarrow \infty} \|(\kappa(\alpha))_g(\xi_x^{(n)}) - \xi_{g \cdot x}^{(n)}\|_{\mathcal{H}_\phi} = 0$ for all $g \in G$ and all $x \in X$.

Fix $n \in \mathbb{N}$. Define a map $\varphi_n: \ell^2(X) \rightarrow \mathcal{H}_\phi$ by $\varphi_n(\delta_x) = \xi_x^{(n)} / \sqrt{r(t-r)}$ for all $x \in X$. Then the maps $(\varphi_n)_{n \in \mathbb{N}}$ are asymptotically equivariant unitaries, witnessing the fact that u_σ is a subrepresentation of the ultrapower of $\kappa(\alpha)^{(0)}$. By Proposition 2.20, the proof is finished. \square

The following definition is standard.

Definition 2.23. Let G be a countable discrete group, let A be a C^* -algebra, and let $\alpha, \beta: G \rightarrow \text{Aut}(A)$ be actions. We say that α and β are *cocycle conjugate*, if there exist an automorphism $\theta \in \text{Aut}(A)$ and a function $u: G \rightarrow \mathcal{U}(M(A))$ satisfying

$$u_{gh} = u_g \alpha_g(u_h) \quad \text{and} \quad \beta_g = \theta \circ (\text{Ad}(u_g) \circ \alpha_g) \circ \theta^{-1}$$

for all $g, h \in G$.

Let $\alpha, \beta: G \rightarrow \text{Aut}(D)$ be actions of a discrete group G on a C^* -algebra D . If α and β are cocycle conjugate, there is in general no relationship between $\kappa(\alpha)$ and $\kappa(\beta)$, even if they both preserve the same trace. This can be seen, for example, by letting $\alpha: \mathbb{Z}_2 \rightarrow \text{Aut}(M_2)$ be the trivial action, and $\beta: \mathbb{Z}_2 \rightarrow \text{Aut}(M_2)$ be the inner action determined by the order two unitary $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. (In this case, with respect to the unique trace, $\kappa(\alpha)$ is conjugate to $\bigoplus_{j=1}^4 1_{\mathbb{Z}_2}$, while $\kappa(\beta)$ is conjugate to $\bigoplus_{j=1}^2 \lambda_{\mathbb{Z}_2}$.) In particular, if $\kappa(\alpha)$ contains a given unitary representation, we cannot conclude that so does $\kappa(\beta)$. A special case when this is indeed true is given in the following proposition.

We remark that the following proposition is the only instance in this work where traces are really necessary, specifically when showing that $\tau \circ \theta$ is β -invariant.

Proposition 2.24. Let G be a countable discrete group, let A be a C^* -algebra, let $\tau \in \mathcal{S}(A)$ be a tracial state, and let $\alpha, \beta: G \rightarrow \text{Aut}(A)$ be τ -preserving actions which are cocycle conjugate, and let $\theta \in \text{Aut}(A)$ and $u: G \rightarrow \mathcal{U}(M(A))$ be as in Definition 2.23. Let $G \curvearrowright^\sigma X$ be an action, and suppose that α has the σ -Rokhlin property with respect to τ . Then $\tau \circ \theta^{-1}$ is β -invariant, and β has the σ -Rokhlin property with respect to it. Moreover,

$$u_\sigma \oplus 1_G \prec_{\mathbb{Z}} \kappa_\tau(\beta) \quad \text{and} \quad u_\sigma \prec_{\mathbb{Z}} \kappa_\tau^{(0)}(\beta).$$

Proof. Find projections $\{p_x^{(n)}: x \in X, n \in \mathbb{N}\}$ as in Definition 2.15, and set $q_x^{(n)} = \theta(p_x^{(n)})$ for $x \in X$ and $n \in \mathbb{N}$. Using asymptotic centrality of these projections, it is easy to verify that the family $\{q_x^{(n)}: x \in X, n \in \mathbb{N}\}$ witnesses the fact that β has the σ -Rokhlin property with respect to $\tau \circ \theta^{-1}$.

