THE COMPLEXITY OF CONJUGACY, ORBIT EQUIVALENCE, AND VON NEUMANN EQUIVALENCE OF ACTIONS OF NONAMENABLE GROUPS

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Abstract. Building on work of Popa, Ioana, and Epstein–Törnquist, we show that, for every nonamenable countable discrete group Γ, the relations of conjugacy, orbit equivalence, and von Neumann equivalence of free ergodic (or weak mixing) measure preserving actions of Γ on the standard atomless probability space are not Borel, thus answering questions of Kechris. This is an optimal and definitive result, which establishes a neat dichotomy with the amenable case, since any two free ergodic actions of an amenable group on the standard atomless probability space are orbit equivalent by classical results of Dye and Ornstein–Weiss. The statement about conjugacy solves the nonamenable case of Halmos' conjugacy problem in Ergodic Theory, originally posed by Halmos in 1956 for ergodic transformations.

In order to obtain these results, we study ergodic (or weak mixing) class-bijective extensions of a given ergodic countable probability measure preserving equivalence relation R. When R is nonamenable, we show that the relations of isomorphism and von Neumann equivalence of extensions of R are not Borel. When R is amenable, all the extensions of R are again amenable, and hence isomorphic by classical results of Dye and Connes–Feldman–Weiss. This approach allows us to extend the results about group actions mentioned above to the case of nonamenable locally compact unimodular groups, via the study of their cross-section equivalence relations.

1. Introduction

Let Γ be countable discrete group, and let (Y, ν) be a standard atomless probability space. A *probability-measure-preserving* (pmp) *action* of Γ on (Y, ν) is a homomorphism $\theta\colon \Gamma \to \mathrm{Aut}(Y, \nu)$ from Γ to the group of measure-preserving Borel automorphisms of (Y, ν). Two pmp actions θ and θ' of Γ on (Y, ν) are *conjugate* if there exists $T \in \mathrm{Aut}(Y, \nu)$ such that $T \circ \theta_\gamma = \theta'_\gamma \circ T$ for every $\gamma \in \Gamma$. The *conjugacy problem* in ergodic theory, initially formulated by Halmos for $\mathbb{Z}$-actions [28], asks whether there exists a *method* to determine whether two given pmp actions of Γ on (Y, ν) are conjugate.

By Halmos' own admission, this is a vague question, but it can be given a precise meaning in the context of Borel complexity. The space $\mathrm{Act}_\Gamma(Y, \nu)$ of pmp actions of Γ on (Y, ν) is endowed with a canonical Polish topology [38, Section 10], and the subset $\mathrm{FE}_\Gamma(Y, \nu)$ of free ergodic actions is a Polish space with the induced topology. The same holds for the subset $\mathrm{FWM}_\Gamma(Y, \nu)$ of free weak mixing actions. An instance of Halmos' conjugacy problem is the following:

Question. *(Kechris, [38, 18.(IVb)]). Let Γ be a countable discrete group and let (Y, ν) be a standard atomless probability space.*

(1) Is the relation of conjugacy of free ergodic pmp actions of Γ on (Y, ν) a Borel subset of $\mathrm{FE}_\Gamma(Y, \nu) \times \mathrm{FE}_\Gamma(Y, \nu)$ endowed with the product topology?

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(2) *Is the relation of conjugacy of free weak mixing pmp actions of* Γ *on* (Y,ν) *a Borel subset of* $\mathrm{FWM}_\Gamma(Y,\nu)\times\mathrm{FWM}_\Gamma(Y,\nu)$ *endowed with the product topology?*

Observe that a negative answer for part (2) implies a negative answer for part (1), since $\mathrm{FWM}_\Gamma(Y,\nu)$ is a Borel subset of $\mathrm{FE}_\Gamma(Y,\nu)$. The question above has been addressed in [21] in the case of $\mathbb{Z}$-actions, where it is shown that the relation of conjugacy of free ergodic pmp transformations is *not* Borel. Epstein and Törnquist showed in [17] that when Γ is a group containing a nonabelian free group as an (almost) normal subgroup, then the relation of conjugacy of free weak mixing actions of Γ on (Y,ν) is not Borel.

In the present paper, we give a complete answer to both parts of the question above for actions of arbitrary nonamenable groups:

Theorem A. *Let* Γ *be a nonamenable countable group and let* (Y,ν) *be a standard atomless probability space. Then the relations of conjugacy of free weak mixing (or ergodic) pmp actions of* Γ *on* (Y,ν) *is not Borel.*

In recent years, ergodic theory has focused on classification of actions up to other two equivalence relations: orbit equivalence and von Neumann equivalence. A *countable Borel equivalence relation* on a standard probability space (Y,ν) is a Borel equivalence relation R on Y such that the R-class of almost every point in Y is countable. A countable Borel equivalence relation R is said to be *probability-measure-preserving* (pmp) if every Borel automorphisms of Y that maps almost everywhere R-classes to R-classes is automatically a measure-preserving automorphism of (Y,ν). A countable pmp Borel equivalence relation R is called *ergodic* if every R-invariant Borel subset of Y is either null or co-null. One can associate with a countable pmp Borel equivalence relation R a von Neumann algebra $L(R)$, which is a II_1 factor if and only if (Y,ν) is atomless and R is ergodic [18, 19].

Two ergodic countable Borel equivalence relations R and R' on the standard atomless probability space are:

- *isomorphic* if there is a measure-preserving automorphism of (Y,ν) that maps almost everywhere R-classes to R'-classes;
- *stably isomorphic* if there exist Borel subsets X and X' of Y that meet every class of R and, respectively, R', and a measure-preserving Borel isomorphism $\varphi : X\to X'$ that maps R-classes to R'-classes;
- *von Neumann equivalent* if the II_1 factors $L(R)$ and $L(R')$ are isomorphic;
- *stably von Neumann equivalent* if the II_1 factors $L(R)$ and $L(R')$ are stably isomorphic, i.e. there exist nonzero projections $p\in L(R)$ and $q\in L(R')$ such that the corners $pL(R)p$ and $qL(R')q$ are isomorphic.

Given a pmp action θ of a countable discrete group Γ on a standard atomless probability space (Y,ν), one can define its *orbit equivalence relation* R on (Y,ν) by setting xRy if and only if there exists $\gamma\in\Gamma$ such that $\theta_\gamma(x)=y$. When θ is free and ergodic, the von Neumann algebra $L(R)$ is a II_1 factor isomorphic to the group-measure space construction $\Gamma\ltimes^\theta L^\infty(Y,\nu)$ of Murray and von Neumann [43]. Two free ergodic action θ and θ' of Γ on the standard atomless probability space are:

- *(stably) orbit equivalent* if their orbit equivalence relations are (stably) isomorphic,
- *(stably) von Neumann equivalent* if their orbit equivalence relations are (stably) von Neumann equivalent.

When Γ is an *amenable group*, it follows from results of Dye [13] and Ornstein–Weiss [45] that any two free ergodic actions of Γ on (Y,ν) are orbit equivalent. On the other hand, when Γ is nonamenable, Epstein and Ioana [16, 31] showed that the relations of orbit equivalence and von Neumann equivalence for free, ergodic, pmp actions have uncountably many classes. This has motivated Kechris to ask the following:

Question. *(Kechris, [38, 18.(IVb)]). Let* Γ *be a nonamenable countable group, and let* (Y,ν) *be a standard atomless probability space. Are the relations of orbit equivalence and von Neumann equivalence for free ergodic actions of* Γ *on* (Y,ν) *Borel?*

The results, as well as the proofs, from [16, 31] that such relations have uncountably many classes do not address the question above. Indeed, in these papers one encodes irreducible representations of $\mathbb{F}_2$ up to unitary equivalence within free ergodic actions of Γ up to orbit equivalence; see also [32]. However, the relation of unitary equivalence of irreducible representations of $\mathbb{F}_2$ *is* Borel. Therefore, a new set of ideas and techniques is needed to answer this question. In this paper, we completely settle the matter:

Theorem B. *Let Γ be the nonamenable countable discrete group, and let (Y, ν) be the standard atomless probability space. Then the relations of orbit equivalence, stable orbit equivalence, von Neumann equivalence, and stable von Neumann equivalence of free weak mixing (or ergodic) pmp actions of Γ on (Y, ν) are not Borel.*

Together with the results of Dye and Ornstein–Weiss for amenable groups recalled above, Theorem B implies the following dichotomy.

Corollary C. *Let Γ be a countable discrete group, and let (Y, ν) be the standard atomless probability space. Suppose that E is one of the following equivalence relations: orbit equivalence, stable orbit equivalence, von Neumann equivalence, or stable von Neumann equivalence of free weak mixing (or ergodic) pmp actions of Γ on (Y, ν).*

(a) If Γ is amenable, then E has a single equivalence class.
(b) If Γ is not amenable, then E is not Borel.

The methods that we use to obtain Theorem A and Theorem B contain two main innovations. First, we introduce the notion of property (T) for a *triple* $\Delta \leq \Lambda \leq \Gamma$ consisting a group Γ together with nested subgroups $\Delta \subseteq \Lambda \subseteq \Gamma$. The classical notion of property (T) for pairs $\Lambda \leq \Gamma$ corresponds to the case when Δ is the trivial subgroup. In this context, we present an extension of Popa's cocycle superrigidity theorem for malleable actions [51], which plays a crucial role in our construction. Specifically, these techniques allow us to construct a family, parametrized by *countable abelian groups*, of free ergodic actions of $\mathbb{F}_2$. The same construction applies after replacing $\mathbb{F}_2$ with more general nonamenable groups.

The next fundamental ingredient, to carry over the argument from free groups to arbitrary amenable groups, is the combination of the Gaboriau–Lyons measurable solution to von Neumann's problem from [24] with the theory of coinduction for actions of groupoids, initially considered for principal groupoids by Epstein in [16]. The case of groups containing a free group is technically easier, as it does not require the Gaboriau–Lyons theorem, and it only uses the theory of coinduction for group actions. Since this case contains *in nuce* all the fundamental tools and ideas needed to prove the general result, while also being significantly less technical, a sketch of its proof is presented in Section 2.

Rigidity questions have also been intensively studied for actions of locally compact, second countable, unimodular groups. For example, it follows from [10] that all free pmp ergodic actions of an amenable locally compact, second countable, unimodular group on standard probability spaces are orbit equivalent. On the other hand, the nonamenable case was considerably more difficult to approach in comparison with the discrete case. In [61], Zimmer showed that any connected semisimple Lie group with real rank at least two, finite center, and no compact quotients, admits uncountably many pairwise not orbit equivalent free ergodic pmp actions. In a very recent breakthrough, Bowen, Hoff, and Ioana [8] showed that the same conclusion holds for any nonamenable locally compact, second countable, unimodular group. However, their proof offers no information on whether the relation of orbit equivalence is Borel or not. Here, we also settle this matter:

Theorem D. *Let G be a nonamenable locally compact, second countable, unimodular group, and let (Y, ν) be the standard atomless probability space. Then the relations of conjugacy, orbit equivalence, stable orbit equivalence, von Neumann equivalence, and stable von Neumann equivalence of free ergodic pmp actions of G on (Y, ν) are not Borel.*

We will obtain Theorem D as a consequences of our main result, Theorem E, which we proceed to motivate. A *pmp class-bijective extension* of a countable pmp Borel equivalence relation R on a standard probability space (Y,ν) is a pair $(\widehat{R},\pi)$, where $\widehat{R}$ is a countable pmp equivalence relation on a standard probability space (X,μ), and $\pi\colon X\to Y$ is a Borel map with $\pi_*(\mu)=\nu$, which maps $[x]_{\widehat{R}}$ bijectively onto $[\pi(x)]_R$ for almost every $x\in X$; see [8, 20, 36]. Ergodicity and weak mixing for (class-bijective extensions of) countable pmp equivalence relations are defined in a natural way; see Subsection 3.10. We consider a natural strengthening of the relation of isomorphism for pmp class-bijective extensions of a given countable pmp Borel equivalence relation R.

Definition 1.1. Let $(\widehat{R}_1,\pi_1)$ and $(\widehat{R}_2,\pi_2)$ be pmp class-bijective extensions of a countable Borel equivalence relation R on (Y,ν). Then $(\widehat{R}_1,\pi_1)$ and $(\widehat{R}_2,\pi_2)$ are *isomorphic relatively to* R if there exists $\theta\in\mathrm{Aut}(Y,\nu)$ such that, up to discarding a null set, $\pi_2\circ\theta=\pi_1$, and θ maps $\widehat{R}_1$-classes to $\widehat{R}_2$-classes.

Countable pmp equivalence relations and class-bijective extensions arise naturally in the context of pmp actions of locally compact groups, as we briefly explain. A free, ergodic, pmp action of a locally compact, second countable group G on a standard probability space (X,μ) admits a *cocompact cross section* [40, Theorem 4.2]; see also [22, Proposition 2.10]. This is a Borel subset $Y\subseteq X$ for which there exist an open neighborhood U of the identity in G and a compact subset K of G such that the restriction of the action $U\times Y\to X$ is injective, and the image $K\cdot Y$ of $K\times Y$ under the action is a G-invariant and conull subset of X. One then defines its associated *cross section equivalence relation* as the restriction to Y of the orbit equivalence relation of G, which is a countable Borel equivalence relation on Y. It is proved in [40, Proposition 4.3] that there exists a unique R-invariant probability measure ν on Y such that the push-forward measure of $\lambda|_U\times\nu$ by the restriction of the action $U\times Y\to X$ is equal to a multiple of $\mu|_{U\cdot Y}$; see also [8, Remark 8.2]. It is also shown in [8, Proposition 8.3] that if G is unimodular, then every countable class-bijective pmp extension of R is isomorphic to the cross section equivalence relation of some other free ergodic pmp action of G.

It follows from classical results of Dye [13], and of Connes–Feldman–Weiss [10], that any two ergodic *amenable* countable pmp equivalence relations on the standard atomless probability space are isomorphic. In particular, any two ergodic class-bijective pmp extensions of an ergodic amenable countable pmp equivalence relation are isomorphic, since these are automatically amenable. On the other hand, it has recently been shown in [8] that a *nonamenable* ergodic countable pmp equivalence relation has uncountably many pairwise not stably isomorphic ergodic class-bijective pmp extensions. Their arguments do not give information on whether the relation of (stable) isomorphism of ergodic class-bijective pmp extensions of a given nonamenable ergodic pmp equivalence relation is Borel. In this paper, we strengthen their result by solving this problem:

Theorem E. *Let R be an ergodic nonamenable countable pmp equivalence relation on a standard probability space (X,μ), and let (Y,ν) be the standard atomless probability space. Then the relations of isomorphism relative to R, isomorphism, stable isomorphism, von Neumann equivalence, and stable von Neumann equivalence of weak mixing (or ergodic) class-bijective extensions of R on (Y,ν) are not Borel.*

One technical difference between our approach and the one used in [8] is that we regard equivalence relations as groupoids, and that we work with certain discrete subgroups of the full groups. This allows us, in some sense, to work with discrete groups throughout, which gives us access to many arguments that would otherwise not be available.

Together with the results of Dye and Connes–Feldman–Weiss recalled above, Theorem E implies the following dichotomy.

Corollary F. *Let R be an ergodic pmp countable Borel equivalence relation on a standard Borel space (X,μ), and let (Y,ν) be the standard atomless probability space. Suppose that E is one of the following equivalence relations: isomorphism, stable isomorphism, von Neumann equivalence, or stable von Neumann equivalence of class-bijective pmp extensions of R on (Y,ν).*

(a) If R is amenable, then E has a single equivalence class.
(b) If R is not amenable, then E is not Borel.

The present paper is divided into four sections, apart from this introduction. In Section 2 we introduce the notion of property (T) for triples of groups, and use it to prove Theorem A and Theorem B in the case when the given nonamenable group Γ contains a nonabelian free group. Although not necessary for the rest of the paper, we decided to include a self-contained proof of this particular case to illustrate the main ideas involved in the proof of Theorem A and Theorem B. Additionally, it provides a different proof of the main result of [31]. The main difference in our approach is the use of an analog of Popa's superrigidity theorem in the context of triples of groups with property (T). This grants us access to a weak form of rigidity for groups containing a free group, which is not available in the context of the previously known superrigidity results.

In Section 3 we prove several facts concerning discrete pmp groupoids, their actions and representations. For future reference, we present these facts in a slightly more general form than strictly needed in this paper. In Section 4, we develop a coinduction theory for actions of groupoids, which can be seen as a common generalization of coinduction for groups and for countable pmp equivalence relations as defined by Epstein [16]. In Section 5, we present the proof of E, and show how to deduce from it Theorems A, B, and D.

Notation. *Let I be an index set, and let $(H_i)_{i\in I}$ be a family of Hilbert spaces with distinguished unit vectors $\xi_i \in H_i$, for $i \in I$. Define H to be the tensor product $\bigotimes_{i\in I}(H_i, \xi)$ as defined in [5, Section III.3]. For every $i_0 \in I$, there is a canonical isometric inclusion $H_{i_0} \to \bigotimes_{i\in I}(H_i, \xi_i)$. For $\eta \in H_{i_0}$, we denote by $\eta_{(i_0)} \in H$ the image of η under such an inclusion. We will adopt similar notations for tensor products of von Neumann algebras with respect to a distinguished normal trace as defined in [5, Section III.3]. (In particular, we denote von Neumann algebraic tensor products by $\otimes$ instead of the more standard $\overline{\otimes}$.) More generally, we will adopt the leg-numbering notation for linear operators on tensor products; see [57, Notation 7.1.1].*

Throughout the paper, we will tacitly use Borel selection theorems for Borel relations with countable fibers; see [37, Section 18.C].

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2. Actions of groups containing a nonabelian free subgroup

2.1. **Cocycles for measure preserving actions.** Let Γ be a countable discrete group, and let (X, μ) be the standard atomless probability space. In the following, we will identify a pmp action of Γ on (X, μ) with the corresponding action on the tracial von Neumann algebra $(M, \tau) = L^\infty(X, \mu)$. (By a *tracial von Neumann algebra* we mean a finite von Neumann algebra endowed with a distinguished faithful normal tracial state.) Here M is endowed with the canonical tracial state $\tau(f) = \int f d\mu$ associated with the measure μ. We also identify the group $\mathrm{Aut}(X, \mu)$ of measure-preserving automorphisms of (X, μ) with the group $\mathrm{Aut}(M, \tau)$ of trace-preserving automorphisms of M.

If $\theta\colon \Gamma \to \mathrm{Aut}(M, \tau)$ is an action on a tracial von Neumann algebra (M, τ), a (scalar, unitary) *1-cocycle for* θ is a function $w\colon \Gamma \to U(M)$ satisfying $w_\gamma \theta_\gamma(w_\rho) = w_{\gamma\rho}$ for $\gamma, \rho \in \Gamma$. The trivial cocycle for θ is the cocycle constantly equal to 1. Given $a, b \in M$, we write $a = b \bmod \mathbb{C}$ if there exists a nonzero $\lambda \in \mathbb{C}$ such that $a = \lambda b$. Two cocycles w, w' for θ are *weakly cohomologous* if there exists a unitary $z \in M$ such that

$$w'_\gamma \theta_\gamma(z) = z w_\gamma \bmod \mathbb{C}$$

for every $\gamma \in \Gamma$. Given two cocycles w, w', we will denote by ww' the cocycle $\gamma \mapsto w_\gamma w'_\gamma$.

Definition 2.1. Let θ be an action of a discrete group Γ on a standard probability space (X, μ), and let $\Delta \leq \Lambda \leq \Gamma$ be nested subgroups.

(1) A θ-cocycle w is said to be Δ-*invariant* if $w_\delta = 1$ and $\theta_\delta(w_\gamma) = w_\gamma$ for every $\gamma \in \Gamma$ and $\delta \in \Delta$. We write $Z^1_{:\Delta,\mathrm{w}}(\theta)$ for the set of Δ-invariant 1-cocycles for θ, which we consider as a topological space with respect to the topology of pointwise convergence in 2-norm.

(2) Two θ-cocycles w and w' are said to be Λ-*relatively weakly cohomologous* if there exists a unitary z in $L^\infty(X,\mu)$ such that $w'_\lambda\theta_\lambda(z) = zw_\lambda \mod \mathbb{C}$ for every $\lambda \in \Lambda$.
(3) The Δ-*invariant weak* 1-*cohomology group* $H^1_{:\Delta,\mathrm{w}}(\theta)$ is the space of weak cohomology classes of Δ-invariant cocycles for θ, endowed with the group operation defined by $[w][w'] = [ww']$.
(4) The Δ-*invariant* Λ-*relative weak* 1-*cohomology group* $H^1_{:\Delta,\Lambda,\mathrm{w}}(\theta)$ is the space of Λ-relative weak cohomology classes of Δ-invariant cocycles for θ endowed with the group operation defined as above.

It is clear that the Δ-invariant Λ-relative weak 1-cohomology group $H^1_{:\Delta,\Lambda,\mathrm{w}}(\theta)$ of an action θ is an invariant of θ up to conjugacy.

2.2. **Property (T) for triples of groups.** We fix a countable discrete group Γ and nested subgroups $\Delta \leq \Lambda \leq \Gamma$. Given a unitary representation $\pi\colon \Gamma \to U(H)$ of Γ on a Hilbert space H, a subset $F \subseteq \Gamma$, and $\varepsilon > 0$, a unit vector $\xi \in H$ is said to be (F,ε)-*invariant* if $\|\pi_\gamma(\xi) - \xi\| < \varepsilon$ for every $\gamma \in F$. Moreover, ξ is said to be Δ-*invariant* for π if $\pi_\delta(\xi) = \xi$ for every $\delta \in \Delta$. The representation π is said to have *almost invariant* Δ-*invariant vectors* if for every finite subset $F \subseteq \Gamma$ and every $\varepsilon > 0$, there exists an (F,ε)-invariant Δ-invariant unit vector for π.

Definition 2.2. The triple $\Delta \leq \Lambda \leq \Gamma$ has *property (T)* if every unitary representation of Γ with almost invariant Δ-invariant unit vectors has a Λ-invariant unit vector.

When $\Delta \leq \Lambda \leq \Gamma$ has property (T), we also say that Λ has the Δ-invariant relative property (T) in Γ. Moreover, if $\Delta \leq \Lambda \leq \Lambda$ has property (T), we say that Λ has the Δ-invariant property (T).

Example 2.3. *It is clear that when Δ is the trivial group, the triple $\Delta \leq \Lambda \leq \Gamma$ has properly (T) if and only if the pair $\Lambda \leq \Gamma$ has the relative property (T) in the usual sense; see [11, 33, 35].*

Example 2.4. *It is also easy to observe that if Δ is a normal subgroup of Λ such that Λ/Δ has property (T), then Λ has the Δ-invariant property (T). Indeed, under these assumptions, for any unitary representation π for Λ, the space of Δ-invariant vectors is Λ-invariant.*

As in the case of property (T) for pairs of groups, one can provide several equivalent reformulations of the notion of property (T) for triples of groups. We collect some in Proposition 2.6 below.

Definition 2.5. Let $\psi\colon \Gamma \to \mathbb{C}$ be a function.

- ψ is said to be of *conditionally negative type* if it satisfies $\psi(\gamma^{-1}) = \overline{\psi(\gamma)}$ for $\gamma \in \Gamma$ and
$$\sum_{i,j=1}^{n} \overline{\alpha}_i\alpha_j\psi(\gamma_i^{-1}\gamma_j) \leq 0$$
for every $n \geq 1$, elements $\gamma_1, \ldots, \gamma_n \in \Gamma$, and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ satisfying $\alpha_1 + \cdots + \alpha_n = 0$.
- ψ is said to be of *positive type* if it satisfies
$$\sum_{i,j=1}^{n} \overline{\alpha}_i\alpha_j\psi(\gamma_i^{-1}\gamma_j) \geq 0$$
for every $n \geq 1$, elements $\gamma_1, \ldots, \gamma_n \in \Gamma$, and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$.
- *normalized* if $\psi(1) = 1$, and Δ-*invariant* if $\psi(\gamma\delta) = \psi(\delta\gamma) = \psi(\gamma)$ for every $\gamma \in \Gamma$ and $\delta \in \Delta$.

Observe that if π is a unitary representation of Γ on H and ξ is a Δ-invariant unit vector in H, then the function $\psi(\gamma) = \langle \pi_\gamma(\xi), \xi\rangle$ is a normalized Δ-invariant function of positive type on Γ. Conversely, the GNS construction for functions of positive type shows that any normalized Δ-invariant function of positive type on Γ arises in this fashion; see [3, Appendix C].

The same proof as [33, Theorem 1.2] allows one to prove the following characterization of property (T) for triples of groups.

Proposition 2.6. *Let $\Delta \leq \Lambda \leq \Gamma$ be nested discrete groups. Then the following are equivalent:*

(1) $\Delta \leq \Lambda \leq \Gamma$ has property (T);

(2) there exist $\varepsilon > 0$ and a finite subset $F \subseteq \Gamma$ such that whenever π is a unitary representation of Γ, if π has an (F, ε)-invariant Δ-invariant unit vector, then the restriction of π to Λ contains a nonzero finite-dimensional subrepresentation;

(3) the restriction to Λ of every Δ-invariant complex-valued function on Γ which is conditionally of negative type is bounded;

(4) for every $\varepsilon > 0$, there exist a finite subset $F \subseteq \Gamma$ and $\delta > 0$ such that whenever π is a unitary representation of Γ, if π has an (F, δ)-invariant Δ-invariant unit vector ξ, then there is a Λ-invariant vector η satisfying $\|\xi - \eta\| < \varepsilon$;

(5) if $(\psi_n)_{n\in\mathbb{N}}$ is a sequence of normalized Δ-invariant functions of positive type on Γ which converges pointwise to 1, then $(\psi_n|_\Lambda)_{n\in\mathbb{N}}$ converges uniformly to 1.

Towards proving an extension of Popa's cocycle superrigidity theorem from [51], we present the following lemma, which is a natural generalization of [51, Lemma 4.2].

Lemma 2.7. Let $\Delta \leq \Lambda \leq \Gamma$ be a triple with property (T), and let $\theta\colon \Gamma \to \mathrm{Aut}(M, \tau)$ be an action on a tracial von Neumann algebra (M, τ). If $w\colon \Gamma \to U(M)$ is a Δ-invariant cocycle for θ, then for every $\varepsilon > 0$ there exists a neighborhood Ω of w in $Z^1_{:\Delta}(\theta)$ such that for all $w' \in \Omega$ there exists a partial isometry $v \in M$ with $w'_h\sigma_\lambda(v) = vw_\lambda$ for all $\lambda \in \Lambda$ and $\|v - 1\|_2 \leq \varepsilon$. Define the w-perturbation θ^w of θ to be the action $\theta^w(\gamma) = \mathrm{Ad}\,(w_\gamma) \circ \theta_\gamma$ for all $\gamma \in \Gamma$. If θ^w is ergodic, then the restriction to Λ of any Δ-invariant cocycle in Ω is cohomologous to $w|_\Lambda$.

