THE RAMSEY PROPERTY FOR BANACH SPACES, CHOQUET SIMPLICES, AND THEIR NONCOMMUTATIVE ANALOGS

DANA BARTOŠOVÁ, JORDI LÓPEZ-ABAD, MARTINO LUPINI, AND BRICE MBOMBO

Abstract. We show that the Gurarij space $\mathbb{G}$ and its noncommutative analog $\mathbb{NG}$ both have extremely amenable automorphism group. We also compute the universal minimal flows of the automorphism groups of the Poulsen simplex $\mathbb{P}$ and its noncommutative analogue $\mathbb{NP}$. The former is $\mathbb{P}$ itself, and the latter is the state space of the operator system associated with $\mathbb{NP}$. This answers a question of Conley and Törnquist. We also show that the pointwise stabilizer of any closed proper face of $\mathbb{P}$ is extremely amenable. Similarly, the pointwise stabilizer of any closed proper biface of the unit ball of the dual of the Gurarij space (the Lusky simplex) is extremely amenable.

These results are obtained via the Kechris–Pestov–Todorcevic correspondence, by establishing the approximate Ramsey property for several classes of finite-dimensional operator spaces and operator systems (with distinguished linear functionals), including: Banach spaces, exact operator spaces, function systems with a distinguished state, and exact operator systems with a distinguished state. This is the first direct application of the Kechris–Pestov–Todorcevic correspondence in the setting of metric structures. The fundamental combinatorial principle that underpins the proofs is the Dual Ramsey Theorem of Graham and Rothschild.

In the second part of the paper, we obtain factorization theorems for colorings of matrices and Grassmannians over $\mathbb{R}$ and $\mathbb{C}$, which can be considered as continuous versions of the Dual Ramsey Theorem for Boolean matrices and of the Graham-Leeb-Rothschild Theorem for Grassmannians over a finite field.

Recall that given a topological group G, a compact G-space or G-flow is a compact Hausdorff space X endowed with a continuous action of G. Such a G-flow X is called minimal when every orbit is dense. There is a natural notion of morphism between G-flows, given by a (necessarily surjective) G-equivariant continuous map (factor). A minimal G-flow is universal if it factors onto any other minimal G-flow. It is a classical fact that any topological group G admits a unique (up to isomorphism of G-flows) universal minimal flow, usually denoted by $M(G)$ [18, 33]. For any locally compact non compact Polish group G, the universal minimal G-flow is nonmetrizable. At the opposite end, non locally compact topological groups often have metrizable universal minimal flows, or even reduced to a single point. A topological group for which $M(G)$ is a singleton is called extremely amenable. (Amenability of G is equivalent to the assertion that every compact G-space has an invariant Borel measure. Thus any extremely amenable group is, in particular, amenable.)

The universal minimal flow has been explicitly computed for many topological groups, typically given as automorphism groups of naturally arising mathematical structures. Examples of extremely amenable Polish groups include the group of order automorphisms of $\mathbb{Q}$ [61], the group of unitary operators on the separable infinite-dimensional Hilbert space [30], the automorphism group of the hyperfinite II_1 factor and of infinite type UHF C^*-algebras [14, 21], or the isometry group of the Urysohn space [62]. Examples of nontrivial metrizable universal minimal flows include the universal minimal flow of the group of orientation preserving homeomorphisms of the circle, which is equivariantly homeomorphic to the circle itself [61], the universal minimal flow of the group S_∞ of permutations of $\mathbb{N}$, which can be identified with the space of linear orders on $\mathbb{N}$ [22], and the universal minimal flow of the homeomorphism group $\mathrm{Homeo}(2^{\mathbb{N}})$ of the Cantor set $2^{\mathbb{N}}$, which can be seen as the canonical action of $\mathrm{Homeo}(2^{\mathbb{N}})$ on the space of maximal chains of closed subsets of $2^{\mathbb{N}}$ [23, 37, 70].

There are essentially two known ways to establish extreme amenability of a given topological group. The first method involves the phenomenon of *concentration of measure*, and can be applied to topological groups that admit an increasing sequence of compact subgroups with dense union [63, Chapter 4]. The second method

2000 *Mathematics Subject Classification.* Primary 05D10, 46B04; Secondary 47L25, 46A55.

Key words and phrases. Gurarij space, Poulsen simplex, extreme amenability, Ramsey property, Banach space, Choquet simplex, operator space, operator system, oscillation stability, Dual Ramsey Theorem.

D.B. was supported by the grant FAPESP 2013/14458-9. J.L.-A. was partially supported by the grant MTM2012-31286 (Spain) and the Fapesp Grant 2013/24827-1 (Brazil). M.L. was partially supported by the NSF Grant DMS-1600186. R. Mbombo was supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) postdoctoral grant, processo 12/20084-1. This work was initiated during a visit of J.L.-A. to the Universidade de Sao Pãulo in 2014, and continued during visits of D.B. and J.L.-A. to the Fields Institute in the Fall 2014, a visit of M.L. to the Instituto de Ciencias Matemáticas in the Spring 2015, and a visit of all the authors at the Banff International Research Station in occasion of the Workshop on Homogeneous Structures in the Fall 2015. The hospitality of all these institutions is gratefully acknowledged.

applies to automorphism groups of discrete ultrahomogeneous structures or, more generally, approximately ultrahomogeneous metric structures [63, Chapter 6]. A metric structure is approximately ultrahomogeneous if any partial isomorphism between finitely generated substructures is the pointwise limit of maps that are restriction of automorphisms. It is worth noting that any Polish group can be realized as the automorphism group of an approximately ultrahomogeneous metric structure [52, Theorem 6]. For the automorphism group $\mathrm{Aut}(M)$ of an approximately ultrahomogeneous structure M, extreme amenability is equivalent to the approximate Ramsey property of the class of finitely generated substructures of M. This criterion is known as the Kechris–Pestov–Todorcevic (KPT) correspondence, first established in [37] for discrete structures, and recently generalized to the metric setting in [54]. The discrete KPT correspondence has been used extensively in the last decade. In this paper the KPT correspondence is directly used to establish new extreme amenability results. This is the first examples of such applications, apart from the example of finite metric spaces with a distinguished unary 1-Lipschitz predicate considered in [54].

In all the known examples of computation of metrizable universal minimal flows, the argument hinges on extreme amenability of a suitable subgroup and the following well known fact. Suppose that G is a topological group with an extremely amenable closed subgroup H. If the completion X of the homogeneous space G/H endowed with the quotient of the right uniformity on G is a minimal compact G-space, then X is the universal minimal flow of G. It was recently shown in [7, 53] that, whenever the universal minimal flow of G is metrizable, it can be realized as the completion of G/H for a suitable closed subgroup H of G.

In this paper we compute the universal minimal flows of the automorphism groups of structures coming from functional analysis and Choquet theory: the Gurarij space $\mathbb{G}$, the Poulsen simplex $\mathbb{P}$, and their noncommutative analogs. Recall that the Gurarij space is the unique separable approximately ultrahomogeneous Banach space that contains ℓ_n^∞ for every $n \in \mathbb{N}$ [49], while $\mathbb{P}$ is the unique nontrivial metrizable Choquet simplex with dense extreme boundary [44]. The group $\mathrm{Aut}(\mathbb{G})$ of surjective linear isometries of the Gurarij space is shown to be extremely amenable by establishing the approximate Ramsey property of the class of finite-dimensional Banach spaces. Similarly, the stabilizer $\mathrm{Aut}_p(\mathbb{P})$ of an extreme point p of $\mathbb{P}$ is proven to be extremely amenable by establishing the approximate Ramsey property of the class of Choquet simplices with a distinguished point. It is then deduced from this that the universal minimal flow of $\mathrm{Aut}(\mathbb{P})$ is $\mathbb{P}$ itself, endowed with the canonical action of $\mathrm{Aut}(\mathbb{P})$. This answers Question 4.4 from [13]. More generally, we prove that for any closed face F of $\mathbb{P}$, the pointwise stabilizer $\mathrm{Aut}_F(\mathbb{P})$ is extremely amenable. The analogous result holds in the Banach space setting as well. A *Lazar simplex* is a compact absolutely convex set that arises as the unit ball of the dual of a Lindenstrauss space. The *Lusky simplex* $\mathbb{L}$ is the Lazar simplex that arises in this fashion from the Gurarij space. The group $\mathrm{Aut}(\mathbb{G})$ can be identified with the group $\mathrm{Aut}(\mathbb{L})$ of symmetric affine homeomorphisms of $\mathbb{L}$. It is proved in [46, Theorem 1.2] that $\mathbb{L}$ plays the same role among Lazar simplices as the Poulsen simplex plays in the class of Choquet simplices, where closed faces are replaced with closed bifaces. We prove that, for any closed proper biface H of $\mathbb{L}$, the corresponding pointwise stabilizer $\mathrm{Aut}_H(\mathbb{L})$ is extremely amenable. In the particular case when H is the trivial biface, this recovers the extreme amenability of $\mathrm{Aut}(\mathbb{G})$.

We also provide the natural noncommutative analogs of the results above, formulated in the categories of operator spaces and operator systems. Operator spaces are the closed subspaces of C^*-algebras [8, Subsection 1.2.2]. The natural notion of morphism between operator spaces are completely contractive linear maps [8, Subection 1.2.1]. Any Banach space is endowed with a canonical operator space structure (minimal quantization) [17, Section 3.3]. The operator spaces that arise in this way are precisely the subspaces of *abelian* C^*-algebras, i.e. C*-algebras where the multiplication is commutative. Thus arbitrary operator spaces can be regarded as the noncommutative analog of Banach spaces. The *noncommutative Gurarij space* $\mathbb{NG}$, introduced in [57] and proved to be unique in [48], is the unique *nuclear* approximately ultrahomogeneous operator spaces that contains the space $M_q(\mathbb{C})$ of $q \times q$ complex matrices for every $q \in \mathbb{N}$. We will prove in this paper that the group $\mathrm{Aut}(\mathbb{NG})$ of surjective complete linear isometries of $\mathbb{NG}$ is extremely amenable. We will also establish in this setting the natural noncommutative analog of extreme amenability of closed bifaces of the Lusky simplex.

Similarly, one can define the noncommutative analog of compact convex sets and simplices in the setting of operator systems. An operator system X is a unital closed self-adjoint subspace of a unital C^*-algebra A [8, Subsection 1.3.2]. The natural notion of morphism between operator systems are unital completely contractive maps [8, Subsection 1.2.1]. The operator systems that can be represented in an *abelian* unital C^*-algebra are called *function systems*. Any function system X can be identified with the space $A(K)$ of continuous complex-valued affine functions on K, where K is the compact convex set of unital contractive linear functionals on X (*states*) [2, Theorem II.1.8]. Furthermore, the map $K \mapsto A(K)$ is a contravariant isomorphism of categories from the category of compact convex sets and continuous affine maps, to the category of function systems and unital linear contractions (*Kadison correspondence*). A metrizable compact convex set K is a simplex if and only if $A(K)$ is a separable Lindenstrauss space, which means that the identity map of $A(K)$ is the pointwise limit of a sequence of unital completely contractive maps that factor through finite-dimensional (abelian) C^*-algebras. The function system $A(\mathbb{P})$ corresponding to the Poulsen simplex is the unique separable approximately ultrahomogeneous function system that contains unital copies of ℓ_n^∞ for $n \in \mathbb{N}$ [46, Theorem 1.1]. The automorphism group $\mathrm{Aut}(A(\mathbb{P}))$ can be identified with the group of affine homeomorphisms of $\mathbb{P}$. The Poulsen simplex $\mathbb{P}$ is then equivariantly homeomorphic to the state space of $A(\mathbb{P})$.

The correspondence between function systems and compact convex sets generalizes to a correspondence between operator systems and compact matrix convex sets [72]. A *compact matrix convex set* is a sequence (K_n) of compact convex sets endowed with a notion of *matrix convex combination.* This is an expression of the form $\alpha_1^* p_1 \alpha_1 + \cdots + \alpha_\ell^* p_\ell \alpha_\ell$ where $\alpha_i \in M_{n,n_i}(\mathbb{C})$ and $p_i \in K_{n_i}$ for $i = 1, 2, \ldots, \ell$ are such that $\alpha_1^* \alpha_1 + \cdots + \alpha_\ell^* \alpha_\ell$. The notions of affine map and extreme point admit natural matrix analogs, where convex combinations are replaced with matrix convex combinations. To any operator system X one can canonically assign a compact matrix convex set: the *matrix state space* $\boldsymbol{S}(X)$. It is proved in [72] that any compact matrix convex set arises in this way. Furthermore the assignment $X \mapsto \boldsymbol{S}(X)$ is a contravariant equivalence of categories from the category of operator systems and unital completely positive maps to the category of compact matrix convex sets and continuous matrix affine maps. One can establish a similar correspondence between operator spaces and *compact rectangular matrix convex sets* [20], generalizing the correspondence between Banach spaces and compact absolutely convex sets.

The compact convex sets that arise from (separable) *nuclear* operator systems can be seen as the noncommutative analog of (metrizable) Choquet simplices. The *noncommutative Poulsen simplex* $\mathbb{NP} = (\mathbb{NP}_n)$ is a metrizable noncommutative Choquet simplex with dense matrix extreme boundary [46]. The operator system $A(\mathbb{NP})$ associated with the noncommutative Poulsen simplex can be realized as the Fraïssé limit of the class of finite-dimensional exact operator systems [46, Theorem 1.3]. The group $\mathrm{Aut}(A(\mathbb{NP}))$ of surjective unital complete isometries of $A(\mathbb{NP})$ can be identified with the space of matrix affine homeomorphisms of $\mathbb{NP}$. We will prove below that the universal minimal $\mathrm{Aut}(\mathbb{NP})$-space is the state space $\mathbb{NP}_1$ of $A(\mathbb{NP})$ endowed with the canonical action of $\mathrm{Aut}(\mathbb{NP})$. The natural noncommutative analog of extreme amenability of the pointwise stabilizers of closed faces of $\mathbb{P}$ will be proved as well. Our techniques will also apply to other Fraïssé classes of operator spaces and operator systems, including the classes of operator sequence spaces [46, Subsection 6.5], q-minimal operator spaces [46, Subsection 6.6], and q–minimal operator systems [46, Subsection 6.7]. This will allow us to compute the universal minimal flows of the automorphisms groups of the corresponding Fraïssé limits.

The main tool to establish the results mentioned above will be the Dual Ramsey Theorem of Graham and Rothschild [29]. This is a powerful pigeonhole principle known to imply many other results, such as the Hales–Jewett theorem, and the Ramsey theorem. It can be seen to be equivalent to a factorization result for colorings of Boolean matrices, which implies the celebrated Graham–Leeb–Rothschild theorem on Grassmannians over a finite field [28]. In fact, this is again a particular case of a factorization for colorings of matrices over a finite field, stating that the coloring of matrices only depends on the invertible matrix needed to transform a given matrix into one in reduced column echelon form. In this paper, we provide factorization theorems for colorings of matrices and Grassmannians over the real or complex numbers. We prove in particular that colorings of matrices depend only on the canonical norm that a given matrix determines, while colorings of Grassmannians are determined by the Banach-Mazur type of the given subspace.

The paper is organized as follows. We start in Section 1 with an exposition of the basic concepts used in the paper: the Dual Ramsey Theorem, extreme amenability, operator spaces and operator systems, Fraïssé classes and Fraïssé limits, and the KPT correspondence. In Section 2, we introduce operator spaces that are Fraïssé limits of various of classes of operator spaces, and then establish the approximate Ramsey property for each of these classes. In Section 3, we prove the approximate Ramsey property for operator spaces with a distinguished functional, while Section 4 deals with the Ramsey property for operator systems with a distinguished state. In Section 5, we present many equivalent versions of the Dual Ramsey Theorem in terms of factorization of colorings of Boolean matrices, and we extend these statements to matrices with values in an arbitrary finite field. This leads to Section 6, where we prove factorization theorems for colorings of matrices and Grassmannians over the real or complex field. We conclude by providing explicit estimates on the Ramsey numbers for finite-dimensional Banach spaces, and by presenting a new proof of the approximate Ramsey property of finite metric spaces [60, 62].

Acknowledgments. We are grateful to Itaï, Ben Yaacov, Clinton Conley, Valentin Ferenczi, Alexander Kechris, Matt Kennedy, Julien Melleray, Lionel Nguyen Van Thé, Vladimir Pestov, Slawomir Solecki, Stevo Todorcevic, and Todor Tsankov for several helpful conversations and remarks.

1. Basic notions

We introduce some terminology to be used in the following. A *metric coloring* of a pseudo-metric space M is 1-Lipschitz map from M to a metric space (N, d_N). When the target space of the coloring is compact we will say that the coloring is a *compact*, and when the target space of the coloring is the unit interval $[0,1]$ we will say that the coloring is *continuous* (as in [54]). The *oscillation* $\mathrm{osc}_F(c)$ of a compact coloring $c : M \to (K, d_K)$ on a subset F of M is the supremum of $d_K(c(y), c(y'))$ where y, y' range within F. If $\mathrm{osc}_F(c) \leq \varepsilon$, then we also say that c *ε-stabilizes* on F. A *finite* coloring of M is a function from c from M to a finite set X. When the target space is a natural number r (identified with the set $\{0, 1, \dots, r-1\}$ of its predecessors), we will say that c is an r-coloring. A subset F of M is *monochromatic* for c if $c(x) = c(y)$ for every $x, y \in F$, and *ε-monochromatic* for c if there exists some $x \in X$ such that $F \subseteq (c^{-1}(x))_\varepsilon$, that is, for every $p \in F$ there is $q \in E$ such that $c(q) = x$ and $d_M(p, q) \leq \varepsilon$. If F is ε-monochromatic, then we also say that c ε-stabilizes on F.

If M is endowed with an action of a Polish group G by isometries, then we say that a compact coloring c is *finitely oscillation stable* [63, Definition 1.1.8] if for every finite subset F of M and $\varepsilon > 0$ there exists $g \in G$ such that c *ε-stabilizes* on $g \cdot F$. We say that the action of G on M is *finitely oscillation stable* if every continuous coloring of M is finitely oscillation stable [63, Definition 1.1.11].

Given a Polish group G and a continuous action $G \curvearrowright M$ of G on a metric space (M, d_M) by isometries, we write $[p]_G$ to denote the closure of the G-orbit of $p \in M$, and $M /\!\!/ G$ to denote the space of closures of G orbits of M. Since G acts by isometries the formula

$$\widehat{d}_{G,M}([p],[q]) := \inf\{d_M(\bar{p}, \bar{q}) \,:\, \bar{p} \in [p],\, \bar{q} \in [q]\}$$

defines the quotient pseudometric induced by the quotient map $\pi_{M,G} : M \mapsto M /\!\!/ G$, and since we consider *closures* of orbits, $\widehat{d}_{G,M}$ is a metric. It is easy to see that $\widehat{d}_{G,M}$ is complete when d_M is complete.

Given a compact metric space (K, d_K) let $\mathrm{Lip}((M, d_M), (K, d_K))$ be the collection of all K-colorings of M, that is, 1-Lipschitz maps from (M, d_M) to (K, d_K). With its topology of pointwise convergence $\mathrm{Lip}((M, d_M), (K, d_K))$ is a compact space, which is metrizable when (M, d_M) is separable. The continuous action $G \curvearrowright (M, d_M)$ induces a natural continuous action $G \curvearrowright \mathrm{Lip}((M, d_M), (K, d_K))$, defined by setting $(g \cdot c)(p) := c(g^{-1} \cdot p)$ for every $c \in \mathrm{Lip}((M, d_M), (K, d_K))$ and $p \in M$.

Recall that a topological group G is called *extremely amenable* if every continuous action of G on a compact Hausdorff space has a fixed point. The following characterization of extreme amenability will be used extensively in this paper.

Proposition 1.1. *Suppose that G is a Polish group, and d_G is a left-invariant compatible metric on G. The following assertions are equivalent.*

(1) G is extremely amenable.
(2) The left translation of G on (G, d_G) is finitely oscillation stable.
(3) For every action $G \curvearrowright M$ of G on a metric space (M, d_M), and for every compact coloring $c : (M, d_M) \to (K, d_K)$ of (M, d_M), there exists a compact coloring $\widehat{c} : M /\!\!/ G \to K$ such that for every finite $F \subseteq M$ and every $\varepsilon > 0$ there is some $g \in G$ such that $d_K(c(p), \widehat{c}([p]_G)) < \varepsilon$ for every $p \in g \cdot F$.
(4) The same as (3) where F is compact.