By Proposition 2.22, we have

$$u_\sigma \oplus 1_G \prec_{\mathbb{Z}} \kappa_{\tau \circ \theta^{-1}}(\beta) \quad \text{and} \quad u_\sigma \prec_{\mathbb{Z}} \kappa_{\tau \circ \theta^{-1}}^{(0)}(\beta).$$

Since the Koopman representations of β with respect to τ and $\tau \circ \theta^{-1}$ are unitarily equivalent, the result follows. \square

3. ACTIONS INDUCED BY FINITE SUBQUOTIENTS

In this section, we specialize to Bernoulli shifts associated with a particular class of actions which are constructed from quasi-regular representations (see Definition 1.1). If H is a subgroup of a discrete group G , then the quasiregular representation $\lambda_{G/H}$ induces a unital homomorphism $\pi_H: C^*(G) \rightarrow \mathcal{B}(\ell^2(G/H))$, by universality of $C^*(G)$. When H is normal in G , then the image of π_H is $C_r^*(G/H)$.

Definition 3.1. Let F be a discrete group, let $\mathcal{P} \subseteq \mathbb{N}$ be a subset of relatively prime numbers. A family $\{H_p\}_{p \in \mathcal{P}}$ of subgroups of F is said to be *separated* if it satisfies the following properties:

- (S.1) $[F : H_p] = p$ for all $p \in \mathcal{P}$, and
- (S.2) for $p \in \mathcal{P}$ and $Q \subseteq \mathcal{P}$, if $\bigcap_{q \in Q} \ker(\pi_{H_q}) \subseteq \ker(\pi_{H_p})$, then $p \in Q$.

We say that F is *separated over* \mathcal{P} if it contains a separated family of subgroups indexed over \mathcal{P} . Finally, we say that F is *infinitely separated* if it is separated over an infinite set \mathcal{P} of relatively prime numbers.

Observe that we do not require the subgroups H_p in the definition above to be normal. Conditions (S.1) and (S.2) seem rather restrictive, but we can always take $\mathcal{P} = \{1\}$ and $H_1 = F$. This relatively trivial choice will be enough to prove Theorem B. Not every (nonamenable) group admits nontrivial choices, but, as we show below, the free group \mathbb{F}_∞ does.

Lemma 3.2. The free group \mathbb{F}_∞ is infinitely separated.

Proof. Let $\{x_n : n \in \mathbb{N}\}$ be free generators of \mathbb{F}_∞ , and let $\mathcal{P} \subseteq \mathbb{N}$ denote the set of prime numbers. For $p \in \mathcal{P}$, let H_p be the normal subgroup generated by

$$\{x_1, \dots, x_{p-1}, x_p^p, x_{p+1}, \dots\} \subseteq \mathbb{F}_\infty.$$

It is clear that $[\mathbb{F}_\infty : H_p] = p$ for all $p \in \mathcal{P}$, so property (S.1) is satisfied.

We proceed to check property (S.2). For $p \in \mathcal{P}$, the quotient map

$$\pi_{H_p}: C^*(\mathbb{F}_\infty) \rightarrow C^*(\mathbb{Z}_p) \cong \mathbb{C}^p$$

can be described as follows. Identify $C^*(\mathbb{F}_\infty)$ with the full free product $*_{n=1}^\infty C(S^1)$ amalgamated over \mathbb{C} (identified with the algebra of scalar multiples of the identity in $C(S^1)$) by sending x_n to the canonical generator of the n -th free factor $C(S^1)$. Given $f \in C(S^1)$ and $n \in \mathbb{N}$, set $f^{(n)} = 1 * \dots * f * 1 \dots \in *_{n=1}^\infty C(S^1)$, where the nontrivial entry is in the n -th position. Then

$$\pi_{H_p}(f^{(n)}) = \begin{cases} (f(1), \dots, f(1)), & \text{if } n \neq p \\ (f(1), f(e^{2\pi i/p}), \dots, f(e^{2\pi i(p-1)/p})) & \text{if } n = p \end{cases}$$