Proof. The proof is similar to that of [51, Lemma 4.2]. Let $F \subseteq \Gamma$ and $\delta > 0$ be as in part (4) of Proposition 2.6 for $\varepsilon^2/4$. Set

$$\Omega = \{w' \in Z^1_{:\Delta}(\theta)\colon \|w_\gamma - w'_\gamma\|_2 \leq \delta \text{ for all } \gamma \in F\}.$$

Given $w' \in \Omega$, define a representation $\pi\colon \Gamma \to U(L^2(M, \tau))$ by $\pi_\gamma(x) = w'_\gamma\theta_\gamma(x)w^*_\gamma$ for all $\gamma \in \Gamma$ and all $x \in L^2(M, \tau)$ coming from M. Then $\|\pi_\gamma(1) - 1\|_2 \leq \delta$ for all $\gamma \in F$. Moreover, the unit vector $1 \in L^2(M, \tau)$ is Δ-invariant because w and w' are trivial on Δ. By part (4) of Proposition 2.6, there exists a Λ-invariant unit vector $\xi \in L^2(M, \tau)$ satisfying $\|\xi - 1\| \leq \varepsilon^2/4$. Let $\xi_0 \in L^2(M, \tau)$ denote the vector of minimal norm in the weakly-closed convex hull of $\{\pi_\lambda(\xi)\colon \lambda \in \Lambda\}$. By convexity, ξ_0 is also a unit vector satisfying $\|\xi_0 - 1\| \leq \varepsilon^2/4$. By construction, we have $w'_\lambda\theta_\lambda(\xi_0)w^*_\lambda = \xi_0$ for all $\lambda \in \Lambda$, so if $v \in M$ is the partial isometry in the polar decomposition of ξ_0 [51, Subsection 1.2], then v is Δ-invariant and we have $w'_\lambda\theta_\lambda(v) = vw_\lambda$ for all $\lambda \in \Lambda$, as desired.

Finally, if θ^w is ergodic, then v^*v, which belongs to its fixed point algebra, must be a scalar. Thus v is a unitary and the restrictions of w' and w to Λ are cohomologous. ∎

Recall that the *centralizer* $\mathrm{Aut}(M)^\theta$ of an action θ of Γ on M is the subgroup of $\mathrm{Aut}(M)$ consisting of the elements α such that $\alpha \circ \theta_\gamma = \theta_\gamma \circ \alpha$ for every $\gamma \in \Gamma$.

Definition 2.8. We say that $\theta\colon \Gamma \to \mathrm{Aut}(M, \tau)$ is *malleable* if the connected component of the identity in $\mathrm{Aut}(M \otimes M)^{\theta\otimes\theta}$ contains an element of the form $(\alpha \otimes \beta) \circ \sigma$, where $\alpha, \beta \in \mathrm{Aut}(M)^\theta$ and $\sigma \in \mathrm{Aut}(M \otimes M)$ is the flip automorphism; see [51, Definition 2.9].

Example 2.9. *The standard example of a malleable action is the Bernoulli action $\beta_{\Gamma\curvearrowright I}$ associated with an action $\Gamma \curvearrowright I$ of Γ on an infinite set I; see [51, Example 4.4 and Lemma 4.5] and [59, Section 3]. Such an action is weak mixing whenever the action $\Gamma \curvearrowright I$ has infinite orbits; see [51, Lemma 4.5] and [39, Proposition 2.1]. It is moreover free whenever the action $\Gamma \curvearrowright I$ is faithful; see [51, Lemma 4.5] and [39, Proposition 2.4].*

The following result is proved similarly to [51, Theorem 5.2], where [51, Lemma 4.2] is replaced with Lemma 2.7; see also [59, Lemma 4.9 and Lemma 4.10], [23, Section 3, Section 4], [38, Section 30].

Theorem 2.10 (Popa). *Let $\Delta \leq \Lambda \leq \Gamma$ be a triple with property (T), and let θ be a malleable action of Γ on the standard probability space (X, μ) such that $\theta|_\Lambda$ is weak mixing. Then the Δ-invariant Λ-relative weak 1-cohomology group $H^1_{:\Delta,\Lambda,\mathrm{w}}(\theta)$ is trivial.*

Proof. The way that property (T) enters in Popa's argument in [51, Theorem 5.2] is exclusively via [51, Lemma 4.6], where property (T) is used through [51, Lemma 4.2]. In our context, the analog of Popa's Lemma 4.6 for triples of groups with property (T) and Δ-invariant cocycles can be proved using Lemma 2.7. The proof of the present theorem then follows from [51, Theorem 3.1 and Proposition 3.5]. ∎

2.3. **Actions with prescribed cohomology.** Theorem 2.10 allows one to construct, in the presence of property (T), actions of groups with prescribed cohomology.

Theorem 2.11. *Let $\Delta \leq \Lambda \leq \Gamma$ be a triple with property (T), and assume that Δ has infinite index in Λ. Then there exists an assignment $A \mapsto \alpha_A$ from countably infinite discrete abelian groups to free weak mixing actions of Γ on the standard atomless probability space (X, μ) such that:*

(1) $\alpha_A|_\Lambda$ is weak mixing;
(2) A is isomorphic to A' if and only if α_A is conjugate to $\alpha_{A'}$;
(3) for every action ρ of Γ on a standard probability space (Y, ν) such that $\rho|_\Delta$ is weak mixing, the Δ-invariant, Λ-relative weak 1-cohomology group $H^1_{:\Delta,\Lambda,\mathrm{w}}(\alpha_A \otimes \rho)$ is isomorphic to A.

Proof. Fix a countably infinite abelian group A, and let G denote its Pontryagin dual, endowed with its Haar (probability) measure ν. Set $(M, \tau) = L^\infty(G, \nu)$ endowed with the trace-preserving action $\mathtt{Lt}\colon G \to \mathrm{Aut}(M, \tau)$ given by left translation. Set

$$\beta = \beta_{\Gamma \curvearrowright \Gamma/\Delta} \otimes \beta_{\Gamma \curvearrowright \Gamma}\colon \Gamma \curvearrowright M^{\otimes \Gamma/\Delta} \otimes M^{\otimes \Gamma},$$

where Γ acts on Γ/Δ and on Γ by left translation. Define also the action

$$\mathtt{Lt}^{\otimes \Gamma/\Delta} \otimes \mathrm{id}_{M^{\otimes\Gamma}}\colon G \curvearrowright M^{\otimes \Gamma/\Delta} \otimes M^{\otimes \Gamma},$$

and let M_A denote its fixed point algebra, which can be identified with

$$M_A = (M^{\otimes \Gamma/\Delta} \otimes M^{\otimes \Gamma})^{\mathtt{Lt}^{\otimes \Gamma/\Delta} \otimes \mathrm{id}_{M^{\otimes\Gamma}}} = (M^{\otimes \Gamma/\Delta})^{\mathtt{Lt}^{\otimes \Gamma/\Delta}} \otimes M^{\otimes \Gamma}.$$

Notice that $M_A = L^\infty(X_A, \mu_A)$ for a standard atomless probability space (X_A, μ_A).

The actions defined above commute, so β restricts to an action $\alpha_A\colon \Gamma \to \mathrm{Aut}(M_A)$. Since Λ/Δ is infinite, $\beta|_\Lambda$ is weak mixing, and thus $\alpha_A|_\Lambda$ is weak mixing as well. It is clear that the conjugacy class of α_A only depends from the isomorphism class of A. This proves (1) and (2), so we check (3).

Let (Y, ν) be a standard probability space, and let $\rho\colon \Gamma \curvearrowright L^\infty(Y, \nu)$ be a trace-preserving action such that $\rho|_\Delta$ is weak mixing. We claim that $H^1_{:\Delta,\Lambda,\mathrm{w}}(\alpha_A \otimes \rho)$ is isomorphic to A. Once we prove this, the proof of the theorem will be complete.

Let w be a Δ-invariant cocycle for $\alpha_A \otimes \rho$. Then w is also a Δ-invariant cocycle for $\beta \otimes \rho$. Since $\rho|_\Delta$ is weak mixing and Δ is infinite, [59, Proposition D.2] implies that

$$(M^{\otimes \Gamma/\Delta} \otimes M^{\otimes \Gamma} \otimes L^\infty(Y, \nu))^{(\beta_{\Gamma \curvearrowright \Gamma/\Delta} \otimes \beta_{\Gamma \curvearrowright \Gamma} \otimes \rho)|_\Delta} = M^{\otimes \Gamma/\Delta} \otimes \mathbb{C} \otimes \mathbb{C} \subseteq M^{\otimes \Gamma/\Delta} \otimes M^{\otimes \Gamma} \otimes L^\infty(Y, \nu).$$

Therefore $w_\lambda \in M^{\otimes \Gamma/\Delta} \otimes \mathbb{C} \otimes \mathbb{C}$ for every $\lambda \in \Lambda$. We will identify $M^{\otimes \Gamma/\Delta} \otimes \mathbb{C} \otimes \mathbb{C}$ with $M^{\otimes \Gamma/\Delta}$. Since Λ/Δ is infinite, the shift $\beta_{\Lambda \curvearrowright \Gamma/\Delta}$ is weak mixing. By Theorem 2.10, there exists a unitary $v \in M^{\otimes \Gamma/\Delta}$ such that

$$v^*(\beta_{\Gamma \curvearrowright \Gamma/\Delta})_\lambda(v) = w_\lambda \bmod \mathbb{C}$$

for every $\lambda \in \Lambda$. Fix $g \in G$. Then $\mathtt{Lt}_g^{\otimes \Gamma/\Delta}(w_\lambda) = w_\lambda$ for every $\lambda \in \Lambda$, and hence

$$v^*\mathtt{Lt}_g^{\otimes \Gamma/\Delta}(v) = (\beta_{\Gamma \curvearrowright \Gamma/\Delta} \otimes \beta_{\Gamma \curvearrowright \Gamma})_\lambda(v^*\mathtt{Lt}_g(v)) \bmod \mathbb{C}.$$

In other words, $v^*\mathtt{Lt}_g^{\otimes \Gamma/\Delta}(v)$ generates a one-dimensional subspace which is invariant under $\beta_{\Gamma \curvearrowright \Gamma/\Delta} \otimes \beta_{\Gamma \curvearrowright \Gamma}$. Since this action is weak mixing, there exists $\chi_w\colon G \to \mathbb{T}$ such that $\mathtt{Lt}_g^{\otimes \Gamma/\Delta}(v) = \chi_w(g)v$ for every $g \in G$. It is straightforward to verify that χ_w is a group homomorphism, so that it can be regarded as an element of $A \cong \widehat{G}$.

Claim: If $w, w' \in Z^1_{:\Delta,\mathrm{w}}(\alpha_A \otimes \rho)$ are Λ-relatively weakly cohomologous, then $\chi_w = \chi_{w'}$.

Given w and w' as above, find a unitary $z \in M_A \otimes L^\infty(Y, \nu)$ with $w'_\lambda = z^* w_\lambda (\alpha_A \otimes \rho)_\lambda(z) \bmod \mathbb{C}$ for every $\lambda \in \Lambda$. Use Δ-invariance of w and w' to get $(\alpha_A \otimes \rho)_\delta(z) = z$ for all $\delta \in \Delta$. This implies that $z \in M^{\otimes\Gamma/\Delta}$ and $w'_\lambda = z^* w_\lambda (\beta_{\Gamma\curvearrowright\Gamma/\Delta})_\lambda(z) \bmod \mathbb{C}$ for every $\lambda \in \Lambda$. As above, there exists a unitary $v \in M^{\otimes\Gamma/\Delta}$ such that

$$w_\lambda = v^*(\beta_{\Gamma\curvearrowright\Gamma/\Delta})_\lambda(v) \bmod \mathbb{C} \quad \text{and} \quad \mathtt{Lt}_g^{\otimes\Gamma/\Delta}(v) = \chi_w(g)v$$

for every $\lambda \in \Lambda$ and every $g \in G$. Thus $w'_\lambda = (vz)^*(\beta_{\Gamma\curvearrowright\Gamma/\Delta})_\lambda(vz) \bmod \mathbb{C}$ for every $\lambda \in \Lambda$, and

$$\mathtt{Lt}_g^{\otimes\Gamma/\Delta}(vz) = \mathtt{Lt}_g^{\otimes\Gamma/\Delta}(v)\mathtt{Lt}_g^{\otimes\Gamma/\Delta}(z) = \chi_w(g)vz$$

for every $g \in G$. This shows that $\chi_{w'} = \chi_w$.

In particular, there is a well-defined assignment $\kappa\colon H^1_{:\Delta,\Lambda,\mathrm{w}}(\alpha_A \otimes \rho) \to \widehat{G}$ given by $\kappa([w]) = \chi_w$ for all $[w] \in H^1_{:\Delta,\Lambda,\mathrm{w}}(\alpha_A \otimes \rho)$. The map κ is easily seen to be a group homomorphism, so it remains to show that it is a bijection.

Claim: κ is injective.

Let $w, w' \in Z^1_{:\Delta,\mathrm{w}}(\alpha_A \otimes \rho)$ satisfy $\chi_w = \chi_{w'} = \chi \in G$. Then there exist unitaries $v, v' \in M^{\otimes\Gamma/\Delta}$ such that

$$v^*(\beta_{\Gamma\curvearrowright\Gamma/\Delta})_\lambda(v) = w_\lambda \bmod \mathbb{C} \quad \text{and } v'^*(\beta_{\Gamma\curvearrowright\Gamma/\Delta})_\lambda(v') = w'_\lambda \bmod \mathbb{C}$$

for every $\lambda \in \Lambda$, and

$$\mathtt{Lt}_g^{\otimes\Gamma/\Delta}(v) = \chi(g)v \quad \text{and} \quad \mathtt{Lt}_g^{\otimes\Gamma/\Delta}(v') = \chi(g)v'$$

for every $g \in G$. Therefore, the unitary $z = v^*v' \in M_A$ satisfies $z^* w_\lambda (\beta_{\Gamma\curvearrowright\Gamma/\Delta})_\lambda(z) = w'_\lambda \bmod \mathbb{C}$ for every $\lambda \in \Lambda$. This shows that w and w' are Λ-relatively weakly cohomologous, proving the claim.

Claim: κ is surjective.

Fix $\chi \in \widehat{G}$, and regard it as a (spectral) unitary in $C(G) \subseteq L^\infty(G,\nu) = M$. This gives a Δ-invariant unitary

$$v_\chi \in M^{\otimes\Gamma/\Delta} \otimes \mathbb{C} \otimes \mathbb{C} \subset M^{\otimes\Gamma/\Delta} \otimes M^{\otimes\Gamma} \otimes L^\infty(Y,\nu)$$

satisfying $\mathtt{Lt}_g^{\otimes\Gamma/\Delta}(v_\chi) = \chi(g)v_\chi$ for every $g \in G$. Define $w_\chi(\lambda) = v_\chi^*\beta_\lambda(v_\chi)$ for all $\lambda \in \Lambda$. Then w_χ is a Δ-invariant cocycle, and one checks that $\chi_{w_\chi} = \chi$, as desired. ∎

We will often use Theorem 2.11 in the particular case when $\Gamma = \Lambda$.

2.4. **Countable to one homomorphism.** Let Γ be a countable discrete group, let Λ be a subgroup, and let $\Lambda \curvearrowright^\alpha (X,\mu)$ be an action on the atomless standard probability space (X,μ). We recall the construction of the coinduced action $\mathrm{CInd}_\Lambda^\Gamma(\alpha)$ of Γ as defined in [12, 41] and used in [17, 31].

Definition 2.12. Adopt the notation above. Set

$$Y = \{f\colon \Gamma \to X\colon f(\gamma\lambda) = \alpha_{\lambda^{-1}}(f(\gamma)) \text{ for every } \gamma \in \Gamma \text{ and } \lambda \in \Lambda\},$$

endowed with the measure ν induced by the product measure on X. Then Y is an atomless standard probability space. The *coinduced action* $\widehat{\alpha} = \mathrm{CInd}_\Lambda^\Gamma(\alpha)$ is the action $\Gamma \curvearrowright^{\widehat{\alpha}} Y$ defined by setting $(\widehat{\alpha}_{\gamma_0}(f))_\gamma = f(\gamma_0^{-1}\gamma)$ for every $\gamma, \gamma_0 \in \Gamma$.

Remark 2.13. Consider the map $\pi\colon Y \to X$ given by evaluation at the unit of Γ. It is easy to see that $\{f \in Y\colon \pi(\widehat{\alpha}_\gamma(f)) = \pi(f)\}$ is ν-null for every $\gamma \in \Gamma \setminus \{1\}$.

For $2 \leq d \leq \infty$, fix a subgroup $\Lambda \leq \mathrm{SL}_2(\mathbb{Z})$ isomorphic to $\mathbb{F}_d$. Then the canonical action of $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{Z}^2$ by group automorphisms induces, by passing to the dual group, a free weak mixing pmp action $\Lambda \curvearrowright^\rho \mathbb{T}^2$. This action has been initially studied by Popa [49], and used by many authors due to its rigidity properties; see [16, 17, 25, 31, 32, 58]. A crucial property of such an action has been established by Ioana in [31, Theorem 1.3] building on previous work from [9, 25, 49]; see also [17, Theorem 6.9] and [8, Lemma 7.4].

Theorem 2.14 (Ioana). *Let Γ be a countable discrete group containing a nonabelian free group as a subgroup. Let $\mathcal{S}$ be a class of free, ergodic, pmp actions of Γ on the standard probability space, and let $\mathcal{S}|_{\mathbb{F}_d}$ be the class of restrictions of actions from $\mathcal{S}$ to $\mathbb{F}_d$. Suppose that*

(1) the actions in $\mathcal{S}$ are pairwise von Neumann equivalent,

(2) the actions in $\mathcal{S}|_{\mathbb{F}_d}$ are pairwise not conjugate;

(3) *every action* $\sigma\colon \Gamma \curvearrowright X_\sigma$ *in* $\mathcal{S}$ *has the property that* ρ *is a factor of* $\sigma|_\Lambda$ *as witnessed by a Borel map* $\pi_\sigma\colon X_\sigma \to \mathbb{T}^2$ *such that* $\{x \in X_\sigma\colon \pi_\sigma(\sigma_\gamma(x)) = \pi_\sigma(x)\}$ *is null for every* $\gamma \in \Gamma \setminus \{1\}$.

Then $\mathcal{S}$ *is countable.*

We now prove Theorems A and B from the introduction, in the case when Γ contains $\mathbb{F}_2$.

Theorem 2.15. *Let* Γ *be a group containing a nonabelian free group as a subgroup* Λ*, and fix an infinite normal subgroup* Δ *of* Λ *such that the quotient* Λ/Δ *is an infinite property (T) group. Then there exists an assignment* $A \mapsto \theta_A$ *from countably infinite abelian groups to free weak mixing actions of* Γ *on the standard atomless probability space such that:*

(1) $\theta_A|_\Lambda$ *is weak mixing;*
(2) *Two groups* A *and* A' *are isomorphic if and only if* θ_A *and* $\theta_{A'}$ *are conjugate;*
(3) *if* $\mathcal{A}$ *is a family of pairwise nonisomorphic countably infinite abelian groups such that the actions* $\{\theta_A\colon A \in \mathcal{A}\}$ *are pairwise von Neumann equivalent, then* $\mathcal{A}$ *is countable.*

Proof. Let $\Lambda \curvearrowright^\rho \mathbb{T}^2$ be the rigid action described above, and set $\widehat{\rho} = \mathrm{CInd}_\Lambda^\Gamma(\rho)$. Let $A \mapsto \alpha_A$ be the assignment from countably infinite abelian groups to weak mixing measure preserving actions of Γ obtained in Theorem 2.11 for the triple $\Delta \le \Lambda \le \Gamma$. (Observe that $\Delta \le \Lambda \le \Gamma$ has property (T) by Example 2.4.) By construction, $\alpha_A|_\Lambda$ is weak mixing, and hence $\widehat{\rho}|_\Delta$ is weak mixing by [17, Lemma 6.8].

Fix a countably infinite discrete abelian group A, and set $\theta_A = \alpha_A \otimes \widehat{\rho}$. Then $\theta_A|_\Lambda$ is a weak mixing extension of ρ. If A and A' are isomorphic, then α_A and $\alpha_{A'}$ are conjugate, and hence θ_A and $\theta_{A'}$ are conjugate. Conversely, if θ_A and $\theta_{A'}$ are conjugate, then $A \cong H^1_{:\Delta,\mathrm{w}}(\theta_A|_\Lambda) \cong H^1_{:\Delta,\mathrm{w}}(\theta_{A'}|_\Lambda) \cong A'$ by Theorem 2.11.

Suppose now that $\mathcal{A}$ is a family of pairwise nonisomorphic countably infinite abelian groups such that $\{\theta_A\colon A \in \mathcal{A}\}$ are pairwise von Neumann equivalent. By Theorem 2.11, we have $H^1_{:\Delta,\mathrm{w}}(\theta_A|_\Lambda) \cong A$ for every $A \in \mathcal{A}$. Therefore the actions $\{\theta_A|_\Lambda\colon A \in \mathcal{A}\}$ are pairwise not conjugate weak mixing extensions of ρ. By Theorem 2.14, we conclude that $\mathcal{A}$ is countable. ∎

The proof in the general case of nonamenable Γ follows similar ideas, but a lot more work is required. The induction argument presented in this section is not sufficient for our purposes, and this forces us to replace countable groups with countable pmp groupoids. The proof of Theorem A and Theorem B can then be obtained by combining this with Gaboriau-Lyons' measurable solution to von Neumann's problem [24] and Epstein's coinduction construction [16]; the main idea is to produce actions of an arbitrary nonamenable group whose (weak, invariant, relative) 1-cohomology can be prescribed via superrigidity. The abstract setting of pmp groupoids can in fact be used to prove the more general Theorem D and Theorem E, from which Theorems A and B follow.

3. Actions of groupoids

3.1. **Groupoids.** We recall here some basic definitions concerning groupoids as can be found, for instance, in [2, 7, 47, 53]. A *groupoid* G is a small category where every morphism (also called *arrow*) is invertible. We denote by G^0 the set of objects of G, which are also called *units* of G, while G^0 is called the *unit space* of G. We identify each unit of G with the corresponding identity arrow. Consistently, we regard G^0 as a subset of G. The source and range maps are denoted by $s, r\colon G \to G^0$, respectively. We let G^2 denote the set the pairs of *composable arrows*:

$$G^2 = \{(\gamma, \rho) \in G\colon s(\gamma) = r(\rho)\}.$$

A *bisection* for a groupoid G is a subset $t \subseteq G$ such that source and range maps are injective on t. Given subsets $A, B \subseteq G$, we set

$$AB = \{\gamma\rho\colon (\gamma, \rho) \in G^2 \cap (A \times B)\} \quad \text{and} \quad A^{-1} = \{\gamma^{-1}\colon \gamma \in A\}.$$

For $\gamma \in G$ and $A \subseteq G$, we set $\gamma A = \{\gamma\}A$ and $A\gamma = A\{\gamma\}$. In particular, $Ax = \{\gamma \in A\colon s(\gamma) = x\}$ and $xA = \{\gamma \in A\colon r(\gamma) = x\}$ for $x \in G^0$. If A and B are bisections, then so are AB and A^{-1}.

Definition 3.1. Let G and H be groupoids with $H \subseteq G$.

- We say that G is a *standard Borel groupoid* if it is endowed with a standard Borel structure such that G^0 is a Borel subset of G, and composition and inversion of arrows, as well as the source and range maps, are Borel functions.
- We say that H is a *Borel subgroupoid* of G if $H^0 = G^0$, and the inclusion map $G \hookrightarrow H$ is a Borel homomorphism.
- We say that G is a *discrete measurable groupoid* [7, 8, 14] (also called *countable Borel groupoid* [1, 54]), if it is a standard Borel groupoid whose source and range maps are countable-to-one.
- We say that G is a *discrete probability-measure-preserving* (pmp) *groupoid* if it is a discrete measurable groupoid endowed with a Borel measure μ_G satisfying
$$\mu_G(A) = \int_{x\in G^0} |xA| d\mu_G(x) = \int_{x\in G^0} |Ax| d\mu_G(x)$$
for every Borel subset A of G.
- If G is a discrete pmp groupoid, we say that H is a *discrete pmp subgroupoid* if it is a Borel subgroupoid with $\mu_H = \mu_G$.
- A Borel subset $A \subseteq G^0$ is said to be *invariant* if $r(GA) = A$.
- A discrete pmp groupoid G is said to be *ergodic* if every non-null invariant Borel subset of G^0 has full measure.

For a non-null invariant Borel subset A of G^0 one can define the *reduction* $G|_A = AGA$, which is a discrete pmp groupoid with unit space A with respect to the measure $\frac{1}{\mu_G(A)}\mu_G|_A$. If A has full measure, such a reduction is called *inessential.* In the following, we will identify two discrete pmp groupoids whenever they have isomorphic inessential reductions.

Remark 3.2. Let G be a discrete pmp groupoid with measure μ_G. Then μ_G is completely determined by its restriction μ_{G^0}. Moreover, μ_G induces a measure μ_{G^2} on G^2 defined by
$$\mu_{G^2}(A) = \int_{x\in G^0} |A \cap (Gx \times xG)| d\mu_G(x).$$

Definition 3.3. Let G be a discrete pmp groupoid. We define the *inverse semigroup of partial automorphisms* of G to be the set $[[G]]$ of Borel bisections of G (identified when they agree almost everywhere).

Similarly, the *full group* $[G]$ of G is defined as the group, under composition, of Borel bisections t of G (identified when they agree almost everywhere) with the property that source and range maps restricted to t are onto.

One checks that the topology induced by the metric $d(t_0, t_1) = \mu(t_0 \triangle t_1)$ turns $[G]$ into a Polish group. For a Borel bisection t of G and $x \in G^0$, we let tx be the unique element of t with source x, and xt be the unique element of t with range x. We say that a subset S of $[[G]]$ *covers* G if the union of S is co-null in G.