Proof. The equivalence of *(1)* and *(2)* can be found in [63, Theorem 2.1.11]. The implication *(3)⇒(2)* is immediate, since orbit space $G /\!\!/ G$ is one point. We now establish the implication *(1)⇒(4)*: Fix a 1-Lipschitz $c : (M, d_M) \to (K, d_K)$. Let L be the closure of the G-orbit of c in $\mathrm{Lip}((M, d_M), (K, d_K))$. By the extreme amenability of G, there is some $c_\infty \in L$ such that $G \cdot c_\infty = \{c_\infty\}$, so we can define $\widehat{c} : M /\!\!/ G \to K$ by $\widehat{c}([p]_G) := c_\infty(p)$. It is clear that $\widehat{c}$ is 1-Lipschitz. Given a compact subset F of M, let $g \in G$ be such that $\max_{p \in F} d_K(c_\infty(p), c(g \cdot p)) < \varepsilon$. If $x \in F$, then $d_K(c(g \cdot x), \widehat{c}([g \cdot x]_G)) = d_K(c(g \cdot x), c_\infty(x)) < \varepsilon$. □

We now to recall the *Dual Ramsey Theorem* (DRT) of Graham and Rothschild [29]. For convenience, we present its formulation in terms of rigid surjections between finite linear orderings. Given two linear orderings $(R, <_R)$ and $(S, <_S)$, a surjective map $f : R \to S$ is called a *rigid surjection* when $\min f^{-1}(s_0) < \min f^{-1}(s_1)$ whenever $s_0 < s_1$. We let $\mathrm{Epi}(R, S)$ be the collection of rigid surjections from R to S.

Theorem 1.2 (Graham–Rothschild). *For every finite linear orderings R and S such that $|R| < |S|$ and every $r \in \mathbb{N}$ there exists an integer $n > |S|$ such that, considering n naturally ordered, every r-coloring of* $\mathrm{Epi}(n, R)$ *has a monochromatic set of the form* $\mathrm{Epi}(S, R) \circ \gamma = \{\sigma \circ \gamma : \sigma \in \mathrm{Epi}(S, R)\}$ *for some* $\gamma \in \mathrm{Epi}(n, S)$.

We will see in Section 5 several equivalent reformulations of the Dual Ramsey Theorem. We now recall some fundamental notions and results from the theory of operator spaces. The monographs [8, 17, 66] provide good introductions to this subject. An *operator space* E is a closed linear subspace of the space $B(H)$ of bounded linear operators on some complex Hilbert space H. The inclusion $E \subset B(H)$ induces *matrix norms* on each $M_n(E)$, $n \in \mathbb{N}$, the space of $n \times n$ matrices with entries in E. The norm of an element $[x_{ij}]$ of $M_n(E)$ is defined as the operator norm of $[x_{ij}]$ when regarded in the canonical way as an linear operator on the n-fold Hilbertian direct sum of H by itself. The ∞-*sum* of two operator spaces $E \subset B(H_0)$ and $F \subset B(H_1)$ is the space $E \oplus_\infty F \subset B(H_0 \oplus H_1)$ of operators of the form

$$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$$

for $x \in E$ and $y \in F$. One can equivalently define operator spaces as the closed subspaces of unital C*-algebras. A unital C*-algebra is a closed subalgebra of $B(H)$ containing the identity operator and closed under taking adjoints. Unital C*-algebras can be abstractly characterized as the complex Banach algebras with multiplicative identity and involution satisfying the C*-identity $\|a^*a\| = \|a\|^2$. Operator spaces also admit an abstract characterization, in terms of Ruan's axioms for the matrix norms [59, Theorem 13.4]. Precisely, a matrix normed complex vector space is an operator space if and only if the matrix norms satisfy the identity

$$\|\alpha_1^* x_1 \beta_1 + \cdots + \alpha_\ell^* x_\ell \beta_\ell\| \leq \|\alpha_1^* \alpha_1 + \cdots + \alpha_\ell^* \alpha_\ell\| \max\{\|x_1\|, \ldots, \|x_\ell\|\} \|\beta_1^* \beta_1 + \cdots + \beta_\ell^* \beta_\ell\|.$$

The abstract characterization of C*-algebras shows that, whenever K is a compact Hausdorff space, $C(K)$ with the pointwise operations and the supremum norm is a unital C*-algebra. The unital C*-algebras of these form are precisely the *abelian* ones. Any complex Banach space E has a canonical operator space structure, obtained by representing E isometrically as a subspace of $C(\mathrm{Ball}(E^*))$, where the unit ball $\mathrm{Ball}(E^*)$ of E^* is endowed with its weak*-topology. This operator space structure on E is called its *minimal quantization* [17] and the corresponding operator space is denoted by $\mathrm{MIN}(E)$. The matrix norms on $\mathrm{MIN}(E)$ are defined by $\|[x_{ij}]\| = \sup_{\phi \in \mathrm{Ball}(E^*)} \|[\phi(x_{ij})]\|$ for $[x_{ij}] \in M_n(E)$. The operator spaces that arise in this fashion are called *minimal operator spaces*. These are precisely the operator spaces that can be represented inside an *abelian* unital C*-algebra. Arbitrary operator spaces can be thought of as the noncommutative analog of Banach spaces.

If $\phi : E \to F$ is a linear map between operator spaces, then one can consider its amplifications $\phi^{(n)} : M_n(E) \to M_n(F)$ obtained by applying ϕ entrywise. The *completely bounded norm* $\|\phi\|_{\mathrm{cb}}$ of ϕ is the supremum of $\|\phi^{(n)}\|$ for $n \in \mathbb{N}$. A linear map ϕ is *completely bounded* if $\|\phi\|_{\mathrm{cb}}$ is finite, and *completely contractive* if $\|\phi\|_{\mathrm{cb}}$ is at most 1. We regard operator spaces as the objects of a category having completely contractive linear maps as morphisms. An isomorphism in this category is a surjective *complete isometry*, which is an invertible completely contractive linear map with completely contractive inverse. If E is an operator space, then its automorphism group is the group of invertible completely contractive linear maps from E to itself. When E is separable, this is a Polish group when endowed with the topology of pointwise convergence. The *dual operator space* of an operator space E is a canonical operator space structure on the space E^* of (completely) bounded linear functionals on E, obtained by identifying completely isometrically $M_n(E^*)$ with the space of completely bounded linear maps from E to $M_n(\mathbb{C})$; see [17, §3.2].

When E, F are Banach spaces, and $\phi : E \to F$ is a linear map, then ϕ is bounded if and only if it is completely bounded when E and F are endowed with their minimal operator space structure. Furthermore, in this case one has the equality of norms $\|\phi : E \to F\| = \|\phi : \mathrm{MIN}(E) \to \mathrm{MIN}(F)\|_{\mathrm{cb}}$. Thus the category of Banach spaces and contractive linear maps can be seen as a full subcategory of the category of operator spaces and completely contractive linear maps. In particular, the group of surjective linear isometries of a Banach space E can be identified with the group of surjective linear complete isometries of $\mathrm{MIN}(E)$. We will identify a Banach space E with the corresponding minimal operator space $\mathrm{MIN}(E)$.

There is a natural class of geometric objects that correspond to operator spaces, generalizing the correspondence between Banach spaces and compact absolutely convex sets. A *compact rectangular matrix convex set* is a sequence $(K_{n,m})$ of compact convex sets endowed with a notion of rectangular convex combination. This is an expression $\alpha_1^* p_1 \beta_1 + \cdots + \alpha_\ell^* p_\ell \beta_\ell$ for $p_i \in K_{n_i,m_i}$, $\alpha_i \in M_{n_i,n}$, and $\beta_i \in M_{m_i,m}$ satisfying $\|\alpha_1^* \alpha_1 + \cdots + \alpha_\ell^* \alpha_\ell\| \leq 1$ and $\|\beta_1^* \beta_1 + \cdots + \beta_\ell^* \beta_\ell\| \leq 1$. The notion of an affine map and extreme points admit natural rectangular matrix analogs, where usual convex combinations are replaced with rectangular matrix convex combinations. When E is an operator space, let $\mathrm{CBall}(E^*)$ be the sequence $(K_{n,m})$, where $K_{n,m} = M_{n,m}(E^*)$. It is proved in [20] that any compact rectangular matrix convex set arises in this way. Furthermore the correspondence $E \mapsto \mathrm{CBall}(E^*)$ is a contravariant equivalence of categories form the category of operator spaces and completely contractive maps to the category of compact rectangular matrix convex sets and continuous rectangular affine maps.

An *operator system* is a closed subspace X of the algebra $B(H)$ for some Hilbert space H that is *unital* and *self-adjoint*, i.e. contains the identity operator and is closed under taking adjoints. In particular, the space $M_n(\mathbb{C})$ has a natural operator system structure, obtained by identifying $M_n(\mathbb{C})$ with $B(\ell_2^n)$. An operator system X inherits from the inclusion $X \subset B(H)$ an involution $x \mapsto x^*$, which corresponds to taking adjoints, and a distinguished element 1 (the *unit*), which corresponds to the identity operator. Furthermore, for every $n \in \mathbb{N}$, $M_n(X)$ has a canonical norm and a notion of *positivity*, obtained by setting $[x_{ij}] \geq 0$ if and only if $[x_{ij}]$ is positive when regarded as an operator on the n-fold Hilbertian sum of H by itself. The self-adjoint part X_{sa} of X is the unital subspace of X containing those $x \in X$ such that $x = x^*$. A linear map $\phi : X \to Y$ between operator systems is *unital* if it maps the unit of X to the unit of Y, *positive* if it maps positive elements to positive elements, and *completely positive* if every amplification $\phi^{(n)}$ is positive. We abbreviate "unital completely positive linear map" as "*ucp map*". It is well known that a unital linear map ϕ is completely positive if and only if it is completely contractive. A unital complete isometry $\phi : X \to Y$ is called a *complete order embedding.* A surjective complete order embedding is a *complete order isomorphism.* One can abstractly characterize the pairs $(X, 1)$ where X is an operator space and $1 \in X$ that are operator systems, in the sense that there exists a complete isometry $\phi : X \to B(H)$ mapping 1 to the identity operator and X onto a closed self-adjoint subspace of $B(H)$ [9]. An earlier abstract characterization of operator systems in terms of the matrix positive cones is due to Choi and Effros [11].

1.1. **Fraïssé classes and limits; topological dynamics of automorphisms and Ramsey properties.** We introduce in this subsection Fraïssé classes and Fraïssé limits in the setting of operator spaces and operator systems. These can all be seen as particular instances of Fraïssé classes and Fraïssé limits of metric structures in the sense of [6, 54]. However, for the sake of concreteness, we will introduce all the notions in this particular case.

Let $\mathtt{Osp}$ be the class of operator spaces. Given $X, Y \in \mathtt{Osp}$ and $\delta \geq 0$, let $\mathrm{Emb}_\delta^{\mathtt{Osp}}(X, Y)$ be the space of δ-embeddings form X into Y, that is, injective complete contractions $\phi : X \to Y$ such that $\|\phi^{-1}\|_{\mathrm{cb}} \leq 1 + \delta$. We call 0-embeddings simply embeddings. Let $\mathrm{Aut}(X)$ be the group of surjective embeddings from X to itself. Given an operator space R, by an *R-operator space* we mean a pair $\boldsymbol{X} = (X, s_X)$, where X is an operator space and $s_X : X \to R$ is a complete contraction. Let $\mathtt{Osp}^R$ be the collection of R-operator spaces. Given $\boldsymbol{X} = (X, s_X)$ and $\boldsymbol{Y} = (Y, s_Y)$ in $\mathtt{Osp}^R$, and $\delta \geq 0$, let $\mathrm{Emb}_\delta^{\mathtt{Osp}^R}(\boldsymbol{X}, \boldsymbol{Y})$ be the space of δ-embeddings $\phi : X \to Y$ such that $\|s_Y \circ \phi - s_X\|_{\mathrm{cb}} \leq \delta$. This is a metric space when we consider the metric $d(\phi, \psi) := \|\phi - \psi\|_{\mathrm{cb}}$. Note that when R is the trivial operator space $\{0\}$, R-operator spaces can be identified with operator spaces.

Similarly as above, we let $\mathtt{Osy}$ be the class of operator systems. Given $X, Y \in \mathtt{Osy}$, and $\delta \geq 0$, let $\mathrm{Emb}_\delta^{\mathtt{Osy}}(X, Y)$ be the collection of all injective *unital* complete contractions $\phi : X \to Y$ such that $\left\|\phi^{-1}\right\|_{\mathrm{cb}} \leq 1+\delta$. For a fixed operator system R, let $\mathtt{Osy}^R$ be the class of *R-operator systems*, that is pairs $\boldsymbol{X} = (X, s_X)$ where X is a operator system and $s_X : X \to R$ is an unital complete contraction. We define $\mathrm{Emb}_\delta^{\mathtt{Osy}^R}(\boldsymbol{X}, \boldsymbol{Y})$ to be the collection of all injective unital complete contractions $\phi : X \to Y$ such that $\left\|\phi^{-1}\right\|_{\mathrm{cb}} \leq 1 + \delta$ and $\|s_X - s_Y \circ \phi\|_{\mathrm{cb}} \leq \delta$, endowed with the metric $d_{\mathrm{cb}}(\phi, \psi) := \|\phi - \psi\|_{\mathrm{cb}}$. We also define $\mathrm{Aut}^{\mathtt{Osy}}(X)$ to be the group of unital surjective complete isometries from X to itself, and $\mathrm{Aut}^{\mathtt{Osy}^R}(\boldsymbol{X})$ to be the group of unital surjective complete isometries ϕ from X to itself such that $s_X \circ \phi = s_X$.

Notice that, for any of the collections $\mathtt{C}$ considered above, one can deduce from the small perturbation lemma in operator space theory [66, Lemma 2.13.2] that $\mathrm{Emb}_\delta^{\mathtt{C}}(X, Y)$ is a compact metric space whenever X, Y are finite-dimensional objects of $\mathtt{C}$. We also have that, when X is separable, $\mathrm{Aut}^{\mathtt{C}}(X)$ is a Polish group when endowed with the topology of pointwise convergence. We write $\mathrm{Emb}^{\mathtt{C}}(X, Y)$ instead of $\mathrm{Emb}_0^{\mathtt{C}}(X, Y)$. The members of $\mathrm{Emb}^{\mathtt{C}}(X, Y)$ are called $\mathtt{C}$-*embeddings* from X into Y. Whenever there is no possible confusion we will use $\mathrm{Emb}_\delta(X, Y)$ and $\mathrm{Aut}(X)$ instead of $\mathrm{Emb}_\delta^{\mathtt{C}}(X, Y)$ and $\mathrm{Aut}^{\mathtt{C}}(X)$, respectively. Also, when $\boldsymbol{X} = (X, s_X)$, $\boldsymbol{Y} = (Y, s_Y)$ are R-operator spaces or R-operator systems, we write $\boldsymbol{X} \subseteq \boldsymbol{Y}$ to denote that $X \subseteq Y$ and $s_Y \upharpoonright_X = s_X$.

In the rest of this section, we assume that $\mathtt{C}$ is one of the classes $\mathtt{Osp}^R$, $\mathtt{Osy}$, or $\mathtt{Osy}^R$, and we refer to the corresponding notion of δ-embedding.

Definition 1.3 (Gromov-Hausdorff metric)**.** The *Gromov-Hausdorff distance* $d_{\mathtt{C}}(X, Y)$ of two finite-dimensional $X, Y \in \mathtt{C}$ is the infimum of all $\delta > 0$ such that there exist $f \in \mathrm{Emb}_\delta^{\mathtt{C}}(X, Y)$ and $g \in \mathrm{Emb}_\delta^{\mathtt{C}}(Y, X)$ such that $\|g \circ f - \mathrm{Id}_X\|_{\mathrm{cb}} < \delta$ and $\|f \circ g - \mathrm{Id}_Y\|_{\mathrm{cb}} < \delta$.

Definition 1.4. Given $\mathcal{A} \subseteq \mathtt{C}$, we write $[\mathcal{A}]$ to denote the class of structures X in $\mathtt{C}$ such that every finite-dimensional $Y \subseteq X$ in $\mathtt{C}$ is a limit, with respect to the Gromov-Hausdorff distance in $\mathtt{C}$, of a sequence of *substructures* of structures in $\mathcal{A}$. Let $\langle \mathcal{A} \rangle$ be the collection of all finite-dimensional elements of $[\mathcal{A}]$.

In the following, we let $\varpi : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing function, continuous at 0, and vanishing at 0.

Definition 1.5 (Stable Fraïssé class)**.** $\mathcal{A} \subseteq \mathtt{C}$ is a *stable Fraïssé class* with stability modulus ϖ when

(1) $\mathcal{A}$ is separable with respect to the Gromov-Hausdorff metric $d_{\mathtt{C}}$;
(2) $\mathcal{A}$ satisfies the *stable amalgamation property (SAP)*: for every $X, Y, Z \in \mathcal{A}$, $\delta \geq 0$, $\varepsilon > 0$, $\phi \in \mathrm{Emb}_\delta^{\mathtt{C}}(X, Y)$, and $\psi \in \mathrm{Emb}_\delta^{\mathtt{C}}(X, Z)$, there exist $V \in \mathcal{A}$, $i \in \mathrm{Emb}^{\mathtt{C}}(Y, V)$, and $j \in \mathrm{Emb}^{\mathtt{C}}(Z, V)$ such that $\|i \circ \phi - j \circ \psi\|_{\mathrm{cb}} \leq \varpi(\delta) + \varepsilon$;
(3) $\mathcal{F}$ satisfies the *joint embedding property*: for every $X, Y \in \mathcal{F}$ there exists $Z \in \mathcal{F}$ such that $\mathrm{Emb}^{\mathrm{C}}(X, Z)$ and $\mathrm{Emb}^{\mathrm{C}}(Y, Z)$ are nonempty.

Definition 1.6 (Stable Homogeneity)**.** Let $\mathcal{A} \subseteq \mathtt{C}$. We say that $M \in \mathtt{C}$ satisfies the $\mathcal{A}$-stable homogeneity property (or it is of almost universal disposition for $\mathcal{A}$) with modulus ϖ if:

(1) $\mathrm{Emb}^{\mathtt{C}}(X, M)$ is nonempty for every $X \in \mathcal{A}$;
(2) for every $X \in \mathcal{A}$, $\delta \geq 0$, $\varepsilon > 0$, and $f, g \in \mathrm{Emb}_\delta^{\mathtt{C}}(X, M)$ there is $\alpha \in \mathrm{Aut}^{\mathtt{C}}(M)$ such that $\|\alpha \circ g - f\|_{\mathrm{cb}} \leq \varpi(\delta) + \varepsilon$.

When $\mathcal{A}$ is the collection of all finite-dimensional $X \subseteq M$ in $\mathtt{C}$, we say that M is *stably homogeneous* with modulus ϖ.

Definition 1.7 (Fraïssé limit)**.** Let $\mathcal{A} \subseteq \mathtt{C}$. The stable *Fraïssé limit* of $\mathcal{A}$ (with modulus ϖ), denoted by $\mathrm{FLim}\,\mathcal{A}$, if it exists, is the unique separable object in $[\mathcal{A}]$ that is $\mathcal{A}$-stably homogeneous (with modulus ϖ).

A usual back-and-forth argument shows the following; see for instance [46, Subsection 2.6].

Proposition 1.8. *Suppose that $\mathcal{A} \subseteq \mathtt{C}$ is a stable Fraïssé class, and $M \in \mathtt{C}$ is separable. Then the Fraïssé limit* $\mathrm{FLim}\,\mathcal{A}$ *exists. Furthermore the following assertions are equivalent:*

(1) $M = \mathrm{FLim}\,\mathcal{A}$
(2) M is stably homogeneous with modulus ϖ;
(3) the class of all finite-dimensional $X \in \mathtt{C}$ such that $X \subseteq M$ is a stable Fraïssé class with modulus ϖ.

Notice that whenever $\mathrm{FLim}\langle \mathcal{A} \rangle$ exists, $\mathrm{FLim}\,\mathcal{A}$ exists and it must be equal to $\mathrm{FLim}\langle \mathcal{A} \rangle$. Stable Fraïssé classes and stable Fraïssé limits are in particular Fraïssé classes and Fraïssé limits as *metric structures* in the sense of [5]. One can realize $\mathrm{FLim}\,\mathcal{A}$ as the limit of an inductive sequence of elements of $\mathcal{A}$ with $\mathtt{C}$-embeddings as connective maps, and it can be proved that every separable structure in $[\mathcal{A}]$ admits a completely isometric $\mathtt{C}$-embedding into $\mathrm{FLim}\,\mathcal{A}$.