Now $p \in \mathcal{P}$ and $Q \subseteq \mathcal{P}$ and suppose that $p \notin Q$. Let $f \in C(S^1)$ be any function satisfying $f(1) = 0$ and $f(e^{2\pi i/p}) \neq 0$. Then $f^{(p)}$ belongs to $\ker(\pi_{H_q})$ for all $q \in Q$, but not to $\ker(\pi_{H_p})$. Thus (S.2) is satisfied as well. \square

From now on, we fix a discrete group G , a subset $\mathcal{P} \subseteq \mathbb{N}$ of relatively prime numbers, and a subgroup $F \leq G$ which is separated over \mathcal{P} , as witnessed by a family $\{H_p\}_{p \in \mathcal{P}}$ of subgroups of F .

Notation 3.3. Given $p \in \mathcal{P}$, we establish the following notations:

- we write $G_p = G/H_p$ and $F_p = F/H_p$ for the coset spaces;

- we write $G \curvearrowright^{\sigma_G^p} G_p$ and $F \curvearrowright^{\sigma_F^p} F_p$ for the canonical translation actions.

Definition 3.4. For a (possibly empty) subset $P \subseteq \mathcal{P}$, set

$$X_P^G = \bigsqcup_{n \in \mathbb{N}} G \sqcup \bigsqcup_{n \in \mathbb{N}} \bigsqcup_{p \in P} G_p \quad \text{and} \quad X_P^F = \bigsqcup_{n \in \mathbb{N}} F \sqcup \bigsqcup_{n \in \mathbb{N}} \bigsqcup_{p \in P} F_p,$$

and define actions $G \curvearrowright^{\sigma_P^G} X_P^G$ and $F \curvearrowright^{\sigma_P^F} X_P^F$ as follows:

- σ_P^G acts on each copy of G via σ_G , and on each copy of G_p via σ_G^p ;
- σ_P^F acts on each copy of F via σ_F , and on each copy of F_p via σ_F^p .

For later use, we identify here the restriction of σ_P^G to F with (some amplification of) σ_P^F . Recall that G is countable by assumption.

Lemma 3.5. Let the notation be as above, and let $P \subseteq \mathcal{P}$. Then the restriction $\sigma_P^G|_F$ of σ_P^G to F is conjugate to σ_P^F .

Proof. We denote by $F \backslash G$ the right coset space of F in G . It is enough to show that $\sigma_G|_F$ is conjugate to $\bigsqcup_{x \in F \backslash G} \sigma_F$, and that $\sigma_G^p|_F$ is conjugate to $\bigsqcup_{x \in F \backslash G} \sigma_F^p$ for

all $p \in \mathcal{P}$. For the first one, choose a section $t: F \backslash G \rightarrow G$, and define a map $f: G \rightarrow F \backslash G \times F$ by $f(g) = (Fg, g \cdot t(Fg)^{-1})$ for all $g \in G$. It is clear that f is a bijection, and we claim that it intertwines $\sigma_G|_F$ and $\text{id}_{F \backslash G} \times \sigma|_F$. Given $k \in F$ and $g \in G$, we have

$$\begin{aligned} f(\sigma_G(k)g) &= f(kg) \\ &= (Fkg, kg \cdot t(Fkg)^{-1}) = (Fg, kg \cdot t(Fg)^{-1}) \\ &= (\text{id}_{F \backslash G} \times \sigma_F(k))(f(g)), \end{aligned}$$

as desired. The result follows, since $\text{id}_{F \backslash G} \times \sigma|_F$ is precisely $\bigsqcup_{x \in F \backslash G} \sigma_F^p$.

The proof for σ_P^G is completely analogous, using right coclasses. \square

Recall the notation u_σ from Definition 2.6 and $\kappa_\sigma^{\mathcal{H}}$ from Notation 2.10.