Definition 3.4. Let G be a discrete pmp groupoid, and let H is a standard Borel groupoid. A *homomorphism* $\pi\colon G \to H$ is a Borel functor such that for a.e. $\gamma, \rho \in G$, $\pi(s(\gamma)) = s(\pi(\gamma))$, $\pi(r(\gamma)) = r(\pi(\gamma))$, and $\pi(\gamma\rho) = \pi(\gamma)\pi(\rho)$. (By [52, Lemma 5.2], π can be seen as a Borel functor from an inessential reduction of G to H.)

Let G and H be discrete pmp groupoids, and let $\pi\colon G \to H$ be a homomorphism. Following [7, Section 4], [8, Definition 2.3], [36, Section 16], we say that π is:

- a *pmp extension* if μ_{H^0} is equal to the push-forward measure $\pi_*(\mu_{G^0})$;
- *class-bijective* if it maps Gx bijectively onto $H\pi(x)$ for almost every $x \in G^0$.

Definition 3.5. The *orbit equivalence relation* E_G of a discrete pmp groupoid G is the countable pmp equivalence relation on G^0 obtained as the image of the map $(r, s)\colon G \to G^0 \times G^0$. The groupoid G is called *principal* if such a map is one-to-one. In this case, one can identify G with E_G.

In the following, we will regard any countable pmp equivalence relation as a discrete pmp groupoid. For every ergodic countable pmp equivalence relation R on a standard probability space (X,μ), there exists $\theta \in [R]$ acting ergodically on (X,μ) [38]. Therefore, a groupoid G is ergodic if and only if there exists $t \in [G]$ such that the map $x \mapsto r(tx)$ acts ergodically on G^0. Furthermore, when G is ergodic, there exist countably many pairwise essentially disjoint elements of $[G]$ which cover G, and if $A, B \subseteq G^0$ are Borel sets satisfying $\mu_{G^0}(A) = \mu_{G^0}(B)$, then there exists $t \in [G]$ such that $r(tx) \in B$ for almost every $x \in A$ and $s(yt) \in A$ for almost every $y \in B$.

3.2. **Bundles of metric spaces and Hilbert spaces.** Let X be a standard probability space. We say that a standard Borel space Z is *fibered over* X if it is endowed with a Borel surjection $q\colon Z \to X$. In this case, we denote by $Z_x = q^{-1}\{x\}$ the *fiber over* $x \in X$. A (Borel) *section* for Z is a Borel function $\sigma\colon X \to Z$ such that $q \circ \sigma = \mathrm{id}_X$. For a section σ and $x \in X$, we write $\sigma_x = \sigma(x)$.

Definition 3.6. Let Z and Z' be standard Borel bundles over X.

(1) The *fibered product* $Z * Z'$ is the standard Borel space fibered over X defined by
$$Z * Z' = \{(z,z') \in Z \times Z' \colon q(z) = q'(z')\}.$$
(2) A Borel *fibered function* $f\colon Z \to Z'$ is a Borel map satisfying $q' \circ f = q$ almost everywhere.

When (Z,λ) is a standard probability space, we say that (Z,λ) is *fibered over the standard probability space* (X,μ) with respect to the Borel surjection $q\colon Z \to X$ if $q_*(\lambda) = \mu$. In this case, one can consider the disintegration $(\lambda_x)_{x\in X}$ of λ with respect to μ. This turns each fiber (Z_x, λ_x) into a standard probability space.

Definition 3.7. A *(Borel) bundle of metric spaces* over a standard Borel space X is a standard Borel space $\mathcal{Z}$ fibered over X endowed with a Borel function $d\colon \mathcal{Z} * \mathcal{Z} \to \mathbb{R}$ with the following properties:

- for every $x \in X$, the restriction d_x of d to $\mathcal{Z}_x \times \mathcal{Z}_x$ is a metric on $\mathcal{Z}_x$;
- there exists a sequence $(\sigma_n)_{n\in\mathbb{N}}$ of sections for $\mathcal{Z}$ such that $\{\sigma_{n,x}\colon n \in \mathbb{N}\}$ is a dense subset of $\mathcal{Z}_x$, for every $x \in X$.

When (X,μ) is a standard probability space, the space $S(X,\mathcal{Z})$ of sections for $\mathcal{Z}$ has a canonical topology induced by the metric
$$d(b,b') = \int_X \frac{d_x(b_x,b'_x)}{1+d_x(b_x,b'_x)} d\mu(x).$$

When X is the unit space of a discrete pmp groupoid G, we let $S(G,\mathcal{Z})$ denote the space of Borel functions $z\colon G \to \mathcal{Z}$ satisfying $z_\gamma \in \mathcal{Z}_{r(\gamma)}$ for all $\gamma \in G$, endowed with the canonical topology induced by the pseudometrics
$$d_t(z,z') = \int_X \frac{\|z_{tx} - z'_{tx}\|}{1+\|z_{tx} - z'_{tx}\|} d\mu(x),$$
for $z, z' \in S(G,\mathcal{Z})$, where t ranges within (a dense subset of) $[G]$.

Definition 3.8. A *(Borel) Hilbert bundle* over a standard Borel space X is a standard Borel space $\mathcal{H}$ fibered over X endowed with Borel fibered functions $\mathbf{0}\colon X \to \mathcal{H}$, $+\colon \mathcal{H} * \mathcal{H} \to \mathcal{H}$, $\cdot\colon \mathcal{H} * \mathcal{H} \to \mathcal{H}$, and $\mathbb{C} \times \mathcal{H} \to \mathcal{H}$, together with a sequence of sections $(\sigma_n)_{n\in\mathbb{N}}$ such that, for every $x \in X$, the fiber $\mathcal{H}_x$ is a Hilbert space when endowed with the operations induced by the given Borel fibered functions, and $(\sigma_{n,x})_{n\in\mathbb{N}}$ enumerates a subset of the unit sphere of $\mathcal{H}_x$ with dense linear span.

We denote the Hilbert bundle $\mathcal{H}$ also by $\bigsqcup_{x\in X} \mathcal{H}_x$, and the space of sections by $S(X,\mathcal{H})$.

Remark 3.9. The Grahm-Schmidt orthogonalization process shows that one can always assume that $(\sigma_{n,x})_{n\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}_x$. In this case, the sequence $(\sigma_n)_{n\in\mathbb{N}}$ is called an *orthonormal basic sequence* for $\mathcal{H}$. The fibered function $\mathcal{H} * \mathcal{H} \to \mathbb{R}$ given by $(x,y) \mapsto \|x-y\|$ turns $\mathcal{H}$ into a bundle of metric spaces over X.

We denote by $L^\infty(X,\mathcal{H})$ the space of (essentially) bounded sections of $\mathcal{H}$, and by $L^2(X,\mathcal{H})$ the space of square-integrable sections of $\mathcal{H}$. The latter is a Hilbert space with respect to the inner product $\langle \xi,\eta\rangle = \int_X \langle \xi_x,\eta_x\rangle d\mu(x)$ for all $\xi,\eta\in L^2(X,\mathcal{H})$. A section $\xi\colon X\to\mathcal{H}$ is a *unit section* if $\|\xi_x\|=1$ for almost every $x\in X$.

3.3. **Representations of groupoids.** Given a Hilbert bundle $\mathcal{H}$ over a standard probability space X, its *unitary groupoid* $U(\mathcal{H})$ is the set of unitary operators $U^{(s,t)}\colon \mathcal{H}_s\to\mathcal{H}_t$ for $s,t\in X$. This is a standard Borel groupoid when endowed with the standard Borel structure generated by the source and range maps together with the functions $U^{(s,t)}\mapsto \langle \sigma_{n,t}, U\sigma_{m,s}\rangle$ for $n,m\in\mathbb{N}$. The unit space of $U(\mathcal{H})$ can be identified with X.

Definition 3.10. Let G be a discrete pmp groupoid, and let $\mathcal{H}$ be a Hilbert bundle over G^0. A (unitary) *representation* of G on $\mathcal{H}$ is a homomorphism $\pi\colon G\to U(\mathcal{H})$ that fixes G^0 pointwise. Moreover, a vector $\xi\in L^2(G^0,\mathcal{H})$ is said to be *π-invariant* if $\pi_\gamma(\xi_{s(\gamma)})=\xi_{r(\gamma)}$ for almost every $\gamma\in G$. The space of π-invariant vector is denoted by $L^2(G^0,\mathcal{H})^\pi$.

A representation $\pi\colon G\to U(\mathcal{H})$ is said to be

1. *ergodic*, if $L^2(G^0,\mathcal{H})^\pi$ is the trivial subspace of $L^2(G^0,\mathcal{H})$.
2. *weak mixing*, if for every $\varepsilon>0$, every $n\in\mathbb{N}$, and sections $\xi_1,\ldots,\xi_n$ for $\mathcal{H}$, there exists $t\in[G]$ such that $\int_{G^0}|\langle \xi_{j,x},\pi_{xt}(\xi_{i,s(xt)})\rangle| d\mu_{G^0}(x)\leq\varepsilon$ for every $i,j=1,\ldots,n$.

Definition 3.11. Let $\mathcal{H}$ be a Hilbert bundle over a standard probability space (X,μ). A sub-bundle $\mathcal{K}$ of $\mathcal{H}$ is a Hilbert bundle $\mathcal{K}$ over X such that $\mathcal{K}_x$ is a subspace of $\mathcal{H}_x$ for every $x\in X$. Moreover, we say that $\mathcal{K}$ is

- *finite-dimensional* if there exists $d\in\mathbb{N}$ such that $\dim\mathcal{K}_x\leq d$ for almost every $x\in X$;
- *d-dimensional* for some $d\in\mathbb{N}\cup\{\infty\}$ if $\dim\mathcal{K}_x=d$ for almost every $x\in X$;
- *nonzero* if there exists a non-null Borel subset A of X such that $\dim\mathcal{K}_x>0$ for every $x\in A$.

When G is a groupoid with $G^0=X$ and $\pi\colon G\to U(\mathcal{H})$ is a representation, we say that $\mathcal{K}$ is *π-invariant* if $\pi_\gamma(\mathcal{K}_{s(\gamma)})=\mathcal{K}_{r(\gamma)}$ for every $\gamma\in G$. We further denote by $\mathrm{Proj}_{\mathcal{K}_x}$ the orthogonal projection from $\mathcal{H}_x$ onto $\mathcal{K}_x$ for $x\in X$.

The restriction of π to a (finite-dimensional) nonzero π-invariant sub-bundle is called a *(finite-dimensional) subrepresentation*.

Let $\pi\colon G\to U(\mathcal{H})$ and $\sigma\colon G\to U(\mathcal{K})$ be representations. Define their fiber-wise tensor product as follows. Set

$$\mathcal{H}\otimes\mathcal{K}=\bigsqcup_{x\in G^0}(\mathcal{H}_x\otimes\mathcal{K}_x),$$

which is also a Hilbert bundle over G^0, and define $\pi\otimes\sigma\colon G\to U(\mathcal{H}\otimes\mathcal{K})$ by $(\pi\otimes\sigma)_\gamma=\pi_\gamma\otimes\sigma_\gamma$ for $\gamma\in G$. One can also define the conjugate representation $\overline{\pi}$ of G on $\overline{\mathcal{H}}=\bigsqcup_{x\in\mathcal{H}_x}\overline{\mathcal{H}}_x$ in a similar way.

Remark 3.12. In the context of the comments above, $\mathcal{H}\otimes\overline{\mathcal{K}}$ can be identified with the Hilbert-Schmidt bundle $\mathrm{HS}(\mathcal{K},\mathcal{H})=\bigsqcup_{x\in G}\mathrm{HS}(\mathcal{K}_x,\mathcal{H}_x)$, where $\mathrm{HS}(\mathcal{K}_x,\mathcal{H}_x)$ denotes the space of Hilbert-Schmidt operators $\mathcal{K}_x\to\mathcal{H}_x$. If $|\xi\rangle\langle\eta|\colon\mathcal{K}_x\to\mathcal{H}_x$ is the rank-one operator $|\zeta\rangle\mapsto\langle\eta,\zeta\rangle|\xi\rangle$, then the isomorphism is induced by the assignment $\xi\otimes\overline{\eta}\mapsto|\xi\rangle\langle\eta|$ for $\eta\in\mathcal{K}_x$ and $\xi\in\mathcal{H}_x$.

Under this identification, the representation $\pi\otimes\overline{\sigma}$ can be identified with the representation on $\mathrm{HS}(\mathcal{K},\mathcal{H})$ defined by $T\mapsto\pi_\gamma T\sigma_\gamma^*$ for $\gamma\in G$ and $T\in\mathrm{HS}(\mathcal{K}_{s(\gamma)},\mathcal{H}_{s(\gamma)})$.

Definition 3.13. Let G be a discrete pmp groupoid, and let π be a representation of G on a Hilbert bundle $\mathcal{H}$. Then π induces a (group) representation $[\pi]\colon[G]\to U(L^2(G^0,\mathcal{H}))$ defined by $[\pi]_t(\xi)=(\pi_{xt}(\xi_{s(xt)}))_{x\in G^0}$ for all $t\in[G]$ and all $\xi\in L^2(G^0,\mathcal{H})$.

Similarly, π induces a (semigroup) representation $[[\pi]]\colon[[G]]\to U(L^2(G^0,\mathcal{H}))$ defined by

$$([[\pi]]_\sigma\xi)_x=\begin{cases}\pi_{x\sigma}(\xi_{s(x\sigma)}), & \text{if } x\in\sigma\sigma^{-1};\\ 0, & \text{otherwise.}\end{cases}$$

for all $\sigma\in[[G]]$, all $\xi\in L^2(G^0,\mathcal{H})$, and all $x\in G^0$.

Remark 3.14. One can check that a representation π of a groupoid on a Hilbert bundle is weak mixing in the sense of Definition 3.10 if and only if the associated representation $[\pi]$ is weak mixing in the usual sense.

The proof of the following lemma is immediate, so we leave it to the reader.

Lemma 3.15. Let π be a representation of an ergodic discrete pmp groupoid G on a Hilbert bundle $\mathcal{H}$. For an element $\xi = (\xi_x)_{x\in G^0}$ of $L^2(G^0, \mathcal{H})$, the following assertions are equivalent:

(1) ξ is fixed by $[\pi]$;
(2) there exists a countable subset $S \subseteq [G]$ that covers G such that $[\pi]_t(\xi) = \xi$ for every $t \in S$;
(3) $\pi_\gamma(\xi_{s(\gamma)}) = \xi_{r(\gamma)}$ for almost every $\gamma \in G$.

Several standard facts about representations of discrete groups admit natural generalizations to the setting of representations of discrete pmp groupoids. We present here some of them, together with the main ideas used in their proofs.

Proposition 3.16. *Let G be an ergodic pmp groupoid, and let $\pi\colon G \to U(H)$ be a representation.*

(1) If there exist $c > 0$ and a unit section $\xi \in L^2(G^0, \mathcal{H})$ such that $\langle \xi, [\pi]_t\xi\rangle \geq c$ for every $t \in [G]$, then π contains a nonzero invariant section.
(2) The following assertions are equivalent:
  (2.a) The representation $\pi \otimes \overline{\pi}$ contains invariant vectors;
  (2.b) The representation $\pi \otimes \sigma$ contains invariant vectors for some representation σ of G;
  (2.c) The representation π contains a finite-dimensional subrepresentation.
(3) The following assertions are equivalent:
  (3.a) π is weak mixing;
  (3.b) π has no finite-dimensional subrepresentations;
  (3.c) If $\mathcal{K}$ is a finite-dimensional sub-bundle of $\mathcal{H}$, then for every $\varepsilon > 0$, there exists $t \in [G]$ such that, for every unit section ξ for $\mathcal{K}$, $\int_{G^0} ||\mathrm{Proj}_{\mathcal{K}_x}(\pi_{xt}\xi_{s(xt)})||\, d\mu_{G^0}(x) \leq \varepsilon$.

Proof. (1). In view of Lemma 3.15, this is a particular instance of [48, Proposition 1.5.2].

(2). (2.a)⇒(2.b) Trivial. (2.b)⇒(2.c) Let σ be a representation of G on $\mathcal{H}'$ such that $\pi\otimes\overline{\sigma}$ contains invariant vectors. By Remark 3.12, we can identify $\pi\otimes\overline{\sigma}$ with a representation of G on $\mathrm{HS}(\mathcal{H}', \mathcal{H})$. By assumption, there exists $T \in L^2(G^0, \mathrm{HS}(\mathcal{H}', \mathcal{H}))$ nonzero such that $\pi_\gamma T_{s(\gamma)}\sigma^*_\gamma = T_{r(\gamma)}$ for every $\gamma \in G$. Thus $\pi_\gamma T_{s(\gamma)}T^*_{s(\gamma)}\pi^*_\gamma = T_{r(\gamma)}T^*_{r(\gamma)}$ for every $\gamma \in G$. Since T is nonzero, there exists $d > 0$ such that $T_xT^*_x$ has eigenvalue d for almost every $x \in G^0$. If $\mathcal{K}_x \subseteq \mathcal{H}_x$ is the eigenspace of $T_xT^*_x$ corresponding to d for almost every $x \in G^0$, then $\mathcal{K}$ defines a nonzero finite-dimensional π-invariant sub-bundle of $\mathcal{H}$.

(2.c)⇒(2.a) Suppose that $\mathcal{K}$ is a nonzero finite-dimensional π-invariant sub-bundle of $\mathcal{H}$. Then $x \mapsto \mathrm{Proj}_{\mathcal{K}_x}$ determines a $\pi\otimes\overline{\pi}$-invariant element of $\mathrm{HS}(\mathcal{H}) = \mathcal{H}\otimes\overline{\mathcal{H}}$.

(3). (3.a)⇒(3.b) Suppose, by contradiction, that $\mathcal{K}$ is a finite-dimensional π-invariant nonzero sub-bundle of $\mathcal{H}$, of dimension $d \in \mathbb{N}$. Fix sections $\xi_1, \ldots, \xi_d\colon G^0 \to \mathcal{K}$ such that $\{\xi_{1,x}, \ldots, \xi_{d,x}\}$ is an orthonormal basis of $\mathcal{K}$ for almost every $x \in G^0$. Fix $\varepsilon, \delta > 0$. By assumption, there exists $t \in [G]$ such that $|\langle \xi_i, [\pi]_t(\xi_j)\rangle| < \varepsilon$ for $1 \leq i, j \leq d$. By choosing $\varepsilon > 0$ small enough, this guarantees that there exists a Borel subset A of G^0 of measure at least $1 - \delta$ such that $|\langle \xi_{i,x}, \pi_{xt}(\xi_{j,s(tx)})\rangle| < \delta$ for all $x \in A$ and $1 \leq i, j \leq n$. Therefore,

$$1 = \|\pi_{xt}(\xi_{1,s(xt)})\|^2 = \sum_{i=1}^{d} |\langle \xi_{i,x}, \pi_{xt}(\xi_{j,xt})\rangle|^2 < \delta^2 d$$

for almost every $x \in A$. Choosing $\delta = d^{-1/2}$, we reach a contradiction.

(3.b)⇒(3.c) Let $\mathcal{K}$ be a finite-dimensional π-invariant sub-bundle of $\mathcal{H}$, and suppose that the conclusion fails. Then there exists $c > 0$ such that for every $t \in [G]$, there exists a unit section $\xi\colon G^0 \to \mathcal{K}$ with $\langle \xi, [\pi]_t(\xi)\rangle \geq c$. Consider the section $\mathrm{Proj}_{\mathcal{K}}\colon G^0 \to \mathcal{H}\otimes\overline{\mathcal{H}}$ given by $\mathrm{Proj}_{\mathcal{K}}(x) = \mathrm{Proj}_{\mathcal{K}_x}$ for all $x \in G^0$. Then $\langle \mathrm{Proj}_{\mathcal{K}}, [\pi\otimes\overline{\pi}]_t(\mathrm{Proj}_{\mathcal{K}})\rangle \geq c$ for all $t \in [G]$. By part (1), $\pi\otimes\overline{\pi}$ has invariant vectors. Therefore π has a finite-dimensional subrepresentation by part (2), contradicting the hypothesis.

(3.c)⇒(3.a) Suppose that $\mathcal{K}$ is a finite-dimensional π-invariant sub-bundle of $\mathcal{H}$, and let $\varepsilon > 0$. Then there exists $t \in [G]$ such that

$$\int_{G^0} \left\| \mathrm{Proj}_{\mathcal{K}_x}(\pi_{xt}(\xi_{s(xt)})) \right\| \, d\mu_{G^0}(x) \leq \varepsilon$$

for every unit section ξ for $\mathcal{K}$. Thus, if η and ξ are unit sections for $\mathcal{K}$, then

$$\langle \eta, [\pi]_t(\xi) \rangle \leq \int_{G^0} \left\| \mathrm{Proj}_{\mathcal{K}_x}(\pi_{xt}(\xi_{s(xt)})) \right\| \, d\mu_{G^0}(x) \leq \varepsilon.$$

This concludes the proof that π is weak mixing. ∎

Corollary 3.17. *Let π be a representation of an ergodic discrete pmp groupoid G on a Hilbert bundle $\mathcal{H}$. Then the following assertions are equivalent:*

(1) π is weak mixing;
(2) $\pi \otimes \sigma$ is weak mixing for every representation σ;
(3) $\pi \otimes \overline{\pi}$ is weak mixing;
(4) $\pi \otimes \overline{\pi}$ is ergodic;
(5) the representation $[\pi \otimes \overline{\pi}]$ of $[G]$ on $L^2(G^0, \mathcal{H} \otimes \mathcal{H})$ is ergodic.

Proof. (1)⇒(2) Suppose that there exists a unitary representation σ such that $\pi \otimes \sigma$ is not weak mixing. Then $\pi \otimes \sigma$ contains a finite-dimensional subrepresentation, by part (3) of Proposition 3.16. Therefore $\pi \otimes \sigma \otimes \pi \otimes \sigma$ has an invariant vector by part (2) of Proposition 3.16, and thus π has a finite-dimensional subrepresentation again by part (2) of Proposition 3.16. Therefore π is not weak mixing by part (3) of Proposition 3.16.

(2)⇒(3)⇒(4) Obvious.

(4)⇒(1) Suppose that π is not weak mixing. Then π has a finite-dimensional sub-representation by part (3) of Proposition 3.16, and $\pi \otimes \overline{\pi}$ contains invariant vectors by part (2) of Proposition 3.16. Thus $\pi \otimes \overline{\pi}$ is not ergodic.

(4)⇔(5) This follows from Lemma 3.15. ∎

3.4. **Bundles of tracial von Neumann algebras.** Let $\mathcal{H}$ be a Hilbert bundle over a standard Borel space X, and set $B(\mathcal{H}) = \bigsqcup_{x \in X} B(\mathcal{H}_x)$, which is a standard Borel space over X. The Borel structure on $B(\mathcal{H})$ is generated by the functions $(T \in B(\mathcal{H}_x)) \mapsto x$ and $(T \in B(\mathcal{H}_x)) \mapsto \langle \sigma_{n,x}, T(\sigma_{m,x}) \rangle$ for $n, m \in \mathbb{N}$. Moreover, $B(\mathcal{H})$ is canonically endowed with the following fibered functions: operator norm, composition, adjoint, sum, and scalar multiplication.

Definition 3.18. A *C*-bundle* over the Hilbert bundle $\mathcal{H}$ is a Borel subset $\mathcal{A}$ of $B(\mathcal{H})$ such that, for every $x \in X$, the corresponding fiber $\mathcal{A}_x = \mathcal{A} \cap B(\mathcal{H}_x)$ is a C*-subalgebra of $B(\mathcal{H}_x)$, and for which there exists a sequence $(a_n)_{n \in \mathbb{N}}$ of sections in $\mathcal{A}$ such that $(a_{n,x})_{n \in \mathbb{N}}$ is a subset of the unit ball of $\mathcal{A}_x$ that generates a norm-dense *-subalgebra of $\mathcal{A}_x$. We also denote the bundle $\mathcal{A}$ by $\bigsqcup_{x \in X} \mathcal{A}_x$.

Definition 3.19. A *tracial von Neumann bundle* over a standard Borel space X is a Borel subset $\mathcal{M}$ of $B(\mathcal{H})$ together with a Borel function $\tau \colon \mathcal{M} \to \mathbb{C}$ such that, for every $x \in X$, the corresponding fiber $\mathcal{M}_x = \mathcal{M} \cap B(\mathcal{H}_x)$ is a von Neumann algebra and the restriction τ_x of τ to $\mathcal{M}_x$ is a faithful tracial state on $\mathcal{M}_x$, and for which there exists a sequence of sections $(a_n)_{n \in \mathbb{N}}$ of $\mathcal{M}$ such that $(a_{n,x})_{n \in \mathbb{N}}$ is a subset of the unit ball of $\mathcal{M}_x$ that generates a *-subalgebra of $\mathcal{M}_x$ whose operator-norm unit ball is dense in the operator-norm unit ball of $\mathcal{M}_x$ with respect to the 2-norm $\|a\|_2 = \tau_x(a^*a)^{1/2}$, and there exists a sequence of sections $(\xi_n)_{n \in \mathbb{N}}$ of $\mathcal{H}$ such that $\tau_x(a) = \sum_{n \in \mathbb{N}} \langle \xi_{n,x}, a(\xi_{n,x}) \rangle$ for $x \in X$ and $a \in \mathcal{M}_x$. (In particular, this implies that τ_x is a normal tracial state on $\mathcal{M}_x$.) We also denote the bundle $(\mathcal{M}, \tau)$ by $\bigsqcup_{x \in X} (\mathcal{M}_x, \tau_x)$. We say that $(\mathcal{M}, \tau)$ is *abelian* if $\mathcal{M}_x$ is an abelian von Neumann algebra for almost every $x \in X$, .

Let $\mathcal{M}$ be a tracial von Neumann bundle over a standard probability space (X, μ). We let $L^2(\mathcal{M}, \tau)$ be the Hilbert bundle $\bigsqcup_{x \in X} L^2(\mathcal{M}_x, \tau_x)$ over X. Given $a \in \mathcal{M}_x$, we let $|a\rangle$ be the corresponding element of $L^2(\mathcal{M}_x, \tau_x)$. We identify $\mathcal{M}_x$ with a subalgebra of $B(L^2(\mathcal{M}_x, \tau_x))$, by identifying $a \in \mathcal{M}_x$ with its associated multiplication operator. We define $L^\infty(X, \mathcal{M})$ to be the algebra of essentially bounded sections of $\mathcal{M}$, which is a tracial von Neumann algebra with respect to the normal tracial state $\tau = \int_X \tau_x d\mu_X(x)$. Thus the inclusion $L^\infty(X) \subseteq L^\infty(X, \mathcal{M})$ is a trace-preserving embedding. The GNS construction $L^2(L^\infty(X, \mathcal{M}), \tau)$ associated with τ can be identified with the space $L^2(X, L^2(\mathcal{M}, \tau))$ associated with the Hilbert bundle $L^2(\mathcal{M}, \tau)$ as defined in Subsection 3.2. We will denote this space simply by $L^2(X, \mathcal{M}, \tau)$.