Several structures in functional analysis arise as the Fraïssé limit of a suitable class $\mathcal{A}$. For example the class of finite-dimensional operator Hilbert spaces is a Fraïssé class, and its corresponding limit is the separable operator Hilbert space OH introduced and studied in [65]. Another natural example of a family with the stable amalgamation property is the collection of finite-dimensional Banach spaces $\{\ell_p^n\}_{n \geq 0}$ for every $p \in [1, +\infty)$. In the case $p = 2$ one can use the polar decomposition for bounded operators on a Hilbert space to deduce that every δ-embedding between Hilbert spaces is close to an embedding. The other cases are treated in the work in preparation [19]. In this case one uses a result by Schechtman in [68] stating that for every such $p \neq 2$ there exists a function $\varpi_p : \mathbb{R}_+ \to \mathbb{R}_+$ continuous at 0 and vanishing at 0, with the property that if $\phi : \ell_p^k \to \ell_p^m$ is a δ-embedding for some $\delta > 0$, then there exist $n \in \mathbb{N}$, $I \in \mathrm{Emb}(\ell_p^m, \ell_p^n)$, and $J \in \mathrm{Emb}(\ell_p^k, \ell_p^n)$ such that $\|J - I \circ \phi\| \leq \varpi_p(\delta)$. The corresponding Fraïssé limit $\mathrm{FLim}\{\ell_p^n\}_n$ of $\{\ell_p^n\}_n$ is the Lebesgue space $L_p[0, 1]$. When p is an even integer other than 2, the space $L_p[0, 1]$ is not stably homogeneous or, equivalently, the class $\langle \{\ell_p^n\}_n \rangle$ does not have the stable amalgamation property. In fact, in this case, $L_p[0, 1]$ is not even *approximately ultrahomogeneous* as shown in [50]; see also [19]. An operator space M is approximately ultrahomogeneous when for every finite-dimensional $X \subseteq M$, every complete isometry $\phi : X \to M$, and every $\varepsilon > 0$ there is a surjective linear complete isometry $\alpha : M \to M$ such that $\|\alpha \restriction_X - \phi\|_{\mathrm{cb}} \leq \varepsilon$. Obviously, stably homogeneous spaces are approximately ultrahomogeneous. Lusky proved in [50] that the space $L_p[0, 1]$ is approximately ultrahomogeneous when $p \in [1, +\infty)$ is not an even integer. This fact will be used later in Subsection 6. The case $p = \infty$ requires different methods. Ramsey properties of the following classes, proved to be Fraïssé in [46], are the main subject of the present paper.

Theorem 1.9. *Let* $\mathtt{C}$ *be either* $\mathtt{Osp}$ *or* $\mathtt{Osy}$*. Suppose that* $\mathcal{I}$ *is a countable class of finite-dimensional injective elements of* $\mathtt{C}$ *which is closed under* ∞*-sums. Then* $\mathcal{I}$ *and* $\langle\mathcal{I}\rangle$ *are stable Fraïssé classes, with stability modulus* $\varpi(\delta)=\delta$ *if* $\mathtt{C}$ *is* $\mathtt{Osp}$*, and* $\varpi(\delta)=2\delta$ *if* $\mathtt{C}$ *is* $\mathtt{Osy}$.

Recall that an operator space X is *injective* if for every operator spaces $Y\subseteq Z$, every complete contraction from Y to X can be extended to a complete contraction from Z to X. One defines injective operator systems similarly, by replacing complete contractions with unital complete contractions.

Definition 1.10 (Approximate and stable Ramsey Property)**.** Let $\mathcal{A}\subseteq\mathtt{C}$.

- $\mathcal{A}$ satisfies the *approximate Ramsey property (ARP)* if for any $X,Y\in\mathcal{A}$, $\varepsilon>0$ there exists $Z\in\mathcal{A}$ such that any continuous coloring of $\mathrm{Emb}^{\mathtt{C}}(X,Z)$ ε-stabilizes on $\gamma\circ\mathrm{Emb}^{\mathtt{C}}(X,Y)$ for some $\gamma\in\mathrm{Emb}^{\mathtt{C}}(Y,Z)$;
- $\mathcal{A}$ satisfies the *stable Ramsey property (SRP)* with stability modulus ϖ if for any $X,Y\in\mathcal{A}$, $\varepsilon>0$, $\delta\geq 0$ there exists $Z\in\mathcal{A}$ such that every continuous coloring of $\mathrm{Emb}_{\delta}^{\mathtt{C}}(X,Z)$ $(\varepsilon+\varpi(\delta))$-stabilizes on $\gamma\circ\mathrm{Emb}_{\delta}^{\mathtt{C}}(X,Y)$ for some $\gamma\in\mathrm{Emb}^{\mathtt{C}}(Y,Z)$.

The *discrete* (ARP) and the *discrete* (SRP) are defined as the (ARP) and the (SRP), respectively, by replacing continuous colorings with finite colorings.

The *compact* (ARP) and the *compact* (SRP) are defined as the (ARP) and the (SRP), respectively, by replacing continuous colorings with compact colorings.

It is not difficult to see that the (ARP) as in Definition 1.10 is equivalent to the one considered in [54, Definition 3.3] when operator spaces are regarded as structures in the logic for metric structures [6] as in [26, Appendix B] or [46, §8.1]. The following proposition provides reformulations of the (ARP) in terms of discrete or compact colorings.

Proposition 1.11. *The following are equivalent for a class* $\mathcal{A}\subseteq\mathtt{C}$*:*

1. *$\mathcal{A}$ satisfies the (ARP);*
2. *$\mathcal{A}$ satisfies the* discrete *(ARP);*
3. *$\mathcal{A}$ satisfies the* compact *(ARP).*

Proof. The compact (ARP) obviously implies the (ARP).

Suppose that $\mathcal{A}$ satisfies the (ARP), and let us prove that $\mathcal{A}$ satisfies the discrete (ARP). This is done by induction on $r\in\mathbb{N}$. The case $r=1$ is trivial. Suppose that we have shown that $\mathcal{A}$ satisfies the discrete (ARP) for r-colorings. Consider $X,Y\in\mathcal{A}$ and $\varepsilon>0$. Then by the inductive hypothesis, there is $Z_0\in\mathcal{A}$ such that every r-coloring of $\mathrm{Emb}^{\mathtt{C}}(X,Z_0)$ ε-stabilizes on $\gamma\circ\mathrm{Emb}^{\mathtt{C}}(X,Y)$ for some $\gamma\in\mathrm{Emb}^{\mathtt{C}}(Y,Z_0)$. Since by the assumption $\mathcal{A}$ satisfies the (ARP), there is $Z\in\mathcal{A}$ such that every continuous coloring of $\mathrm{Emb}^{\mathtt{C}}(X,Z)$ $\varepsilon/2$-stabilizes on $\gamma\circ\mathrm{Emb}^{\mathtt{C}}(X,Z_0)$ for some $\gamma\in\mathrm{Emb}^{\mathtt{C}}(Z_0,Z)$. We claim that Z witnesses that $\mathcal{A}$ satisfies the discrete (ARP) for $(r+1)$-colorings. Indeed, suppose that c is an $(r+1)$-coloring of $\mathrm{Emb}^{\mathtt{C}}(X,Z)$. Define $f:\mathrm{Emb}^{\mathtt{C}}(X,Z)\to[0,1]$ by $f(\phi):=\frac{1}{2}d_{\mathrm{cb}}\left(\phi,c^{-1}(r)\right)$. This is a continuous coloring, so by the choice of Z there exists $\gamma\in\mathrm{Emb}^{\mathtt{C}}(Z_0,Z)$ such that f $\varepsilon/2$-stabilizes on $\gamma\circ\mathrm{Emb}^{\mathtt{C}}(Z_0,Z)$. Now, if there is some $\phi\in\mathrm{Emb}^{\mathtt{C}}(X,Z_0)$ such that $c(\gamma\circ\phi)=r$, then $\gamma\circ\mathrm{Emb}^{\mathtt{C}}(X,Z_0)\subseteq(c^{-1}(r))_{\varepsilon}$, so choosing an arbitrary $\bar{\gamma}\in\mathrm{Emb}^{\mathtt{C}}(Y,Z_0)$ we obtain that c ε-stabilizes on $\gamma\circ\bar{\gamma}\circ\mathrm{Emb}^{\mathtt{C}}(X,Y)$. Otherwise, $(\gamma\circ\mathrm{Emb}^{\mathtt{C}}(X,Z_0))\cap c^{-1}(r)=\emptyset$, so defining $\bar{c}(\phi):=c(\gamma\circ\phi)$ for $\phi\in\mathrm{Emb}^{\mathtt{C}}(X,Z_0)$ gives an r-coloring of $\mathrm{Emb}^{\mathtt{C}}(X,Z_0)$. By the choice of Z_0 there exists $\bar{\gamma}\in\mathrm{Emb}^{\mathtt{C}}(Y,Z_0)$ such that $\bar{c}$ ε-stabilizes on $\bar{\gamma}\circ\mathrm{Emb}^{\mathtt{C}}(X,Y)$. Therefore c ε-stabilizes on $\gamma\circ\bar{\gamma}\circ\mathrm{Emb}^{\mathtt{C}}(X,Y)$. This concludes the proof that the (ARP) implies the discrete (ARP).

Finally, the discrete (ARP) implies the compact (ARP). Indeed, given $\varepsilon>0$ and a compact metric space K, one can find a finite ε-dense subset $D\subseteq K$. Thus if $Z\in\mathcal{A}$ witnesses the discrete (ARP) for X,Y, ε and D, then given a 1-Lipschitz $f:\mathrm{Emb}^{\mathtt{C}}\to K$ we can define a coloring $c:\mathrm{Emb}^{\mathtt{C}}(X,Z)\to D\subseteq K$ such that $d_K(c(\phi),f(\phi))\leq\varepsilon$ for every $\phi\in\mathrm{Emb}^{\mathtt{C}}(X,Z)$. In this way, if c ε-stabilizes on $\gamma\circ\mathrm{Emb}^{\mathtt{C}}(X,Y)$, then f 3ε-stabilizes on $\gamma\circ\mathrm{Emb}^{\mathtt{C}}(X,Y)$. ☐

The following is a useful property of classes with the (SAP). It can be easily proved by induction, using the fact that the spaces $\mathrm{Emb}_{\delta}(X,Y)$ for finite-dimensional $X,Y\in\mathcal{A}$ are compact.

Proposition 1.12. *Suppose that $\mathcal{A}$ satisfies the stable amalgamation property with modulus ϖ, $X_1,\dots,X_{n+1}\in\mathcal{A}$ and $\varepsilon,\delta>0$. Then:*

1. *there are $Z\in\mathcal{A}$ and $J\in\mathrm{Emb}(X_{n+1},Z)$ such that $J\circ\mathrm{Emb}_{\delta}(X_i,X_{n+1})\subseteq(\mathrm{Emb}(X_i,Z))_{\varepsilon+\varpi(\delta)}$ for every $1\leq i\leq n$;*
2. *there are $Z\in\mathcal{A}$ and $I\in\mathrm{Emb}(X_2,Z)$ such that for every $\phi,\psi\in\mathrm{Emb}_{\delta}(X_1,X_2)$ there is $J\in\mathrm{Emb}(X_2,Z)$ such that $\|I\circ\phi-J\circ\psi\|_{\mathrm{cb}}\leq\varepsilon+\varpi(\delta)$.*

Proposition 1.13. *Suppose that $\mathcal{A} \subseteq \mathtt{C}$ satisfies the stable amalgamation property with modulus ϖ. Then the following assertions are equivalent:*

(1) $\mathcal{A}$ satisfies the (ARP);
(2) $\mathcal{A}$ satisfies the (SRP) with modulus ϖ:
(3) $\mathcal{A}$ satisfies the discrete (SRP) with modulus ϖ;
(4) $\mathcal{A}$ satisfies the compact (SRP) with modulus ϖ.

Proof. A simple modification of the proof of the Proposition 1.11 gives that the compact (SRP) with modulus ϖ implies the (SRP) with modulus ϖ, which in turn implies the discrete (SRP) with modulus ϖ. Trivially, the discrete (SRP) with modulus ϖ implies the discrete (ARP), and this one is equivalent to the (ARP) by Proposition 1.11. We will now show that the (ARP) implies the compact (SRP) with modulus ϖ. Suppose that $\mathcal{A}$ satisfies the (ARP). Fix $X, Y \in \mathcal{A}$, $\delta, \varepsilon > 0$ and a compact metric space K. We use Proposition 1.12 *(2)* to find $Y_0 \in \mathcal{A}$ such that for every $\phi, \psi \in \mathrm{Emb}^{\mathtt{C}}_\delta(X, Y)$ there are $i, j \in \mathrm{Emb}^{\mathtt{C}}(Y, Y_0)$ such that $\|i \circ \phi - j \circ \psi\|_{\mathrm{cb}} \le \varepsilon + \varpi(\delta)$. Here we regard the space $\mathrm{Lip}(\mathrm{Emb}^{\mathtt{C}}_\delta(X, Y), K)$ of 1-Lipschitz maps from $\mathrm{Emb}^{\mathtt{C}}_\delta(X, Y)$ to K as a compact metric space, endowed with the metric $d(f, g) = \sup\left\{d_K(f(\phi), g(\phi)) : \phi \in \mathrm{Emb}^{\mathtt{C}}_\delta(X, Y)\right\}$. By Proposition 1.11, $\mathcal{A}$ satisfies the compact (ARP). Thus there exists some $Z \in \mathcal{A}$ such that every $\mathrm{Lip}(\mathrm{Emb}^{\mathtt{C}}_\delta(X, Y), K)$-coloring of $\mathrm{Emb}^{\mathtt{C}}(Y, Z)$ ε-stabilizes on $\gamma \circ \mathrm{Emb}^{\mathtt{C}}(Y, Y_0)$ for some $\gamma \in \mathrm{Emb}^{\mathtt{C}}(Y_0, Z)$. We claim that Z works, so let $c : \mathrm{Emb}^{\mathtt{C}}_\delta(X, Z) \to K$ be 1-Lipschitz. We can then define a $\mathrm{Lip}(\mathrm{Emb}^{\mathtt{C}}_\delta(X, Y), K)$-coloring $\widehat{c}$ of $\mathrm{Emb}^{\mathtt{C}}(Y, Z)$ by setting, for $\gamma \in \mathrm{Emb}^{\mathtt{C}}(Y, Z)$, $\widehat{c}(\gamma) : \mathrm{Emb}^{\mathtt{C}}_\delta(X, Y) \to K$, $\phi \mapsto c(\gamma \circ \phi)$. By the choice of Z, there exists $\gamma \in \mathrm{Emb}^{\mathtt{C}}(Y_0, Z)$ be such that $\widehat{c}$ ε-stabilizes on $\gamma \circ \mathrm{Emb}^{\mathtt{C}}(Y, Y_0)$. Choose an arbitrary $\varrho \in \mathrm{Emb}^{\mathtt{C}}(Y, Y_0)$. We claim that c $(\varepsilon + \varpi(\delta))$-stabilizes on $\gamma \circ \varrho \circ \mathrm{Emb}^{\mathtt{C}}_\delta(X, Y)$. Let $\phi, \psi \in \mathrm{Emb}^{\mathtt{C}}_\delta(X, Y)$. By the choice of Y_0 there are $i, j \in \mathrm{Emb}^{\mathtt{C}}(Y, Y_0)$ such that $d_{\mathrm{cb}}(i \circ \phi, j \circ \psi) \le \varepsilon + \varpi(\delta)$. Since $d(\widehat{c}(\gamma \circ \varrho), \widehat{c}(\gamma \circ i)), d(\widehat{c}(\gamma \circ \varrho), \widehat{c}(\gamma \circ j)) \le \varepsilon$, it follows that $d_K(c(\gamma \circ \varrho \circ \phi), c(\gamma \circ i \circ \phi)), d_K(c(\gamma \circ \varrho \circ \psi), c(\gamma \circ j \circ \psi)) \le \varepsilon$. This implies that $d_K(c(\gamma \circ \varrho \circ \phi), c(\gamma \circ \varrho \circ \psi)) \le 3\varepsilon + \varpi(\delta)$. □

It is unclear whether the characterization of the (ARP) provided in 1.13 holds for an arbitrary class $\mathcal{A} \subseteq \mathtt{C}$. Let now, as before, $\mathtt{C}$ be one of the classes $\mathtt{Osp}^R$, $\mathtt{Osy}$, and $\mathtt{Osy}^R$, with corresponding notion of $\mathtt{C}$-embedding. We suppose that $\mathcal{A} \subseteq \mathtt{C}$ is such that $\langle \mathcal{A} \rangle$ is a stable Fraïssé class with modulus ϖ, whose Fraïssé limit is E. For a fixed $X \subseteq E$, we let d_X be the pseudometric on $\mathrm{Aut}^{\mathtt{C}}(E)$ defined by $d_X(\alpha, \beta) := \|(\alpha - \beta) \restriction_X\|_{\mathrm{cb}}$. Observe that, by the stable homogeneity property of E, the restriction map $\alpha \mapsto \alpha \restriction_X$ is an isometry from $(\mathrm{Aut}^{\mathtt{C}}(E), d_X)$ onto a dense subset of $\mathrm{Emb}^{\mathtt{C}}(X, E)$. In particular a continuous coloring of $(\mathrm{Aut}^{\mathtt{C}}(E), d_X)$ induces a continuous coloring of $\mathrm{Emb}^{\mathtt{C}}(X, E)$. The next characterization of extreme amenability is an immediate consequence of this, Proposition 1.1, and [63, Theorem 1.1.18]; see also [54, Proposition 3.9].

Lemma 1.14. *The following statements are equivalent:*

(1) $\mathrm{Aut}^{\mathtt{C}}(E)$ *is extremely amenable;*
(2) for every $X \subseteq E$ that belongs to $\mathcal{A}$ the left translation action of $\mathrm{Aut}^{\mathtt{C}}(E)$ *on* $(\mathrm{Aut}^{\mathtt{C}}(E), d_X)$ *is finitely oscillation stable;*
(3) the same as (2) where $X \subseteq E$ is arbitrary finite-dimensional structure in $\mathtt{C}$;
(4) for any r-coloring of $\mathrm{Aut}^{\mathtt{C}}(E)$*, finite subset F of* $\mathrm{Aut}^{\mathtt{C}}(E)$*, finite-dimensional $X \subseteq E$, and $\varepsilon > 0$, there exists $\alpha \in \mathrm{Aut}^{\mathtt{C}}(E)$ and $i \in \{0, \dots, r-1\}$ such that for any $\gamma \in F$ there exists $\beta \in \mathrm{Aut}^{\mathtt{C}}(E)$ such that $c(\beta) = i$ and $\|(\alpha \circ \gamma - \beta) \restriction_X\|_{\mathrm{cb}} \le \varepsilon$;*
(5) same as (4) where c is a 2-coloring and $X \in \mathcal{A}$.

We refer the reader to [63, Chapter 1] for more information on oscillation stability.

The following proposition can be seen as the analogue in this context of the celebrated Kechris-Pestov-Todorcevic correspondence from [37]. For a general version of such a correspondence in the setting of metric structures, the reader is referred to [54].