Theorem 3.6. Let (\mathcal{H}, η) be a separable Hilbert space with a distinguished unit vector. Let $P \subseteq \mathcal{P}$ be a (possibly empty) subset, and let σ_P^G be as in the discussion above. Following Notation 2.10, we abbreviate $\kappa_{\sigma_P^G}^{\mathcal{H}}$ to simply $\kappa_{\sigma_P^G}$. Then

$$\kappa_{\sigma_P^G}|_F \subseteq u_{\sigma_P}|_F \oplus 1_G \quad \text{and} \quad \kappa_{\sigma_P^G}^{(0)}|_F \subseteq u_{\sigma_P}|_F.$$

Proof. We begin by providing an alternative description of the restriction $\kappa_{\sigma_P}|_F$ of κ_{σ_P} to F . Find an orthonormal basis $\{\eta_n: n \in \mathbb{N}\}$ of \mathcal{H} with $\eta_0 = \eta$, and set

$$\mathcal{F} = \{\xi: X_P \rightarrow \mathbb{N}: \xi(x) = 0 \text{ for all but finitely many } x \in X_P\}.$$

By Lemma 2.3, \mathcal{F} is an orthonormal basis for $\bigotimes_{x \in X_P} \mathcal{H}$. Moreover, the unitary representation $\kappa_{\sigma_P}|_F$ restricts to an action of F on \mathcal{F} , which is given by

$$(\kappa_{\sigma_P}(g)(\xi))(x) = \xi(g^{-1} \cdot x)$$

for $g \in F$, for $\xi \in \mathcal{F}$ and for $x \in X$. We denote by $\xi_0 \in \mathcal{F}$ the function which is identically zero, and write \mathcal{F}_0 for $\mathcal{F} \setminus \{\xi_0\}$. Then \mathcal{F}_0 is an orthonormal basis for the orthogonal complement of η , and it is also invariant under $\kappa_{\sigma_P}|_F$. (In fact, the restriction of $\kappa_{\sigma_P}|_F$ to $\overline{\text{span}}(\mathcal{F}_0)$ is $\kappa_{\sigma_P}^{(0)}|_F$.)

Denote by \mathcal{G} and \mathcal{G}_0 the F -orbit spaces of \mathcal{F} and \mathcal{F}_0 , respectively. For $\xi \in \mathcal{F}$, we write $[\xi] \in \mathcal{G}$ for its equivalence class, and $\text{Stab}_F(\xi)$ for the stabilizer subgroup of F . It follows that

$$\kappa_{\sigma_P^G}|_F \cong \bigoplus_{[\xi] \in \mathcal{G}} \lambda_{F/\text{Stab}_F(\xi)} \quad \text{and} \quad \kappa_{\sigma_P^{(0)}}|_F \cong \bigoplus_{[\xi] \in \mathcal{G}_0} \lambda_{F/\text{Stab}_F(\xi)}.$$

It is clear that $\lambda_{F/\text{Stab}_F(\xi_0)}$ is the trivial representation of G .

Claim: Fix $\xi \in \mathcal{F}_0$. Then $\lambda_{F/\text{Stab}_F(\xi)}$ is unitarily contained in $u_{\sigma_P}|_F$.

By part (1) of Lemma 2.12 and Lemma 3.5, the representation $u_{\sigma_P}|_F$ is unitarily equivalent to $\bigoplus_{n \in \mathbb{N}} \lambda_F \oplus \bigoplus_{n \in \mathbb{N}} \bigoplus_{p \in P} \lambda_{F_p}$. We divide the proof into two cases.