The GNS representation of $L^\infty(X, \mathcal{M})$ on $L^2(X, \mathcal{M}, \tau)$ maps an element $a \in L^\infty(X, \mathcal{M})$ to the corresponding *decomposable operator* on $L^2(X, \mathcal{M}, \tau)$ defined by $a(\xi) = (a_x \xi_x)_{x \in X}$ for $\xi = (\xi_x)_{x \in X}$. The canonical conditional expectation $E_{L^\infty(X)} \colon L^\infty(X, \mathcal{M}) \to L^\infty(X)$ is given by $E_{L^\infty(X)}(a) = (\tau_x(a_x))_{x \in X}$, for $a = (a_x)_{x \in X}$. This gives to $L^\infty(X, \mathcal{M})$ the structure of pre-C*-module over $L^\infty(X)$.

A *von Neumann sub-bundle* of a von Neumann bundle $(\mathcal{M}, \tau)$ over a standard Borel space X is a Borel subset $\mathcal{N} \subseteq \mathcal{M}$ such that $\mathcal{N}_x$ is a w*-closed subalgebra of $\mathcal{M}_x$, for all $x \in X$. For every $x \in X$, the unique trace-preserving conditional expectation $\mathcal{M}_x \to \mathcal{N}_x$ is denoted by $E_{\mathcal{N}_x}$. This defines a trace-preserving expectation $E_{\mathcal{N}} \colon L^\infty(X, \mathcal{M}) \to L^\infty(X, \mathcal{N})$, given by $E_{\mathcal{N}}(a) = (E_{\mathcal{N}_x}(a_x))_{x \in X}$.

Notation 3.20 (Orthogonal complements)**.** *Let $\mathcal{N}$ be a von Neumann sub-bundle of a von Neumann bundle $\mathcal{M}$. We let $L^2(\mathcal{M}, \tau) \cap \mathcal{N}^\perp$ be the sub-bundle $\bigsqcup_{x \in X} (L^2(\mathcal{M}_x, \tau_x) \cap \mathcal{N}_x^\perp)$. Given a subalgebra A of $L^\infty(X, \mathcal{M})$, we let $L^2(X, \mathcal{M}, \tau) \cap A^\perp$ be the orthogonal complement of A inside $L^2(X, \mathcal{M}, \tau)$, where A is canonically identified with a subspace of $L^2(X, \mathcal{M}, \tau)$.*

A particular example of a von Neumann sub-bundle is the *trivial sub-bundle* $\bigsqcup_{x \in X} \mathbb{C}1_x$, where 1_x is the unit of $\mathcal{M}_x$. We define the *center* of $\mathcal{M}$ to be the sub-bundle $Z(\mathcal{M}) = \bigsqcup_{x \in X} Z(\mathcal{M}_x)$.

3.5. **Actions of groupoids.**

Definition 3.21. Given a C*-bundle $\mathcal{A}$ over a standard Borel space X, we define the *automorphism groupoid* of $\mathcal{A}$ as

$$\mathrm{Aut}(\mathcal{A}) = \{\alpha \colon \mathcal{A}_x \to \mathcal{A}_y \ \ast\text{-isomorphism, for } x, y \in X\}.$$

Similarly, given a tracial von Neumann bundle $(\mathcal{M}, \tau)$, its *(tracial) automorphism groupoid* is

$$\mathrm{Aut}(\mathcal{M}, \tau) = \{\alpha \colon \mathcal{M}_x \to \mathcal{M}_y \text{ trace preserving } \ast\text{-isomorphism, for } x, y \in X\}.$$

In the following, we will identify the unit space of $\mathrm{Aut}(\mathcal{A})$ and $\mathrm{Aut}(\mathcal{M}, \tau)$ with X.

Remark 3.22. The automorphism groupoid of a C*-bundle $\mathcal{A}$ is naturally a standard Borel groupoid, endowed with the standard Borel structure generated by the source and range maps together with the functions $(\alpha : \mathcal{A}_x \to \mathcal{A}_y) \mapsto \|\alpha(a_{n,x})\|$ for $n \in \mathbb{N}$, where $(a_n)_{n \in \mathbb{N}}$ are the sections of $\mathcal{A}$ as in the definition of a C*-bundle. The same applies to $\mathrm{Aut}(\mathcal{M}, \tau)$ for a tracial von Neumann bundle $(\mathcal{M}, \tau)$ when one replaces the operator norm with the 2-norm defined by τ.

Definition 3.23. An *action* of a discrete pmp groupoid G with unit space X on a tracial von Neumann bundle $(\mathcal{M}, \tau)$ over X is a homomorphism $\alpha \colon G \to \mathrm{Aut}(\mathcal{M}, \tau)$ that fixes the unit space. The action α induces a (group) action $[\alpha] \colon [G] \to \mathrm{Aut}((L^\infty(X, \mathcal{M}), \tau))$ defined by

$$[\alpha]_t(a) = (\alpha_{xt}(a_{s(xt)})_{x \in G^0})$$

for $t \in [G]$ and $a = (a_x)_{x \in G^0} \in L^\infty(G^0, \mathcal{M})$. It also induces an action of $[[G]]$, defined similarly.

The proof of the following lemma is immediate.

Lemma 3.24. Let α be a representation of an ergodic discrete pmp groupoid G on a tracial von Neumann bundle $\mathcal{M}$. For $a = (a_x)_{x \in G^0}$ of $L^\infty(G^0, \mathcal{M})$, the following assertions are equivalent:

(1) a is fixed by $[\alpha]$;
(2) there exists a countable subset $S \subseteq [G]$ that covers G such that $[a]_t(a) = a$ for every $t \in S$;
(3) $\alpha_\gamma(a_{s(\gamma)}) = a_{r(\gamma)}$ for almost every $\gamma \in G$.

3.6. **Actions of groupoids on spaces.** Let (G, μ_G) be a discrete pmp groupoid, and let (Z, λ) be a standard probability space fibered over G^0. Let $(\lambda_x)_{x\in G^0}$ be the disintegration of λ with respect to μ_G. Then $\bigsqcup_{x\in G^0} L^2(Z_x, \lambda_x)$ is a Hilbert bundle over G^0, and $\bigsqcup_{x\in G^0}(L^\infty(Z_x), \lambda_x)$ is a tracial von Neumann bundle over G^0

Definition 3.25. A *pmp action* of a discrete pmp groupoid G on a standard probability space (Z, λ) fibered over G^0 is an action of G on the tracial von Neumann bundle $(\mathcal{M}, \tau) = \bigsqcup_{x\in G^0}(L^\infty(Z_x), \lambda_x)$.

Remark 3.26. The automorphism groupoid $\mathrm{Aut}\left(\bigsqcup_{x\in G^0}(L^\infty(Z_x), \tau_x)\right)$ can be identified with the groupoid $\mathrm{Aut}\left(\bigsqcup_{x\in G^0}(Z_x, \lambda_x)\right)$ consisting of all Borel isomorphisms $\eta\colon Z_s \to Z_t$, for $s, t \in G^0$, satisfying $\eta_*(\lambda_s) = \lambda_t$. Thus, a pmp action of G on (Z, λ) can be seen as a Borel groupoid homomorphism $G \to \mathrm{Aut}(\bigsqcup_{x\in G^0}(Z_x, \lambda_x))$ fixing the unit space.

Lemma 3.27. Let G be a discrete pmp groupoid. Then pmp actions of G on standard probability spaces can be canonically identified with class-bijective pmp extensions of G. In other words, there are natural assignments $\alpha \mapsto (\pi_\alpha, H_\alpha)$ and $(\pi, H) \mapsto \alpha_{\pi,H}$ between the classes of pmp actions and class-bijective extensions of G satisfying $\alpha_{(\pi_\alpha, H_\alpha)} = \alpha$ and $(\pi_{\alpha_{(\pi,H)}}, H_{\alpha_{(\pi,H)}}) = (\pi, H)$.

Proof. Let H be a pmp groupoid and let $\pi\colon H \to G$ be a class-bijective pmp extension. Then π turns (H^0, μ_{H^0}) into a standard probability space fibered over G^0, and we let $(H^0_x, \mu_{H^0,x})_{x\in G^0}$ be the corresponding disintegration. One can then consider the pmp action $\alpha_{\pi,H}$ of G on (H^0, μ_{H^0}) given by, for $\gamma \in G$ and $y \in H_{s(\gamma)}$,

$$(\alpha_{\pi,H})_\gamma(y) = r\left((\pi|_{Hy})^{-1}(\gamma)\right).$$

Conversely, let $\alpha\colon G \to \mathrm{Aut}(\bigsqcup_{x\in G^0}(Z_x, \lambda_x))$ be an action of G on (Z, λ). Define the corresponding *action groupoid* $G \ltimes^\alpha Z$ as follows: the set of objects of $G \ltimes^\alpha Z$ is Z, while the arrows of $G \ltimes^\alpha Z$ are pairs (γ, z) with $\gamma \in G$ and $z \in Z_{s(\gamma)}$. We represent the pair (γ, z), regarded as an element of $G \ltimes^\alpha Z$, by $\gamma \ltimes^\alpha z$. Then the following operations turn $G \ltimes^\alpha Z$ into a discrete pmp groupoid:

$$s(\gamma \ltimes^\alpha z) = z, \quad r(\gamma \ltimes^\alpha z) = \alpha_\gamma(z), \quad (\gamma \ltimes z)^{-1} = (\gamma^{-1} \ltimes^\alpha \alpha_\gamma(z)), \quad (\rho \ltimes^\alpha w)(\gamma \ltimes^\alpha z) = \rho\gamma \ltimes^\alpha z,$$

the last one whenever $\alpha_\gamma(z) = w$. Moreover, the map $\pi_\alpha\colon G \ltimes^\alpha Z \to G$ given by $\pi_\alpha(\gamma \ltimes^\alpha z) = \gamma$ for all $\gamma \ltimes^\alpha z \in G \rtimes^\alpha Z$ is a class-bijective pmp extension of G.

It is not difficult to verify that the constructions described above are inverse of each other, and this concludes the proof. ∎

In view of the above lemma, from now on we will identify pmp actions and class-bijective pmp extensions of a discrete pmp groupoid.

3.7. **Tensor products.** Let (M, τ) be a tracial von Neumann algebra, and let $N \subseteq M$ be a von Neumann subalgebra. We denote by $E_N : M \to N$ the unique trace-preserving conditional expectation, and set $\langle a, b\rangle_N = E_N(a^*b)$ for all $a, b \in M$. This pairing turns M into a (right) pre-C*-module over N, whose completion is a C*-module over N. As explained in [6, Subsection 8.5.32], the weak closure of this C*-module within its linking algebra has a canonical structure of W*-module over N, which we denote by $L^2(M, E_N)$. By [6, Lemma 8.5.4], $L^2(M, E_N)$ has a unique predual that makes the N-valued inner product separately w*-continuous. We consider the w*-topology on $L^2(M, E_N)$ to be the one defined by its unique predual. As shown in [6, Section 8.5], a bounded N-bimodule map on $L^2(M, E_N)$ is automatically adjointable, and the space $B_N(L^2(M, E_N))$ of such maps is a von Neumann algebra. Given $a \in M$, we let $|a\rangle_N \in L^2(M, E_N)$ be the corresponding operator. Assigning to an element $a \in M$ the corresponding multiplication operator on $L^2(M, E_N)$ defines a faithful representation $M \to B_N(L^2(M, E_N))$.

Definition 3.28. Let (M_0, τ_0) and (M_1, τ_1) be tracial von Neumann algebras with a common subalgebra N contained in their centers. Consider the *external tensor product* of W*-modules $L^2(M_0, E_N) \otimes L^2(M_1, E_N)$. Given $a_0 \in M_0$ and $a_1 \in M_1$, denote by $a_0 \otimes_N a_1$ the operator on $L^2(M_0, E_N) \otimes L^2(M_1, E_N)$ given by

$$(a_0 \otimes_N a_1)(|b_0\rangle_N \otimes |b_1\rangle_N) = |a_0 b_0\rangle_N \otimes |a_1 b_1\rangle_N$$

for $|b_0\rangle_N \otimes |b_1\rangle_N \in L^2(M_0, E_N) \otimes L^2(M_1, E_N)$. The *tensor product* $M_0 \otimes_N M_1$ *relative to* N is the w*-closed subalgebra of $B(L^2(M_0, E_N) \otimes L^2(M_1, E_N))$ generated by the N-bimodule operators of the form $a_0 \otimes_N a_1$ for some $a_0 \in M_0$ and $a_1 \in M_1$. We define a tracial state τ on it by setting $\tau(a) = \tau(\langle 1 \otimes_N 1|a|1 \otimes_N 1\rangle_N)$ for $a \in M_0 \otimes_N M_1$.

It is easy to see that $a \otimes_N 1 = 1 \otimes_N a$ whenever $a \in N$. When N is the trivial subalgebra of M_0, M_1, then the conditional expectations E_N coincide with the given tracial states of M_0 and M_1, and the tensor product relative to N coincides with the tensor product $M_0 \otimes M_1$ with respect to the given traces as defined in [5, Section III.3].

Definition 3.29. Adopt the notation of the previous definition. Let Γ be a group, and let $\alpha_j \colon \Gamma \to \mathrm{Aut}(M_j, \tau_j)$, for $j = 0, 1$, be actions. We say that α_0 and α_1 *agree on* N if N is α_j-invariant for $j = 0, 1$, and the restrictions of α_0 and α_1 to N are equal. In this case, the actions α_0 and α_1 canonically induce an action $\alpha_0 \otimes_N \alpha_1$ of Γ on $(M_0 \otimes_N M_1, \tau)$.

Remark 3.30. Since we are assuming that N is a central subalgebra of M_0 and M_1, their tensor product $M_0 \otimes_N M_1$ can be equivalently described as follows. Given a presentation $N = L^\infty(Z, \lambda)$, the algebras M_0 and M_1 admit direct integral decompositions

$$M_0 = \int_Z^\oplus M_{0,z} d\lambda(z) \quad \text{and} \quad M_1 = \int_Z^\oplus M_{1,z} d\lambda(z);$$

see [34, Chapter 14] or [60, Appendix F]. Then $M_0 \otimes_N M_1$ can be identified with $\int_Z^\oplus (M_{0,z} \otimes M_{1,z}) d\lambda(z)$.

Definition 3.31. Given two tracial von Neumann bundles $(\mathcal{M}_0, \tau_0)$ and $(\mathcal{M}_1, \tau_1)$ with a common central sub-bundle $\mathcal{N}$, their tensor product $\mathcal{M}_0 \otimes_{\mathcal{N}} \mathcal{M}_1$ is the tracial von Neumann bundle $\bigsqcup_{x \in X} (\mathcal{M}_{0,x} \otimes_{\mathcal{N}_x} \mathcal{M}_{1,x})$. Similarly as before, one can define the tensor product of actions on $\mathcal{M}_0$ and $\mathcal{M}_1$ that agree on $\mathcal{N}$.

In the context above, $L^\infty(X, \mathcal{M}_0 \otimes_{\mathcal{N}} \mathcal{M}_1)$ can be identified with $L^\infty(X, \mathcal{M}_0) \otimes_{L^\infty(X, \mathcal{N})} L^\infty(X, \mathcal{M}_1)$; see [46, Section 1.3] and [55]

3.8. **The Koopman representation.**

Definition 3.32. Let G be a discrete pmp groupoid, and let $\alpha \colon G \to \mathrm{Aut}(\mathcal{M}, \tau)$ be an action on a von Neumann bundle $(\mathcal{M}, \tau)$ over G^0. The *Koopman representation* associated with α is the representation $\kappa^\alpha \colon G \to U(L^2(\mathcal{M}, \tau))$ defined by $\kappa^\alpha_\gamma |a\rangle = |\alpha_\gamma(a)\rangle$ for $\gamma \in G$ and $a \in \mathcal{M}_{s(\gamma)}$.

We denote by κ^α_0 the restriction of κ^α to the invariant sub-bundle $\bigsqcup_{x \in G^0} (L^2(\mathcal{M}_x, \tau_x) \cap (\mathbb{C}|1_x\rangle)^\perp)$. More generally, if $(\mathcal{N}, \tau)$ is an α-invariant central von Neumann sub-bundle of $(\mathcal{M}, \tau)$, we define a representation $\kappa^{\alpha,\mathcal{N}} \colon G \to U(L^2(\mathcal{M}, E_{\mathcal{N}}))$ by setting $\kappa^{\alpha,\mathcal{N}}_\gamma(|a\rangle_{\mathcal{N}_{s(\gamma)}}) = |\alpha_\gamma(a)\rangle_{\mathcal{N}_{r(\gamma)}}$ for all $\gamma \in G$ and all $|a\rangle_{\mathcal{N}_{s(\gamma)}} \in L^2(\mathcal{M}_{s(\gamma)}, E_{\mathcal{N}_{s(\gamma)}})$. We define then $\kappa^{\alpha,\mathcal{N}}_0$ to be the restriction of $\kappa^{\alpha,\mathcal{N}}$ to the sub-bundle $\bigsqcup_{x \in G^0} (L^2(\mathcal{M}_x, E_{\mathcal{N}_x}) \cap \mathcal{N}_x^\perp)$.

Remark 3.33. Adopt the notation from the above definition. Then the canonical identification of $L^2(L^\infty(G^0, \mathcal{M}), \tau)$ with $L^2(G^0, L^2(\mathcal{M}, \tau))$ allows one to identify the Koopman representation $\kappa^{[\alpha]}$ associated with the action $[\alpha]$ of $[G]$ on $(L^\infty(G^0, \mathcal{M}), \tau)$ with the representation $[\kappa^\alpha]$ of $[G]$ on $L^2(G^0, L^2(\mathcal{M}, \tau))$ induced by κ^α. More generally, the canonical identification of $L^2(G^0, L^2(\mathcal{M}, E_{\mathcal{N}}))$ with $L^2(L^\infty(G^0, \mathcal{M}), E_{L^\infty(G^0, \mathcal{N})})$ allows one to identify $[\kappa^{\alpha,\mathcal{N}}]$ with $\kappa^{[\alpha], L^\infty(G^0, \mathcal{N})}$.

Lemma 3.34. Let G be a discrete pmp groupoid. For $j = 1, 2$, let $(\mathcal{M}_j, \tau_j)$ be von Neumann bundles over G^0 with common central sub-bundle $\mathcal{N}$, and let $\alpha_j \colon G \to \mathrm{Aut}(\mathcal{M}_j, \tau)$ be actions agreeing on $\mathcal{N}$.

(1) For $j = 1, 2$, the representation $\kappa_0^{\alpha_j,\mathcal{N}}$ is conjugate to $\overline{\kappa}_0^{\alpha_j,\mathcal{N}}$.
(2) The representation $\kappa_0^{\alpha_1 \otimes_{\mathcal{N}} \alpha_2, \mathcal{N}}$ is conjugate to $(\kappa_0^{\alpha_1,\mathcal{N}} \otimes \kappa_0^{\alpha_2,\mathcal{N}}) \oplus \kappa_0^{\alpha_1,\mathcal{N}} \oplus \kappa_0^{\alpha_2,\mathcal{N}}$.

Proof. (1). We fix $j \in \{1, 2\}$ and write α for α_j. Then the canonical anti-unitary $J \colon L^2(\mathcal{M}, E_{\mathcal{N}}) \mapsto \overline{L^2(\mathcal{M}, E_{\mathcal{N}})}$, $|a\rangle \mapsto |a^*\rangle$ induces an isomorphism from $L^2(\mathcal{M}, E_{\mathcal{N}}) \cap \mathcal{N}^\perp$ to $\overline{L^2(\mathcal{M}, E_{\mathcal{N}}) \cap \mathcal{N}^\perp}$ that intertwines $\kappa_0^{\alpha,\mathcal{N}}$ and $\overline{\kappa}_0^{\alpha,\mathcal{N}}$.

(2). The map $\mathcal{M}_1 \otimes_{\mathcal{N}} \mathcal{M}_2 \to (\mathcal{M}_1 \otimes_{\mathcal{N}} \mathcal{M}_2) \oplus \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{N}$ defined by

$$a \otimes_{\mathcal{N}} b \mapsto ((a - E_{\mathcal{N}}(a)) \otimes_{\mathcal{N}} (b - E_{\mathcal{N}}(b)), E_{\mathcal{N}}(b)a, E_{\mathcal{N}}(a)b, E_{\mathcal{N}}(a)E_{\mathcal{N}}(b))$$

induces an isomorphism from $L^2(\mathcal{M}_1 \otimes_{\mathcal{N}} \mathcal{M}_2, E_{\mathcal{N}}) \cap \mathcal{N}^\perp$ to

$$(L^2(\mathcal{M}_1, E_{\mathcal{N}}) \cap \mathcal{N}^\perp) \otimes (L^2(\mathcal{M}_2, E_{\mathcal{N}}) \cap \mathcal{N}^\perp) \oplus L^2(\mathcal{M}_1, E_{\mathcal{N}}) \oplus L^2(\mathcal{M}_2, E_{\mathcal{N}})$$

that witnesses the desired conjugacy. ∎

3.9. **Ergodicity.** Recall that a unitary representation π of a Polish group Γ on a Hilbert space H is said to be *ergodic* if the space H^π of π-invariant vectors is trivial.

Lemma 3.35. Let α be an action of a countable discrete group Γ on a tracial von Neumann algebra (M, τ), and let N be an α-invariant central subalgebra of (M, τ). Then the fixed point algebra M^α is contained in N if and only $\kappa_0^{\alpha,N}$ is ergodic.

Proof. Assume that $M^\alpha \subseteq N$, and consider the direct integral decomposition

$$(M, \tau) = \int_X (M_x, \tau_x) d\mu(x)$$

with respect to the central subalgebra $N = L^\infty(X, \mu) \subseteq M$; see [60, Appendix F]. With $\mathcal{H}$ denoting the Hilbert bundle $\bigsqcup_{x \in X} L^2(M_x, \tau_x)$, one can identify $L^2(M, \tau)$ with $L^2(X, \mathcal{H}, \tau)$. Let $\xi \in L^2(X, \mathcal{H}, \tau)$ be a κ^α-invariant unit vector. Identifying $L^2(M_x, \tau_x)$ with the space of affiliated operators with M_x—see [51, Section 1.2]—let $\xi_x = v_x|\xi_x|$ be the polar decomposition of ξ_x, where $v_x \in M_x$ is a partial isometry, and the spectral resolution $(e_{s,x})_{s>0}$ of $|\xi_x|$ is contained in M_x; see [51, Subsection 2.1]. Suppose, by contradiction, that ξ is orthogonal to N. Find $s > 0$, a non-null Borel subset $A \subseteq X$, and projections $e_{s,x}$, for $x \in X$, satisfying $0 < \tau_x(e_{s,x}) < 1$. By the essential uniqueness of the spectral resolution, the element $e_s = (e_{x,s})_{x \in X} \in M$ is α-invariant. Since $e \in N$ by assumption, we have $\tau_x(e_{s,x}) \in \{0, 1\}$ for almost every $x \in X$, which is a contradiction. The converse is obvious. ∎

Recall that a discrete pmp groupoid G is ergodic if a non-null invariant Borel subset of G^0 has full measure.

Definition 3.36. Let G be an ergodic discrete pmp groupoid, and let $(\mathcal{M}, \tau)$ be a tracial von Neumann bundle over G^0. An action $\alpha\colon G \to \mathrm{Aut}(\mathcal{M}, \tau)$ is said to be *ergodic* if whenever $a \in L^\infty(G^0, \mathcal{M})$ satisfies $\alpha_\gamma(a_{s(\gamma)}) = a_{r(\gamma)}$ for almost every $\gamma \in G$, then $a_x \in \mathbb{C}1_x$ for almost every $x \in G^0$.

Lemma 3.37. Adopt the notation from the above definition, and let $(\mathcal{N}, \tau)$ be an α-invariant central sub-bundle of $(\mathcal{M}, \tau)$. Then the following assertions are equivalent:

(1) $L^\infty(G^0, \mathcal{M})^\alpha \subseteq L^\infty(G^0, \mathcal{N})$;
(2) $L^\infty(G^0, \mathcal{M})^{[\alpha]|_\Gamma} \subseteq L^\infty(G^0, \mathcal{N})$ for every countable subgroup $\Gamma \subseteq [G]$ that covers G;
(3) $L^\infty(G^0, \mathcal{M})^{[\alpha]|_\Gamma} \subseteq L^\infty(G^0, \mathcal{N})$ for some countable subgroup $\Gamma \subseteq [G]$ that covers G;
(4) $\kappa_0^{\alpha,\mathcal{N}}$ is ergodic;
(5) $[\kappa_0^{\alpha,\mathcal{N}}]|_\Gamma$ is ergodic for every countable subgroup $\Gamma \subseteq [G]$ that covers G;
(6) $[\kappa_0^{\alpha,\mathcal{N}}]|_\Gamma$ is ergodic for some countable subgroup $\Gamma \subseteq [G]$ that covers G.

Proof. The equivalences (1)⇔(2)⇔(3) follow from Lemma 3.24. The equivalences (4)⇔(5)⇔(6) follow from Lemma 3.15. Finally, the equivalence (2)⇔(5) follows from Lemma 3.35, since for a countable subgroup $\Gamma \subseteq [G]$ one can identify $[\kappa^{\alpha,\mathcal{N}}]|_\Gamma$ with $\kappa^{[\alpha]|_\Gamma, L^\infty(G^0,\mathcal{N})}$; see Remark 3.33. ∎

Corollary 3.38. *Let α be an action of an ergodic discrete pmp groupoid G on a tracial von Neumann bundle $(\mathcal{M}, \tau)$ over G^0. Then the following assertions are equivalent:*

(1) α is ergodic;
(2) $[\alpha]$ is ergodic;
(3) for every countable subgroup Γ of $[G]$ that covers G, the action $[\alpha]|_\Gamma$ is ergodic;
(4) for some countable subgroup Γ of $[G]$ that covers G, the action $[\alpha]|_\Gamma$ is ergodic.
(5) the representation $[\kappa_0^\alpha]$ is ergodic;

(6) for every countable subgroup Γ of $[G]$ that covers G, the representation $[\kappa_0^\alpha]|_\Gamma$ is ergodic;
(7) for some countable subgroup Γ of $[G]$ that covers G, the representation $[\kappa_0^\alpha]|_\Gamma$ is ergodic.