Proposition 1.15 (Correspondence between (ARP), (SRP), and extreme amenability)**.** *Suppose that $\mathcal{A} \subseteq \mathtt{C}$ is such that $\mathcal{A}$ and $\langle \mathcal{A} \rangle$ satisfy the stable amalgamation property with stability modulus ϖ, and let E be its corresponding Fraïssé limit. The following assertions are equivalent:*

(1) $\mathrm{Aut}^{\mathtt{C}}(E)$ *is extremely amenable;*
(2) for every finite-dimensional $X, Y \subset E$ in $\mathtt{C}$, $\delta \ge 0$, $\varepsilon > 0$, every compact coloring of $\mathrm{Emb}^{\mathtt{C}}_\delta(X, E)$ *has an $(\varepsilon + \varpi(\delta))$-monochromatic set of the form $\alpha \circ \mathrm{Emb}^{\mathtt{C}}_\delta(X, Y)$ for some $\alpha \in \mathrm{Aut}^{\mathtt{C}}(E)$;*
(3) for every finite-dimensional $X, Y \subset E$ in $\mathtt{C}$, $\varepsilon > 0$, every finite coloring of $\mathrm{Emb}^{\mathtt{C}}(X, E)$ *has an ε-monochromatic set of the form $\alpha \circ \mathrm{Emb}^{\mathtt{C}}(X, Y)$ for some $\alpha \in \mathrm{Aut}^{\mathtt{C}}(E)$;*

(4) *for every $X, Y \subset E$ that belong to $\mathcal{A}$, $\varepsilon > 0$, every finite coloring of* $\mathrm{Emb}^{\mathtt{C}}(X, E)$ *has an ε-monochromatic set of the form $\alpha \circ \mathrm{Emb}^{\mathtt{C}}(X, Y)$ for some $\alpha \in \mathrm{Aut}^{\mathtt{C}}(E)$;*
(5) *$\langle \mathcal{A} \rangle$ satisfies the (ARP);*
(6) *$\mathcal{A}$ satisfies the (ARP);*
(7) *For every $X, Y \in \mathcal{A}$, every $r \in \mathbb{N}$ and every $\varepsilon > 0$ there is $Z \in \langle \mathcal{A} \rangle$ such that every r-coloring of* $\mathrm{Emb}^{\mathtt{C}}(X, Z)$ *has an ε-monochromatic set of the form $\gamma \circ \mathrm{Emb}^{\mathtt{C}}(X, Y)$ for some $\gamma \in \mathrm{Emb}^{\mathtt{C}}(Y, Z)$.*

Proof. The proof uses standard arguments; see for example [54, Proposition 3.4 and Theorem 3.10]. We see first show that properties *(1)–(4)* are equivalent, and then we show that properties (4)–(7) are equivalent.
(1) implies *(2)*: Fix all data, in particular, let $c : \mathrm{Emb}^{\mathtt{C}}_\delta(X, E) \to K$ be a compact coloring. We use Proposition 1.1 *(3)* to find a compact coloring $\widehat{c} : \mathrm{Emb}^{\mathtt{C}}_\delta(X, E) /\!/ \mathrm{Aut}^{\mathtt{C}}(E) \to K$ and a $\mathtt{C}$-automorphism α of E such that $d_K(b(\alpha \circ \phi), \widehat{b}([\phi]_{\mathrm{Aut}^{\mathtt{C}}(E)})) < \varepsilon/2$ for every $\phi \in \mathrm{Emb}^{\mathtt{C}}_\delta(X, Y)$. Since E has the stable homogeneity property with modulus ϖ, $\mathrm{Emb}^{\mathtt{C}}_\delta(X, E) /\!/ \mathrm{Aut}^{\mathtt{C}}(E)$ has diameter $\leq \varpi(\delta)$. Thus c $(\varepsilon + \varpi(\delta))$-stabilizes on $\mathrm{Emb}^{\mathtt{C}}_\delta(X, Y)$.
(2) implies *(3)*: Given a finite coloring $c : \mathrm{Emb}^{\mathtt{C}}_\delta(X, E) \to r$, let K be the ball of ℓ_∞^k centered at 0 and of radius $2(1+\delta)$, and let $f : \mathrm{Emb}^{\mathtt{C}}_\delta(X, E) \to K$ by defined by $f(\sigma) := (d_{\mathrm{cb}}(\sigma, c^{-1}(i)))_{i<r}$. This is a compact coloring, so by hypothesis, there is $\alpha \in \mathrm{Aut}^{\mathtt{C}}(E)$ such that the oscillation of f on $\alpha \circ \mathrm{Emb}^{\mathtt{C}}_\delta(X, Y)$ is at most $\varepsilon + \varpi(\delta)$. Then $\alpha \circ \mathrm{Emb}^{\mathtt{C}}_\delta(X, Y)$ is $(\varepsilon + \varpi(\delta))$-monochromatic for c. Indeed, fix $\phi \in \mathrm{Emb}^{\mathtt{C}}_\delta(X, Y)$, and let $i := c(\alpha \circ \phi)$. Then $(f(\alpha \circ \phi))_i = 0$, so for every $\bar{\phi} \in \mathrm{Emb}^{\mathtt{C}}_\delta(X, Y)$ one has that $d_{\mathrm{cb}}(\alpha \circ \bar{\phi}, c^{-1}(i)) \leq \varepsilon + \varpi(\delta)$.
(3) implies *(4)* trivially.
(4) implies *(1)*: We verify condition *(5)* from Lemma 1.14. Fix $X \subseteq E$ that belongs to $\mathcal{A}$, a finite subset F of $\mathrm{Aut}^{\mathtt{C}}(E)$ and $\varepsilon > 0$. Let $Y_0 \subseteq E$ be a structure in $\mathcal{A}$ such that for every $\beta \in F$ there is $\sigma_\beta \in \mathrm{Emb}^{\mathtt{C}}_\varepsilon(X, Y_0)$ such that $\|\beta - \sigma_\beta\|_{\mathrm{cb}} \leq \varepsilon$. Since $\mathcal{A}$ satisfies the stable amalgamation property with modulus ϖ, it follows from Proposition 1.12 *(1)* that there is some $Y_1 \in \mathcal{A}$ and $\gamma_0 \in \mathrm{Emb}^{\mathtt{C}}(Y_0, Y_1)$ such that $\gamma_0 \circ \mathrm{Emb}^{\mathtt{C}}_\varepsilon(X, Y_0) \subseteq (\mathrm{Emb}^{\mathtt{C}}(X, Y_1))_{2\varepsilon}$. Let $Y \subseteq E$ be completely isometric to Y_1, $\theta : Y_1 \to Y$ be a complete isometry. Finally, let $\alpha_0 \in \mathrm{Aut}^{\mathtt{C}}(E)$ be such that $\|\theta \circ \gamma_0 - \alpha_0 \restriction_{Y_0}\|_{\mathrm{cb}} \leq \varepsilon$. Given a 2-coloring c of $\mathrm{Aut}^{\mathtt{C}}(E)$, we can define $\widehat{c} : \mathrm{Emb}^{\mathtt{C}}(X, E) \to \{0, 1\}$ by declaring $\widehat{c}(\gamma) = c(\beta_\gamma)$ where $\beta_\gamma \in \mathrm{Aut}^{\mathtt{C}}(E)$ is chosen such that $\|\gamma - \beta_\gamma \restriction_X\|_{\mathrm{cb}} \leq \varepsilon$. By hypothesis there is $\alpha_1 \in \mathrm{Aut}^{\mathtt{C}}(E)$ and $i \in \{0, 1\}$ such that for every $\phi \in \mathrm{Emb}^{\mathtt{C}}(X, Y)$ there is some $\beta_\phi \in \mathrm{Aut}^{\mathtt{C}}(E)$ such that $c(\beta) = i$ and $\|\alpha_1 \circ \phi - \beta_\phi \restriction_X\|_{\mathrm{cb}} \leq 2\varepsilon$. We claim that $\alpha_1 \circ \alpha_0$ and i work. Let $\varrho \in F$. Let $\psi \in \mathrm{Emb}^{\mathtt{C}}(X, Y_1)$ be such that $\|\gamma_0 \circ \varrho \restriction_X - \psi\|_{\mathrm{cb}} \leq 2\varepsilon$. Set $\phi := \theta \circ \psi \in \mathrm{Emb}^{\mathtt{C}}(X, Y)$. Then $c(\beta_\phi) = i$ and

$$
\begin{aligned}
\|\alpha_1 \circ \alpha_0 \circ \varrho \restriction_X - \beta_\phi \restriction_X\|_{\mathrm{cb}} \leq & \|\alpha_1 \circ \alpha_0 \circ \varrho \restriction_X - \alpha_1 \circ \phi\|_{\mathrm{cb}} + \|\alpha_1 \circ \phi - \beta_\phi \restriction_X\|_{\mathrm{cb}} \leq \|\alpha_0 \circ \varrho \restriction_X - \theta \circ \psi\|_{\mathrm{cb}} + 2\varepsilon \\
\leq & \|\alpha_0 \circ \varrho \restriction_X - \theta \circ \gamma_0 \circ \varrho \restriction_X\|_{\mathrm{cb}} + \|\theta \circ \gamma_0 \circ \varrho \restriction_X - \theta \circ \psi\|_{\mathrm{cb}} + 2\varepsilon \leq \\
\leq & \|\gamma_0 \circ \varrho \restriction_X - \psi\|_{\mathrm{cb}} + 3\varepsilon \leq 5\varepsilon.
\end{aligned}
$$

(2) implies *(5)*: Suppose by contradiction that $\langle \mathcal{A} \rangle$ does not satisfies the (ARP). Then by Proposition 1.13 and Proposition 1.11, $\langle \mathcal{A} \rangle$ does not satisfies the (SRP) with modulus ϖ. Therefore there exist $X \subset Y \subset E$ in $\langle \mathcal{A} \rangle$, $\delta \geq 0$, and $\varepsilon_0 > 0$ witnessing that $\langle \mathcal{A} \rangle$ does not satisfies the (SRP) with modulus ϖ. Let $D_X \subseteq \mathrm{Emb}^{\mathtt{C}}_\delta(X, E)$ and $D_Y \subseteq \mathrm{Emb}^{\mathtt{C}}(Y, E)$ be countable dense subsets, and let $(Z_n)_n$ be an increasing sequence of finite-dimensional, $Z_n \subset E$, such that $Y \subseteq Z_0$ and such that for every $\phi \in D_X$ and $\psi \in D_Y$ there is n such that $\mathrm{Im}(\phi) \cup \mathrm{Im}(\psi) \subseteq Z_n$. This implies that $\bigcup_n \mathrm{Emb}^{\mathtt{C}}_\delta(X, Z_n)$ and $\bigcup_n \mathrm{Emb}^{\mathtt{C}}(Y, Z_n)$ are dense in $\mathrm{Emb}^{\mathtt{C}}_\delta(X, E)$ and $\mathrm{Emb}^{\mathtt{C}}(Y, E)$, respectively. Choose for each n a "bad" continuous coloring $c_n : (\mathrm{Emb}^{\mathtt{C}}_\delta(X, Z_n), d_{\mathrm{cb}}) \to [0, 1]$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. We define $c : \mathrm{Emb}^{\mathtt{C}}_\delta(X, E) \to [0, 1]$ by choosing for a given $\phi \in \mathrm{Emb}^{\mathtt{C}}_\delta(X, E)$ a sequence $(\phi_n)_n$, each $\phi_n \in \mathrm{Emb}^{\mathtt{C}}_\delta(X, Z_n)$, such that $\lim_n \|\phi - \phi_n\|_{\mathrm{cb}} = 0$, and then by declaring $c(\phi) := \mathcal{U} - \lim c_n(\phi_n)$. We claim that c is well defined. Indeed, if $(\psi_n)_n$ is another sequence such that $\lim_n \|\psi_n - \phi\|_{\mathrm{cb}} = 0$, then $\lim_n \|\psi_n - \phi_n\|_{\mathrm{cb}} = 0$. Using the fact that each c_n is 1-Lipschitz, this implies that $\lim_n |c_n(\psi_n) - c_n(\phi_n)| = 0$. Since $\mathcal{U}$ is nonprincipal, we conclude that $\mathcal{U} - \lim c_n(\phi_n) = \mathcal{U} - \lim c_n(\psi_n)$. Let $\varepsilon > 0$ be arbitrary. Clearly c is 1-Lipschitz, so by assumption there exists $\alpha \in \mathrm{Aut}^{\mathtt{C}}(E)$ such that c $(\varepsilon + \varpi(\delta))$-stabilizes on $\alpha \circ \mathrm{Emb}^{\mathtt{C}}_\delta(X, Y))$. Let $A \subseteq \mathrm{Emb}^{\mathtt{C}}_\delta(X, Y)$ be a finite ε-dense subset. Choose $n \in \mathbb{N}$ such that:

- for every $\phi \in A$ there is $\bar{\phi} \in \mathrm{Emb}^{\mathtt{C}}_\delta(X, Z_n)$ such that $\|\alpha \circ \phi - \bar{\phi}\|_{cb} \leq \varepsilon$ and such that for every $\phi, \psi \in A$ one has that $|c_n(\bar{\phi}) - c_n(\bar{\psi})| \leq \varepsilon + \varpi(\delta)$;
- there is $\gamma \in \mathrm{Emb}^{\mathtt{C}}(Y, Z_n)$ such that $\|\alpha \restriction_Y - \gamma\|_{\mathrm{cb}} \leq \varepsilon$.

It follows that $\|\bar{\phi} - \gamma \circ \phi\|_{\mathrm{cb}} \leq 2\varepsilon$ for every $\phi \in A$. Hence, $|c_n(\gamma \circ \phi) - c_n(\gamma \circ \psi)| \leq 5\varepsilon + \varpi(\delta)$ for every $\phi, \psi \in A$. Consequently, c_n has oscillation at most $7\varepsilon + \varpi(\delta)$ on $\gamma \circ \mathrm{Emb}^{\mathtt{C}}_\delta(X, Y)$. Since ε is an arbitrary positive real number, this contradicts the assumption that c_n is a bad coloring.
(5) implies *(6)*: By Proposition 1.11, it suffices to show that $\mathcal{A}$ satisfies the discrete (ARP). We know by hypothesis that $\langle \mathcal{A} \rangle$ satisfies the discrete (ARP). Now, fix X, Y in $\mathcal{A}$, $r \in \mathbb{N}$, and $\varepsilon > 0$. Fix a structure $Z_0 \in \langle \mathcal{A} \rangle$

containing isometric copies of Y and such that every r-coloring of $\mathrm{Emb}^{\mathrm{C}}(X, Z_0)$ has an ε-monochromatic subset of the form $\gamma \circ \mathrm{Emb}^{\mathrm{C}}(X, Y)$ for some $\gamma \in \mathrm{Emb}^{\mathrm{C}}(Y, Z_0)$. Let $Z_1 \in \mathcal{A}$ be such that there is $Z_2 \subseteq Z_1$ and a contraction $\theta : Z_0 \to Z_2$ with $\|\theta^{-1}\|_{\mathrm{cb}} \leq 1 + \varepsilon$. By the (SAP) of $\mathcal{A}$ we can find $Z \in \mathcal{A}$ and $I \in \mathrm{Emb}(Z_1, Z)$ such that for every $\phi \in \mathrm{Emb}^{\mathrm{C}}_{\varepsilon}(X, Z_1)$ there is $\bar{\phi} \in \mathrm{Emb}^{\mathrm{C}}(X, Z)$ such that $\|I \circ \phi - \bar{\phi}\|_{\mathrm{cb}} \leq \varepsilon$ and similarly for the elements of $\mathrm{Emb}^{\mathrm{C}}_{\varepsilon}(Y, Z_1)$. We claim that Z works. Fix a coloring $c : \mathrm{Emb}^{\mathrm{C}}(X, Z) \to r$. Define $b : \mathrm{Emb}^{\mathrm{C}}(X, Z_0) \to r$, by choosing for each $\phi \in \mathrm{Emb}^{\mathrm{C}}(X, Z_0)$ an element $\bar{\phi} \in \mathrm{Emb}^{\mathrm{C}}(X, Z)$ such that $\|I \circ \theta \circ \phi - \bar{\phi}\|_{\mathrm{cb}} \leq \varepsilon$ and declaring $b(\phi) := c(\bar{\phi})$. Let $\alpha \in \mathrm{Emb}^{\mathrm{C}}(Y, Z)$ and $i < r$ be such that $\alpha \circ \mathrm{Emb}^{\mathrm{C}}(X, Y) \subseteq (b^{-1}(i))_{\varepsilon}$. Let $\alpha \in \mathrm{Emb}^{\mathrm{C}}(Y, Z)$ be such that $\|I \circ \theta \circ \alpha - \bar{\alpha}\|_{\mathrm{cb}} \leq \varepsilon$. We claim that $\bar{\alpha} \circ \mathrm{Emb}^{\mathrm{C}}(X, Y) \subseteq (c^{-1}(i))_{\varepsilon}$: Fix $\phi \in \mathrm{Emb}^{\mathrm{C}}(X, Y)$. Let $\sigma \in \mathrm{Emb}^{\mathrm{C}}(X, Z_0)$ be such that $b(\sigma) = i$ and $\|\alpha \circ \phi - \sigma\|_{\mathrm{cb}} \leq \varepsilon$. By definition, we can find $\bar{\sigma} \in \mathrm{Emb}^{\mathrm{C}}(X, Z)$ such that $c(\bar{\sigma}) = i$ and such that $\|I \circ \theta \circ \sigma - \bar{\sigma}\|_{\mathrm{cb}} \leq \varepsilon$. Then,

$$\|\bar{\alpha} \circ \phi - \bar{\sigma}\|_{\mathrm{cb}} \leq \|\bar{\alpha} \circ \phi - I \circ \theta \circ \alpha \circ \phi\|_{\mathrm{cb}} + \|I \circ \theta \circ \alpha \circ \phi - I \circ \theta \circ \sigma\|_{\mathrm{cb}} + \|I \circ \theta \circ \sigma - \bar{\sigma}\|_{\mathrm{cb}} \leq 3\varepsilon.$$

(6) implies *(7)*: This is a consequence of Proposition 1.11.
(7) implies *(4)*: Fix $X \subset Y \subset E$ that belong to $\mathcal{A}$, $\varepsilon > 0$, and $r \in \mathbb{N}$. Let $Z \in \langle \mathcal{A} \rangle$ be witnessing that *(7)* holds for the given data. Since E is universal for $\langle \mathcal{A} \rangle$, we may assume that $Z \subset E$. Given an r-coloring c of $\mathrm{Emb}^{\mathrm{C}}(X, E)$, we can take its restriction to $\mathrm{Emb}^{\mathrm{C}}(X, Z)$, and then find $\gamma \in \mathrm{Emb}^{\mathrm{C}}(Y, Z)$ such that $\gamma \circ \mathrm{Emb}^{\mathrm{C}}(X, Y)$ is ε-monochromatic for c. Finally, let $\alpha \in \mathrm{Aut}^{\mathrm{C}}(E)$ be such that $\|\alpha \restriction_Y - \gamma\|_{\mathrm{cb}} \leq \varepsilon$. It follows that c 2ε-stabilizes on $\alpha \circ \mathrm{Emb}^{\mathrm{C}}(X, Y)$. □

2. The Ramsey property for Banach spaces and operator spaces

The goal of this section is to prove the extreme amenability of the automorphism group of operator spaces which are Fraïssé limits of certain classes of finite-dimensional injective operator spaces. Recall that an operator space E is *injective* if it is injective in the category of operator spaces; that is, whenever $X \subset Y$ are operator spaces, any completely contractive map $\phi : X \to E$ can be extended to a completely contractive map $\psi : Y \to E$. The finite-dimensional injective operator spaces are precisely the ones of the form $M_{q_1,s_1} \oplus_\infty \cdots \oplus_\infty M_{q_n,s_n}$ for $n, q_1, s_1, \ldots, q_n, s_n \in \mathbb{N}$. Here $M_{q,s}$ is the space of $q \times s$ matrices with complex entries, regarded as a space of operators on the $(q+s)$-dimensional Hilbert space of the form

$$\begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix},$$

where the diagonal blocks have size $q \times q$ and $s \times s$. The operator spaces $M_{q,1}$ and $M_{1,q}$ are called the q-dimensional *column operator Hilbert space* and the q-dimensional *row operator Hilbert space*, respectively. The space $M_{q,q}$ of $q \times q$ matrices will be simply denoted by M_q, and the n-fold ∞-sum of $M_{q,s}$ by itself will be denoted by $\ell^n_\infty(M_{q,s})$. It is known that the class of finite-dimensional injective operator spaces coincides with the class of finite-dimensional ternary rings of operator; see [69]. The finite-dimensional *commutative* ternary rings of operators are precisely the spaces ℓ^n_∞ for $n \in \mathbb{N}$ [8, Subsection 8.6.4], which are precisely the finite-dimensional minimal injective operator spaces.

Definition 2.1 (Injective classes)**.** We say that a family of finite-dimensional operator spaces is an *injective class* of operator spaces if it is one of the following families $\mathbb{I}_1 := \{\ell^n_\infty\}_{n \in \mathbb{N}}$, $\mathbb{I}_q := \{\ell^n_\infty(M_q)\}_{n \in \mathbb{N}}$, $\mathbb{I}_{\mathrm{c}} := \{\ell^n_\infty(M_{q,1})\}_{n,q \in \mathbb{N}}$, $\mathbb{I}_{\mathrm{e}} := \{M_q\}_{q \in \mathbb{N}}$, and $\mathbb{I}_{\mathrm{inj}} = \{M_{q_1,s_1} \oplus_\infty \cdots \oplus_\infty M_{q_n,s_n}\}_{n,q_1,s_1,\ldots,q_n,s_n \in \mathbb{N}}$.

It follows from Theorem 1.9 that all the classes considered in Definition 2.1 are stable Fraïssé classes with modulus $\varpi(\delta) = \delta$.

Definition 2.2 (Spaces locally approximated by injective classes)**.**

- $[\mathbb{I}_1]$ is the class of *minimal operator spaces*, i.e. the class of *Banach spaces.*
- $[\mathbb{I}_q]$ is the class of *q-minimal operator spaces* (see [43]);
- $[\mathbb{I}_{\mathrm{c}}]$ is the class of *operator sequence spaces* (see [40]);
- $[\mathbb{I}_{\mathrm{e}}] = [\mathbb{I}_{\mathrm{inj}}]$ is the class of *exact operator spaces* (see [64], [66, Theorem 17.1]).

Observe that Banach spaces (endowed with their minimal operator space structure) coincide with 1-minimal operator spaces. The corresponding limits are the following.

Definition 2.3.

- $\mathrm{FLim}\,\mathbb{I}_1$ is the *Gurarij space* $\mathbb{G}$ [5, 32, 39, 49];
- $\mathrm{FLim}\,\mathbb{I}_q$ is the *q-minimal Gurarij space* $\mathbb{G}_q$ [46, §6.5];
- $\mathrm{FLim}\,\mathbb{I}_{\mathrm{c}}$ is the *Gurarij column space* $\mathbb{GC}$ [46, §6.3];

- $\mathrm{FLim}\,\mathbb{I}_{\mathrm{e}}$ is the *noncommutative Gurarij space* $\mathbb{NG}$ [46, §8.1].

Observe that the Gurarij space $\mathbb{G}$ coincides with $\mathbb{G}_1$ in the notation of Definition 2.3. The goal of this section is to prove that the operator spaces in Definition 2.3 have extremely amenable automorphism group.

2.1. **The approximate Ramsey property and extreme amenability.** Fix $q, s \in \mathbb{N}$. The goal of this part is to prove that various classes of finite-dimensional exact operator spaces satisfy the stable Ramsey property with modulus $\varpi(\delta) = \delta$. To do this, by Proposition 1.15, Proposition 1.13, and Theorem 1.9, it suffices to prove the discrete approximate Ramsey property of appropriate subclasses of $\{\ell_\infty^d(M_{q,s})\}_{d,q,s\in\mathbb{N}}$.