Assume first that the support $\text{supp}(\xi)$ of ξ meets one of the copies of G in X_P . If $k \in \text{Stab}_F(\xi)$, then $k \cdot \text{supp}(\xi) = \text{supp}(\xi)$. By restricting this equality to the copy of G which $\text{supp}(\xi)$ intersects, we deduce that only finitely many group elements $k \in F$ satisfy this identity, so $\text{Stab}_F(\xi)$ is finite. (In fact, in this case even $\text{Stab}_G(\xi)$ is finite.) Thus $\lambda_{F/\text{Stab}_F(\xi)} \subseteq \lambda_F$ by Lemma 1.3, so $\lambda_{F/\text{Stab}_F(\xi)} \subseteq u_{\sigma_P^G}|_F$.

Assume now that the support of ξ does not meet any of the copies of G in X_P . Let (q_1, \dots, q_m) be a minimal tuple of elements of P (potentially with repetitions) such that $\text{supp}(\xi) \subseteq \bigsqcup_{j=1}^m G_{q_j}$. Denote by p_1, \dots, p_n the distinct values of q_1, \dots, q_m , and set $H = H_{p_1} \cap \dots \cap H_{p_n}$. Then $\text{Stab}_F(\xi) \supseteq H$. In particular, $\text{Stab}_F(\xi)$ has finite index in F . Using Lemma 1.3 at the first step, and part (2) of Lemma 1.2 at the second, we conclude that

$$\lambda_{F/\text{Stab}_F(\xi)} \subseteq \lambda_{F/H} \cong \lambda_{F_{p_1}} \oplus \dots \oplus \lambda_{F_{p_n}} \subseteq u_{\sigma_P^G}|_F.$$

The claim is proved. It follows that

$$\kappa_{\sigma_P^{(0)}}|_F \cong \bigoplus_{[\xi] \in \mathcal{G}_0} \lambda_{F/\text{Stab}_F(\xi)} \subseteq u_{\sigma_P^G}|_F,$$

and thus also $\kappa_{\sigma_P^G}|_F \subseteq u_{\sigma_P^G}|_F \oplus 1_G$, as desired. \square

The following theorem connects weak equivalence of certain representations to cycle conjugacy of the associated generalized Bernoulli shifts (see Definition 2.6). In particular, it implies that for a *nonamenable* countable discrete group G , the generalized Bernoulli actions obtained from different subsets of \mathcal{P} are not cocycle equivalent.

Theorem 3.7. Let D be a tracial unital C*-algebra or a tracial von Neumann algebra, let G be a discrete group containing a nonamenable subgroup F which is separated over some set $\mathcal{P} \subseteq \mathbb{N}$, and adopt the notation from before Theorem 3.6. For $P, Q \subseteq \mathcal{P}$, the following are equivalent:

- (1) $P = Q$.
- (2) $u_{\sigma_P^G} \cong u_{\sigma_Q^G}$.
- (3) $u_{\sigma_P^G}|_F \sim_Z u_{\sigma_Q^G}|_F$
- (4) $\beta_{\sigma_P^G, D}$ is cocycle conjugate to $\beta_{\sigma_Q^G, D}$.
- (5) For every (normal) tracial state τ on D for which \overline{D}^τ is separable and not one-dimensional, the action $\beta_{\sigma_P^G, \overline{D}^\tau}$ is cocycle conjugate to $\beta_{\sigma_Q^G, \overline{D}^\tau}$.

Proof. The implications (1) \Rightarrow (2) \Rightarrow (3), and (1) \Rightarrow (4) \Rightarrow (5) are immediate. It therefore suffices to prove that (5) implies (3), and that (3) implies (1).

(5) \Rightarrow (3). Assume that $\beta_{\sigma_P^G, \overline{D}}$ is cocycle conjugate to $\beta_{\sigma_Q^G, \overline{D}}$. We apply Proposition 2.22 at the first step with $\alpha = \beta_{\sigma_P^G, \overline{D}}$, $\beta = \beta_{\sigma_Q^G, \overline{D}}$ and $\sigma = \sigma_P^G$ (and restricting to the subgroup F), and Theorem 3.6 at the second step, to deduce that

$$u_{\sigma_P^G}|_F \prec_Z \kappa_{\sigma_Q^G, \overline{D}}^{(0)}|_F \subseteq u_{\sigma_Q^G}|_F.$$

In view of Proposition 2.24, by reversing the roles of P and Q , we conclude that $u_{\sigma_P^G}|_F \sim_Z u_{\sigma_Q^G}|_F$, as desired.