Lemma 3.39. Let G be an ergodic discrete pmp groupoid, let $A \subseteq G^0$ be a non-null Borel subset, and let $\pi\colon G \to U(\mathcal{H})$ be a representation. Then π is ergodic if and only if $\pi|_{AGA}$ is ergodic.

Proof. Since the "if" implication is obvious, we show the "only if" direction. We prove the contrapositive. Suppose that $\pi|_{AGA}$ is not ergodic. Then there exists an AGA-invariant unit section ξ for $\mathcal{H}|_A = \bigsqcup_{x\in A}\mathcal{H}_x$. Choose $\sigma_1,\ldots,\sigma_n \in [[G]]$ such that $\sigma_i^{-1}\sigma_i = A = \sigma_0$ for $i = 0,1,\ldots,n$, and $(\sigma_i\sigma_i^{-1})_{i=1}^n$ is a partition of G^0. Define $\eta \in \mathcal{H}$ by setting $\eta_{r(x\sigma_i)} = \pi_{x\sigma_i}(\xi_{s(x\sigma_i)})$ for $i = 0,1,\ldots,n$ and $x \in \sigma_i\sigma_i^{-1}$.

We claim that η is a G-invariant (unit) section. Indeed, for $\gamma \in G$ with $s(\gamma) \in \sigma_i\sigma_i^{-1}$ and $r(\gamma) \in \sigma_j\sigma_j^{-1}$, we use AGA-invariance of ξ to get

$$\pi_{\sigma_j^{-1}r(\gamma)}\pi_\gamma\pi_{s(\gamma)\sigma_i}(\xi_{s(s(\gamma)\sigma_i)}) = \pi_{\sigma_j\gamma\sigma_i}(\xi_{s(\sigma_j^{-1}\gamma\sigma_i)}) = \xi_{r(\sigma_j^{-1}\gamma\sigma_i)} = \xi_{s(r(\gamma)\sigma_j)}.$$

Therefore

$$\pi_\gamma(\eta_{s(\gamma)}) = \pi_\gamma\pi_{s(\gamma)\sigma_i}(\xi_{s(s(\gamma)\sigma_i)}) = \pi_{r(\gamma)\sigma_j}(\xi_{s(r(\gamma)\sigma_j)}) = \eta_{r(\gamma)}.$$

This shows that η is a G-invariant unit section, and hence π is not ergodic. ∎

3.10. **Weak mixing actions.** Recall that a representation $\pi\colon \Gamma \to U(H)$ of a Polish group Γ is *weak mixing* if for every $\varepsilon > 0$, every $n \in \mathbb{N}$, and every $\xi_1,\ldots,\xi_n \in H$, there exists $\gamma \in \Gamma$ such that $|\langle \xi_i, \pi_\gamma(\xi_j)\rangle| < \varepsilon$ for $i,j = 1,2,\ldots,n$. Also recall that an action $\alpha\colon \Gamma \to \mathrm{Aut}(M,\tau)$ on a tracial von Neumann algebra is said to be *weak mixing* if for any $\varepsilon > 0$, every $n \in \mathbb{N}$, and every $a_1,\ldots,a_n \in M$, there exists $\gamma \in \Gamma$ such that $|\tau(a_i\alpha_\gamma(a_j)) - \tau(a_i)\tau(a_j)| < \varepsilon$ for $i,j = 1,\ldots,n$. More generally, given an α-invariant central subalgebra N on M with trace-preserving conditional expectation $E_N\colon M \to N$, the action α is said to be *weak mixing relatively to* N if for every $\varepsilon > 0$, every $n \in \mathbb{N}$, and every $a_1,\ldots,a_n \in M$, there exists $\gamma \in \Gamma$ such that $\|E_N(a_i\alpha_\gamma(a_j)) - E_N(a_i)E_N(a_j)\|_2 < \varepsilon$ for $i,j = 1,\ldots,n$; see [51, Lemma 2.11].

Definition 3.40. Let G be an ergodic discrete pmp groupoid, let $\mathcal{H}$ be a Hilbert bundle over G^0, and let $(\mathcal{M},\tau)$ be a tracial von Neumann bundle over $\mathcal{H}$. Let $\alpha\colon G \to \mathrm{Aut}(\mathcal{M},\tau)$ be an action, and let $(\mathcal{N},\tau)$ be an α-invariant central sub-bundle. We say that

- α is *weak mixing relative to* $\mathcal{N}$ if the action $[\alpha]\colon [G] \to \mathrm{Aut}(L^\infty(G^0,\mathcal{M}))$ is weak mixing relative to $L^\infty(G^0,\mathcal{N})$;
- the groupoid G is *weak mixing* if the canonical action of $[G]$ on G^0 is weak mixing.

Lemma 3.41. Let G be an ergodic discrete pmp groupoid, let $\mathcal{H}$ is a Hilbert bundle over G^0, and let $(\mathcal{M},\tau)$ be a tracial von Neumann bundle over $\mathcal{H}$. Let $\alpha\colon G \to \mathrm{Aut}(\mathcal{M},\tau)$ be an action, and let $(\mathcal{N},\tau)$ be an α-invariant central sub-bundle. Then the following assertions are equivalent:

(1) $\kappa_0^{\alpha,\mathcal{N}}$ is weak mixing;
(2) $L^\infty(G^0,\mathcal{M}\otimes_\mathcal{N}\mathcal{M})^{\alpha\otimes_\mathcal{N}\alpha}$ is contained in $L^\infty(G^0,\mathcal{N})$;
(3) for some countable subgroup $\Gamma \subseteq [G]$ that covers G, the action $[\alpha]|_\Gamma$ is weak mixing relatively to $L^\infty(G^0,\mathcal{N})$;
(4) for every countable subgroup $\Gamma \subseteq [G]$ that covers G, the action $[\alpha]|_\Gamma$ is weak mixing relative to $L^\infty(G^0,\mathcal{N})$;
(5) α is weak mixing relative to $(\mathcal{N},\tau)$;
(6) for every action β of G on a tracial von Neumann bundle $(\mathcal{M}',\tau')$ containing $(\mathcal{N},\tau)$ as a β-invariant central sub-bundle, $L^\infty(G^0,\mathcal{M}\otimes_\mathcal{N}\mathcal{M}')^{\alpha\otimes\beta}$ is contained in $L^\infty(G^0,\mathcal{N}\otimes_\mathcal{N}\mathcal{M}')^{\alpha\otimes\beta}$.

Proof. (1)⇒(2) Suppose that $\kappa_0^{\alpha,\mathcal{N}}$ is weak mixing. Then $\kappa_0^{\alpha\otimes_\mathcal{N}\alpha,\mathcal{N}}$ is isomorphic to $(\kappa_0^{\alpha,\mathcal{N}} \otimes \kappa_0^{\alpha,\mathcal{N}}) \oplus \kappa_0^{\alpha,\mathcal{N}} \oplus \kappa_0^{\alpha,\mathcal{N}}$ by part (2) of Lemma 3.34. Furthermore, $\kappa_0^{\alpha,\mathcal{N}} \otimes \kappa_0^{\alpha,\mathcal{N}}$ is ergodic by Corollary 3.17, so the conclusion then follows from Lemma 3.37.

(2)⇒(1) By Lemma 3.37, $\kappa_0^{\alpha\otimes_{\mathcal{N}}\alpha,\mathcal{N}}$ is ergodic. Since $\kappa_0^{\alpha\otimes_{\mathcal{N}}\alpha,\mathcal{N}}$ is conjugate to $(\kappa_0^{\alpha,\mathcal{N}}\otimes\overline{\kappa}_0^{\alpha,\mathcal{N}})\oplus\kappa_0^{\alpha,\mathcal{N}}\oplus\kappa_0^{\alpha,\mathcal{N}}$ by part (2) of Lemma 3.34, we conclude that $\kappa_0^{\alpha,\mathcal{N}}\otimes\overline{\kappa}_0^{\alpha,\mathcal{N}}$ is ergodic. Hence $\kappa_0^{\alpha,\mathcal{N}}$ is weak mixing by Corollary 3.17.

(1)⇔(3)⇔(4): This follows from the equivalence (1)⇔(2) together with the equivalence of items (1),(2),(3) in Lemma 3.37.

(5)⇒(6) This is the same as (i)⇒(ii) in [50, Proposition 2.4.2].

The implications (4)⇒(5) and (6)⇒(2) are obvious. ∎

Corollary 3.42. *Let α be an action of an ergodic discrete pmp groupoid G on a tracial von Neumann bundle $(\mathcal{M},\tau)$ over G^0. The following assertions are equivalent:*

(1) κ_0^α is weak mixing;
(2) $\alpha\otimes\alpha$ is ergodic;
(3) For every countable subgroup Γ of $[G]$ that covers G, the action $[\alpha]|_\Gamma$ is weak mixing relatively to $L^\infty(G^0)$;
(4) For some subgroup Γ of $[G]$ that covers G, the action $[\alpha]|_\Gamma$ is weak mixing relatively to $L^\infty(G^0)$;
(5) α is weak mixing relative to the trivial sub-bundle;
(6) $L^\infty(G^0,\mathcal{M}\otimes\mathcal{N})^{\alpha\otimes\beta}=1\otimes L^\infty(G^0,\mathcal{N})^\beta$ for every action β of G on a tracial von Neumann bundle $\mathcal{N}$;
(7) $\alpha\otimes\beta$ is ergodic for every ergodic action β of G on a tracial von Neumann bundle.

Proof. This immediately follows from Lemma 3.41, after observing that, for any action β of G, $\alpha\otimes\beta$ is equal, by definition, to $\alpha\otimes_{\mathcal{N}}\beta$ where $\mathcal{N}$ is the trivial sub-bundle of $(\mathcal{M},\tau)$. ∎

Corollary 3.43. *Let $(\alpha^{(n)})_{n\in\mathbb{N}}$ be a sequence of actions of an ergodic discrete pmp groupoid on tracial von Neumann bundles $(\mathcal{M}^{(n)},\tau^{(n)})_{n\in\mathbb{N}}$, each of which is weak mixing relative to the trivial sub-bundle. Then the action $\bigotimes_n\alpha^{(n)}$ on $(\bigotimes_n\mathcal{M}^{(n)},\bigotimes_n\tau^{(n)})$ is weak mixing relative to the trivial sub-bundle.*

Lemma 3.44. Suppose that α is an action of a groupoid G on a tracial von Neumann bundle $(\mathcal{M},\tau)$. Let Γ be a subgroup of $[G]$ that covers G such that the canonical action of Γ on G^0 is weak mixing. If α is weak mixing relatively to the trivial sub-bundle, then the action $[\alpha]|_\Gamma$ of Γ on $L^\infty(G^0,\mathcal{M},\tau)$ is weak mixing.

Proof. Fix $\varepsilon>0$ and $a_1,\ldots,a_n\in L^\infty(G^0,\mathcal{M},\tau)$. Use weak mixing of the action of Γ on G^0 to find $\gamma_0\in\Gamma$ such that, for $1\le i,j\le n$,

$$\left|\tau(\mathrm{E}_{L^\infty(G^0)}(a_i)[\alpha]_{\gamma_0}\mathrm{E}_{L^\infty(G^0)}(a_j))-\tau(a_i)\tau(a_j)\right|<\varepsilon.$$

Since α is weak mixing relatively to the trivial sub-bundle, there exists $\gamma_1\in\Gamma$ such that

$$\left\|\mathrm{E}_{L^\infty(G^0)}(a_i[\alpha]_{\gamma_1\gamma_0}(a_j))-\mathrm{E}_{L^\infty(G^0)}(a_i)\mathrm{E}_{L^\infty(G^0)}([\alpha]_{\gamma_0}a_j)\right\|_2<\varepsilon,$$

for $1\le i,j\le n$, and hence

$$\left|\tau(a_i[\alpha]_{\gamma_1\gamma_0}(a_j))-\tau(\mathrm{E}_{L^\infty(G^0)}(a_i)\mathrm{E}_{L^\infty(G^0)}([\alpha]_{\gamma_0}a_j))\right|<\varepsilon.$$

Since $\mathrm{E}_{L^\infty(G^0)}([\alpha]_{\gamma_0}a_j)=[\alpha]_{\gamma_0}\mathrm{E}_{L^\infty(G^0)}(a_j)$ for $1\le j\le n$, we conclude that

$$|\tau(a_i[\alpha]_{\gamma_1\gamma_0}(a_j))-\tau(a_i)\tau(a_j)|<2\varepsilon,$$

for $1\le i,j\le n$. Since $\gamma_0\gamma_1\in\Gamma$, this concludes the proof that the action $[\alpha]|_\Gamma$ is weak mixing. ∎

3.11. **Finite index subgroupoids.**

Definition 3.45. Let G be an ergodic discrete pmp groupoid and let $H\subseteq G$ be a subgroupoid. We define a countable equivalence relation $\sim_H$ on G by $\gamma\sim_H\gamma'$ if and only $\gamma=h\gamma'$ for some $h\in H$.

By ergodicity of G, the number of $\sim_H$-classes of H contained in Gx is constant for almost every $x\in G^0$. We define the *index* $[G:H]\in[1,\infty]$ of H in G to be this number.

Remark 3.46. In the context of the above definition, suppose additionally that H is ergodic as well. Then one can find elements $(\sigma_n)_{n=1}$ in $[G]$ such that $(H\sigma_n)_{n=1}$ is a partition of G; see [32, Section 2] and [30, Lemma 1.1]. We call this a *coset selection* for H in G.

Lemma 3.47. Let G be an ergodic discrete pmp groupoid and let $\alpha\colon G \to \mathrm{Aut}(\mathcal{M}, \tau)$ be an action with an α-invariant sub-bundle $(\mathcal{N}, \tau)$. Then α is weak mixing relative to $\mathcal{N}$ if and only if the restriction of α to any finite index subgroupoid is weak mixing relative to $\mathcal{N}$.

Proof. Since the "if" implication is obvious, we prove the converse. Let H be a subgroupoid of G with index $n < \infty$, and fix a coset selection $(\sigma_1, \ldots, \sigma_n)$ for H in G. Let Λ be a countable subgroup of $[H]$ that covers H. Then $[\alpha]|_\Lambda$ is weak mixing. Furthermore, the subgroup $\Gamma \subseteq [G]$ generated by Λ and $\sigma_1, \ldots, \sigma_n$ contains Λ as a finite index subgroup. Therefore $[\alpha]|_\Gamma$ is weak mixing. Since Γ covers G, it follows that α is weak mixing by [48, Corollary 2.2.12]. ∎

3.12. **The groupoid von Neumann algebra.**

Definition 3.48. Let G be a discrete pmp groupoid. The *left regular representation* of G is the representation $\lambda\colon G \to U(\bigsqcup_{x\in G^0} \ell^2(xG))$ defined by $\lambda_\gamma(\delta_\rho) = \delta_{\gamma\rho}$ for $(\gamma, \rho) \in G^2$. Similarly, the *right regular representation* of G is the representation $\rho\colon G \to U(\bigsqcup_{x\in G^0} \ell^2(Gx))$ defined by $\rho_\gamma \delta_\rho = \delta_{\rho\gamma^{-1}}$ for $\gamma \in G$ and $\rho \in Gs(\gamma)$.

Fix an action α of G on a tracial von Neumann bundle $(\mathcal{M}, \tau)$, and consider the Hilbert bundle $\mathcal{H} = \bigsqcup_{x\in G^0}(\ell^2(xG) \otimes L^2(\mathcal{M}_x))$. Define a representation $\pi^\alpha = \lambda \otimes \kappa^\alpha$ of G on $\mathcal{H}$ by $\pi^\alpha_\gamma(\delta_\rho \otimes |a\rangle) = \delta_{\gamma\rho} \otimes |\alpha_\gamma(a)\rangle$ for $(\gamma, \rho) \in G^2$ and $a \in \mathcal{M}_{s(\gamma)}$. There is also a canonical normal *-representation of $L^\infty(G^0, \mathcal{M})$ on $L^2(G^0, \mathcal{H}, \tau)$ defined by $a(\xi) = (a_x \xi_x)_{x\in G^0}$ for $a = (a_x)_{x\in G^0} \in L^\infty(G^0, \mathcal{M})$ and $\xi = (\xi_x)_{x\in G^0} \in L^2(G^0, \mathcal{H}, \tau)$, where $a_x(\delta_\rho \otimes |b\rangle) = \delta_\rho \otimes |a_x b\rangle$ for $x \in G^0$, $\rho \in xG$, and $b \in \mathcal{M}_x$.

Definition 3.49. Let G be a pmp ergodic groupoid, and let $\alpha\colon G \to \mathrm{Aut}(\mathcal{M}, \tau)$ be an action on a von Neumann bundle. The *crossed product* $G \ltimes^\alpha (\mathcal{M}, \tau)$ is the von Neumann subalgebra of $B(L^2(G^0, \mathcal{H}, \tau))$ generated by $\{[\pi^\alpha]_\sigma \colon \sigma \in [G]\} \cup L^\infty\left(G^0, \mathcal{M}\right)$, endowed with a canonical faithful normal tracial state defined by $\tau(x) = \langle 1|x|1\rangle$, where $|1\rangle$ denotes the element $(\delta_x \otimes |1_x\rangle)_{x\in G^0}$ of $L^2(G^0, \mathcal{M}, \tau)$.

The *groupoid von Neumann algebra* $L(G)$ is the crossed product of the action of G on G^0.

Remark 3.50. The action of G on G^0 can be seen as an action of G on the trivial von Neumann bundle $\mathcal{M}$ over G^0. In this case, in the notations above, we have $\mathcal{H} = \bigsqcup_{x\in G^0} \ell^2(xG)$, and $L^2(G^0, \mathcal{H}) = L^2(G)$. Hence, $L(G)$ coincides with the von Neumann subalgebra of $B(L^2(G))$ generated by $\{[[\lambda]]_\sigma \colon \sigma \in [[G]]\}$.

Let $\xi_0 \in L^2(G)$ be the element corresponding to the characteristic function of G^0. Then $(L^2(G), \xi_0)$ is the pointed Hilbert space obtained from $(L^\infty(G), \tau)$ via the GNS construction. This allows one to define the canonical anti-unitary $J\colon L^2(G) \to L^2(G)$ by $J|a\rangle = |a^*\rangle$ for $a \in L^\infty(G)$. The same proof as in the case of countable pmp equivalence relations gives the following; see [19].

Proposition 3.51. *Let G be a discrete pmp groupoid, and let Σ be a countable set of pairwise essentially disjoint elements of $[[G]]$ that covers G. Then:*

1. *The characteristic function $|1\rangle \in L^2(G)$ of G^0 is a cyclic separating vector for $L(G)$.*
2. *The vector state $x \mapsto \langle 1|x|1\rangle$ is a faithful normal tracial state on $L(G)$.*
3. *For $\sigma \in [[G]]$ one has $J[[\lambda]]_\sigma J = [[\rho]]_\sigma$ and hence $L(G)' = \{[[\rho]]_\sigma \colon \sigma \in [[G]]\}$.*
4. *An element $a \in L(G)$ can be written uniquely as $a = \sum_{\sigma\in\Sigma} a_\sigma [[\lambda]]_\sigma$, with $a_\sigma \in L^\infty(G^0)$ and the convergence is in 2-norm. In this case, $a_\sigma = \mathrm{E}_{L^\infty(G_0)}(x[[\lambda]]_{\sigma^{-1}})$ and $\|a\|_2^2 = \sum_{\sigma\in\Sigma} \|a_\sigma\|_2^2$.*

When G is an ergodic principal discrete pmp groupoid such that Gx is infinite for almost every $x \in G^0$, then $L(G)$ is a II_1 factor which contains $L^\infty(G^0)$ as a maximal abelian subalgebra; see [19].

4. Coinduction theory for groupoids

4.1. **Bernoulli actions of groupoids.** Let G be a discrete pmp groupoid. A *bundle of countable sets* over G is a standard Borel space $\mathcal{I}$ fibered over G^0 with countable fibres, such that there exists a sequence $(i_n)_{n\in\mathbb{N}}$ of sections for $\mathcal{I}$ such that $\{i_{n,x} \colon x \in G^0\}$ enumerates $\mathcal{I}_x$ for every $x \in G^0$. One can then define the standard Borel groupoid $\mathrm{Sym}(\mathcal{I})$ with unit space G^0 consisting of bijections $\sigma\colon I_x \to I_y$ for $x, y \in G^0$. The Borel structure on $\mathrm{Sym}(\mathcal{I})$ is generated by the source and range maps together with the subsets $\{\sigma\colon I_x \to I_y \colon x, y \in G^0, \sigma(i_{n,x}) = i_{m,y}\}$ for $n, m \in \mathbb{N}$. An *action* of G on $\mathcal{I}$ is a homomorphism from G to $\mathrm{Sym}(\mathcal{I})$.

Let (M,τ) be a tracial von Neumann algebra with separable predual, and let $L^2(M,\tau)$ be the corresponding Hilbert space obtained via the GNS construction, with cyclic vector $|1\rangle$. Define a Hilbert bundle $L^2(M,\tau)^{\otimes\mathcal{I}}$ by $L^2(M,\tau)^{\otimes\mathcal{I}} = \bigsqcup_{x\in G^0}(L^2(M,\tau),|1\rangle)^{\otimes\mathcal{I}_x}$. The standard Borel structure on $L^2(M,\tau)^{\otimes\mathcal{I}}$ can be described as follows. Fix a $\|\cdot\|_2$-dense sequence $(d_n)_{n\in\mathbb{N}}$ in M, and for every $n \in \mathbb{N}$, $\bar{k} = (k_1,\ldots,k_n) \in \mathbb{N}^n$ and $\bar{\ell} = (\ell_1,\ldots,\ell_n) \in \mathbb{N}^n$ with $\ell_1 < \ell_2 < \cdots < \ell_n$, define a section $\sigma^{(\bar{k},\bar{\ell})}$ for $L^2(M,\tau)^{\otimes\mathcal{I}}$ by $\sigma_x^{(\bar{k},\bar{\ell})} = (|d_{k_1}\rangle \otimes \cdots \otimes |d_{k_n}\rangle)_{(i_{\ell_1,x} i_{\ell_2,x} \cdots i_{\ell_n,x})} \in (L^2(M,\tau),|1\rangle)^{\otimes\mathcal{I}_x}$ for $x \in G^0$; see Notation 1. Then the Borel structure on $L^2(M,\tau)^{\otimes\mathcal{I}}$ is generated by the maps $\xi \mapsto \langle \xi, \sigma_x^{(\bar{k},\bar{\ell})}\rangle$, where $\xi \in (L^2(M,\tau),|1\rangle)^{\otimes\mathcal{I}_x}$, for $k, \ell \in \mathbb{N}$.

We define the tracial von Neumann bundle $(M,\tau)^{\otimes\mathcal{I}} = \bigsqcup_{x\in G^0}(M,\tau)^{\otimes\mathcal{I}_x}$, and endow it with a Borel structure defined similarly as above.

Definition 4.1. The *Bernoulli action* $\beta_{G\curvearrowright\mathcal{I}}$ of G with base (M,τ) associated with the action $G \curvearrowright \mathcal{I}$ is the action of G on $(M,\tau)^{\otimes\mathcal{I}}$ defined by $\beta_{G\curvearrowright\mathcal{I},\gamma}(a_{(i)}) = a_{(\gamma\cdot i)}$ for $\gamma \in G$, for $i \in \mathcal{I}_{s(\gamma)}$ and $a \in M$.

When (M,τ) is an abelian tracial von Neumann algebra, one obtains the notion of Bernoulli action of G on a standard probability space.

Example 4.2. *Let G be a discrete pmp groupoid, let K be an ergodic subgroupoid, and let $G/K = \bigsqcup_{x\in G^0} xG/K$ be the corresponding quotient. Given a subgroupoid $H \subseteq G$, consider the canonical action $H \curvearrowright G/K$. If (M,τ) is a tracial von Neumann algebra, we obtain a Bernoulli action $\beta_{H\curvearrowright G/K}$ on $(M,\tau)^{\otimes G/K} = \bigsqcup_{x\in G^0}(M,\tau)^{\otimes xG/K}$.*

Lemma 4.3. Let Λ be a countable discrete group, and let $\Delta \le \Lambda$ be an infinite index subgroup. Let G be a principal discrete pmp groupoid with unit space X, and let $\Lambda \curvearrowright^{\theta} X$ be a free ergodic action satisfying $\{\theta_\lambda(x) \colon \lambda \in \Lambda\} \subseteq [x]_G$ for almost every $x \in X$. Let $H = \Lambda \ltimes^{\theta} X$ be the corresponding action groupoid, which can be regarded as a subgroupoid of G, and set $K = \Delta \ltimes^{\theta|_\Delta} X \le H$. Given a tracial von Neumann algebra (M,τ), the Bernoulli action $\beta_{H\curvearrowright G/K}$ of H on $(M,\tau)^{G/K}$ is weak mixing relatively to the trivial sub-bundle.

Proof. Since G, H and K are a principal groupoids, we will identify them with their corresponding orbit equivalence relations. Since H is ergodic, we can define the index $N \in \mathbb{N}\cup\{\infty\}$ of H in G. There exists a coset selection $\{\sigma_n \colon n \le N\} \subseteq [G]$ of H in G such that G is the disjoint union of $\{\sigma_n H \colon n \le N\}$. One can identify xG/K with the set $[x]_{G/K} = \{[y]_K \colon y \in [x]_G\}$ of K-classes contained inside the G-class of x. Furthermore, one can identify σ_n with a Borel function $\sigma_n \colon X \to X$ for which $[x]_G$ is the disjoint union of $\{[\sigma_n(x)]_H \colon n \le N\}$ for every $x \in X$. Moreover, $[x]_{G/K}$ is the disjoint union of $\{[\sigma_n(x)]_{H/K} \colon n \le N\}$ for every $x \in X$. This gives an isomorphism of $(M,\tau)^{\otimes G/K}$ with the bundle $(\mathcal{M},\tau) = \bigsqcup_{x\in X}\bigotimes_{n\le N}(M,\tau)^{\otimes[\sigma_n(x)]_{H/G}}$.