Lemma 2.4 (ARP of $\{\ell_\infty^d(M_{q,s})\}_{d\in\mathbb{N}}$)**.** *Fix $q, s \in \mathbb{N}$. For any $d, m, r \in \mathbb{N}$ and $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for any r-coloring of $\mathrm{Emb}(\ell_\infty^d(M_{q,s}), \ell_\infty^n(M_{q,s}))$ there exists $\gamma \in \mathrm{Emb}(\ell_\infty^m(M_{q,s}), \ell_\infty^n(M_{q,s}))$ such that $\gamma \circ \mathrm{Emb}(\ell_\infty^d(M_{q,s}), \ell_\infty^m(M_{q,s}))$ is ε-monochromatic.*

Rather than proving Lemma 2.4 directly, we will establish its natural *dual statement*, which is Lemma 2.5 below. Given two operator spaces X and Y, let $\mathrm{CQ}(X, Y)$ be the set of *completely contractive complete quotient* mappings $\phi : X \to Y$, i.e. $\phi : X \to Y$ such that each amplification $\phi^{(n)}$ is a contractive quotient mapping [17, §2.2]. Notice that this is equivalent to the assertion that the dual map $\phi^* : X^* \to Y^*$ is a completely isometric embedding. We denote by $T_{s,q}$ the operator space dual of $M_{q,s}$. This can be regarded as the space of $s \times q$ matrices with norm $\|\alpha\| = \mathrm{Tr}((\alpha^*\alpha)^{\frac{1}{2}})$ [17, §1.2], where α^* denotes the adjoint of α, and Tr denotes the normalized trace on the space of $q \times q$ matrices. The duality between $M_{q,s}$ and $T_{s,q}$ is implemented by the paring $(\alpha, \beta) \mapsto \mathrm{Tr}(\alpha^t\beta)$, there α^t denotes the transpose of α. One can then canonically identify the operator space dual of $\ell_\infty^d(M_{q,s})$ with the 1-sum $\ell_1^d(T_{s,q})$ of d copies of $T_{s,q}$. From this it is easy to see that Lemma 2.4 and the following dual statement of it are equivalent.

Lemma 2.5. *Fix $q, s \in \mathbb{N}$. For any $d, m, r \in \mathbb{N}$ and $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for any r-coloring of $\mathrm{CQ}(\ell_1^d(T_{s,q}), \ell_1^n(T_{s,q}))$ there exists $\gamma \in \mathrm{CQ}(\ell_1^n(T_{s,q}), \ell_1^m(T_{s,q}))$ such that $\mathrm{CQ}(\ell_1^m(T_{s,q}), \ell_1^d(T_{s,q})) \circ \gamma$ is ε-monochromatic.*

In order to prove Lemma 2.5 we will need the following fact about linear complete isometries, which is an immediate consequence of [48, Lemma 5.17]; see also [15, Lemma 3.6].

Lemma 2.6. *Suppose that $q, s, q_1, s_1, \ldots, q_n, s_n \in \mathbb{N}$, $\phi_i : M_{q,s} \to M_{q_i,s_i}$ are completely contractive linear maps for $i = 1, 2, \ldots, n$, and that $\phi : M_{q,s} \to M_{q_1,s_1} \oplus_\infty \cdots \oplus_\infty M_{q_n,s_n}$ is the linear map $x \mapsto (\phi_1(x), \ldots, \phi_n(x))$. Then ϕ is completely isometric if and only if ϕ_i is completely isometric for some $i \leq n$.*

The proof of Lemma 2.5 relies on the Dual Ramsey Theorem; see Theorem 1.2.

Proof of Lemma 2.5. Fix $d, m, r \in \mathbb{N}$ and $\varepsilon > 0$. We identify a linear map ϕ from $\ell_1^n(T_{s,q})$ to $\ell_1^d(T_{s,q})$ with a $d \times n$ matrix $[\phi_{ij}]$ where $\phi_{ij} : T_{s,q} \to T_{s,q}$ is a linear map. It follows from (the dual of) Lemma 2.6 that ϕ is a completely contractive complete quotient mapping if and only if every row of $[\phi_{ij}]$ has an entry that is a surjective complete isometry of $T_{s,q}$, and every column is a complete contraction from $T_{s,q}$ to $\ell_1^d(T_{s,q})$. This implies that if a column has an entry that is a surjective complete isometry of $T_{s,q}$, then all the other entries of the column are zero. Let now $\mathcal{P}$ be a finite set of complete contractions from $T_{s,q}$ to $\ell_1^d(T_{s,q})$ with the following properties:

(i) the zero map belongs to $\mathcal{P}$;
(ii) for every $i \leq d$ the canonical embedding of $T_{s,q}$ into the i-th coordinate of $\ell_1^d(T_{s,q})$ belongs to $\mathcal{P}$;
(iii) for every nonzero complete contraction $\phi : T_{s,q} \to \ell_1^d(T_{s,q})$ there exists a nonzero element ϕ_0 of $\mathcal{P}$ such that $\|\phi - \phi_0\|_{\mathrm{cb}} < \varepsilon$ and $\|\phi_0\|_{\mathrm{cb}} < \|\phi\|$.

Fix $\varepsilon_0 > 0$ small enough and a finite ε_0-dense subset U of the group of automorphisms of $T_{s,q}$. Let $\mathcal{Q}$ be the (finite) set of linear complete isometries from $\ell_1^d(T_{s,q})$ to $\ell_1^m(T_{s,q})$ such that every row contains exactly one nonzero entry, every column contains at most one nonzero entry, and every nonzero entry is an automorphism of $T_{s,q}$ that belongs to U. Fix any linear order on $\mathcal{Q}$, and a linear order on $\mathcal{P}$ with the property that $\phi < \phi'$ whenever $\|\phi\|_{\mathrm{cb}} < \|\phi'\|_{\mathrm{cb}}$. Endow $\mathcal{Q} \times \mathcal{P}$ with the corresponding antilexicographic order. An element of $\mathrm{Epi}(n, \mathcal{P})$ is an n-tuple $\overline{v} = (v_1, \ldots, v_n)$ of elements of $\mathcal{P}$. We associate with such an n-tuple the element $\alpha_{\overline{v}}$ of $\mathrm{CQ}(\ell_1^n(T_{s,q}), \ell_1^d(T_{s,q}))$ whose representative matrix has v_i as i-th column for $i = 1, 2, \ldots, n$. Similarly an element of $\mathrm{Epi}(n, \mathcal{Q} \times \mathcal{P})$ is an n-tuple $(\overline{B}, \overline{w}) = (B_1, w_1, \ldots, B_n, w_n)$. We associate with such an n-tuple the element $\alpha_{(\overline{B},\overline{w})}$ of $\mathrm{CQ}(\ell_1^n(T_{s,q}), \ell_1^m(T_{s,q}))$ with B_iw_i as i-th column for $i = 1, 2, \ldots, n$. Suppose now that $n \in \mathbb{N}$ is obtained from $\mathcal{P}$ and $\mathcal{Q} \times \mathcal{P}$ by applying Theorem 1.2. We claim that such an n satisfies the desired conclusions. Suppose that c is an r-coloring of $\mathrm{CQ}(\ell_1^n(T_{s,q}), \ell_1^d(T_{s,q}))$. The map $\overline{v} \mapsto \alpha_{\overline{v}}$ from $\mathrm{Epi}(\mathcal{P}, n)$ to

$\mathrm{CQ}(\ell_1^n(T_{s,q}), \ell_1^d(T_{s,q}))$ induces an r-coloring on $\mathrm{Epi}(\mathcal{P}, n)$. By the choice of n there exists an element $(\overline{B}, \overline{w})$ of $\mathrm{Epi}(\mathcal{Q} \times \mathcal{P}, n)$ such that any rigid surjection from n to $\mathcal{P}$ that factors through $(\overline{B}, \overline{w})$ has a fixed color $i \in r$. To conclude the proof it remains to show that the set of completely contractive complete quotient mappings from $\ell_1^n(T_{s,q})$ to $\ell_1^d(T_{s,q})$ that factors through $\alpha_{(\overline{B},\overline{w})}$ is ε-monochromatic. By our choice of n this will follow once we show that, given any $\rho \in \mathrm{CQ}(\ell_1^m(T_{s,q}), \ell_1^d(T_{s,q}))$ there exists $\tau \in \mathrm{Epi}(\mathcal{P}, \mathcal{Q} \times \mathcal{P})$ such that $\left\| \alpha_{\tau(\overline{B},\overline{w})} - \rho \circ \alpha_{\overline{(B,\overline{w})}} \right\|_{\mathrm{cb}} \leq \varepsilon$, where we denoted by $\tau(\overline{B}, \overline{w})$ the element $(\tau(B_1, w_1), \ldots, \tau(B_n, w_n))$ of $\mathrm{Epi}(\mathcal{P}, n)$. If ρ has representative matrix A, then this is equivalent to the assertion that, for every $i \leq n$, $\tau(B_i, w_i)$ has cb-distance at most ε from ABw_i. We proceed to define such a rigid surjection τ from $\mathcal{Q} \times \mathcal{P}$ to $\mathcal{P}$. By the structure of completely contractive complete quotient mappings from $\ell_1^m(T_{s,q})$ to $\ell_1^d(T_{s,q})$ recalled above, there exists $A^\dagger \in \mathcal{Q}$ such that $\left\| AA^\dagger - \mathrm{Id}_{T_{s,q}} \right\|_{\mathrm{cb}} \leq \varepsilon$, provided that ε_0 is small enough (depending only from ε). Define now $\tau : \mathcal{Q} \times \mathcal{P} \to \mathcal{P}$ by letting, for $B \in \mathcal{Q}$ and $w \in \mathcal{P}$, $\tau(B, w) = 0$ if $w = 0$, $\tau(B, w) = w$ if $B = A^\dagger$, and otherwise $\tau(B, w) \in \mathcal{P}$ such that $0 < \|\tau(B, w)\|_{\mathrm{cb}} < \|ABw\|_{\mathrm{cb}}$ and $\|\tau(B, w) - ABw\|_{\mathrm{cb}} < \varepsilon$. It is clear from the definition that $\tau(B, w)$ has distance at most ε from ABw. We need to verify that τ is indeed a rigid surjection from $\mathcal{Q} \times \mathcal{P}$ to $\mathcal{P}$. Observe that τ is onto, and the pairs $(B, 0)$ are the only elements of $\mathcal{Q} \times \mathcal{P}$ that are mapped by τ to zero. It is therefore enough to prove that, for every $w \in \mathcal{P}$, $(A^\dagger, w)$ is the minimum of the preimage of w under τ. Suppose that (B', w') is an element of the preimage of w under τ. Then by definition of τ we have that

$$\|w\|_{\mathrm{cb}} < \|ABw'\|_{\mathrm{cb}} \leq \|w'\|_{\mathrm{cb}}.$$

By our assumptions on the ordering of $\mathcal{P}$, it follows that $w < w'$ and hence $(A^\dagger, w) < (B', w')$. This concludes the proof. □

Using the general facts about the approximate Ramsey property from Proposition 1.15, one can bootstrap the approximate Ramsey property from the class considered in Lemma 2.5 to other classes of operator spaces. In fact one can obtain the compact stable Ramsey property with modulus $\varpi(\delta) = \delta$; see Definition 1.10.

Theorem 2.7. *The following classes of finite-dimensional operator spaces satisfy the compact (SRP) with modulus $\varpi(\delta) = \delta$:*

1. *for every $q \in \mathbb{N}$, the class of finite-dimensional q-minimal operator spaces, and in particular the class of finite-dimensional Banach spaces;*
2. *the class of finite-dimensional operator sequence spaces;*
3. *the class of finite-dimensional exact operator spaces;*
4. *any of the injective classes from Definition 2.1.*

Proof. In each of the cases, it is enough to verify that the given class satisfies the (ARP) in view of Proposition 1.15, Proposition 1.13, and Theorem 1.9.

(1): It follows from Lemma 2.4 for $q = r$ that $\mathbb{I}_q$ satisfies the (ARP). Hence, by the equivalence of *(5)* and *(6)* in Proposition 1.15, the same applies to $\langle \mathbb{I}_q \rangle$, which is the class of finite-dimensional q-minimal operator spaces.

(2): It can be easily deduced from Lemma 2.4 using Proposition 1.15 that $\mathbb{I}_{\mathrm{c}}$ satisfies the (ARP). By the equivalence of *(5)* and *(6)* in Proposition 1.15, the same applies to $\langle \mathbb{I}_{\mathrm{c}} \rangle$, which is the class of finite-dimensional operator sequence spaces.

(3): We verify that $\mathbb{I}_{\mathrm{e}}$ satisfies the (ARP). By the equivalence of *(6)* and *(7)* in Proposition 1.15, it suffices to show that for every positive integer p, q, r such that $p \leq q$, and every $\varepsilon > 0$ there is some finite-dimensional exact operator space Z such that every r-coloring of $\mathrm{Emb}(M_p, Z)$ ε-stabilizes on $\gamma \circ \mathrm{Emb}(M_p, M_q)$ for some $\gamma \in \mathrm{Emb}(M_q, Z)$. Now, M_p and M_q are q-minimal, so by *(1)* there is such a Z which is a finite-dimensional q-minimal operator space. Since every q-minimal operator space is exact, this concludes that $\mathbb{I}_{\mathrm{e}}$ satisfies the (ARP). The same argument shows that $\mathbb{I}_{\mathrm{inj}}$ satisfies the (ARP). It follows from this and the equivalence of *(5)* and *(6)* in Proposition 1.15 that the same applies to $\langle \mathbb{I}_{\mathrm{e}} \rangle$, which is the class of finite-dimensional exact operator spaces.

(4): This has already been proved in *(1)*, *(2)*, *(3)* above. □

The limits of the Fraïssé classes mentioned in Definition 2.3 have extremely amenable automorphism groups in view of Proposition 1.15 and Theorem 2.7.

Theorem 2.8. *The following operator spaces have extremely amenable automorphism groups:*

1. *each q-minimal Gurarij space $\mathbb{G}_q$, and in particular the Gurarij space $\mathbb{G}$;*
2. *the column Gurarij space $\mathbb{CG}$;*

(3) the noncommutative Gurarij space $\mathbb{NG}$*.*

One can also give a direct, quantitative proof of the ARP for finite-dimensional Banach spaces using Lemma 2.5 when $r = q = 1$ and the injective envelope construction [8, Subsection 4.2]. Such a proof yields an explicit bound of the Ramsey numbers for the class of finite-dimensional Banach spaces in terms of the Ramsey numbers for the Dual Ramsey Theorem. Furthermore the same argument applies with no change in the case of real Banach spaces, yielding extreme amenability of the group of surjective linear isometries of the real Gurarij space.

3. The Ramsey property for operator spaces with a distinguished functional

The natural geometric object associated with a Banach space X is the w^*-compact absolutely convex space $\mathrm{Ball}(X^*)$ of contractive linear functionals on X. As discussed in the introduction, the noncommutative analog of such a correspondence involves the notion of matrix functionals. Given an operator space X, a matrix functional on X is a linear function from X to $M_{q,r}$ for some $q, r \in \mathbb{N}$. The space $\mathrm{CBall}(X^*)$ is the sequence $(K_{q,r})_{q,r\in\mathbb{N}}$, where $K_{q,r}$ is the w^*-compact convex set of completely contractive matrix functionals from X to $M_{q,r}$. The space $\mathrm{CBall}(X^*)$ is endowed with a notion of rectangular matrix convex combinations that makes it a compact rectangular matrix convex set [20]. Furthermore any compact rectangular convex set arises in this way. It should be clear from this that matrix functionals provide the right noncommutative analog of functionals on Banach spaces.

More generally, suppose that R is a separable *nuclear* operator space, that is, the identity map of R is the pointwise limit of completely contractive maps that factor through finite-dimensional injective operator spaces. When R is in addition a minimal operator space (i.e. a Banach space), this is equivalent to the assertion that R is a *Lindenstrauss* Banach space [8, Subsection 8.6.4]. A classical result of Wojtaszczyk [73] asserts that the separable Lindenstrauss spaces are precisely the separable Banach spaces that are isometric to the range of a contractive projection on the Gurarij space $\mathbb{G}$. The noncommutative analog of such a result asserts that the separable nuclear operator spaces are precisely the separable operator spaces that are completely isometric to the range of a completely contractive projection on the noncommutative Gurarij space [47]. A similar result holds for operator sequence spaces in terms of the column Gurarij space [46, Subsection 6.5]. Notice that injective *finite-dimensional* operator spaces are always nuclear, but the converse does not hold. The following result can be found in [46, §6.3, §6.5].

Proposition 3.1. *Let R be a separable operator space.*

(1) If R is q-minimal, then R is nuclear if and only if the identity on R is the pointwise limit of completely contractive maps that factor through some space in $\mathbb{I}_q$.
(2) If R is an operator sequence space, then R is nuclear if and only if the identity on R is the pointwise limit of completely contractive maps that factor through some space in $\mathbb{I}_c$.

Definition 3.2 (R-functionals)**.** For operator space X and a separable nuclear operator space R, an *R-functional* on X is a completely bounded linear operator from X to R. Let $\mathrm{CC}(X, R)$ be the space of completely contractive R-functionals on X, considered as a Polish space with respect to the topology of pointwise convergence. Let $\mathrm{Aut}(X) \curvearrowright \mathrm{CC}(X, R)$ be the continuous action $(\alpha, s) \mapsto s \circ \alpha^{-1}$. Finally, given $s \in \mathrm{CC}(X, R)$, let $\mathrm{Aut}(X, s) \subseteq \mathrm{Aut}(X)$ be the stabilizer of s with respect to such an action.

Given a family $\mathcal{A}$ of operator spaces and a nuclear separable operator space R, let $\mathcal{A}^R$ be the collection of R-operator spaces $\boldsymbol{X} = (X, s_X)$ where $X \in \mathcal{A}$. The following result is established in [46, Section 5].

Proposition 3.3. *Let $\mathbb{I}$ be an injective class of operator spaces, and R be a separable nuclear operator space.*

(1) $\mathbb{I}^R$ and $\langle\mathbb{I}\rangle^R$ are stable Fraïssé classes with stability modulus $\varpi(\delta) = 3\delta$.
(2) The Fraïssé limit of $\mathbb{I}^R$ is the R-operator space $(\mathrm{FLim}\,\mathbb{I}, \Omega^R_{\mathrm{FLim}\,\mathbb{I}})$.

The R-functional $\Omega^R_{\mathrm{FLim}\,\mathbb{I}}$ as in Proposition 3.3 is called the *generic* completely contractive R-functional on $\mathrm{FLim}\,\mathbb{I}$. The name is justified by the fact that the $\mathrm{Aut}(\mathrm{FLim}\,\mathbb{I})$-orbit of $\Omega^R_{\mathrm{FLim}\,\mathbb{I}}$ is a dense G_δ subset of the space $\mathrm{CC}(\mathrm{FLim}\,\mathbb{I}, R)$ of completely contractive R-functionals on $\mathrm{FLim}\,\mathbb{I}$.

3.1. **The approximate Ramsey property and extreme amenability.**

We present the approximate Ramsey properties of several classes of R-operator spaces, and the corresponding extreme amenability of the automorphism group of their Fraïssé limits.

Theorem 3.4. *The following classes of finite-dimensional R-operator spaces satisfy the compact (SRP) with stability modulus $\varpi(\delta) = 3\delta$:*

(1) for a q-minimal separable nuclear operator space R, the class of $\langle \mathbb{I}_q \rangle^R$ finite-dimensional q-minimal R-operator spaces; in particular, for a separable Lindenstrauss space R, the class of R-Banach spaces;
(2) for a separable nuclear operator sequence space R, the class $\langle \mathbb{I}_c \rangle^R$ of finite-dimensional R-operator sequence spaces;
(3) for a separable nuclear operator space R, the class $\langle \mathbb{I}_e \rangle^R$ of finite-dimensional exact R-operator spaces.