(3) \Rightarrow (1). Let $p \in P$, and suppose that

$$\lambda_{F_p} \prec_Z u_{\sigma_Q^G}|_F \cong \bigoplus_{n \in \mathbb{N}} \lambda_F \oplus \bigoplus_{n \in \mathbb{N}} \bigoplus_{q \in Q} \lambda_{F_q}.$$

Let μ be an irreducible subrepresentation of λ_{F_p} . Then it follows from Proposition 2.21 that either $\mu \prec_Z \bigoplus_{n \in \mathbb{N}} \lambda_F$ or $\mu \prec_Z \bigoplus_{n \in \mathbb{N}} \bigoplus_{q \in Q} \lambda_{F_q}$. Since clearly $1_F \prec_Z \mu$, the first case would imply that F is amenable, which contradicts our assumptions. Thus, we must have $\mu \prec_Z \bigoplus_{n \in \mathbb{N}} \bigoplus_{q \in Q} \lambda_{F_q}$. Since this applies to every irreducible subrepresentation of λ_p , and λ_p is unitarily conjugate to their sum, it follows that $\lambda_{F_p} \prec_Z \bigoplus_{n \in \mathbb{N}} \bigoplus_{q \in Q} \lambda_{F_q}$.

Considering the induced representations of the full group C^* -algebra $C^*(F)$ (using the notation from Definition 3.1), and taking kernels, we deduce that

$$\bigcap_{q \in Q} \ker(\pi_{H_q}) \subseteq \ker(\pi_{H_p}).$$

By the properties of the family $\{H_p : p \in \mathcal{P}\}$ (specifically, by (S.2)), we have $p \in Q$. Since $p \in P$ was arbitrary, this shows that $P \subseteq Q$, as desired. \square

In order to deal with \mathcal{Z} -stable C^* -algebras, or with McDuff factors, we will also need the following variation of Theorem 3.7.

Theorem 3.8. Let D and A be both tracial C^* -algebras or tracial von Neumann algebras, with D unital, let G be a nonamenable countable discrete group, and adopt the notation from before Theorem 3.6. For $P, Q \subseteq \mathcal{P}$, the following are equivalent:

- (1) $P \cup \{1\} = Q \cup \{1\}$.
- (2) $\text{id}_A \otimes \beta_{\sigma_P^G, D}$ is cocycle conjugate to $\text{id}_A \otimes \beta_{\sigma_Q^G, D}$.
- (3) For every (normal) tracial states $\tau \in \mathcal{S}(D)$ and $\tau_A \in \mathcal{S}(A)$ for which \overline{D}^τ and \overline{A}^{τ_A} are separable and not one-dimensional, the action $\text{id}_{\overline{A}^{\tau_A}} \otimes \beta_{\sigma_P^G, \overline{D}}$ is cocycle conjugate to $\text{id}_{\overline{A}^{\tau_A}} \otimes \beta_{\sigma_Q^G, \overline{D}}$.

Proof. The proof is almost identical to that of Theorem 3.7, using the fact that the Koopman representation of $\text{id}_A \otimes \beta_{\sigma_P^G, D}$ is conjugate to that of $\beta_{\sigma_{P \cup \{1\}}^G, D}$. \square

4. MAIN RESULTS

In this section, we prove Theorem B and Theorem C from the introduction, which make significant contributions to part (2) of Conjecture A.

Definition 4.1. Let D be a tracial C^* -algebra, and let $\theta \in \text{Aut}(D)$. We say that θ is *strongly outer* if for every $\tau \in T(D)$ satisfying $\tau \circ \theta = \tau$, the weak extension

$\bar{\theta}^\tau \in \text{Aut}(\bar{D}^\tau)$ is outer. An action $\alpha: G \rightarrow \text{Aut}(D)$ of a discrete group G is said to be *strongly outer* if α_g is a strongly outer automorphism of D for every $g \in G \setminus \{1\}$.