For $x \in X$ and $n \le N$, there is an isomorphism $(M,\tau)^{\otimes\Lambda/\Delta} \to (M,\tau)^{\otimes[\sigma_n(x)]_{H/K}}$ associated with the bijection $\Lambda/\Delta \to [\sigma_n(x)]_{H/K}$ given by $\Delta\lambda \mapsto [\theta_\lambda(\sigma_n(x))]_K$. These maps determine an isomorphism

$$L^\infty(X,(M,\tau)^{G/K}) \cong L^\infty(X) \otimes ((M,\tau)^{\otimes\Lambda/\Delta})^{\otimes N}.$$

Let $a_1,\ldots,a_\ell \in L^\infty(X,(M,\tau)^{G/K})$ be contractions with $\mathrm{E}_{L^\infty(X)}(a_i) = 0$ for $i = 1,2,\ldots,\ell$, and fix $\varepsilon > 0$. Find $n \in \mathbb{N}$, a finite subset $F \subseteq \Lambda/\Delta$, and contractions $a_1',\ldots,a_\ell' \in L^\infty(X,(M,\tau)^{G/K})$ with

$$\|a_i - a_i'\|_2 < \varepsilon,\ \mathrm{E}_{L^\infty(X)}(a_i') = 0, \quad\text{and}\quad a_{i,x}' \in \bigotimes_{k\le N}(M,\tau)^{\otimes_{\gamma\in F}[\theta_\gamma(\sigma_k(x))]_K}$$

for $1 \le i \le \ell$, and almost every $x \in X$. Since Δ has infinite index in Λ, there exists $\lambda \in \Lambda$ such that $\lambda F \cap F = 0$. Identifying λ with the corresponding element of $[H]$, we have $\mathrm{E}_{L^\infty(X)}([\beta_{H\curvearrowright G/K}]_\lambda(a_i')a_j') = 0$, and therefore

$$\mathrm{E}_{L^\infty(X)}([\beta_{H\curvearrowright G/K}]_\lambda(a_i)a_j) < 2\varepsilon$$

for every $1 \le i,j \le d$. Thus $\beta_{H\curvearrowright G/K}$ is weak mixing relative to the trivial sub-bundle. ∎

4.2. **Coinduction for groupoids.** Let G be a discrete pmp groupoid, and let $H \leq G$ be a subgroupoid. Recall that the equivalence relation $\sim_H$ on G is defined by $\gamma \sim_H \gamma'$ if and only if $\gamma H = \gamma' H$. Let X be the unit space of G and H, and let Λ be a countable discrete group. Assume that there exists a free action $\Lambda \curvearrowright^\theta X$ such that $H = \Lambda \ltimes^\theta X$.

Let $G \ltimes G/H$ be the countable Borel groupoid associated with the canonical action $G \curvearrowright G/H$, i.e.

$$G \ltimes G/H = \{(\rho, \gamma H) \in G \times G/H \colon s(\rho) = r(\gamma)\}.$$

Let $T\colon G/H \to G$ be a *Borel selector* for $\sim_H$, i.e. a Borel map satisfying $T(\gamma H)H = \gamma H$ for every $\gamma \in G$. We will furthermore assume that $T(xH) = x$ for every $x \in G^0$.

Lemma 4.4. Let the notation be as before, and define a map $c\colon G \ltimes G/H \to \Lambda$ by letting $c(\rho, \gamma H)$ be the unique element of Λ such that $T(\rho\gamma H)^{-1}\rho T(\gamma H) = c(\rho, \gamma H) \ltimes^\theta x$ for some $x \in X$. Then c is a homomorphism.

Proof. We need only check that c is multiplicative. Given $(\rho_0, \rho_1) \in G^2$, we have

$$T(\rho_1\rho_0 H)(c(\rho_1, \rho_0\gamma H)c(\rho_0, \gamma H) \ltimes^\theta x) = \rho_1 T(\rho_0\gamma H)(c(\rho_0, \gamma H) \ltimes^\theta x) = \rho_1\rho_0 T(\gamma H),$$

as desired. ∎

Recall that we use the leg-numbering notation for linear operators on tensor product.

Definition 4.5. Adopt the notation of the discussion above, and let $\alpha\colon \Lambda \to \mathrm{Aut}(M, \tau)$ be an action. The *coinduced action* $\widehat{\alpha} = \mathrm{CInd}_H^G(\alpha)$ is the action of G on the tracial von Neumann bundle $(\mathcal{M}, \tau) = \bigsqcup_{x\in X}(M, \tau)^{\otimes xG/H}$ defined (on elementary tensors) by $\widehat{\alpha}_\gamma(a_{(\rho H)}) = \alpha_{c(\gamma,\rho H)}(a_{(\gamma\rho H)})$ for $\gamma \in G$ and $a_{(\rho H)} \in (M, \tau)$ with $(\gamma, \rho) \in G^2$; see Notation 1.

Example 4.6. *When α is the trivial action on (M, τ), then $\mathrm{CInd}_H^G(\alpha)$ is the Bernoulli shift $\beta_{G\curvearrowright G/H}$ as defined in the previous subsection.*

Remark 4.7. Let (Y, ν) be a standard probability space, and suppose that $(M, \tau) = L^\infty(Y, \nu)$, so that one can regard α as an action of Λ on Y. Then $\widehat{\alpha} = \mathrm{CInd}_H^G(\alpha)$ can be seen as the action of G on $\bigsqcup_{x\in X} Y^{xG/H}$ defined by setting

$$(\hat{\alpha}_{\gamma^{-1}}(\omega))(\rho H) = \alpha_{c(\gamma,\rho H)^{-1}}(\omega(\gamma\rho H))$$

for $\gamma \in G$, $\rho H \in s(\gamma)G/H$, and $\omega \in Y^{s(\gamma)G/H}$.

Proof. For $a \in L^\infty(Y)$, we can consider $a_{(\rho H)} \in L^\infty(Y^{s(\rho)G/H})$. Then $\hat{\alpha}_\gamma(a_{(\rho H)}) \in L^\infty(Y^{s(\gamma)G/H})$ is defined by setting, for $\omega \in Y^{s(\gamma)G/H}$,

$$\hat{\alpha}_\gamma(a_{(\rho H)})(\omega) = \alpha_{\sigma(\gamma,\rho H)}(a_{(\gamma\rho H)})(\omega) = a(\alpha_{c(\gamma,\rho H)^{-1}}(\omega(\gamma\rho H)))$$

On the other hand we have that

$$\hat{\alpha}_\gamma(a_{(\rho H)})(\omega) = a_{(\rho H)}(\hat{\alpha}_{\gamma^{-1}}(\omega)) = a((\hat{\alpha}_{\gamma^{-1}}(\omega))(\rho H)).$$

Since this holds for every $a \in L^\infty(Y)$, $(\hat{\alpha}_{\gamma^{-1}}(\omega))(\rho H) = \alpha_{c(\gamma,\rho H)^{-1}}(\omega(\gamma\rho H))$. ∎

Remark 4.8. Let Γ be a subgroup of $[G]$, and assume that the canonical action $\Gamma \curvearrowright G^0$ is free. If α is an action of Λ on an abelian tracial von Neumann algebra (M, τ), then $[\widehat{\alpha}]|_\Gamma$ is also free. Indeed, $\Gamma \curvearrowright G^0$ can be identified with the restriction of $[\widehat{\alpha}]|_\Gamma$ to $L^\infty(G^0) \subseteq L^\infty(G^0, \mathcal{M})$.

Suppose now that H is ergodic, and let N be the index of H in G, and $\Sigma = \{\sigma_n : n \leq N\} \subseteq [G]$ be a coset selection for H in G with $\sigma_1 = G^0$. For every $x \in X$, the set xG is the disjoint union of $x\sigma_n H$, for $n \leq N$. For $\gamma \in G$, let $\sigma_\gamma \in \Sigma$ be the unique element satisfying $\gamma \in \sigma_\gamma H$. Then the function $G/H \to X \times \Sigma$ given by $\gamma H \mapsto (r(\gamma), \sigma_\gamma)$, is a Borel isomorphism.

In the next proposition, we describe the form that T, α and $\widehat{\alpha}$ take under the isomorphism $G/H \cong X \times \Sigma$ described above. Its proof is straightforward, so we omit it.

Proposition 4.9. *Under the Borel isomorphism $G/H \cong X \times \Sigma$ as above, we have the following identifications:*

(1) *The Borel selector* $T\colon G/H \to G$ *can be seen as the function* $T(x,\sigma) = x\sigma$, *for* $(x,\sigma) \in X \times \Sigma$.
(2) *The action* $G \curvearrowright G/H$ *can be realized as follows. Define a homomorphism* $\pi\colon G \to \mathrm{Sym}(\Sigma)$ *by letting* $\pi_\gamma(\sigma)$ *be the unique element of* Σ *such that* $\pi_\gamma(\sigma)^{-1}\gamma\sigma \in H$. *Then* $G \curvearrowright G/H$ *can be identified with* $\gamma \cdot (x,\sigma) = (s(\gamma), \pi_\gamma(\sigma))$ *for* $\gamma \in G$ *and* $(x,\sigma) \in X \times \Sigma$.
(3) *The homomorphism* $c\colon G \ltimes G/H \to \Lambda$ *can be seen to be given by* $c(\gamma,\sigma) = \delta_{\gamma,\sigma}$, *where*

$$\delta_{\gamma,\sigma} \ltimes^\theta s(\gamma\sigma) = \pi_\gamma(\sigma)^{-1}\gamma\sigma$$

for $\gamma \in G$ *and* $\sigma \in \Sigma$.
(4) *The coinduced action* $\widehat{\alpha}$ *can be identified with the action* G *on* $\bigsqcup_{x\in X}(M,\tau)^{\otimes\Sigma}$ *defined by* $\widehat{\alpha}_\gamma(d_{(\sigma)}) = \alpha_{\delta_{\gamma,\sigma}}(d_{(\pi_\gamma(\sigma))})$ *for* $d \in M$.
(5) *When* $(M,\tau) = L^\infty(Y,\nu)$, *we can regard* $\widehat{\alpha}$ *as the action on* $\bigsqcup_{x\in X} Y^\Sigma$ *defined by* $\widehat{\alpha}_\gamma(\omega)(\sigma) = \alpha_{\delta^{-1}_{\gamma^{-1},\sigma}}(\omega(\pi_\gamma(\sigma)))$ *for all* $\omega \in Y^\Sigma$ *and all* $\gamma \in G$.

Lemma 4.10. We keep the notation from the beginning of this subsection. Let $\alpha\colon \Lambda \to \mathrm{Aut}(M,\tau)$ be an action on a tracial von Neumann algebra (M,τ), and set $\mathrm{CInd}_H^G(\alpha) = \widehat{\alpha}$.
(1) If α is ergodic and G is ergodic, then $\widehat{\alpha}$ is ergodic.
(2) If α is weak mixing, then $\widehat{\alpha}$ is weak mixing relatively to the trivial sub-bundle.

Proof. (1) Set $(\mathcal{M},\tau) = \bigsqcup_{x\in X}(M,\tau)^{\otimes\Sigma}$. The proof of [8, Proposition 7.2](2) shows that $L^\infty(X,\mathcal{M})^{\widehat{\alpha}} \subseteq L^\infty(X)$ whenever α is ergodic. Thus if G is ergodic, then $L^\infty(X,\mathcal{M})^{\widehat{\alpha}}$ is trivial.

(2) The action $\widehat{\alpha}\otimes\widehat{\alpha}$ can be identified with $\widehat{\alpha\otimes\alpha}$. If α is weak mixing, then $\alpha\otimes\alpha$ is ergodic. Hence, $L^\infty(X,\mathcal{M}\otimes\mathcal{M})^{\widehat{\alpha}\otimes\widehat{\alpha}}$ is contained in $L^\infty(X)$ by [8, Proposition 7.2](2), so the result follows from Lemma 3.41. ∎

Lemma 4.11. Let G be an ergodic discrete pmp groupoid such that G_x is infinite for almost every $x \in G^0$, and let $F\colon G^0 \to \{A \subseteq G \colon A \text{ is finite}\}$ be a Borel assignment such that $F_x \subseteq xG$ for every $x \in G^0$. Let $\varepsilon > 0$. Then there exists $t \in [G]$ such that $\mu\left(\{x \in G^0 \colon xt \in F_x\}\right) < \varepsilon$.

Proof. Fix a sequence $(t_n)_{n\in\mathbb{N}}$ in $[G]$ whose union is equal to G, and choose a Borel partition $(X_n)_{n\in\mathbb{N}}$ of G^0 such that $F_x \subseteq \{xt_1,\ldots,xt_n\}$ and $\mu(X_n) \leq \frac{\varepsilon}{n+1}$ for every $n \in \mathbb{N}$, and almost every $x \in X_n$. Inductively define, for $k \in \mathbb{N}$, pairwise disjoint subsets $A_k \subseteq G^0$ and elements $\rho_k \in [G]$ such that $s(x\rho_k) \in A_k$ and $x\rho_k \notin F_x$ for $x \in X_k$. The construction can proceed as long as $A_1 \cup \cdots \cup A_k$ has measure less than or equal to $1-\varepsilon$. If the construction can proceed for every k, then one can find $t \in [G]$ satisfying $xt = x\rho_j$ for all $x \in X_j$ and $j \in \mathbb{N}$. If the construction stops at some $k \in \mathbb{N}$, then one can choose an arbitrary $\rho \in [G]$ such that $x\rho \in G^0 \setminus (A_1 \cup \cdots \cup A_k)$ for $x \in G^0 \setminus (X_1 \cup \cdots \cup X_k)$, and then choose $t \in [G]$ such that $xt = x\rho_j$ for $x \in X_j$ and $j \leq k$, and $xt = x\rho$ for $x \in X_j$ and $j > k$. In this case, we have $\mu(X_1 \cup \cdots \cup X_k) = \mu(A_1 \cup \cdots \cup A_k) > 1-\varepsilon$, which concludes the proof. ∎

Lemma 4.12. Let G be a principal discrete pmp groupoid with unit space X, let Λ be a countable discrete group, and let $\Delta \leq \Lambda$ be an infinite index infinite subgroup. Let β denote the Bernoulli action $\beta_{\Lambda\curvearrowright\Lambda/\Delta}$ with base $M = L^\infty([0,1])$, and let $\Lambda \curvearrowright^\theta X$ be a free weak mixing action such that $\{\theta_\lambda(x)\colon \lambda \in \Lambda\} \subseteq [x]_G$ for almost every $x \in X$. Identify the action groupoid $H = \Lambda \ltimes^\theta X$ with a subgroupoid of G, and consider the coinduced action $\widehat{\beta} = \mathrm{CInd}_H^G(\beta)$. Then $[\widehat{\beta}]|_\Lambda$ is weak mixing and malleable.

Proof. We first show that $[\widehat{\beta}]|_\Lambda$ is weak mixing. Since the action $\Lambda \curvearrowright^\theta X$ is weak mixing, in view of Lemma 3.44 it suffices to show that $\widehat{\beta}|_H$ is weak mixing relative to the trivial sub-bundle. Let $\Sigma = \{\sigma_n\}_{n\in\mathbb{N}}$ be a coset selection for H in G. We implicitly use the identifications from Proposition 4.9, so we regard $\widehat{\beta}|_H$ as an action on $\bigsqcup_{x\in X}(M^{\otimes\Lambda/\Delta})^{\otimes\Sigma}$. The canonical identification of $(M^{\otimes\Lambda/\Delta})^{\otimes\Sigma}$ with $M^{\otimes(\Lambda/\Delta\times\Sigma)}$ allows one to regard $\widehat{\beta}|_H$ as an action on the bundle $\mathcal{M} = \bigsqcup_{x\in X} M^{\otimes((\Lambda/\Delta)\times\Sigma)}$, which is defined by

$$\widehat{\beta}_{\lambda\ltimes x}(d_{(\lambda_0\Delta,\sigma)}) = d_{(\delta_{\lambda\ltimes x,\sigma}\lambda_0\Delta,\pi_{\lambda\ltimes x}(\sigma))}$$

for $d \in M$, $\lambda_0\Delta \in \Lambda/\Delta$, $\sigma \in \Sigma$, $\lambda \in \Lambda$, and $x \in X$.

Fix $\varepsilon > 0$ and $n \in \mathbb{N}$, and let $a_1, \ldots, a_n \in L^\infty(X, \mathcal{M}) = L^\infty(X) \otimes M^{\otimes((\Lambda/\Delta)\times\Sigma)}$ be contractions satisfying $\mathrm{E}_{L^\infty(X)}(a_j) = 0$ for $j = 1, \ldots, n$. We will show that there exists $t \in [G]$ such that $\mathrm{E}_{L^\infty(X)}([\widehat{\beta}]_t(a_i)a_j) < \varepsilon$ for $i, j \in \{1, \ldots, n\}$. Without loss of generality, we can assume that there exist finite subsets $F \subseteq \Lambda/\Delta$ and $S \subseteq \Sigma$ such that $a_{i,x} \in M^{\otimes(F\times S)}$ for almost every $x \in X$ and every $i \in \{1, \ldots, n\}$.

Fix $x \in X$. We claim that for every $\sigma \in \Sigma$ and $\lambda_0\Delta \in F$, the set

$$\{\lambda \in \Lambda \colon (\delta_{\lambda\ltimes x,\sigma}\lambda_0\Delta, \pi_{\lambda\ltimes x}(\sigma)) \in F \times S\}$$

is finite. Since Δ has infinite index in Λ, it follows that $\lambda\Delta$ is disjoint from Δ for all but finitely many $\lambda \in \Lambda$. Suppose by contradiction that there exists an infinite sequence $(\lambda_k)_{k\in\mathbb{N}}$ in Λ such that, for every $k \in \mathbb{N}$ there exist $\sigma_{n_k} \in \Sigma$ and $\lambda_k\Delta \in F$ such that $(\delta_{\lambda_k\ltimes x,\sigma_{n_k}}\lambda_k\Delta, \pi_{\delta_k\ltimes x}(\sigma_{n_k}))$ belongs to $F \times S$. After passing to a subsequence, we can assume that there exist $\sigma, \sigma' \in \Sigma$ and $\lambda\Delta, \lambda'\Delta \in F$ such that

$$(\delta_{\lambda_k\ltimes x,\sigma}\lambda\Delta, \pi_{\lambda_k\ltimes x}(\sigma)) = (\lambda'\Delta, \sigma')$$

for every $k \in \mathbb{N}$. Recall that $\pi_{\lambda_k\ltimes x}$ is the permutation of Σ defined by letting $\pi_{\lambda_k\ltimes x}(\sigma)$ be the unique element σ' of Σ such that $\sigma'^{-1}(\lambda_k \ltimes x)\sigma \in H$, while $\delta_{\lambda_k\ltimes x,\sigma}$ is the unique element of Λ such that $\delta_{\lambda_k\ltimes x,\sigma} \ltimes (s(x\sigma)) = \sigma'^{-1}(\lambda_k \ltimes x)\sigma$ or, equivalently, $\lambda_k \ltimes x = \sigma'(\delta_{\lambda_k\ltimes x,\sigma} \ltimes (s(x\sigma)))\sigma^{-1}$. Since $(\lambda_k)_{k\in\mathbb{N}}$ is an infinite sequence in Λ, the set $\{\delta_{\lambda_k\ltimes x,\sigma} \colon k \in \mathbb{N}\} \subseteq \Lambda$ is infinite. Since Δ has infinite index in Λ, the coset $\widetilde{\lambda}\lambda\Delta$ is disjoint from $\lambda'\Delta$ for all but finitely many $\widetilde{\lambda} \in \Lambda$. Therefore there exists $k \in \mathbb{N}$ such that $\delta_{\lambda_k\ltimes x,\sigma}\lambda\Delta$ is disjoint from $\lambda'\Delta$. This contradicts the previous conclusion that $\delta_{\lambda_k\ltimes x,\sigma}\lambda\Delta = \lambda'\Delta$ for every $k \in \mathbb{N}$, and proves the claim.

Use Lemma 4.11 to choose $t \in [H]$ and a Borel subset $A \subseteq X$ such that $\mu(A) > 1 - \varepsilon$ and $(\delta_{xt,\sigma}\lambda_j\Delta, \pi_{xt}(\sigma_0)) \notin F \times T$ for every $x \in A$, every $\sigma \in \Sigma$ and whenever $\lambda_j\Delta \in F$. Hence, $\mathrm{E}_{L^\infty(X)}([\widehat{\beta}]_t(a_i)a_j) < \varepsilon$ for $i \in \{1, 2, \ldots, n\}$, since $a_{i,x}, a_{j,x} \in M^{\otimes(F\times T)}$ for almost every $x \in X$. This concludes the proof that $\widehat{\beta}|_H$ is weak mixing relatively to the trivial sub-bundle.

We now show that $[\widehat{\beta}]|_\Lambda$ is malleable. Use malleability of β to choose a continuous path $(\alpha_t)_{t\in[0,1]}$ in $\mathrm{Aut}(M \otimes M)^{\beta\otimes\beta}$ such that $\alpha_0 = \mathrm{id}_{M\otimes M}$ and α_1 is the flip automorphism; see [23, Lemma 4.4]. Let

$$\Phi \colon L^\infty(X) \otimes (M^{\otimes\Lambda/\Delta})^{\otimes\Sigma} \otimes L^\infty(X) \otimes (M^{\otimes\Lambda/\Delta})^{\otimes\Sigma} \to L^\infty(X) \otimes L^\infty(X) \otimes (M \otimes M)^{\otimes((\Lambda/\Delta)\times\Sigma)}$$

be the canonical isomorphism obtained by rearranging the tensor factors. For $t \in [0, 1]$ define

$$\widehat{\alpha}_t \in \mathrm{Aut}(L^\infty(X) \otimes (M^{\otimes\Lambda/\Delta})^{\otimes\Sigma} \otimes L^\infty(X) \otimes (M^{\otimes\Lambda/\Delta})^{\otimes\Sigma})$$

by

$$\widehat{\alpha}_t = \Phi^{-1} \circ (\mathrm{id}_{L^\infty(X)\otimes L^\infty(X)} \otimes \alpha_t^{\otimes((\Lambda/\Delta)\times\Sigma)}) \circ \Phi.$$

Then $(\widehat{\alpha}_t)_{t\in[0,1]}$ is a path in the centralizer of $[\widehat{\beta}]|_\Lambda \otimes [\widehat{\beta}]|_\Lambda$ satisfying $\widehat{\alpha}_0$ is the identity and $\widehat{\alpha}_1$ is the flip automorphism. This concludes the proof. ∎

5. Main result and consequences

5.1. **Expansions of the rigid action.** The notion of *expansion* of countable pmp equivalence relations has been introduced in [8], and we briefly recall it.

Definition 5.1. Let R and $\widehat{R}$ be countable pmp equivalence relations over a standard probability spaces (X, μ) and $(\widehat{X}, \widehat{\mu})$. We say that $\widehat{R}$ is an *expansion* of R if there exists a Borel map $\pi \colon \widehat{X} \to X$ with $\pi_*\widehat{\mu} = \mu$ such that:

- the restriction of π to the $\widehat{R}$-class $[x]_{\widehat{R}}$ of x is one-to-one, for almost every $x \in \widehat{X}$;
- the image of $[x]_{\widehat{R}}$ under π contains the R-class $[\pi(x)]_R$ of $\pi(x)$, for almost every $x \in \widehat{X}$.

If furthermore $\pi([x]_{\widehat{R}}) = [\pi(x)]_R$ for almost every $x \in X$, we say that $\widehat{R}$ is a *class-bijective extension* of R.

As remarked in Subsection 3.6, one can identify a class-bijective extension of R on Y with the action groupoid associated with an action of R on Y. Conversely, the action groupoid associated with an action of R on Y can be seen as a class-bijective extension of R. Therefore, we can identify actions of R on Y and class-bijective extensions of R on Y.

Let Λ be a nonamenable subgroup $\mathrm{SL}_2(\mathbb{Z})$. Then the canonical action of $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{Z}^2$ restricts to an action $\Lambda \curvearrowright \mathbb{Z}^2$, which induces a free weak mixing action $\Lambda \curvearrowright^{\rho} \mathbb{T}^2$ by duality; see Subsection 2.4. It follows from [30, Theorem 0.1] that the orbit equivalence relation of ρ is *rigid* in the sense defined therein. Using the observations above, the same proof as [8, Lemma 7.4] shows the following.

Theorem 5.2 (Bowen–Hoff–Ioana)**.** *Let Λ be a nonamenable subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and let ρ be the induced action $\Lambda \curvearrowright^{\rho} \mathbb{T}^2$. Let G be a principal discrete pmp groupoid with unit space X, and let $\Lambda \curvearrowright^{\theta} X$ be a free weak mixing action such that $\{\theta_\lambda(x) \colon \lambda \in \Lambda\} \subseteq [x]_G$ for almost every $x \in X$. Identify $H = \Lambda \ltimes^{\theta} X$ with a subgroupoid of G, and Λ with a subgroup of $[G]$.*

Suppose that $\mathcal{S}$ is a collection of actions $G \curvearrowright X$, and let $[\mathcal{S}]|_\Lambda$ denote the collection $\{[\alpha]|_\Lambda \colon \alpha \in \mathcal{S}\}$. Assume that:

1. *the elements of $\mathcal{S}$ have stably isomorphic crossed products;*
2. *the elements of $[\mathcal{S}]|_\Lambda$ are pairwise non-conjugate;*
3. *the orbit equivalence relation of any element of $\mathcal{S}$ is an expansion of the orbit equivalence relation of ρ.*

Then $\mathcal{S}$ is countable.

In the next proposition, we show that there exists a free subgroup of $\mathrm{SL}_2(\mathbb{Z})$ that acts freely on $\mathbb{Z}^2 \setminus \{0\}$. For the usual free subgroups of $\mathrm{SL}_2(\mathbb{Z})$ with finite index, the stabilizers are cyclic but not in general trivial. This is, however, not enough for our purposes.

Proposition 5.3. *There exists a subgroup $\Lambda \subseteq \mathrm{SL}_2(\mathbb{Z})$, which is isomorphic to $\mathbb{F}_\infty$, and such that the induced action $\Lambda \curvearrowright \mathbb{Z}^2 \setminus \{0\}$ is free.*

Proof. It is clear that the set of elements of $\mathrm{SL}_2(\mathbb{Z})$ that belong to the stabilizer of a point in $\mathbb{Z}^2 \setminus \{0\}$ is contained in the set of the matrices in $\mathrm{SL}_2(\mathbb{Z})$ with eigenvalues equal to 1, which in turn coincides with set of matrices in $\mathrm{SL}_2(\mathbb{Z})$ with trace 2. Thus, it suffices to find a copy of $\mathbb{F}_\infty$ in $\mathrm{SL}_2(\mathbb{Z})$ whose nontrivial elements have trace other than two.