Proof. In all three cases, by Proposition 1.11, Proposition 1.13, and Proposition 3.3, it suffices to establish the discrete (ARP).
(1): By the equivalence of Proposition 1.11, Proposition 1.13, Proposition 3.3, and Proposition 1.15, it suffices to show that $\mathbb{I}_q^R$ satisfies the discrete (ARP). We first consider the case when $R = \ell_\infty^k(M_q)$ for some $k \in \mathbb{N}$. Let $\boldsymbol{X} = (X, s_X)$ and $\boldsymbol{Y} = (Y, s_Y)$ be structures in $\mathbb{I}_q^R$, $\varepsilon > 0$, and $r \in \mathbb{N}$. Suppose that $Z \in \mathbb{I}_q$ is obtained from X, Y, ε, and r by applying the discrete (ARP) of $\mathbb{I}_q$. We claim that the $\boldsymbol{Z} := (\bar{Z}, \pi_2) \in \mathbb{I}_q^R$, where $\bar{Z} := Z \oplus_\infty R \in \mathbb{I}_q$ and $\pi_2(z, r) := r$ is the canonical projection on the second factor, witnesses that the discrete (ARP) for $\mathbb{I}_q^R$ holds given the parameters $\boldsymbol{X}$, $\boldsymbol{Y}$, ε, and r. Let $f : \mathrm{Emb}(X, Z) \to \mathrm{Emb}(X, \bar{Z})$ be the isometry defined by $f(\gamma)(x) := (\gamma(x), s_X(x))$. Suppose that c is an r-coloring of $\mathrm{Emb}^R(\boldsymbol{X}, \boldsymbol{Z})$. We define the r-coloring $\widehat{c} = c \circ f$ of $\mathrm{Emb}(X, Z)$. By the choice of Z, there exists $I \in \mathrm{Emb}(Y, Z)$ such that $I \circ \mathrm{Emb}(X, Y)$ is ε-monochromatic for $\widehat{c}$. Let $\bar{I} : Y \to Z \oplus_\infty R$, $\bar{I}(z) := (I(z), s_Y(z))$. Then $\bar{I} \in \mathrm{Emb}^R(\boldsymbol{Y}, \boldsymbol{Z})$, and since for every $\gamma \in \mathrm{Emb}^R(\boldsymbol{X}, \boldsymbol{Y})$ one has that $\bar{I} \circ \gamma = f(I \circ \gamma)$, it follows that $\bar{I} \circ \mathrm{Emb}^R(\boldsymbol{X}, \boldsymbol{Y})$ is ε-monochromatic for c.

Now suppose that R is an arbitrary q-minimal separable nuclear operator space. Fix $\boldsymbol{X} = (X, s_X), \boldsymbol{Y} = (Y, s_Y) \in \mathbb{I}_q^R$, $\varepsilon > 0$ and $r \in \mathbb{N}$. By Proposition 3.1 we can find $S \in \mathbb{I}_q$ and complete contractions $i : R \to S$ and $j : S \to R$ such that $d_{\mathrm{cb}}(j \circ i \circ s_X, s_X) \leq \varepsilon$ and $d_{\mathrm{cb}}(j \circ i \circ s_Y, s_Y) \leq \varepsilon$. Set $\boldsymbol{X}_0 := (X, i \circ s_X), \boldsymbol{Y}_0 := (Y, i \circ s_Y) \in \mathbb{I}_q^s$. Let now $\boldsymbol{Z}_0 = (Z_0, s_{z_0}) \in \mathbb{I}_q^s$ be obtained by the discrete (ARP) of $\mathbb{I}_q^S$—established above—from the parameters $\boldsymbol{X}_0, \boldsymbol{Y}_0$, ε and r. Set $\boldsymbol{Z}_1 := (Z_0, j \circ s_{Z_0}) \in \mathbb{I}_q^R$. Since $\mathbb{I}_q^R$ satisfies the (SAP) with modulus $\varpi(\delta) = 3\delta$, we can use Proposition 1.12 to find $\boldsymbol{Z} = (Z, s_Z) \in \langle \mathbb{I}_q \rangle^R$ and $I \in \mathrm{Emb}^R(\boldsymbol{Z}_1, \boldsymbol{Z})$ such that $I \circ \mathrm{Emb}_\varepsilon^R(\boldsymbol{X}, \boldsymbol{Z}_1) \subseteq (\mathrm{Emb}^R(\boldsymbol{X}, \boldsymbol{Z}))_{4\varepsilon}$ and $I \circ \mathrm{Emb}_\varepsilon^R(\boldsymbol{Y}, \boldsymbol{Z}_1) \subseteq (\mathrm{Emb}^R(\boldsymbol{Y}, \boldsymbol{Z}))_{4\varepsilon}$. We claim that $\boldsymbol{Z}$ witnesses the discrete (ARP) for the class of q-minimal R-operator systems for the parameters $\boldsymbol{X}$, $\boldsymbol{Y}$, ε,and r. Suppose that c is an r-coloring of $\mathrm{Emb}^R(\boldsymbol{X}, \boldsymbol{Z})$. We define an r-coloring $\widehat{c}$ of $\mathrm{Emb}^S(\boldsymbol{X}_0, \boldsymbol{Z}_0)$ as follows. Given $\gamma \in \mathrm{Emb}^S(\boldsymbol{X}_0, \boldsymbol{Z}_0)$, it follows from the fact that $d_{\mathrm{cb}}(j \circ i \circ s_X, s_X) \leq \varepsilon$, that $\gamma \in \mathrm{Emb}_\varepsilon^R(\boldsymbol{X}, \boldsymbol{Z}_1)$. Thus by the choice of $\boldsymbol{Z}$ we can find $\phi_\gamma \in \mathrm{Emb}^R(\boldsymbol{X}, \boldsymbol{Z})$ such that $d_{\mathrm{cb}}(\phi_\gamma, I \circ \gamma) \leq 4\varepsilon$ and declare that $\widehat{c}(\gamma) := c(\phi_\gamma)$. By the choice of $\boldsymbol{Z}_0$ there exists $J \in \mathrm{Emb}^S(\boldsymbol{Y}_0, \boldsymbol{Z}_0)$ and $i < r$ be such that $J \circ \mathrm{Emb}^S(\boldsymbol{X}_0, \boldsymbol{Y}_0) \subseteq (\widehat{c}^{-1}(i))_\varepsilon$. Using the fact that $d_{\mathrm{cb}}(j \circ i \circ s_Y, s_Y) \leq \varepsilon$, one can see that $J \in \mathrm{Emb}_\varepsilon^R(\boldsymbol{Y}, \boldsymbol{Z}_1)$. Thus we can find by the choice of $\boldsymbol{Z}$, $\bar{J} \in \mathrm{Emb}^R(\boldsymbol{Y}, \boldsymbol{Z})$ such that $d_{\mathrm{cb}}(\bar{J}, I \circ J) \leq 4\varepsilon$. It is routine then to show that $\bar{J} \circ \mathrm{Emb}^R(\boldsymbol{X}, \boldsymbol{Y}) \subseteq (c^{-1}(i))_{10\varepsilon}$.

The proof of *(2)* and *(3)* are similar. For *(2)*, first one proves the discrete (ARP) for $\mathbb{I}_c^R$ where $R = \ell_\infty^k(M_{q,1})$. Then one uses Proposition 3.1 and the discrete (ARP) of $\mathbb{I}_c^R$ for $R = \ell_\infty^k(M_{q,1})$ to establish the discrete (ARP) for $\mathbb{I}_c^R$ for an arbitrary separable nuclear operator sequence space R.

For *(3)* one proceeds similarly, in this case first proving the discrete (ARP) for $\mathbb{I}_e^R$ where $R = M_k$ and then deducing the (ARP) for $\mathbb{I}_e^R$ where R is an arbitrary separable nuclear operator space R. ☐

From Theorem 3.4 and the characterization of extreme amenability in Proposition 1.15 we obtain new extremely amenable groups.

Corollary 3.5. *The following Polish groups are extremely amenable:*
(1) the stabilizer of the generic contractive R-functional on the Gurarij space for any separable Lindenstrauss Banach space R;
(2) the stabilizer of the generic completely contractive R-functional on the q-minimal Gurarij space for any separable q-minimal nuclear operator space R;
(3) the stabilizer of the generic completely contractive R-functional on the column Gurarij space for any separable nuclear operator sequence R;
(4) the stabilizer of the generic completely contractive R-functional on the noncommutative Gurarij space for any separable nuclear operator space R.

Again, the same proof shows that (1) of Corollary 3.5 also holds when one considers the real Gurarij space and any real separable Lindenstrauss space R.

3.2. **Closed bifaces of the Lusky simplex.** We reformulate Corollary 3.5 in terms of certain convex sets. In this subsection we consider *real* Banach spaces. Similar results hold in the complex case as well. By a compact absolutely convex set we mean a compact subset of a locally convex topological real vector space that is closed under absolutely convex combinations of the form $\mu x + \lambda y$ for $\lambda, \mu \in \mathbb{R}$ such that $|\lambda| + |\mu| \leq 1$. Any compact absolutely convex set K has a canonical involution σ mapping x to $-x$. A real-valued continuous function f on K is symmetric if $f \circ \sigma = -f$. Similarly, a continuous affine function between compact absolutely convex sets is symmetric if it commutes with the given involutions.

There is a natural correspondence between compact absolutely convex sets and Banach spaces. Indeed if X is a Banach space, then the unit ball $\mathrm{Ball}(X^*)$ of the dual space of X is a compact absolutely convex set when endowed with the w*-topology. Any compact absolutely convex set K is of this form, where X is the Banach space $A_\sigma(K)$ of real-valued symmetric affine continuous functions on K endowed with the supremum norm. Furthermore such a correspondence is functorial, and induces an equivalence of categories.

The compact absolutely convex sets that correspond to Lindenstrauss spaces are called *Lazar simplices* in [46, Section 6.1]. They have been characterized by Lazar in [41] in terms of a uniqueness assertion for boundary representing measures, reminiscent of the analogous characterization of Choquet simplices due to Choquet [2, Section II.3]; see also Subsection 4.1 below. The Lazar simplex corresponding to the Gurarij space is denoted by $\mathbb{L}$ and called the *Lusky simplex* in [46, Section 6.1]. It is proved in [46, 49, 51] that $\mathbb{L}$ plays the same role in the category of metrizable Lazar simplices as $\mathbb{P}$ plays in the category of metrizable Choquet simplices. Recall that a closed subset H of a Lazar simplex is a *biface* or *essential face* if it is the absolutely convex hull of a (not necessarily closed) face [42]. This is equivalent to the assertion that the linear span of H inside $A_\sigma(K)^*$ is a w*-closed L-ideal [3, 4]. The Lusky simplex is the *unique* nontrivial metrizable Lazar simplex with dense extreme boundary [49]. Furthermore it is *universal* among metrizable Lazar simplices, in the sense that any metrizable Lazar simplex is symmetrically affinely homeomorphic to a closed biface of the $\mathbb{L}$ [51]. Finally, the Lusky simplex is *homogeneous*: any symmetric affine homeomorphism between proper closed bifaces of $\mathbb{L}$ extends to a symmetric affine homeomorphisms of $\mathbb{L}$ [46, Subsection 6.1].

Suppose that R is a separable Lindenstrauss space, and H an absolutely convex subset of $\mathbb{L}$ symmetrically affinely homeomorphic to $\mathrm{Ball}(R^*)$. Identifying R with $A_\sigma(H)$ and $\mathbb{G}$ with $A_\sigma(\mathbb{L})$, we have that H induces a contractive map $P_H : \mathbb{G} \to R$, $f \mapsto f|_H$. The universality and homogeneity properties of $\mathbb{L}$ can be seen as consequences of the following result, established in [46, Subsection 6.1] using the theory of M-ideals in Banach spaces developed by Alfsen and Effros [3, 4], and the Choi–Effros lifting theorem from [12].

Proposition 3.6. *Let H be a metrizable Lazar simplex, and let s be a contractive $A_\sigma(H)$-functional on the Lusky simplex $\mathbb{L}$. The following assertions are equivalent:*

(1) s belongs to the $\mathrm{Aut}(\mathbb{L})$*-orbit of the generic contractive $A_\sigma(H)$-functional $\Omega^{A_\sigma(H)}_{A_\sigma(\mathbb{L})}$;*

(2) There is a closed proper biface $\bar{H}$ of $\mathbb{L}$ affinely homeomorphic to H and such that s is the map $A_\sigma(\mathbb{L}) \to A(\bar{H})$, $f \mapsto f \restriction_{\bar{H}}$.

Using Proposition 3.6 one can reformulate (1) of Corollary 3.5 as follows.

Theorem 3.7. *Suppose that H is a closed biface of the Lusky simplex $\mathbb{L}$. Then the group* $\mathrm{Aut}_H(\mathbb{L})$ *of symmetric affine homeomorphisms α of $\mathbb{L}$ such that $\alpha(p) = p$ for every $p \in H$ is extremely amenable.*

In the particular case when H is the trivial biface $\{0\}$, such a statement recovers extreme amenability of the group of surjective linear isometries of $\mathbb{G}$.

A similar result holds for complex Banach spaces. In this setting, one considers compact convex sets endowed with a continuous action of the circle group $\mathbb{T}$ (*compact convex circled sets*). The compact convex circled sets corresponding to complex Lindenstrauss spaces (*Effros simplices*) have been characterized by Effros in [16]. Again, the unit ball of the dual space of the complex Gurarij space has canonical uniqueness, universality, and homogeneity properties within the class of Effros simplices; see [46, Subsection 6.2]. Here one consider the natural complex analog of the notion of closed biface (*circled face*). The same argument as above shows that the pointwise stabilizer of any closed circled face of $\mathrm{Ball}(\mathbb{G}^*)$ is extremely amenable.

It is also proved in [46, Subsection 8.1] that whenever R is a separable nuclear operator space, the generic completely contractive R-state $\Omega^R_{\mathbb{NG}}$ on the noncommutative Gurarij space can be regarded as the noncommutative analog of a closed circled face; see [46, Definition 8.1 and Proposition 8.2]. We do not know if every noncommutative closed circled face of $\mathbb{NG}$ belongs to the $\mathrm{Aut}(\mathbb{NG})$-orbit of $\Omega^R_{\mathbb{NG}}$, which would be the natural noncommutative analog of the homogeneity property of $\mathrm{Ball}(\mathbb{G}^*)$.

4. The Ramsey property for Choquet simplices and operator systems

The main goal now is to establish the approximate (dual) Ramsey property for Choquet simplices with a distinguished point, as well as its natural noncommutative counterpart. We will then apply this to compute the universal minimal flows of the automorphisms group of the *Poulsen simplex* and of its noncommutative analog. This will be done by studying operator systems with a distinguished ucp map to a fixed nuclear separable operator system R (R-operator systems).

4.1. **Choquet simplices and operator systems.** Recall that a *compact convex set* K is a compact convex subset of some locally convex topological vector space. In a compact convex set one can define in the usual way the notion of convex combination. The *extreme boundary* $\partial_e K$ of K is the set of *extreme points* of K, that is, points that can not be written in a nontrivial way as a convex combination of points of K. When K is metrizable the boundary $\partial_e K$ is a G_δ subset. In this case, a *boundary measure* on K is a Borel probability measure on K that vanishes off the boundary of K. Choquet's representation theorem asserts that any point in a compact convex set can be realized as the barycenter of a boundary measure on K (*representing measure*). A compact convex set K where every point has a *unique* representing measure is called a *Choquet simplex.* In particular, any standard finite-dimensional simplex Δ_n for $n \in \mathbb{N}$ is a Choquet simplex.

The class of standard finite-dimensional simplices Δ_n for $n \in \mathbb{N}$ naturally form a projective Fraïssé class in the sense of [34]; see [38]. The corresponding Fraïssé limit is the *Poulsen simplex* $\mathbb{P}$. Initially constructed by Poulsen in [67], $\mathbb{P}$ is a nontrivial metrizable Choquet simplex with dense extreme boundary. It was later shown in [44] that there exists a *unique* nontrivial metrizable Choquet simplex with this property. Furthemore $\mathbb{P}$ is *universal* among metrizable Choquet simplices, in the sense that any metrizable Choquet simplex is affinely homeomorphic to a closed proper face of $\mathbb{P}$. Also, the Poulsen simplex is homogeneous: any affine homeomorphism between closed proper faces of $\mathbb{P}$ extends to an affine homeomorphism of $\mathbb{P}$.

The Poulsen simplex $\mathbb{P}$ can also be studied from the perspective of direct Fraïssé theory by considering the natural dual category to compact convex sets. For a compact convex set K let $A(K)$ be the space of complex-valued continuous affine functions on K. This is a closed subspace of the space $C(K)$ of complex-valued continuous functions on K, endowed with the supremum norm. Furthermore $A(K)$ contains a distinguished element, its *unit*, that corresponds to the function constantly equal to 1.

Recall that a *function system* is a closed subspace of $C(T)$ for some compact Hausdorff space T containing the function constantly equal to 1. So, $A(K)$ is a function system, and in fact any function system $V \subseteq C(T)$ arises in this way from a suitable compact convex set K. Precisely, K is the compact convex set of *states* of V, that is, the contractive functionals on V that are *unital*, i.e. map the unit of $C(T)$ to 1.

As mentioned in the introduction, the assignment $K \mapsto A(K)$ established a contravariant equivalence of categories from the category of compact convex sets and continuous affine maps to the category of function systems and unital contractive linear maps. The finite-dimensional function systems that are *injective* in such a category are precisely the function systems $A(\Delta_n) = \ell_\infty^n$ corresponding to the standard finite-dimensional simplices Δ_n. The function systems that correspond to Choquet simplices are precisely those that are Lindenstrauss as Banach spaces, or equivalently, the function systems whose identity map is the pointwise limit of *unital* contractive linear maps that factor through finite-dimensional injective operator systems.

The function systems approach has been adopted in the work of Conley and Törnquist [13] and, independently, in [45, 46], where it is shown that the class of finite-dimensional function systems is a Fraïssé class. Its limit can be identified with the function system $A(\mathbb{P})$ corresponding to the Poulsen simplex, which we will call the *Poulsen system.* One can also adopt this approach to define and study the natural noncommutative analog of the Poulsen simplex, in the setting of operator systems.

Any function system V has a canonical operator system structure coming from the inclusion $V \subset C(T)$ in an abelian unital C*-algebra. Explicitly, the matrix norms on $V = A(K)$ can be described as $\|[x_{ij}]\| = \sup_p \|[x_{ij}(p)]\|$ where p ranges in the state space K of V. Function systems are precisely the operator systems that can be represented inside an abelian unital C*-algebra. Arbitrary operator systems can be seen as the noncommutative analog of function systems. The correspondence between compact convex sets and function systems admits a natural noncommutative generalization. A *compact matrix convex set* is a sequence $\boldsymbol{K} = (K_n)$ of sets $K_n \subset M_n(V)$ for some topological vector space V that is matrix convex [72, Definition 1.1]. This means that whenever $\alpha_i \in M_{q_i,q}$ and $v_i \in K_{q_i}$ are such that $\alpha_1^*\alpha_1 + \cdots + \alpha_q^*\alpha_q = 1$, then the *matrix convex combinations* combination $\alpha_1^* v_1 \alpha_1 + \cdots + \alpha_q^* v_q \alpha_q$ belongs to K_q. A continuous matrix affine function $\boldsymbol{\phi} : \boldsymbol{K} \to \boldsymbol{T}$ between compact matrix convex sets is a sequence of continuous functions $\phi_n : K_n \to T_n$ that is matrix affine in the sense that it preserves matrix convex combinations. The group $\mathrm{Aut}(\boldsymbol{K})$ of matrix affine homeomorphisms of $\boldsymbol{K}$ is a Polish group when endowed with the compact-open topology.

To each operator system X one can canonically assign a compact matrix convex set: the *matrix state space* $\boldsymbol{S}(X)$. This is the sequence $(S_n(X))$, where $S_n(X) \subset M_n(X^*)$ is the space of all ucp maps from X to M_n. Conversely, to a compact convex set $\boldsymbol{K}$ one can associate an operator system $A(\boldsymbol{K})$ of matrix-affine functions from $\boldsymbol{K}$ to $\mathbb{R}$. It is proved in [72, Section 3] that these constructions are the inverse of each other, and define an equivalence between the category of operator systems and ucp maps, and the category of compact matrix convex sets and continuous matrix affine functions. In particular if X is an operator system, then the group $\mathrm{Aut}(X)$ of surjective unital complete isometries on X can be identified with the group $\mathrm{Aut}(\boldsymbol{K})$ of matrix affine homeomorphisms of the matrix state space $\boldsymbol{K}$ of X. The notions of matrix extreme point and matrix extreme boundary can be defined in the setting of compact matrix convex sets by using matrix convex combinations [72].

Recall that an operator system X is called *nuclear* if its identity map is the pointwise limit of ucp maps that factor throught finite-dimensional injective operator systems. When $X = A(K)$, this is equivalent to the assertion that the state space K of X is a Choquet simplex. The matrix state spaces of nuclear operator systems can be seen as the noncommutative generalization of Choquet simplices. The natural noncommutative analog of the Poulsen simplex is studied in [46], where it is proved that finite-dimensional exact operator systems form a Fraïssé class. The matrix state space $\mathbb{NP} = (\mathbb{NP}_n)$ of the corresponding Fraïssé limit $A(\mathbb{NP})$ is a nontrivial noncommutative Choquet simplex with dense matrix extreme boundary, which is called the *noncommutative Poulsen simplex* in [46].