For later use, we record here the following standard fact.

Proposition 4.2. Let G be an infinite, countable group, and let D be a tracial unital C^* -algebra. Then the Bernoulli shift $\beta_D: G \rightarrow \text{Aut}(\bigotimes_{g \in G} D)$ is strongly outer. More generally, if A is any C^* -algebra and $\alpha: G \rightarrow \text{Aut}(A)$ is any action, then $\beta_D \otimes \alpha$ is strongly outer.

Theorem 4.3. Let G be a nonamenable countable discrete group, and let D be a tracial C^* -algebra or a tracial von Neumann algebra. Then the Bernoulli shift $\beta_D: G \rightarrow \text{Aut}(\bigotimes_{g \in G} D)$ does not tensorially absorb the trivial action on any unital C^* -algebra (or von Neumann algebra) tensorially. In particular, β_D and $\beta_D \otimes \text{id}_D$ are two non-cocycle conjugate, strongly outer actions of G on $\bigotimes_{n \in \mathbb{N}} D$.

Proof. Take $P = \emptyset$ and $Q = 1$, so that, in the notation from before Proposition 2.22, one has $\sigma_P^G = \lambda_G^\infty$ and $\sigma_Q^G = \lambda_G^\infty \oplus 1_G^\infty$. Choose a (normal) state $\phi \in \mathcal{S}(D)$ for which \bar{D}^ϕ is separable and not one-dimensional. It then follows from the equivalence between parts (1) and (5) in Theorem 3.7 that $\beta_{\sigma_P^G, D} = \beta_D$ is not cocycle conjugate to $\beta_{\sigma_Q^G, D} = \beta_D \otimes \text{id}_D$. Finally, these actions are strongly outer by Proposition 4.2. \square

We have arrived at Theorem B.

Theorem 4.4. Let G be a countable group, and let \mathcal{D} be a finite strongly self-absorbing C^* -algebra. Then the following are equivalent:

- (1) G is amenable;
- (2) The Bernoulli shift $\beta_{\mathcal{D}}: G \rightarrow \text{Aut}(\bigotimes_{g \in G} \mathcal{D})$ is cocycle conjugate to $\beta_{\mathcal{D}} \otimes \text{id}_{\mathcal{Z}}$.

Proof. That (1) implies (2) is a consequence of Corollary 4.8 in [9] (and it can also be deduced from the proof of Theorem 1.1 in [19]). The converse follows immediately from Theorem 4.3, since $\bigotimes_{n \in \mathbb{N}} \mathcal{D} \cong \mathcal{D}$. \square

The theorem above complements the results in [19] and [9] quite nicely: while every strongly outer action of an amenable group on a tracial strongly self-absorbing C^* -algebra absorbs the identity on \mathcal{Z} tensorially, this result fails for every nonamenable group. In particular, we deduce a weak version of part (2) of Conjecture A: any nonamenable group admits at least two non-cycle conjugate, strongly outer actions on \mathcal{D} , namely, the Bernoulli shift $\beta_{\mathcal{D}}$ and $\beta_{\mathcal{D}} \otimes \text{id}_{\mathcal{Z}}$.

Our strongest result, which will imply Theorem C and Theorem D, assumes the existence of an infinitely separated subgroup, in the sense of Definition 3.1.

Theorem 4.5. Let G be a countable nonamenable group containing an infinitely separated subgroup, and let D and A be both tracial C^* -algebras or tracial von Neumann algebras, with D unital. Then there exist uncountably many pairwise non-cocycle conjugate, strongly outer actions of G on $A \otimes \bigotimes_{n \in \mathbb{N}} D$.