It is shown in [44, Theorem 1] that the matrices

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$$

generate a free subgroup of rank 2 of $\mathrm{SL}_2(\mathbb{Z})$. We adopt the notation from [44, Theorem 1], and for matrices $X, Y \in M_n(\mathbb{Z})$ write $X \gg Y$ if every entry of X is greater than or equal to the absolute value of the corresponding entry of Y. For an integer n, we let $\mathrm{sgn}(n)$ be its sign. It is shown in [44, Lemma 1] that $\mathrm{sgn}(rs)A^rB^s \gg |rs|I$ for every $r, s \in \mathbb{Z} \setminus \{0\}$. Let $n \in \mathbb{N}$ and $r_1, \ldots, r_n, s_1, \ldots, s_n \in \mathbb{Z} \setminus \{0\}$, and set $r = r_1 \cdots r_n$ and $s = s_1 \cdots s_n$. Then

$$\mathrm{sgn}(rs)A^{r_1}B^{s_1} \cdots A^{r_n}B^{s_n} \gg |rs|I.$$

In particular, this shows that

$$|\mathrm{Tr}[A^{r_1}B^{s_1} \cdots A^{r_n}B^{s_n}]| \geq 2|rs|$$

where Tr denotes the canonical trace of 2×2 matrices.

Consider now the elements $x_n = A^{2n}B^{2n}$ for $n \geq 1$. These freely generate a subgroup $\Lambda \cong \mathbb{F}_\infty$ of $\mathrm{SL}_2(\mathbb{Z})$. If λ is a nontrivial element of Λ, then there exist $n \geq 1$ and nonzero *even* integers $r_1, \ldots, r_n$ such that

$$|\mathrm{Tr}[\lambda]| = |\mathrm{Tr}[A^{r_1}B^{r_1} \cdots A^{r_n}B^{r_n}]| \geq 2|r_1 \cdots r_n|^2 \geq 4.$$

This shows that every nontrivial element of Λ has trace different from 2, and hence the canonical action $\Lambda \curvearrowright \mathbb{Z}^2 \setminus \{0\}$ is free. ∎

5.2. **Class-bijective extensions of Borel equivalence relations.** We turn to the main result of this section. A countable pmp equivalence relation is said to be *amenable* if it is amenable as a discrete pmp groupoid; see [2, 10].

We let Λ the group subgroup of $\mathrm{SL}_2(\mathbb{Z})$ provided by Proposition 5.3, and recall that $\Lambda \cong \mathbb{F}_\infty$. We also let Ξ be the cyclic subgroup of Λ generated by $x_1 = A^2B^2$.

Theorem 5.4 (Bowen–Hoff–Ioana [8])**.** *Let R be an ergodic nonamenable countable pmp equivalence relation on the standard probability space (X,μ), let $\beta_{R\curvearrowright R}$ be the Bernoulli action with base space (X,μ), and let G be the corresponding action groupoid, which is a class-bijective pmp extension of R. Then there exists a free weak mixing action $\Lambda \curvearrowright^{\theta} X$ such that $\{\theta_\lambda(x)\colon \lambda\in\Lambda\}\subseteq [x]_G$ for almost every $x\in X$. In particular, the action groupoid $H=\Lambda\ltimes^{\theta} X$ can be canonically identified with a subgroupoid of G. Furthermore, one can assume that the restriction of θ to Ξ is weak mixing.*

Proof. It is enough to observe that the proof of [8, Theorem A] shows that one can choose the action in the statement in such a way that the restriction to Ξ is ergodic, and apply Dye's theorem to Ξ [38, Theorem 3.13]. ∎

In the next proposition, we show that certain coinduction of the rigid action $\Lambda \curvearrowright^{\rho} \mathbb{T}^2$ is weak mixing.

Proposition 5.5. *Let R be an ergodic nonamenable countable pmp equivalence relation on the standard probability space (X,μ), let $\beta_{R\curvearrowright R}$ be the Bernoulli action with base space (X,μ), and let G be the corresponding action groupoid. Let $\Lambda\curvearrowright^{\theta} X$ be the action provided by Theorem 5.4, let Δ be an infinite index subgroup of Λ containing Ξ, and let H be the action groupoid $\Lambda\ltimes^{\theta} X$, which can be identified with a subgroupoid of G. Let ρ be the rigid action $\Lambda\curvearrowright\mathbb{T}^2$, and set $\widehat{\rho}=\mathrm{CInd}_H^G(\rho)$. Then $[\widehat{\rho}]|_\Delta$ is weak mixing.*

Proof. We will use the identifications from Proposition 4.9. Set $K=\Delta\ltimes^{\theta|_\Delta} X$, and regard $\widehat{\rho}|_K$ as an action on the bundle $\mathcal{M}=\bigsqcup_{x\in X} L^\infty(\mathbb{T}^2)^{\otimes\Sigma}$. Since $\theta|_\Delta$ is weak mixing, in view of Lemma 3.44 it suffices to show that $\widehat{\rho}|_K$ is weak mixing relatively to the trivial sub-bundle. By Corollary 3.42, it suffices to show that the Koopman representation $\kappa_0^{\widehat{\rho}|_K}$ is weak mixing.

As observed in [17, Lemma 6.7], the Koopman representation κ_0^{ρ} of ρ can be identified with the representation of Λ on $\ell^2(\mathbb{Z}^2\setminus\{0\})$ obtained from the canonical action of $\Lambda\curvearrowright\mathbb{Z}^2\setminus\{0\}$ as a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Similarly, and letting $\mathcal{F}(\Sigma,\mathbb{Z}^2)_0$ denote the set of non-zero finitely-supported functions $f\colon\Sigma\to\mathbb{Z}^2$, the representation $\kappa_0^{\widehat{\rho}|_K}$ can be seen as the representation on the Hilbert bundle $\mathcal{H}=\bigsqcup_{x\in X}\ell^2(\mathcal{F}(\Sigma,\mathbb{Z}^2)_0)$, defined as follows. Let $\{\delta_f\colon f\in\mathcal{F}(\Sigma,\mathbb{Z}^2)_0\}$ be the canonical orthonormal basis of $\ell^2(\mathcal{F}(\Sigma,\mathbb{Z}^2)_0)$. Consider the action of G on Σ given by $\gamma\cdot\sigma=\pi_\gamma(\sigma)$ for $\gamma\in G$ and $\sigma\in\Sigma$. (Recall that $\gamma\cdot\sigma=\sigma'$ if and only if $\gamma\sigma H\in\sigma' H$.) Define the action of K on $\mathcal{F}(\Sigma,\mathbb{Z}^2)_0$ by setting $(\gamma\cdot f)(\gamma\cdot\sigma)=\delta_{\gamma,\sigma}\cdot f(\sigma)$ for $\gamma\in K$, $\sigma\in\Sigma$, and $f\in\mathcal{F}(\Sigma,\mathbb{Z}^2)_0$. Then $\kappa_{0,\gamma}^{\widehat{\rho}|_K}(\delta_f)=\delta_{\gamma\cdot f}$ for $f\in\mathcal{F}(\Sigma,\mathbb{Z}^2)_0$ and $\gamma\in K$.

Let $\xi_1,\ldots,\xi_n$ be invariant unit sections for $\mathcal{H}$, and let $\varepsilon>0$. We will show that there exists $t\in[K]$ such that

$$\int_X \langle \kappa_{0,xt}^{\widehat{\rho}|_K}(\xi_{i,s(xt)}),\xi_{j,x}\rangle d\mu(x)\leq\varepsilon$$

for $i,j\in\{1,\ldots,n\}$. Without loss of generality, we can assume that there exists a finite subset $F\subseteq\mathcal{F}(\Sigma,\mathbb{Z}^2)_0$ such that $\xi_{i,x}\in\mathrm{span}\{\delta_f\colon f\in F\}$ for every $x\in X$ and $1\leq i\leq n$.

As the action θ is free, we can assume, after discarding a null subset of X, that $\theta_\lambda(x)\neq x$ for every $\lambda\in\Lambda\setminus\{1\}$ and for every $x\in X$. Fix $x\in X$. We claim first that there exists a finite subset $\Delta_x\subseteq\Delta$ such that $\kappa_{0,h\ltimes x}^{\widehat{\rho}|_K}(\xi_{i,x})$ is orthogonal to $\xi_{j,\theta_h(x)}$ for every $h\in\Delta\setminus\Delta_x$, for $i,j\in\{1,\ldots,n\}$. Suppose by contradiction that this is not the case. Then there exist $f,f'\in F$ and an infinite sequence $(h_n)_{n\in\mathbb{N}}$ of pairwise distinct elements of Δ such that $\kappa_{0,h_n\ltimes x}^{\widehat{\rho}|_K}(\delta_f)$ is not orthogonal to $\delta_{f'}$ for all $n\in\mathbb{N}$. By the definition of $\kappa_{0,h\ltimes x}^{\widehat{\rho}|_K}$, after passing to a subsequence of $(h_n)_{n\in\mathbb{N}}$, we can furthermore assume that there exist $\sigma,\sigma'\in\mathrm{supp}(f)$ such that $\delta_{h_n\ltimes x,\sigma}$ belongs to a coset of the stabilizer of $f(\sigma)$ in Λ, and that

$(h_n \ltimes x)\cdot\sigma = \sigma'$ for all $n \in \mathbb{N}$. Since Λ acts freely on $\mathbb{Z}^2 \setminus \{0\}$ by construction, and $f(\sigma) \neq 0$, we have $\delta_{h_n \ltimes x, \sigma} = 1$ for every $n \in \mathbb{N}$. Moreover,

$$\theta_{h_n}(x) = h_n \ltimes^\theta x = \sigma'(\delta_{h_n\ltimes x,\sigma} \ltimes^\theta s(x\sigma))\sigma^{-1} = \sigma'(1 \ltimes^\theta s(x\sigma))\sigma^{-1}$$

for every $n \in \mathbb{N}$. In particular, $\theta_{h_n}(x)$ is independent of n. This contradicts the assumption that $(h_n)_{n\in\mathbb{N}}$ is an infinite sequence of pairwise distinct elements of Δ, and the claim is proved.

Apply Lemma 4.11 to K to obtain $t \in [K]$ such that the set

$$\{x \in X \colon \kappa^{\widehat{\rho}|_K}_{0,xt}(\xi_{i,s(xt)}) \perp \xi_{j,x} \text{ for every } i,j = 1,\dots,n\}$$

has measure at least $1-\varepsilon$. Then $\int_X \langle \kappa^{\widehat{\rho}|_K}_{0,xt}(\xi_{i,s(xt)}), \xi_{j,x}\rangle d\mu(x) \leq \varepsilon$ for $i,j = 1,\dots,n$, as desired. ∎

The following is the main result of this paper, from which we will derive Theorems A, B, D and E from the introduction. In the statement, we say that a class-bijective pmp extension of R is weak mixing if it is weak mixing as a countable pmp groupoid.

Theorem 5.6. *Let R be an ergodic nonamenable countable pmp equivalence relation on the standard probability space (X,μ), and let (Y,ν) be the standard atomless probability space. Then there exists an assignment $A \mapsto R_A$ from countably infinite discrete abelian groups to weak mixing class-bijective pmp extensions of R on (Y,ν) such that:*

1. *if A and A' are isomorphic groups, then R_A and $R_{A'}$ are isomorphic relatively to R;*
2. *if $\mathcal{A}$ is a collection of pairwise nonisomorphic countably infinite abelian groups such that $\{R_A \colon A \in \mathcal{A}\}$ are pairwise stably von Neumann equivalent, then $\mathcal{A}$ is countable.*

Proof. Let Λ be the free subgroup of $\mathrm{SL}_2(\mathbb{Z})$ provided by Proposition 5.3, and let $\Delta \subseteq \Lambda$ be an infinite index normal subgroup containing Ξ such that the quotient $\Omega = \Lambda/\Delta$ is a property (T) group. Then $\Delta \leq \Lambda \leq \Lambda$ has property (T) by Example 2.4. Let ν be the Haar measure on the Pontryagin dual $\widehat{A}$, and set $M = L^\infty(\widehat{A},\nu)$, endowed with the trace-preserving action $\mathtt{Lt}$ of $\widehat{A}$ given by left translation. As in the proof of Theorem 2.11, consider the Bernoulli action $\beta\colon \Lambda \curvearrowright M^{\otimes\Omega}$ associated with $\Lambda \curvearrowright \Omega = \Lambda/\Delta$. Let θ be the free action $\Lambda \curvearrowright^\theta X$ provided by Proposition 5.3, let G be the action groupoid associated to the Bernoulli shift $\beta_{R\curvearrowright R}$ with base (X,μ), and let H be the action groupoid $\Lambda \ltimes^\theta X$, which can be identified with a subgroupoid of G. Let ρ be the rigid action $\Lambda \curvearrowright \mathbb{T}^2$. Set $\widehat{\rho} = \mathrm{CInd}_H^G(\rho)$ and $\widehat{\beta} = \mathrm{CInd}_H^G(\beta)$. Thus $\widehat{\rho}$ is an action of G on $\bigsqcup_{x\in X}(L^\infty(\mathbb{T}^2)^{\otimes\Omega})^{\otimes\Sigma}$, and $\widehat{\beta}$ is an action of G on $\bigsqcup_{x\in X}(M^{\otimes\Omega})^{\otimes\Sigma}$. We will use the identifications of Proposition 4.9.

Let $M_A \subseteq M^{\otimes(\Omega\times\Sigma)}$ denote the fixed point algebra of the action $\mathtt{Lt}^{\otimes(\Omega\times\Sigma)}$, and let $\mathcal{M}_A$ be the $(\widehat{\beta}\otimes\widehat{\rho})$-invariant sub-bundle

$$\mathcal{M}_A = \bigsqcup_{x\in X}(M_A \otimes L^\infty(\mathbb{T}^2)^{\otimes\Sigma})$$

of $\mathcal{M}$. Then there exists a standard atomless probability space (X_A,μ_A) with $M_A \cong L^\infty(X_A,\mu_A)$. Define ζ_A to be the restriction of $\widehat{\beta}\otimes\widehat{\rho}$ to $\mathcal{M}_A$, and let R_A be the orbit equivalence relation of the action ζ_A, which is a class-bijective pmp extension of R on $(X_A,\mu_A) \cong (Y,\nu)$.

The action $[\widehat{\beta}]|_\Lambda$ is weak mixing and malleable by Lemma 4.12, and $[\widehat{\rho}]|_\Delta$ is weak mixing by Proposition 5.5. It follows that R_A is weak mixing. It is clear that isomorphic groups yield pmp class-bijective extensions of R that are isomorphic relatively to R, so that (1) holds.

Claim: R_A is an expansion of the orbit equivalence relation of $\Lambda \curvearrowright^\rho \mathbb{T}^2$.

Denote by $\widehat{\beta}_A$ the restriction of $\widehat{\beta}$ to M_A. By [8, Proposition 7.2 (3)], R_A is an extension of the orbit equivalence relation of $[\widehat{\beta}_A]|_\Lambda \times \rho$ of Λ on $X \times X_A \times \mathbb{T}$. In turn, this equivalence relation is an expansion of the orbit equivalence relation of ρ, as witnessed by the second coordinate projection $X_A \times \mathbb{T} \to \mathbb{T}$. This proves the claim.

Our next goal is to show that there is a group isomorphism $H^1_{:\Delta,\mathrm{w}}([\zeta_A]|_\Lambda) \cong A$. We will prove this in a sequence of claims. Let $w\colon \Lambda \to L^\infty(X) \otimes M_A \otimes L^\infty(\mathbb{T}^2)^{\otimes\Sigma}$ be a Δ-invariant cocycle for $[\zeta_A]|_\Lambda$.

Then w is also a Δ-invariant cocycle for $[\widehat{\beta}\otimes\widehat{\rho}]|_\Lambda$. Since $[\widehat{\rho}]|_\Delta$ is weak mixing and w is Δ-invariant, it follows from Lemma 3.41 that w takes values in the Δ-fixed point subalgebra

$$L^\infty(X)\otimes(M^{\otimes\Omega})^{\otimes\Sigma}\otimes\mathbb{C}\subseteq L^\infty(X)\otimes M^{\otimes(\Omega\times\Sigma)}\otimes L^\infty(\mathbb{T}^2)^{\otimes\Sigma}.$$

Since $\Omega=\Lambda/\Delta$ has property (T), it follows that Λ has the Δ-invariant property (T). Using weak mixing and malleability for $[\widehat{\beta}]|_\Lambda$, we apply Theorem 2.10 to find a unitary $v\in L^\infty(X)\otimes M^{\otimes(\Omega\times\Sigma)}$ such that $v^*[\widehat{\beta}]_\lambda(v)=w_\lambda \bmod \mathbb{C}$ for every $\lambda\in\Lambda$. Fix $g\in\widehat{A}$ and $\lambda\in\Lambda$. Then

$$(\mathrm{id}_{L^\infty(X)}\otimes\mathtt{Lt}_g^{\otimes(\Omega\times\Sigma)})(w_\lambda)=w_\lambda$$

and hence

$$v^*[\widehat{\beta}]_\lambda(v)=(\mathrm{id}_{L^\infty(X)}\otimes\mathtt{Lt}_g^{\otimes(\Omega\times\Sigma)})(v)^*([\widehat{\beta}]_\lambda(\mathrm{id}_{L^\infty(X)}\otimes\mathtt{Lt}_g^{\otimes(\Omega\times\Sigma)})(v)) \bmod \mathbb{C}$$

and

$$(\mathrm{id}_{L^\infty(X)}\otimes\mathtt{Lt}_g^{\otimes(\Omega\times\Sigma)})(v)v^*=[\widehat{\beta}]_\lambda(\mathrm{id}_{L^\infty(X)}\otimes\mathtt{Lt}_g^{\otimes(\Omega\times\Sigma)})(v)v^*) \bmod \mathbb{C}.$$

In other words, the left-hand side of the last equation generates a one-dimensional subspace which is invariant under $[\widehat{\beta}]|_\Lambda$. Since $[\widehat{\beta}]|_\Lambda$ is weak mixing, this subspace must consist of the scalar multiples of the unit, so there exists $\chi_w(g)\in\mathbb{C}$ such that

$$(\mathrm{id}_{L^\infty(X)}\otimes\mathtt{Lt}_g^{\otimes(\Omega\times\Sigma)})(v)=\chi_w(g)v.$$

Claim: The resulting map $\chi_w\colon\widehat{A}\to\mathbb{C}$ is a character. Let $g,g'\in\widehat{A}$. Then

$$\begin{aligned}\chi_w(gg')v&=(\mathrm{id}_{L^\infty(X)}\otimes\mathtt{Lt}_{gg'}^{\otimes(\Omega\times\Sigma)})(v)\\&=(\mathrm{id}_{L^\infty(X)}\otimes\mathtt{Lt}_g^{\otimes(\Omega\times\Sigma)})((\mathrm{id}_{L^\infty(X)}\otimes\mathtt{Lt}_{g'}^{\otimes(\Omega\times\Sigma)})(v)\\&=(\mathrm{id}_{L^\infty(X)}\otimes\mathtt{Lt}_g^{\otimes(\Omega\times\Sigma)})(\chi_w(g')v)\\&=\chi_w(g')\chi_w(g),\end{aligned}$$

as desired. It follows that there is a well-defined map $\chi\colon Z^1_{:\Delta,\mathrm{w}}([\zeta_A]|_\Lambda)\to A$.

Claim: χ is a group homomorphism. Let $w,w'^1_{:\Delta,\mathrm{w}}([\zeta_A]|_\Lambda)$. We want to show that $\chi_{ww'}(g)=\chi_w(g)\chi_{w'}(g)$ for all $g\in\widehat{A}$. As before, find unitaries $v,v'^\infty(X)\otimes M^{\otimes(\Omega\times\Sigma)}$ satisfying

$$v^*[\widehat{\beta}]_\lambda(v)=w_\lambda \bmod \mathbb{C}\quad\text{and}\quad(v'^*[\widehat{\beta}]_\lambda(v')=w'_\lambda \bmod \mathbb{C}$$

for all $\lambda\in\Lambda$. Set $z=vv'$. Then $w_\lambda w'_\lambda=z^*[\widehat{\beta}]_\lambda(z)$ mod $\mathbb{C}$ for all $\lambda\in\Lambda$. Fix $g\in\widehat{A}$. Then

$$\begin{aligned}\chi_{ww'}(g)z&=(\mathrm{id}_{L^\infty(X)}\otimes\mathtt{Lt}_g^{\otimes(\Omega\times\Sigma)})(z)\\&=(\mathrm{id}_{L^\infty(X)}\otimes\mathtt{Lt}_g^{\otimes(\Omega\times\Sigma)})(v)(\mathrm{id}_{L^\infty(X)}\otimes\mathtt{Lt}_g^{\otimes(\Omega\times\Sigma)})(v')\\&=\chi_w(g)v\chi_{w'}(g)v'=\chi_w(g)\chi_{w'}(g)z,\end{aligned}$$

as desired.

Claim: The kernel of χ is the set of relative weak coboundaries for $[\zeta_A]|_\Lambda$. Let $w\in Z^1_{:\Delta,\mathrm{w}}([\zeta_A]|_\Lambda)$ satisfy $\chi_w=1$. We want to show that w is weakly cohomologous to the trivial cocycle. Find $v\in L^\infty(X)\otimes M^{\otimes(\Omega\times\Sigma)}$ such that

$$v^*[\widehat{\beta}]_\lambda(v)=w_\lambda \bmod \mathbb{C}\quad\text{and}\quad(\mathrm{id}_{L^\infty(X)}\otimes\mathtt{Lt}_g^{\otimes(\Omega\times\Sigma)})(v)=v$$

for every $\lambda\in\Lambda$ and every $g\in\widehat{A}$. In particular, v belongs to $L^\infty(X)\otimes M_A\otimes L^\infty(\mathbb{T}^2)$ and $vw_\lambda[\widehat{\beta}]_\lambda(v^*)=1 \bmod \mathbb{C}$ for every $\lambda\in\Lambda$, as desired. The converse is identical, so the claim is proved.

Claim: χ is surjective. Fix $\omega\in A$, which we regard as a unitary in $C(\widehat{A})\subseteq L^\infty(\widehat{A},\nu)=M$. This readily gives a Δ-invariant unitary element v of

$$L^\infty(X)\otimes M^{\otimes(\Omega\times\Sigma)}\otimes\mathbb{C}\subseteq L^\infty(X)\otimes M^{\otimes(\Omega\times\Sigma)}\otimes L^\infty(\mathbb{T}^2)$$

such that $(\mathrm{id}_{L^\infty(X)} \otimes \mathtt{Lt}_g^{\otimes(\Omega\times\Sigma)})(v) = \omega(g)v$ for every $g \in A$. Define a Δ-invariant cocycle z_ω for $[\zeta_A]|_\Lambda$ by $z_\omega(\lambda) = v^*[\beta]_\lambda(v)$ for all $\lambda \in \Lambda$. It is then immediate to show that $\chi_{z_\omega} = \omega$.

It follows that χ induces a group isomorphism from $H^1_{:\Delta,\mathrm{w}}([\zeta_A]|_\Lambda)$ to A, as desired.

We now prove (3). Let $\mathcal{A}$ be a collection of pairwise nonisomorphic countably infinite discrete abelian groups such that the relations $\{R_A : A \in \mathcal{A}\}$ are pairwise stably von Neumann equivalent. Since ζ_A is free, the crossed product of ζ_A is isomorphic to $L(R_A)$. Thus, the actions $\{\zeta_A : A \in \mathcal{A}\}$ have stably isomorphic crossed products. Furthermore, for every $A \in \mathcal{A}$ there is a group isomorphism $H_{:\Delta,\mathrm{w}}([\zeta_A]|_\Lambda) \cong A$. Therefore the actions $\{[\zeta_A]|_\Lambda : A \in \mathcal{A}\}$ are pairwise not conjugate by Theorem 5.2, and hence $\mathcal{A}$ is countable by Theorem 5.2. This concludes the proof. ∎

Corollary 5.7. *Let Γ be a nonamenable countable discrete group and let (Y,ν) be the standard atomless probability space. Then there exists an assignment $A \mapsto \theta_A$ from countably infinite discrete abelian groups to free weak mixing actions $\Gamma \curvearrowright (Y,\nu)$ such that:*

1. *if A and A' are isomorphic, then θ_A and θ'_A are conjugate;*
2. *if $\mathcal{A}$ is a collection of pairwise nonisomorphic countably infinite abelian groups such that $\{\theta_A : A \in \mathcal{A}\}$ are pairwise stably von Neumann equivalent, then $\mathcal{A}$ is countable.*

Proof. Let $\theta\colon \Gamma \curvearrowright (X,\mu)$ be the Bernoulli action of Γ with base $[0,1]$, and let R be the corresponding orbit equivalence relation. Then R is a nonamenable ergodic countable pmp equivalence relation. Observe that if $\hat{R}$ is a class-bijective extension of R on (Y,ν), then $\hat{R}$ is the orbit equivalence relation of a free pmp action $\hat{\theta}$ of Γ on (Y,ν) with a distinguished factor map $\pi\colon Y \to X$ onto θ. Furthermore, $\hat{R}$ is ergodic (respectively, weak mixing) if and only if $\hat{\theta}$ is ergodic (respectively, weak mixing). If $\hat{R}'$ is another class-bijective extension of R on (Y,ν), with corresponding Γ-action θ' and factor map $\pi'\colon Y \to X$, then $\hat{R}$ and $\hat{R}'$ are isomorphic relatively to R if and only $\hat{\theta}$ and $\hat{\theta}'$ are *conjugate relatively to* θ, that is, there exists an automorphism ϕ of (Y,ν) such that, up to discarding a null set, $\pi' \circ \phi = \pi$ and $\phi \circ \hat{\theta}_\gamma = \hat{\theta}'_\gamma \circ \phi$ for every $\gamma \in \Gamma$. The conclusion thus follows from Theorem 5.6. ∎

5.3. **Borel complexity.** We recall here some fundamental notions from Borel complexity theory, which can also be found in [26, 29].

Definition 5.8. Let X and Y be standard Borel spaces, and let E and F be equivalence relations on X and Y, respectively.

1. E is said to be *Borel* if it is a Borel subset of $X \times X$ endowed with the product topology.
2. A *Borel reduction* from E to F is a Borel function $f\colon X \to Y$ such that xEx' if and only if $f(x)Ff(x')$, for every $x, x' \in X$.
3. A *countable-to-one Borel homomorphism* from E to F is a Borel function $f\colon X \to Y$ such that xEx' implies $f(x)Ff(x')$ for every $x, x' \in X$, and if $\mathcal{A}$ is a set of pairwise not E-equivalent elements of X such that $f(\mathcal{A})$ is contained in a single E-class, then $\mathcal{A}$ is countable.

When $X = Y$, if F is contained in E (as subsets of $X \times X$), then we say that E is *coarser* than F and F is *finer* than E.