One can also define a sequence of structures $(\mathbb{P}^{(q)})$ for $q \in \mathbb{N}$ that interpolates between the Poulsen simplex and the noncommutative Poulsen simplex, in the context of q-minimal operator systems. An operator system is q-minimal if it admits a complete order embedding into unital C*-algebra $C(K, M_q)$ for some compact Hausdorff space K [74]. Here we regard the unital selfadjoint subspaces of $C(K, M_q)$ as operator systems, called q-minimal operator systems or M_q-systems. For $q = 1$, these are precisely the function systems. A q-minimal operator system X can be completely recovered from the portion of the matrix state space only consisting of $S_k(X)$ for $k = 1, 2, \dots, q$. Conversely a sequence $(K_1, \dots, K_q)$ of compact convex sets $K_j \subset M_j(V)$ closed under matrix convex combinations $\alpha_1^* v_1 \alpha_1 + \cdots + \alpha_n^* v_n \alpha_n$ for $\alpha_i \in M_{q_i q}$ and $v_i \in M_{q_i}$ and $q_i \leq q$ such that $\alpha_1^* \alpha_1 + \cdots + \alpha_n^* \alpha_n = 1$, uniquely determines a q-minimal operator system $A(K_1, \dots, K_q)$. The finite-dimensional q-minimal operator systems form a Fraïssé class [46, Section 6.7]. The matrix state space $\mathbb{P}^{(q)} = (\mathbb{P}_1^{(q)}, \dots, \mathbb{P}_q^{(q)})$ of the corresponding limit $A(\mathbb{P}^{(q)})$ is the q-minimal Poulsen simplex. The model-theoretic properties of $A(\mathbb{P})$, $A(\mathbb{NP})$, and $A(\mathbb{P}^{(q)})$ have been studied in [25].

We regard operator systems as objects of the category `Osy` which has ucp maps as morphisms. The finite-dimensional *injective* objects in this category are precisely the finite ∞-sums of copies of M_q, which are also the finite-dimensional C*-algebras. The notion of isomorphism in this category coincides with complete order isomorphism. The Gromov-Hausdorff distance of two finite-dimensional operator systems X, Y is the infimum of $\varepsilon > 0$ such that there exist ucp maps $f : X \to Y$ and $g : Y \to X$ such that $\|g \circ f - \mathrm{Id}_X\|_{\mathrm{cb}} < \varepsilon$ and $\|f \circ g - \mathrm{Id}_X\|_{\mathrm{cb}} < \varepsilon$. If X is an operator system, then its automorphism group $\mathrm{Aut}(X)$ is the group of surjective unital complete isometries from X to itself. This is a Polish group when endowed with the topology of pointwise convergence. If X and Y are operator systems, then we let $\mathrm{UCP}(X, Y)$ be the space of ucp maps from X to Y. This is a Polish space when endowed with the topology of pointwise convergence. There is a natural continuous action of $\mathrm{Aut}(X)$ on $\mathrm{UCP}(X, Y)$ defined by $(\alpha, s) \mapsto s \circ \alpha^{-1}$. In particular when $Y = M_q$ we have that $\mathrm{UCP}(X, R) = S_q(X)$.

Given a class of operator systems $\mathcal{A}$, let $[\mathcal{A}]$ be the collection of operator systems E such that every finite-dimensional operator system $X \subseteq E$ is a limit of subspaces of operators systems in $\mathcal{A}$. Let $\langle \mathcal{A} \rangle$ be those operator systems in $[\mathcal{A}]$ that are finite-dimensional.

Definition 4.1 (Injective classes)**.** We say that a family of finite-dimensional operator systems is an *injective class* of operator systems if it is one of the following families $\{\ell_\infty^n\}_{n\in\mathbb{N}}$, $\{\ell_\infty^n(M_q)\}_{n\in\mathbb{N}}$, or $\{M_q\}_{q\in\mathbb{N}}$.

It follows from Theorem 1.9 that all the classes of operator systems considered in Definition 4.1 are stable Fraïssé classes with modulus $\varpi(\delta) = 2\delta$.

Definition 4.2 (Spaces locally approximated by injective classes)**.**

- $[\{\ell_\infty^n\}_{n\in\mathbb{N}}]$ is the class of *function systems*;
- $[\{\ell_\infty^n(M_q)\}_{n\in\mathbb{N}}]$ is the class of *q-minimal operator systems* (see [74]);
- $[\{M_q\}_{q\in\mathbb{N}}]$ is the class of *exact operator systems* (see [36]).

The corresponding limits are the following.

Definition 4.3.

- $\mathrm{FLim}\{\ell_\infty^n\}_{n\in\mathbb{N}}$ is the function system $A(\mathbb{P})$ associated with the Poulsen simplex $\mathbb{P}$ (see [46, Section 6.3]);
- $\mathrm{FLim}\{M_q\}_{q\in\mathbb{N}}$ is the operator system $A(\mathbb{NP})$ associated with the noncommutative Poulsen simplex $\mathbb{NP}$ (see [46, Section 8.2]);

- $\mathrm{FLim}\{\ell_\infty^n(M_q)\}_{n\in\mathbb{N}}$ is the operator system $A(\mathbb{P}^{(q)})$ associated with the q-minimal Poulsen simplex $\mathbb{P}^{(q)}$ (see [46, Section 6.7]).

The main goal of this section is to compute the universal minimal flow of the groups $\mathrm{Aut}(\mathbb{P})$ and $\mathrm{Aut}(\mathbb{NP})$ of affine homeomorphisms of the Poulsen simplex $\mathbb{P}$ and of matrix affine homeomorphisms of the noncommutative Poulsen simplex $\mathbb{NP}$, respectively. We will prove that the minimal compact $\mathrm{Aut}(\mathbb{P})$-space is the Poulsen simplex $\mathbb{P}$ itself endowed with the canonical action of $\mathrm{Aut}(\mathbb{P})$, answering [13, Question 4.4]. (The fact that such an action is minimal is a result of Glasner from [24].) The natural noncommutative analog of such fact is also true: the universal minimal compact $\mathrm{Aut}(\mathbb{NP})$-space is the canonical action of $\mathrm{Aut}(\mathbb{NP})$ on the space $\mathbb{NP}_1$ of (scalar) states on $A(\mathbb{NP})$.

4.1.1. *Operator systems with a distinguished state.* Similarly as in the case of Banach spaces and operator spaces (Section 3), we need to consider operator sequence with a distinguished (matrix) state. Suppose that X is an operator system. Recall that a *state* on X is a ucp map from X to $\mathbb{C}$. More generally, an M_n-state is a ucp map from X to M_n. Even more generally, if R is any separable nuclear operator system, we call a ucp map from X to R an R-state on X. As observed above, the space $\mathrm{UCP}(X,R)$ of R-states on X is a Polish space endowed with a canonical continuous action of $\mathrm{Aut}(X)$. An *R-operator system* is a pair $\mathbf{X} = (X, s_X)$ of a operator system X and an R-state s_X on X. Let $\mathrm{Aut}(X, s_X)$ be the stabilizer of s_X in $\mathrm{Aut}(X)$ with respect to the canonical action $\mathrm{Aut}(X) \curvearrowright \mathrm{UCP}(X,R)$. Given a family $\mathcal{A}$ of operator systems, let $\mathcal{A}^R$ be the collection of R-operator spaces (X, s_X) where $X \in \mathcal{A}$.

The following result is established in [46].

Proposition 4.4. *Let $\mathbb{I}$ be an injective class of operator systems.*
(1) $\mathbb{I}^R$ and $\langle\mathbb{I}\rangle^R$ are stable Fraïssé classes with stability modulus $\varpi(\delta) = 4\delta$;
(2) The Fraïssé limit of $\mathbb{I}^R$ is the R-operator system $(\mathrm{FLim}\,\mathbb{I}, \Omega^R_{\mathrm{FLim}\,\mathbb{I}})$.

As in the case of operator spaces, the R-state $\Omega^R_{\mathrm{FLim}\,\mathbb{I}}$ as in Proposition 4.4 is called the *generic* R-state on $\mathrm{FLim}\,\mathbb{I}$. This is the unique R-state on $\Omega^R_{\mathrm{FLim}\,\mathbb{I}}$ whose $\mathrm{Aut}(\mathrm{FLim}\,\mathbb{I})$-orbit is a dense G_δ subset of the space $\mathrm{UCP}(\mathrm{FLim}\,\mathbb{I}, R)$. The elements of the $\mathrm{Aut}(\mathrm{FLim}\,\mathbb{I})$-orbit of $\Omega^R_{\mathrm{FLim}\,\mathbb{I}}$ can be characterized as follows.

4.2. **Approximate Ramsey property and extreme amenability.** For the rest of this section we fix $q, k \in \mathbb{N}$. We identify as in Subsection 2.1 the dual of $\ell_\infty^d(M_q)$ with $\ell_1^d(T_q)$. Let σ_d be the M_q-state on $\ell_\infty^d(M_q)$ mapping $(x_1, \ldots, x_d)$ to x_d. The space $\ell_1^d(T_q)$ is endowed with a canonical trace defined by

$$\tau(x_1, \ldots, x_d) = \frac{1}{d}(\mathrm{Tr}(x_1) + \cdots + \mathrm{Tr}(x_d))$$

where Tr denotes the canonical normalized trace on the space of $q \times q$ matrices. The canonical dual notion of positivity in T_q—defined by setting $\alpha \geq 0$ if and only if $\mathrm{Tr}(\beta^t \alpha) \geq 0$ for every positive $\beta \in M_q$—coincides with the usual notion of positive semi-definiteness for matrices.

A linear map $\eta : \ell_\infty^d(M_q) \to \ell_\infty^n(M_q)$ is ucp if and only if its dual $\eta^* : \ell_1^n(T_q) \to \ell_1^d(T_q)$ is trace-preserving and completely positive. (Such maps are called *quantum channels* in the quantum information theory literature; see [31, §4.1].) Thus η is a complete order embedding if and only if η^* is a trace-preserving completely contractive complete quotient mapping. A linear map $\eta : \ell_\infty^d(M_q) \to \ell_\infty^n(M)$ has the property that $\sigma_n \circ \eta = \sigma_d$ if and only if $\eta^*(0, \ldots, 0, x) = (0, \ldots, 0, x)$ for every $x \in T_q$. We denote by $\mathrm{TPCQ}^{M_q}(\ell_1^n(T_q), \ell_1^d(T_q))$ the space of trace-preserving completely contractive complete quotient mapping ϕ from $\ell_1^n(T_q)$ to $\ell_1^d(T_q)$ such that $\phi(0, \ldots, 0, x) = (0, \ldots, 0, x)$ for every $x \in T_q$.

Lemma 4.5. *Suppose that $\psi_1, \ldots, \psi_{d-1}, \phi_d : M_q \to M_q$ are completely positive linear maps such that $\|y - 1\| < \varepsilon$ where $y = \psi_1(1) + \cdots + \psi_{d-1}(1) + \phi_d(1)$. Then there exists a completely positive map $\psi_d : M_q \to M_q$ such that*

$$\psi_1(1) + \cdots + \psi_{d-1}(1) + \psi_d(1) = 1$$

and $\|\psi_d - \phi_d\| < \varepsilon$.

Proof. Fix any state s on M_q and define $\psi_d(x) = \phi_d(x) + s(x)(1 - y)$. □

The following proposition can be proved similarly as Lemma 2.5. We present the details for the reader's convenience.

Proposition 4.6. *Fix $q \in \mathbb{N}$. For any $d, m \in \mathbb{N}$ and $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for any r-coloring of $\mathrm{TPCQ}^{M_q}(\ell_1^n(T_q), \ell_1^d(T_q))$ there exists $\gamma \in \mathrm{TPCQ}^{M_q}(\ell_1^n(T_q), \ell_1^m(T_q))$ such that $\mathrm{TPCQ}^{M_q}(\ell_1^m(T_q), \ell_1^d(T_q)) \circ \gamma$ is ε-monochromatic.*

Proof. The proof is analogous to the proof of Lemma 2.5. Fix $d, m, r \in \mathbb{N}$ and $\varepsilon > 0$. We identify a linear map ϕ from $\ell_1^n(M_q)$ to $\ell_1^d(M_q)$ with an $d \times n$ matrix $[\phi_{ij}]$ where $\phi_{ij} : T_q \to T_q$ is a linear map. It follows from Lemma 2.6 that $\phi \in \mathrm{TPCQ}^{M_q}(\ell_1^n(T_q), \ell_1^d(T_q))$ if and only if

- every row of $[\phi_{ij}]$ has an entry that is an automorphism of T_q,
- every column is a trace-preserving completely positive from T_q to $\ell_1^d(T_q)$,
- the last column is $(0, 0, \ldots, 0, \mathrm{Id}_{T_q})$, where Id_{T_q} is the identity map of T_q.

Fix $\varepsilon_0 \in (0, \varepsilon)$ small enough, and a finite ε_0-dense subset U of the group of automorphisms of T_q containing the identity map of T_q. The dual statement of Lemma 4.5 and the small perturbation lemma [66, Lemma 2.13.2] show that one can find a finite set $\mathcal{P}$ of trace-preserving complete contractions from T_q to $\ell_1^d(T_q)$ with the following properties:

(i) for every $i \leq d$ the canonical embedding of T_q into the i-th coordinate of $\ell_1^d(T_q)$ belongs to $\mathcal{P}$;
(ii) for every trace-preserving completely positive map $v = (v_1, \ldots, v_d) : T_q \to \ell_1^d(T_q)$ such that $(v_1, \ldots, v_{d-1})$ is nonzero, there exists a trace-preserving completely positive map $w = (w_1, \ldots, w_d)$ in $\mathcal{P}$ such that $\|w - v\|_{\mathrm{cb}} < \varepsilon_0$, $(w_1, \ldots, w_{d-1})$ is nonzero, and $\|(w_1, \ldots, w_{d-1})\|_{\mathrm{cb}} < \|(v_1, \ldots, v_{d-1})\|_{\mathrm{cb}}$.

Let $\mathcal{Q}$ be the (finite) set of trace-preserving completely contractive complete quotient mappings from $\ell_1^d(T_q)$ to $\ell_1^m(T_q)$ such that the last row is $(0, 0, \ldots, \mathrm{Id}_{T_q})$, every column contains exactly one nonzero entry, every row contains at most one nonzero entry, and every nonzero entry is an automorphism of T_q that belongs to U. Fix any linear order on $\mathcal{Q}$, and a linear order on $\mathcal{P}$ with the property that $v < w$ whenever $\|(v_1, \ldots, v_{d-1})\|_{\mathrm{cb}} < \|(w_1, \ldots, w_{d-1})\|_{\mathrm{cb}}$. Endow $\mathcal{Q} \times \mathcal{P}$ with the corresponding antilexicographic order. An element of $\mathrm{Epi}(n, \mathcal{P})$ is a tuple $\overline{v} = (v^{(1)}, \ldots, v^{(n)})$ of elements of $\mathcal{P}$. We associate with such a tuple the element $\alpha_{\overline{v}}$ of $\mathrm{TPCQ}^{M_q}(\ell_1^{n+1}(T_q), \ell_1^d(T_q))$ whose i-th column is $v^{(i)}$ for $i = 1, 2, \ldots, n$ and the $(n+1)$-th column is $(0, 0, \ldots, \mathrm{Id}_{T_q})$. Similarly an element of $\mathrm{Epi}(n, \mathcal{Q} \times \mathcal{P})$ is an n-tuple $(\overline{B}, \overline{v}) = (B_1, v_1, \ldots, B_n, v_n)$. We associate with such an tuple the completely contractive complete quotient mapping $\alpha_{(\overline{B}, \overline{v})}$ from $\ell_1^{n+1}(T_q)$ to $\ell_1^m(T_q)$ whose i-th column is $B_i v_i$ for $i \leq n$, and $(0, 0, \ldots, 0, \mathrm{Id}_{T_q})$ for $i = n+1$. Suppose now that $n \in \mathbb{N}$ is obtained from $\mathcal{P}$ and $\mathcal{Q} \times \mathcal{P}$ by applying Theorem 1.2. We claim that $n+1$ satisfies the desired conclusions. Suppose that c is an r-coloring of $\mathrm{TPCQ}^{M_q}(\ell_1^{n+1}(T_q), \ell_1^d(T_q))$. The identification of $\mathrm{Epi}(n, \mathcal{P})$ with a subspace of $\mathrm{TPCQ}^{M_q}(\ell_1^{n+1}(T_q), \ell_1^d(T_q))$ described above induces an r-coloring on $\mathrm{Epi}(n, \mathcal{P})$. By the choice of n there exists an element $(\overline{B}, \overline{w})$ of $\mathrm{Epi}(n, \mathcal{Q} \times \mathcal{P},)$ such that any rigid surjection from n to $\mathcal{P}$ that factors through $(\overline{B}, \overline{w})$ has a fixed color $i \in r$. To conclude the proof it remains to show that the set of elements of $\mathrm{TPCQ}^{M_q}(\ell_1^{n+1}(T_q), \ell_1^d(T_q))$ that factor through $(\overline{B}, \overline{w})$ is ε-monochromatic. By our choice of n this will follow once we show that, given any $\rho \in \mathrm{TPCQ}^{M_q}(\ell_1^m(T_q), \ell_1^d(T_q))$ there exists $\tau \in \mathrm{Epi}(\mathcal{Q} \times \mathcal{P}, \mathcal{P})$ such that $\left\|\alpha_{\tau(\overline{B}, \overline{w})} - \rho \circ \alpha_{(\overline{B}, \overline{w})}\right\|_{\mathrm{cb}} \leq \varepsilon$. Here we denoted by $\tau(\overline{B}, \overline{w})$ the rigid surjection from n to $\mathcal{P}$ that one obtains by composing $(\overline{B}, \overline{w})$ and τ. If ρ has representative matrix A, this is equivalent to the assertion that for every $i \leq n$, $\|AB_i w_i - \tau(B_i, w_i)\|_{\mathrm{cb}} \leq \varepsilon$. We proceed to define such a rigid surjection τ from $\mathcal{Q} \times \mathcal{P}$ to $\mathcal{P}$. By the structure of completely contractive complete quotient mappings from $\ell_1^m(T_q)$ to $\ell_1^d(T_q)$ recalled above, there exists $A^\dagger \in \mathcal{Q}$ such that $\left\|AA^\dagger - \mathrm{Id}_{\ell^1(T_q)}\right\| \leq \varepsilon$, provided that ε_0 is small enough. Define now $\tau : \mathcal{Q} \times \mathcal{P} \to \mathcal{P}$ by letting, for $B \in \mathcal{Q}$ and $w = (w_1, \ldots, w_n) \in \mathcal{P}$, if $v = ABw$, $\tau(B, w) := \widetilde{v}$ where

- $\widetilde{v} = w$ if $B = A^\dagger$
- $\|\widetilde{v} - v\|_{\mathrm{cb}} \leq \varepsilon$
- $(\widetilde{v}_1, \ldots, \widetilde{v}_{d-1})$ is nonzero provided that $(v_1, \ldots, v_{d-1})$ is nonzero.

It is clear from the definition that $\|\tau(B, w) - ABw\|_{\mathrm{cb}} \leq \varepsilon$ for every $(B, w) \in \mathcal{Q} \times \mathcal{P}$. We need to verify that τ is indeed a rigid surjection from $\mathcal{Q} \times \mathcal{P}$ to $\mathcal{P}$. Observe that τ is onto, and the pairs $(B, (0, \ldots, 0, v_d))$ are the only elements of $\mathcal{Q} \times \mathcal{P}$ that are mapped by τ to an element of $\mathcal{P}$ of the form $(0, \ldots, 0, v_d)$ of $\mathcal{P}$. It is therefore enough to prove that, for every $v \in \mathcal{P}$ with $(v_1, \ldots, v_{d-1})$ nonzero, (B, v) is the minimum of the preimage of v under τ. Suppose that (B', v') is an element of the preimage of v under τ. Then by definition of τ we have that

$$\|(v_1, \ldots, v_{d-1})\|_{\mathrm{cb}} < \left\|(AB_1' v_1', \ldots, AB_{d-1}' v_{d-1})\right\|_{\mathrm{cb}} \leq \left\|(v_1', \ldots, v_{d-1}')\right\|_{\mathrm{cb}}$$

By our assumptions on the ordering of $\mathcal{P}$, it follows that $v < v'$ and hence $(A^\dagger, v) < (B', v')$. □

We isolate a particular instance of Proposition 4.6 which seems of independent interest. We identify the n-dimensional standard simplex Δ_n with the state space $S(\ell_\infty^{n+1}) \subset \ell_1^{n+1}$. We let $\mathrm{Epi}(\Delta_n, \Delta_d)$ be the space of surjective continuous affine maps from Δ_n to Δ_d endowed with the metric $d(\phi, \psi) = \sup_{p \in \Delta_n} \|\phi(p) - \psi(p)\|_{\ell_1^d}$. We also let $\mathrm{Epi}_0(\Delta_n, \Delta_d)$ be the subspace of $\phi \in \mathrm{Epi}(\Delta_n, \Delta_d)$ such that $\phi(0, \ldots, 0, 1) = (0, \ldots, 0, 1)$. One can identify $\mathrm{Epi}(\Delta_n, \Delta_d)$ isometrically with the space of trace-preserving contractive quotient mappings $\phi : \ell_1^n \to \ell_1^d$, and the space $\mathrm{Epi}_0(\Delta_n, \Delta_d)$ with $\mathrm{TPCQ}^{M_q}(\ell_1^n, \ell_1^d)$. The following statement is therefore the particular instance of Proposition 4.6 when $q = 1$.