Proof. Assume first that both D and A are C^* -algebras. Let F be a subgroup of G and let \mathcal{P} be an infinite subset of \mathbb{N} over which F is separated. For a subset $P \subseteq \mathcal{P}$ containing 1, consider the action $G \curvearrowright^{\sigma_P^G} X_P$ defined before Theorem 3.6. Let $\tau \in \mathcal{S}(D)$ be a tracial state for which \bar{D}^τ is separable and not one-dimensional. By

the equivalence between (1) and (2) in Theorem 3.8, the family $\{\text{id}_A \otimes \beta_{\sigma_P^G, D} : 1 \in P \subseteq \mathcal{P}\}$ consists of pairwise non-cocycle conjugate actions of G on $A \otimes \bigotimes_{n \in \mathbb{N}} D$. Since \mathcal{P} is infinite, this family is uncountable. Finally, these actions are strongly outer by Proposition 4.2, so the proof is finished.

The same argument applies in the case of tracial von Neumann algebras, using the equivalence between (1) and (3) in Theorem 3.8. \square

As an immediate consequence, we obtain the following, which implies Theorem C.

Corollary 4.6. Let G be a nonamenable group containing an infinitely separated subgroup, and let A be a tracial \mathcal{Z} -stable C^* -algebra. Then there exist uncountably many pairwise non-cocycle conjugate, strongly outer actions of G on A , which are pointwise asymptotically inner.

Proof. This follows immediately from Theorem 4.5, by taking $D = \mathcal{Z}$. The statement about pointwise asymptotic innerness follows from the fact that any automorphism of \mathcal{Z} is automatically asymptotically inner. \square

By Lemma 3.2, the corollary above applies to any group containing a free group, and hence implies Theorem C. Also, by Winter's theorem [22], the result above applies to any strongly self-absorbing C^* -algebra, thus proving part (2) of Conjecture A for groups containing a free group. This can be regarded as a the noncommutative analog of Ioana's celebrated result [12].

A similar result holds for McDuff von Neumann algebras, thus strengthening a result of Brothier-Vaes (Theorem B in [2]) for groups containing a free group. Even in the case of \mathcal{R} , our methods offer a simpler and shorter proof of their theorem, which avoids Popa's very advanced theory of spectral gap rigidity; see [18] and [17].

Corollary 4.7. Let G be a nonamenable group containing an infinitely separated subgroup, and let M be a tracial McDuff von Neumann algebra. Then there exist uncountably many pairwise non-cocycle conjugate, outer actions of G on M .

Proof. This follows immediately from Theorem 4.5, by taking $D = \mathcal{R}$. \square

In the measurable setting, Epstein [5] combined Ioana's result from [12] with Gaboriau-Lyons' solution [7] to the von Neumann problem, to show that *any* non-amenable group admits a continuum of non-orbit equivalent free, ergodic actions. In order to prove part (2) of Conjecture A for all nonamenable groups, one could attempt a similar approach of inducing actions from \mathbb{F}_2 to any amenable group. For this approach to work, however, one would need a noncommutative analog of the result of Gaboriau-Lyons. This suggests the following interesting problem:

Problem 4.8. Is there an analog of Gaboriau-Lyon's measurable solution to the von Neumann problem in the context of strongly outer actions on (finite) strongly self-absorbing C^* -algebras? And for outer actions on the hyperfinite II_1 -factor?

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EUSEBIO GARDELLA, WESTFÄLISCHE WILHELMS-UNIVERSITÄT MÜNSTER, FACHBEREICH MATHEMATIK, EINSTEINSTRASSE 62, 48149 MÜNSTER, GERMANY

E-mail address: gardella@uni-muenster.de

URL: <http://wwwmath.uni-muenster.de/u/gardella/>

MARTINO LUPINI, MATHEMATICS DEPARTMENT, CALIFORNIA INSTITUTE OF TECHNOLOGY, 1200 EAST CALIFORNIA BOULEVARD, MAIL CODE 253-37, PASADENA, CA 91125

E-mail address: lupini@caltech.edu

URL: <http://www.lupini.org/>