One can regard the relation $\cong_{\mathrm{AG}}$ of isomorphism of countably infinite discrete abelian groups as an equivalence relation on a standard Borel space in a canonical way. Indeed, assuming without loss of generality that countably infinite discrete groups have $\mathbb{N}$ as universe, a countably infinite discrete group is an element of $2^{\mathbb{N}^3} \times 2^{\mathbb{N}^2} \times \mathbb{N}$, coding the multiplication operation, the inverse map, and the distinguished element representing the identity. It is easy to see that the set AG of elements of $2^{\mathbb{N}^3} \times 2^{\mathbb{N}^2} \times \mathbb{N}$ that arise in this fashion is a Borel subset, and hence a standard Borel space with the induced Borel structure. The following result is proved in [17, Theorem 5.1].

Theorem 5.9 (Epstein–Törnquist)**.** *Let E be an equivalence relation on a standard Borel space. If there exists a countable-to-one Borel homomorphism from $\cong_{\mathrm{AG}}$ to E, then E is not Borel.*

We proceed to explain how to regard the spaces of actions of a discrete pmp groupoid and of class-bijective extensions of a given countable Borel equivalence relation as a standard Borel space. We need some preparation first.

Notation 5.10. *Let (X,μ) and (Y,ν) be standard probability spaces. We denote by $\mathcal{I}_{(X,\mu)}(Y,\nu)$ the space of quadruples (T,A,B,π) such that $A,B\subseteq X$ are Borel sets with $\mu(A)=\mu(B)$, $\pi\colon Y\to X$ is a Borel map such that $\pi_*(\nu)=\mu$, and $T\colon \pi^{-1}(A)\to\pi^{-1}(B)$ is a measure-preserving Borel isomorphism. Two such quadruples (T,A,B,π) and (T',A',B',π') are identified whenever $\pi=\pi'$, the symmetric differences $A\triangle A'$ and $B\triangle B'$ have zero measure, and $T|_{\pi^{-1}(A\cap A')}=T'|_{\pi'^{-1}(A\cap A')}$ almost everywhere.*

Remark 5.11. There is a canonical Polish topology on $\mathcal{I}_{(X,\mu)}(Y,\nu)$. Indeed, set $M=L^\infty(Y,\nu)$ and $N=L^\infty(X,\mu)$, endowed with the canonical traces. Identify $\mathcal{I}_{(X,\mu)}(Y,\nu)$ with the space of quadruples (θ,p,q,η) such that $\eta\colon N\to M$ is an injective trace-preserving normal *-homomorphism, $p,q\in N$ are projections of the same trace, and $\theta\colon \eta(p)M\to\eta(q)M$ is a trace-preserving *-isomorphism. Let $(d_n)_{n\in\mathbb{N}}$ and $(e_n)_{n\in\mathbb{N}}$ be countable 2-dense subsets of the unit balls of M and N, respectively, and set

$$\begin{aligned}\rho((\theta,p,q,\eta),(\theta',p',q',\eta')) =&\|p-p\|_2+\|q-q'\|_2+\sum_{n\in\mathbb{N}}2^{-n}\|\theta(\eta(p)d_n)-\theta'(\eta'(p')d_n)\|_2\\ &+\sum_{n\in\mathbb{N}}2^{-n}\|\theta^{-1}(\eta(q)d_n)-\theta'(\eta'(q')d_n)\|_2+\sum_{n\in\mathbb{N}}2^{-n}\|\eta(e_n)-\eta'(e_n)\|.\end{aligned}$$

Then ρ is complete metric on $\mathcal{I}_{(X,\mu)}(Y,\nu)$. Since it is clear that the corresponding topology is separable, this shows that $\mathcal{I}_{(X,\mu)}(Y,\nu)$ is a Polish space.

Observe that, for a fixed η_0, the space $\mathcal{I}_{(X,\mu),\eta_0}(Y,\nu)$ of quadruples (θ,p,q,η) in $\mathcal{I}_{(X,\mu)}(Y,\nu)$ with $\eta=\eta_0$ is an inverse subsemigroup, whose idempotent semilattice can be identified with the space of projections of $L^\infty(X,\mu)$. If G is a discrete pmp groupoid with $G^0=X$, then the semilattice of projections of $L^\infty(X,\mu)$ can be identified with the idempotent semilattice of $[[G]]$.

Proposition 5.12. *Let (Y,ν) be the standard probability space, let G be a discrete pmp groupoid, and let R be an ergodic countable Borel equivalence relation.*

1. *There is a canonical Polish topology on the space $\mathrm{Act}_G(Y,\nu)$ of actions of G on (Y,ν). With respect to this Polish topology, the space $\mathrm{Erg}_G(Y,\nu)$ of ergodic actions of G on (Y,ν).*
2. *There is a canonical Polish topology on the space $\mathrm{Ext}_R(Y,\nu)$ of class-bijective pmp extensions of R on (Y,ν). With respect to this Polish topology, the space $\mathrm{Erg}_R(Y,\nu)$ of class-bijective pmp extensions of R on (Y,ν) which are ergodic, and the space $\mathrm{WM}_R(Y,\nu)$ of class-bijective pmp extensions of R which are weak mixing, are Borel subsets of $\mathrm{Ext}_R(Y,\nu)$.*

Proof. (1). Set $(X,\mu)=(G^0,\mu_G)$. An action α of G on (Y,ν) is given by a trace-preserving normal unital *-homomorphism $\eta\colon L^\infty(X,\mu)\to L^\infty(Y,\nu)$ together with a Borel groupoid homomorphism $\alpha\colon G\to\mathrm{Aut}(\bigsqcup_{x\in X}L^\infty(Y_x))$ that is the identity on the unit space. In turn, this induces an inverse semigroup homomorphism $t\mapsto[[\alpha]]_t$ from $[[G]]$ to $\mathcal{I}_{(X,\mu),\eta}(Y,\nu)$ which is the identity on the corresponding idempotent semilattices.

Fix now a countable dense inverse semigroup $\Sigma\subseteq[[G]]$ such that the set of elements of Σ with support and range with full measure is dense in $[G]$. Let $\eta\colon L^\infty(X,\mu)\to L^\infty(Y,\nu)$ be a trace-preserving injective *-homomorphism, and let $\theta\colon\Sigma\to\mathcal{I}_{(X,\mu),\eta}(Y,\nu)$ be a inverse semigroup homomorphism which is the identity on the corresponding semilattices. Then the pair (η,θ) gives rise to a unique action α of G on (Y,ν) satisfying $[[\alpha]]=\theta$. Thus one can identify the set of actions of G on (Y,ν) with the set of such pairs (η,θ), which is a closed subset of the countably infinite product $\mathcal{I}_{(X,\mu)}(Y,\nu)^\Sigma$. This yields a canonical Polish topology on the space $\mathrm{Act}_G(Y,\nu)$ of actions of G on (Y,ν).

Fix a 2-norm dense countable subset $\mathcal{P}$ of the set of projections of $L^\infty(Y,\nu)$, and a 2-norm dense countable subset $\mathcal{F}$ of the unit ball of $L^\infty(Y,\nu)$. Denote by $\mathrm{E}_{\eta(L^\infty(X,\mu))}\colon L^\infty(Y,\nu)\to\eta(L^\infty(X,\mu))$ the unique trace-preserving conditional expectation from $L^\infty(Y,\nu)$ onto $\eta(L^\infty(X,\mu))$. Ergodicity of an action α of G on (Y,ν) can be characterized in terms of the associated maps $\eta\colon L^\infty(X,\mu)\to L^\infty(Y,\nu)$ and $\theta\colon\Sigma\to\mathcal{I}_{(X,\mu),\eta}(Y,\nu)$ as follows: α is *ergodic* if and only if for every $p,p'\in\mathcal{P}$ with $\tau(p)=\tau(p')$

and every $\varepsilon > 0$ there exists $\sigma \in \Sigma \cap [G]$ such that $\|\theta_\sigma(p) - \theta_\sigma(p')\|_2 < \varepsilon$. Hence $\mathrm{Erg}_G(Y,\nu)$ is a Borel subset of $\mathrm{Act}_G(Y,\nu)$.

(2). As noted in Subsection 3.6, one can identify class-bijective pmp extensions of R on (Y,ν) with the (necessarily principal) action groupoid associated with a pmp action of R. Thus, the space $\mathrm{Ext}_R(Y,\nu)$ of class-bijective pmp extensions of R is endowed with a canonical Polish topology, obtained by identifying $\mathrm{Ext}_R(Y,\nu)$ with $\mathrm{Act}_R(Y,\nu)$ and using part (1). Since R is assumed to be ergodic, a class-bijective extension of R is ergodic if and only if the associated action of R is ergodic. Therefore it follows from part (1) that $\mathrm{Erg}_R(Y,\nu)$ is a Borel subset $\mathrm{Ext}_R(Y,\nu)$. Similarly, one can see that $\mathrm{WM}_R(Y,\nu)$ is a Borel subset of $\mathrm{Ext}_R(Y,\nu)$ as follows. Fix a countable 2-norm dense subset X of $L^\infty(Y,\nu)$. Then an extension $\widehat{R}$ of R corresponding to an action θ of R, regarded as above an element of $\mathrm{Act}_R(Y,\nu)$, is weak mixing if and only if, for every $n \in \mathbb{N}$ and $f_1,\ldots,f_n \in X$ there exists $\sigma \in \Sigma \cap [G]$ such that $|\tau(f_i\theta_\sigma(f_j)) - \tau(f_i)\tau(f_j)| < 2^{-n}$ for $1 \leq i,j \leq n$. Thus, $\mathrm{WM}_R(Y,\nu)$ is a Borel subset of $\mathrm{Ext}_R(Y,\nu)$. ∎

Remark 5.13. As observed in [36, Section 16], the proof of Rokhlin's Skew Product Theorem [27, Theorem 3.18] shows that any ergodic class-bijective pmp extension of R is isomorphic to a *skew product* of R via a cocycle. This observation allows one to give a different (but equivalent) parametrization of the space $\mathrm{Erg}_R(Y,\nu)$ of ergodic class-bijective pmp extensions of R.

Observing that the construction in Theorem 5.6 is given by a Borel map with respect to the parametrization of class-bijective extensions described above, we obtain:

Theorem 5.14. *Let R be an ergodic nonamenable countable pmp equivalence relation. There is a Borel function $A \mapsto R_A$ from abelian countably infinite groups to weak mixing pmp extensions of R on the standard atomless probability space such that:*

1. *if A and A' are isomorphic groups, then R_A and $R_{A'}$ are isomorphic relatively to R;*
2. *if $\mathcal{A}$ is a collection of pairwise nonisomorphic countably infinite abelian groups such that $\{R_A \colon A \in \mathcal{A}\}$ are pairwise stably von Neumann equivalent, then $\mathcal{A}$ is countable.*

In particular, if E is any equivalence relation for weak mixing pmp extensions of R on the standard atomless probability space that is coarser than isomorphism relative to R and finer than stable von Neumann equivalence, then $A \mapsto R_A$ is a countable-to-one Borel homomorphism from $\cong_{\mathrm{AG}}$ to E.

The following corollary is an immediate consequence of Theorem 5.14 and Theorem 5.9.

Corollary 5.15. *Let R be an ergodic nonamenable countable pmp equivalence relation. Let E be any equivalence relation for weak mixing pmp extensions of R on the standard atomless probability space that is coarser than orbit equivalence and finer than stable von Neumann equivalence. Then E is not Borel.*

Theorem E is then a particular instance of Corollary 5.15.

5.4. **Actions of discrete groups.** In this subsection, we explain how to deduce Theorem A and Theorem B from Theorem E.

Fix a countable group Γ. We consider the space $\mathrm{Aut}(Y,\nu)$ of measure-preserving Borel automorphisms of X as a topological space, endowed with the weak topology. We also let $\mathrm{Aut}(Y,\nu)^\Gamma$ be the corresponding product space, endowed with the product topology. We denote by $\mathrm{FE}_\Gamma(Y,\nu)$ the space of free ergodic pmp actions of Γ on (Y,ν), and by $\mathrm{FWM}_\Gamma(Y,\nu)$ the space of free weak mixing pmp actions of Γ on (Y,ν). It is shown in [38, Section 10] that both $\mathrm{FE}_\Gamma(Y,\nu)$ and $\mathrm{FWM}_\Gamma(Y,\nu)$ are Borel subsets of $\mathrm{Aut}(Y,\nu)^\Gamma$, and hence standard Borel spaces when endowed with the inherited Borel structure. Suppose now that R is the orbit equivalence relation of a free ergodic action θ of Γ on a standard probability space (X,μ). As observed in the proof of Corollary 5.7, every weak mixing class-bijective pmp extension $\hat{R}$ of R on (Y,ν) canonically gives rise to a free weak mixing action $\hat{\theta}$ of Γ on (Y,ν) having $\hat{R}$ as its orbit equivalence relation. This yields a Borel function $\hat{R} \mapsto \hat{\theta}$ from $\mathrm{WM}_R(Y,\nu)$ to $\mathrm{FWM}_\Gamma(Y,\nu)$, which maps class-bijective extensions of R that are isomorphic relatively to R to conjugate actions of Γ. We therefore deduce the following from Theorem 5.14.

Theorem 5.16. *Let Γ be a nonamenable countable group. There is a Borel function $A \mapsto \theta_A$ from abelian countably infinite groups to free weak mixing actions of Γ on the standard atomless probability space such that:*

(1) if A and A' are isomorphic groups, then θ_A and $\theta_{A'}$ are conjugate;

(2) if $\mathcal{A}$ is a collection of pairwise nonisomorphic countably infinite abelian groups such that $\{R_A \colon A \in \mathcal{A}\}$ are pairwise stably von Neumann equivalent, then $\mathcal{A}$ is countable.

In particular, if E is any equivalence relation for free weak mixing actions of Γ on the standard atomless probability space that is coarser than conjugacy and finer than stable von Neumann equivalence, then $A \mapsto \theta_A$ is a countable-to-one Borel homomorphism from $\cong_{\mathrm{AG}}$ to E.

Theorem A and Theorem B are immediate consequences of Theorem 5.16, in view of Theorem 5.9.

5.5. **Actions of locally compact groups.** In this subsection, we explain how to deduce Theorem D from Theorem E. For a locally compact, second countable group G, the space $\mathrm{Act}_G(Y,\nu)$ of pmp actions of G on a standard probability space (Y,ν) is endowed with a canonical Polish topology. For example, it can be regarded as the space of continuous group homomorphisms from G to $\mathrm{Aut}(Y,\nu)$, which is a closed subspace of the Polish space $C(G,\mathrm{Aut}(Y,\nu))$ of continuous functions from G to $\mathrm{Aut}(Y,\nu)$ endowed with the compact-open topology. Indeed, it follows from Mackey's point realization theorem [42] that any continuous group homomorphism from G to $\mathrm{Aut}(Y,\nu)$ arises from a continuous pmp action of G on (Y,ν), and vice versa.

One can consider an equivalent parametrization of pmp actions of G on (Y,ν) in the setting of von Neumann algebras, adopting the perspective from the theory of locally compact quantum groups. Suppose that M,N are von Neumann algebras with separable preduals M_* and N_*, respectively. We can identify the space of σ-weakly continuous linear isometries from M to N with the space of linear contractive quotient mappings from N_* to M_*, which is a Polish space with respect to the topology induced by the operator norm. The space $\mathrm{Hom}(M,N)$ of injective unital *-homomorphisms from M to N is a closed subset of the space of σ-weakly continuous linear isometries from M to N [5, III.2.2.2].

Let $C = L^\infty(G,\lambda))$ be the von Neumann algebra of essentially bounded functions on G with respect to a left Haar measure λ on G. The multiplication operation on G gives rise to a normal injective unital *-homomorphism $\Delta \colon C \to C \otimes C$ (*comultiplication*) given by $\Delta(f)(s,t) = f(st)$ for all $f \in C$ and all $s,t \in G$. Consider now a standard probability space (X,μ), and set $M = L^\infty(X,\mu)$. A continuous pmp action $G \curvearrowright (X,\mu)$ gives rise to a normal injective unital *-homomorphism $\alpha \colon M \to C \otimes M$ given by $\alpha(f)(s,x) = f(g \cdot x)$ for all $f \in C$ and all $s \in G$ and all $x \in X$. Such a map is a *coaction* of C on M, as it satisfies $(\Delta \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \alpha) \circ \alpha$. The G-action on (X,μ) being pmp is equivalent to the identity $(\mathrm{id} \otimes \mu) \circ \alpha = \mathrm{id}$, where we identify μ with a normal state on M.

We can thus identify the space of continuous actions of G on (X,μ) with the space of coactions of C on M satisfying $(\mathrm{id} \otimes \mu) \circ \alpha = \mathrm{id}$, which is a closed subspace of $\mathrm{Hom}(M, C \otimes M)$. The action $G \curvearrowright X$ is free if and only if $(1 \otimes M)\alpha(M)$ has dense linear span inside $C \otimes M$; see [15, Theorem 2.9]. Furthermore, the action $G \curvearrowright X$ is ergodic if and only if the fixed point algebra $M^\alpha = \{x \in M \colon \alpha(x) = x \otimes 1\}$ only contains the scalar multiples of the identity or, equivalently, if $((\lambda_G \otimes \mathrm{id}) \circ \alpha)(x) = \lambda_G(x)1$ for every $x \in M$. This shows that the space of free ergodic pmp actions of G on (X,μ) can be seen as a closed subspace of $\mathrm{Hom}(M, C \otimes M)$.

Definition 5.17. Let G be a locally compact, second countable group, let (X,μ) be a standard probability space, and let $G \curvearrowright Y$ be a free action. A *cross section* for $G \curvearrowright X$ is a Borel subset $Y \subseteq X$ for which there exists an open neighborhood $U \subseteq G$ containing the identity of G, such that the restricted action $U \times Y \to X$ is injective, and the orbit $G \cdot Y$ of Y has full measure in x. The cross section Y is *cocompact* if there is a compact subset $K \subseteq G$ such that $K \cdot Y$ is G-invariant and has full measure in X.

We will use cross section equivalence relations, following [40, Section 4] and [8].

Definition 5.18. Let G be a locally compact, second countable group, let (X,μ) be a standard probability space, let $G \curvearrowright X$ be a free action, and let $Y \subseteq X$ be a cocompact cross section. The *cross* *section equivalence relation* associated to Y is the orbit equivalence relation of the action restricted to Y, that is,

$$R = \{(y, y') \in Y \times Y : y \in G \cdot y'\}.$$

This is a countable Borel equivalence relation on Y [40, Proposition 4.3]. If λ is a Haar measure on G, then there exist a unique R-invariant probability measure ν on Y and a constant $c \in (0, +\infty)$ such that, with $\eta\colon U \times Y \to X$ denoting the restriction of the G-action, one has $\eta_*(\lambda|_U \times \nu) = c\ \mu|_{U \cdot Y}$. Such a measure ν is called the *canonical R-invariant probability measure* on Y. The countable pmp equivalence relation R is ergodic if and only if the action $G \curvearrowright X$ is ergodic by [40, Proposition 4.3].

Recall that a locally compact, second countable group is said to be *unimodular* if its left Haar measure is also right invariant.

Proposition 5.19. *Let G be a locally compact, second countable, unimodular group, let (X, μ) be an atomless standard probability space, and let $G \curvearrowright X$ be a free ergodic action with a cocompact cross section Y. Let R denote the associated cross section equivalence relation, and ν the canonical R-invariant probability measure on Y. Then there is a Borel assignment $S \mapsto \zeta_S$ from ergodic class-bijective pmp extensions of R on a standard atomless probability space $(\bar{Y}, \bar{\nu})$ to free ergodic pmp actions of G on the standard probability space $(\tilde{X}, \tilde{\mu})$ such that, for ergodic class-bijective pmp extensions S and S' of R on $(\bar{Y}, \bar{\nu})$:*

- *if S and S' are isomorphic relatively to R, then ζ_S and $\zeta_{S'}$ are conjugate;*
- *if ζ_S and $\zeta_{S'}$ are stably von Neumann equivalent, then S and S' are stably von Neumann equivalent.*

Proof. Let S be an ergodic class-bijective pmp extension of R on $(\bar{Y}, \bar{\nu})$. A free ergodic pmp action ζ_S of G on $(\tilde{X}, \tilde{\mu})$ is constructed in [8, Proposition 8.3] such that, as proved in [8, Proof of Theorem B] using [40, Lemma 4.5], the group-measure space construction $G \ltimes^{\zeta_S} L^\infty(\tilde{X}, \tilde{\mu})$ is isomorphic to an amplification of the II_1 factor $L(S)$. We review here the construction, since we need some specific details about it.

Fix an open neighborhood U of the identity in G, a compact subset K of G, $n \in \mathbb{N}$, and $g_1, \ldots, g_n \in G$ such that:

- the map $U \times Y \to X$ given by $(g, y) \mapsto g \cdot y$ is injective,
- $K \cdot Y$ is a G-invariant conull subset of X,
- $g_1 = 1$ and $g_1 U \cup \cdots \cup g_n U$ contains K.

After discarding null sets, one can assume without loss of generality that the stabilizer of every point of X is trivial and that $K \cdot Y = X$. One can then define a Borel map $\pi\colon X \to Y$ such that $\pi(x)$ belongs to the G-orbit of x for every $x \in X$, by setting $\pi(g \cdot y) = y$ for $g \in U$ and $y \in Y$, and $\pi(g_i g \cdot y) = y$ for $i = 2, 3, \ldots, n$, $g \in U$, and $y \in Y$ such that $g_i g \cdot y \notin (g_1 U \cdot Y) \cup \cdots \cup (g_{i-1} U \cdot Y)$.

Fix an ergodic class-bijective pmp extension S of R on $(\bar{Y}, \bar{\nu})$, with corresponding factor map $p\colon \bar{Y} \to Y$. After discarding a null set, one can assume that $p|_{[\bar{y}]_S}$ maps $[\bar{y}]_S$ bijectively onto $[p(\bar{y})]_R$ for every $\bar{y} \in \bar{Y}$. One then defines $\tilde{X}$ to be

$$X \times_Y \bar{Y} = \{(x, \bar{y}) \in X \times \bar{Y} : \pi(x) = p(\bar{y})\}.$$

The G-action ζ_S on $\tilde{X}$ is defined by setting $g \cdot (x, \bar{y}) = (g \cdot x, \hat{y})$ where $\hat{y}$ is the unique element of $[\bar{y}]_S$ such that $p(\hat{y}) = \pi(g \cdot x)$ (notice that $\pi(g \cdot x)$ belongs to $[\pi(x)]_R = [p(y)]_R$). The subsets K and U of G as above witness that

$$\tilde{Y} = \{(p(\bar{y}), \bar{y}) : \bar{y} \in \bar{Y}\}$$

is a Borel cross-section for the G-action on $\tilde{X}$. As above, one can define a Borel function $\tilde{\pi}\colon \tilde{X} \to \tilde{Y}$ by setting $\pi(g \cdot \tilde{y}) = \tilde{y}$ for $g \in U$ and $\tilde{y} \in \tilde{Y}$, and $\pi(g_i g \cdot \tilde{y}) = \tilde{y}$ for $i = 2, \ldots, n$, $g \in U$, and $\tilde{y} \in \tilde{Y}$ such that $g_i g \cdot \tilde{y} \notin (g_1 U \cdot \tilde{Y}) \cup \cdots \cup (g_{i-1} U \cdot \tilde{Y})$. The measure $\bar{\nu}$ on $\bar{Y}$ induces via the canonical Borel isomorphism $\bar{Y} \to \tilde{Y}$, given by $\bar{y} \mapsto (p(\bar{y}), \bar{y})$, a probability measure $\tilde{\nu}$ on $\tilde{Y}$. In turn, $\tilde{\nu}$ induces a G-invariant probability measure $\tilde{\mu}$ on X obtained by setting

$$\tilde{\mu}(A) = (\lambda \times \tilde{\nu})\{(g, \tilde{y}) \in G \times \widetilde{Y} : \tilde{\pi}(g\tilde{y}) = \tilde{y} \text{ and } g\tilde{y} \in A\},$$

for any measurable subset $A \subseteq X$. This concludes the construction of the ergodic pmp action ζ_S of G on $(\tilde{X}, \tilde{\mu})$.

Suppose now that one starts with ergodic class-bijective pmp extensions S and S' of R on $(\bar{Y}, \bar{\nu})$. It is clear from the construction above that, if S and S' are isomorphic relatively to R, then the corresponding G-actions ζ_S and $\zeta_{S'}$ are conjugate. Furthermore, if S and S' are stably von Neumann equivalent, then ζ_S and $\zeta_{S'}$ are also stably von Neumann equivalent, as $G \ltimes^{\zeta_S} L^\infty(\tilde{X}, \tilde{\mu})$ is isomorphic to an amplification of $L(S)$ and $G \ltimes^{\zeta_{S'}} L^\infty(\tilde{X}, \tilde{\mu})$ is isomorphic to an amplification of $L(S')$. Finally, one should notice that the assignment $S \mapsto \zeta_S$ is given by a Borel map with respect to the parametrizations of class-bijective pmp extensions of R and free ergodic actions of G as described above. Indeed, it suffices to observe that, once one has fixed the cocompact cross section Y for $G \curvearrowright X$, the canonical R-invariant measure ν on Y, the compact subset K of G and the open neighborhood U of the identity of G witnessing that Y is a cocompact cross section, and the elements $g_1, \ldots, g_n$ of G such that $g_1 = 1$ and $g_1 U \cup \cdots \cup g_n U$ contains K, the construction of the action ζ_S is canonical. ∎

Every locally compact, second countable, unimodular group admits a free ergodic pmp action on the standard probability space, which can be obtained as a Gaussian action; see [4, 56]. Combining Proposition 5.19 with Theorem 5.14 we deduce the following.

Theorem 5.20. *Let G be a locally compact, second countable, unimodular group. There is a Borel function $A \mapsto \zeta_A$ from abelian countably infinite groups to free ergodic actions of G on the standard atomless probability space such that:*

1. *if A and A' are isomorphic groups, then ζ_A and $\zeta_{A'}$ are conjugate;*
2. *if $\mathcal{A}$ is a collection of pairwise nonisomorphic countably infinite abelian groups such that $\{\zeta_A \colon A \in \mathcal{A}\}$ are pairwise stably von Neumann equivalent, then $\mathcal{A}$ is countable.*

In particular, if E is any equivalence relation for free ergodic actions of G on the standard atomless probability space that is coarser than conjugacy and finer than stable von Neumann equivalence, then $A \mapsto \zeta_A$ is a countable-to-one Borel homomorphism from $\cong_{\mathrm{AG}}$ to E.

Theorem D is then an immediate consequence of Theorem 5.20, by Theorem 5.9.

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