Corollary 4.7. *For any $d, m \in \mathbb{N}$ and $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for any r-coloring of the space $\mathrm{Epi}_0(\Delta_n, \Delta_d)$ there exists $\gamma \in \mathrm{Epi}_0(\Delta_n, \Delta_m)$ such that $\mathrm{Epi}_0(\Delta_m, \Delta_d) \circ \gamma$ is ε-monochromatic.* □

The following result can be proved from Proposition 4.6 similarly as Theorem 3.4.

Theorem 4.8. *The following classes of finite-dimensional R-operator systems satisfy the stable Ramsey property with modulus $\varpi(\delta) = 4\delta$:*

(1) for every $q \in \mathbb{N}$ the class $\{(\ell_\infty^d(M_q), s_d)\}_{d\in\mathbb{N}}$ of M_q-operator systems, where $s_d(x_1, \ldots, x_d) = x_d$;
(2) for every $q \in \mathbb{N}$ and q-minimal separable nuclear operator system R, the class of finite-dimensional q-minimal R-operator systems;
(3) for every separable nuclear operator system R, the class of finite-dimensional exact R-operator systems.

The limits of the Fraïssé classes mentioned in Theorem 4.8 have extremely amenable automorphism in view of the correspondence between extreme amenability and the approximate Ramsey property given by Proposition 1.15.

Corollary 4.9. *The following Polish groups are extremely amenable:*

(1) the stabilizer of the generic $A(F)$-state $\Omega_{A(\mathbb{P})}^{A(F)}$ on the Poulsen system $A(\mathbb{P})$ for any metrizable Choquet simplex F;
(2) the stabilizer of the generic R-state $\Omega_{A(\mathbb{NP})}^{R}$ on the noncommutative Poulsen system $A(\mathbb{NP})$ for any separable nuclear operator system R;
(3) the stabilizer of the generic R-state $\Omega_{A(\mathbb{P}^{(q)})}^{R}$ on the q-minimal Poulsen system $A(\mathbb{P}^{(q)})$ for any q-minimal nuclear operator system R.

4.3. **Closed faces of the Poulsen simplex.** Corollary 4.9 can be restated geometrically in terms of a property of the Poulsen simplex. The Poulsen simplex $\mathbb{P}$ has the following universality and homogeneity property for faces: any metrizable Choquet simplex is affinely homeomorphic to a closed proper face of $\mathbb{P}$, and any affine homeomorphism between closed proper faces of $\mathbb{P}$ extends to an affine homeomorphism of $\mathbb{P}$ [44, Theorem 2.3 and Theorem 2.5]. This can be seen as a consequence of existence, uniqueness, and homogeneity of Fraïssé limits together with the following fact, established in [46, Section 6.3].

Proposition 4.10. *Let F be a metrizable Choquet simplex, and let s be an $A(F)$-state on the Poulsen system $\mathbb{P}$. The following assertions are equivalent:*

(1) s belongs to the $\mathrm{Aut}(\mathbb{P})$-orbit of the generic $A(F)$-state of $\Omega_{A(\mathbb{P})}^{A(F)}$;
(2) There is a closed proper face $\bar{F}$ of $\mathbb{P}$ affinely homeomorphic to F and such that s is the map $A(\mathbb{P}) \to A(\bar{F})$, $f \mapsto f\restriction_{\bar{F}}$.

In particular the $\mathrm{Aut}(\mathbb{P})$-orbit of the generic state $\Omega_{A(\mathbb{P})}^{\mathbb{C}} : A(\mathbb{P}) \to \mathbb{C}$ is the set of extreme points of $\mathbb{P}$.

It follows from Proposition 4.10 that Corollary 4.9 *(1)* can be reformulated as follows.

Theorem 4.11. *Suppose that F is a closed proper face of the Poulsen simplex $\mathbb{P}$. Then the pointwise stabilizer $\mathrm{Aut}_F(\mathbb{P})$ of F with respect to the canonical action $\mathrm{Aut}(\mathbb{P}) \curvearrowright \mathbb{P}$ is extremely amenable.*

It is also proved in [46, Section 8.2] that, if $R = A(\boldsymbol{F})$ is a nuclear operator system, then the generic R-state $\Omega_{A(\mathbb{NP})}^{R}$ on the noncommutative Poulsen system can be seen as a "noncommutative closed proper face" $\boldsymbol{F}$ of the noncommutative Poulsen system $\mathbb{NP}$. We do not know at this point if, conversely, any noncommutative closed proper face of $\mathbb{NP}$ as defined in [46, Section 8.2] belongs to the $\mathrm{Aut}(\mathbb{NP})$ orbit of $\Omega_{A(\mathbb{NP})}^{R}$.

4.4. **The universal minimal flows of $\mathbb{P}$ and $\mathbb{NP}$.** Using Corollary 4.9 and Theorem 4.11 we can compute the universal minimal flows of the affine homeomorphism group $\mathrm{Aut}(\mathbb{P})$ of the Poulsen simplex, the matrix affine homeomorphism group $\mathrm{Aut}(\mathbb{NP})$ of the noncommutative Poulsen simplex, and the matrix affine homeomorphism group $\mathrm{Aut}(\mathbb{P}^{(q)})$ of the q–minimal Poulsen simplex.

Theorem 4.12.

(1) The universal minimal flow of $\mathrm{Aut}(\mathbb{P})$ is the canonical action $\mathrm{Aut}(\mathbb{P}) \curvearrowright \mathbb{P}$.
(2) The universal minimal flow of $\mathrm{Aut}(\mathbb{NP})$ is the canonical action $\mathrm{Aut}(\mathbb{NP}) \curvearrowright \mathbb{NP}_1$.
(3) The universal minimal flow of $\mathrm{Aut}(\mathbb{P}^{(q)})$ is the canonical action $\mathrm{Aut}(\mathbb{P}^{(q)}) \curvearrowright \mathbb{P}_1^{(q)}$.

Proof. (1): The action $\mathrm{Aut}(\mathbb{P}) \curvearrowright \mathbb{P}$ is minimal by a result of Glasner from [24]. This can be seen directly using the homogeneity property of $A(\mathbb{P})$ and the fact that for any $\varepsilon > 0$ and $d \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that for any state s on ℓ_∞^d and t on ℓ_∞^m there exists a unital linear isometry $\phi : \ell_\infty^d \to \ell_\infty^m$ such that $\|t \circ \phi - s\| < \varepsilon$. Consider the generic state $\Omega_{A(\mathbb{P})}^{\mathbb{C}}$ on $A(\mathbb{P})$. We know from Proposition 4.10 that the state $\Omega_{A(\mathbb{P})}^{\mathbb{C}}$ is an extreme point of $\mathbb{P}$, whose $\mathrm{Aut}(\mathbb{P})$-orbit is dense. The stabilizer $\mathrm{Aut}(\mathbb{P}, \Omega_{A(\mathbb{P})}^{\mathbb{C}})$ of $\Omega_{A(\mathbb{P})}^{\mathbb{C}}$ is extremely amenable by Corollary 4.9. The canonical $\mathrm{Aut}(\mathbb{P})$-equivariant map from the quotient $\mathrm{Aut}(\mathbb{P})$-space $\mathrm{Aut}(\mathbb{P}) /\!\!/ \mathrm{Aut}(\mathbb{P}, \Omega_{A(\mathbb{P})}^{\mathbb{C}})$ to $\mathbb{P}$ is a uniform equivalence. This follows from the homogeneity property of $(A(\mathbb{P}), \Omega_{A(\mathbb{P})}^{\mathbb{C}})$ as the Fraïssé limit of the class of finite-dimensional function systems with a distinguished state; see also [46, Subsection 5.4]. This allows one to conclude via a standard argument—see [53, Theorem1.2]—that the action $\mathrm{Aut}(\mathbb{P}) \curvearrowright \mathbb{P}$ is the universal minimal compact $\mathrm{Aut}(\mathbb{P})$-space.
The proofs of *(2)* and *(3)* are similar: Minimality of the action $\mathrm{Aut}(\mathbb{NP}) \curvearrowright \mathbb{NP}_1$ is a consequence of the following fact: for any $d \in \mathbb{N}$ and $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that for any $s \in S(M_d)$ and $t \in S(M_m)$ there exists a complete order embedding $\phi : M_d \to M_m$ such that $\|t \circ \phi - s\|_{\mathrm{cb}} < \varepsilon$; see [46, Lemma 8.10] and [46, Proposition 5.8]. Minimality of the action $\mathrm{Aut}(\mathbb{P}^{(q)}) \curvearrowright \mathbb{P}_1^{(q)}$ is a consequence of the similar assertion where M_d and M_m are replaced with $\ell_\infty^d(M_q)$ and $\ell_\infty^m(M_q)$; see [46, Lemma 6.25]. The rest of the argument is entirely analogous. $\square$

It has recently been proved in [7] that the situation in Theorem 4.12 is typical. Whenever G is a Polish group whose universal compact G-space $M(G)$ is metrizable, there exists a closed extremely amenable subgroup H of G such that the completion of the homogeneous quotient G-space G/H is G-equivariantly homeomorphic to $M(G)$.

5. The Dual Ramsey Theorem and matrices over finite fields

There are several equivalent ways to present the dual Ramsey theorem (DRT) of Graham and Rothschild [29]. Among these, there is a factorization result for *Boolean matrices* stated below as Theorem 5.2. Motivated by this, we study Ramsey-theoretical factorization results for colorings of other classes of matrices. We begin with matrices with entries in a finite field, and then conclude, in the next section, with matrices over $\mathbb{R}$ or $\mathbb{C}$.

It is well known that a $n \times m$-matrix A has a unique decomposition $A = \mathrm{red}(A) \cdot \tau(A)$ where $\mathrm{red}(A)$ is in reduced column echelon form and $\tau(A)$ is an invertible $m \times m$-matrix. We prove that when the field is finite any finite coloring of matrices over a finite field is determined, in a precise way, by τ. This can be seen as an extension of the well known result of Graham, Leeb, and Rothschild on Grassmannians over a finite field [28].

We recall some definitions introduced before. Let $Y \subseteq X$ be two subsets and $r \in \mathbb{N}$. An *r-coloring* of X is a mapping $c : X \to r = \{0, 1 \dots, r-1\}$. Y is *c-monochromatic* if c is constant on Y. We say that $\pi : X \to K$ is a *factor* of $c : X \to r$ if there is some $\widehat{c} : K \to r$ such that $c = \widehat{c} \circ \pi$. Finally, π is a *factor of c in $Y \subseteq X$* if $\pi \restriction_Y$ is a factor of $c \restriction_Y$. So, Y is c-monochromatic when the trivial constant map $\pi : X \to \{0\} = 1$ is a factor of c in Y.

Perhaps the most common formulation of the Dual Ramsey Theorem involves partitions of finite sets. Given $k, n \in \mathbb{N}$, let $\mathcal{E}_k(n)$ be the family of all partitions of n into k pieces. Given a partition $\mathcal{P} \in \mathcal{E}_m(n)$, let $\langle \mathcal{P} \rangle_k$ be the set of all partitions $\mathcal{Q}$ of n with k pieces that are coarser than $\mathcal{Q}$, i.e. such that each piece of $\mathcal{Q}$ is a union of pieces of $\mathcal{P}$.

Theorem (DRT, partitions version)**.** *For every $k, m \in \mathbb{N}$ an $m \in \mathbb{N}$ there is some integer $n \geq k$ such that every r-coloring of $\mathcal{E}_k(n)$ has a monochromatic set of the form $\langle \mathcal{P} \rangle_k$ for some $\mathcal{P} \in \mathcal{E}_m(n)$.*

The next formulation of the Dual Ramsey Theorem was already introduced in Theorem 1.2 and it uses the concept of *rigid surjection.* Given two linear orders $(R, <_R)$ and $(S, <_S)$, a surjective map $f : R \to S$ is called a rigid surjection if $\min f^{-1}(s_0) < \min f^{-1}(s_1)$ whenever $s_0 < s_1$.

Theorem (DRT, rigid surjections)**.** *For every finite linearly ordered sets R and S and every $r \in \mathbb{N}$ there is an integer $n := n_{\mathrm{DR}}(R, S, r)$ such that, considering n naturally ordered, every r-coloring of* $\mathrm{Epi}(n, R)$ *has a monochromatic set of the form* $\mathrm{Epi}(S, R) \circ \gamma$ *for some* $\gamma \in \mathrm{Epi}(n, S)$.

The following three reformulations of the Dual Ramsey Theorem are *structural Ramsey results* for finite Boolean algebras.

Theorem (DRT, Boolean algebras)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be finite Boolean algebras, and let $r \in \mathbb{N}$. Then there exists a finite Boolean algebra $\mathcal{C}$ such that every r-coloring of the set $\binom{\mathcal{C}}{\mathcal{A}}$ of isomorphic copies of $\mathcal{A}$ inside $\mathcal{C}$ admits a monochromatic set of the form $\binom{\mathcal{B}_0}{\mathcal{A}}$ for some $\mathcal{B}_0 \in \binom{\mathcal{C}}{\mathcal{B}}$.*

Let $\mathcal{A}$ be a finite Boolean algebra, and let $\mathrm{At}(\mathcal{A})$ be the set of its atoms. Any $a \in \mathcal{A}$ is represented

$$a = \bigvee_{x \in \Gamma_a} x,$$

for a unique set of atoms Γ_a. So, any linear ordering $<$ on $\mathrm{At}(\mathcal{A})$ *extends* to $\mathcal{A}$ by defining $a < b$ iff $\min_<(\Gamma_a \triangle \Gamma_b) \in \Gamma_a$. Following [37], we will say that $(\mathcal{A}, <)$ is a *canonically ordered* Boolean algebra. Given canonically ordered Boolean algebras $\mathcal{A}$ and $\mathcal{B}$, let $\mathrm{Emb}_<(\mathcal{A}, \mathcal{B})$ be the collection of ordering-preserving embeddings from $\mathcal{A}$ into $\mathcal{B}$, respectively.

Theorem 5.1 (DRT, canonically ordered Boolean algebras). *For every canonically ordered finite Boolean algebras $\mathcal{A}$ and $\mathcal{B}$ and every integer r there is a canonically ordered Boolean algebra $\mathcal{C}$ such that every r-coloring of* $\mathrm{Emb}_<(\mathcal{A}, \mathcal{C})$ *has a monochromatic set of the form $\varrho \circ \mathrm{Emb}_<(\mathcal{A}, \mathcal{B})$ for some $\varrho \in \mathrm{Emb}_<(\mathcal{B}, \mathcal{C})$.*

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are finite Boolean algebras with k and n atoms, respectively. Any embedding from $\mathcal{A}$ to $\mathcal{B}$ has a corresponding *representing* $n \times k$ matrix with entries in $\{0, 1\}$. We call the matrices arising in this fashion *Boolean matrices.* The set of $n \times k$ Boolean matrices will be denoted by $M^{\mathrm{ba}}_{n,k}$. This is exactly the set of $n \times k$ matrix with entries in $\{0, 1\}$ whose columns (which can be identified with subsets of n) form a k-partition of n. We let $M^{\mathrm{oba}}_{n,k}$ be the set of Boolean $n \times k$-matrices that correspond to order-preserving embeddings between *canonically ordered* Boolean algebras. These are precisely the set of Boolean matrices whose columns $(P_i)_{i \in k}$ furthermore satisfy $\min P_i < \min P_{i+1}$ for $i < k - 1$.

In the following we identify a permutation σ of k with the associated $k \times k$ permutation matrix. This allows one to identify the group $\mathcal{S}_k$ of permutations of k with a group of unitary matrices. Let $\pi : M^{\mathrm{ba}}_{n,k} \to \mathcal{S}_k$ be the function assigning to a matrix A the unique element $\pi(A)$ of $\mathcal{S}_k$ such that $A \cdot \pi(A)$ is the matrix representing an order-preserving embedding. Given an $n \times m$-matrix A, we let $A \cdot M^{\mathrm{ba}}_{m,k} = \{A \cdot B : B \in M^{\mathrm{ba}}_{m,k}\}$.

Theorem 5.2 (DRT, Boolean matrices). *For every k, m and r there is n such that for every $c : M^{ba}_{n,k} \to r$ there is $R \in M^{oba}_{n,m}$ such that π is a factor of c in $R \cdot M^{ba}_{m,k}$. That is, the color of $R \cdot B$ depends only on $\pi(B)$ for every $B \in M^{ba}_{m,k}$.*

Proof. Let $\mathcal{C}$ be a canonically ordered finite Boolean algebra obtained by applying the Dual Ramsey Theorem for canonically ordered Boolean algebras—Theorem 5.1—to the Boolean algebras $\mathcal{P}(k)$, $\mathcal{P}(m)$ canonically ordered as above by $s < t$ when $\min(s \triangle t) \in s$, and to the number of colors $r^{\mathcal{S}_k}$. Without loss of generality we can assume that $\mathcal{C}$ is equal to $\mathcal{P}(n)$ for some $n \in \omega$, since any canonically ordered finite Boolean algebra is of this form. We claim that such an n satisfies the desired conclusions. Indeed, fix a coloring $c : M^{\mathrm{ba}}_{n,k} \to r$. This induces a coloring $f : \mathrm{Emb}_<(\mathcal{P}(k), \mathcal{P}(n)) \to r^{\mathcal{S}_k}$ as follows. Let γ be an element of $\mathrm{Emb}_<(\mathcal{P}(k), \mathcal{P}(n))$, and let $A_\gamma \in M^{\mathrm{ba}}_{n,k}$ be the corresponding representing matrix. Define then $f(\gamma)$ to be element $(c(A_\gamma \cdot \sigma))_{\sigma \in \mathcal{S}_k}$ of $r^{\mathcal{S}_k}$. By the choice of $\mathcal{C} = \mathcal{P}(n)$ there exists $\varrho \in \mathrm{Emb}_<(\mathcal{P}(m), \mathcal{P}(n))$ such that f is constant on $\varrho \circ \mathrm{Emb}_<(\mathcal{P}(k), \mathcal{P}(m))$. Let now $\widehat{c} \in r^{\mathcal{S}_k}$ be the constant value of f. It is now easy to see that $c(A_\varrho \cdot B) = \widehat{c}(\pi(B))$ for every $B \in M^{\mathrm{ba}}_{m,k}$. $\square$

5.1. **Ramsey properties of matrices over a finite field.** It is natural to consider Ramsey properties of other classes of matrices over a field $\mathbb{F}$. We are going to see that for $\mathbb{F}$ finite there is a factorization result similar to the DRT for Boolean matrices, that extends the well known theorem by Graham, Leeb and Rothschild on Grassmannians $\mathrm{Gr}(k, V)$, the family of all k-dimensional subspaces of a vector space V over $\mathbb{F}$.

In the following, given a sequence (x_i) in a vector space E, we let $\langle x_i \rangle$ be its linear span inside E. We also let $(u_i)_{i<d}$ be the canonical basis of the vector space $\mathbb{F}^d$, where each u_i is the unit vector whose unique nonzero coordinate is equal to 1 and it is in the position i.

Theorem (Graham-Leeb-Rothschild [28]). *For every positive integers d, m and r there is $n \geq d$ such that every r-coloring of the Grassmannians* $\mathrm{Gr}(k, \mathbb{F}^n)$ *has a monochromatic set of the form* $\mathrm{Gr}(k, R)$ *for some* $R \in \mathrm{Gr}(m, \mathbb{F}^n)$.

This result is a particular case of the factorization theorem for injective matrices. Recall that a $p \times q$-matrix $A = (a_{ij})$ is in *reduced row echelon form* (RREF) when there is $p_0 \leq p$ and (a unique) strictly increasing sequence $(j_i)_{i<p_0}$ of integers $< q$ such that

(i) $A \cdot u_{j_i} = u_i$ for every $i < p_0$ and
(ii) $\langle A \cdot u_j \rangle_{j<j_i} = \langle u_l \rangle_{l<i}$ for every $i < p_0$.

Notice that if A has rank q and it is in RREF then the $q \times p$-matrix I_A with entries in $\{0, 1\}$ and whose nonzero entries are in the positions (i, j_i) $(i < q)$ is a right inverse to A, that is, $A \cdot I_A = \mathrm{Id}_q$. A $p \times q$-matrix A

is in *reduced column echelon form (RCEF)* when its transpose A^{t} is in RREF. Let $M^k_{n,m}(\mathbb{F})$ be the set of all $n \times m$-matrices of rank k with entries in $\mathbb{F}$, and $\mathcal{E}_{n,m}(\mathbb{F})$ be the collection of matrices in RCEF.

Definition 5.3. Let $\tau : M^k_{n,k} \to \mathrm{GL}(\mathbb{F}^k)$ be the mapping that assigns to each $A \in M^k_{n,k}(\mathbb{F})$ the unique $k \times k$-invertible matrix $\tau(A)$ such that $A \cdot \tau(A)$ is in RCEF.

Theorem 5.4 (Factorization of full rank matrices over a finite field)**.** *For every $k, m, r \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for every $c : M^k_{n,k}(\mathbb{F}) \to r$ there is $R \in \mathcal{E}_{n,m}(\mathbb{F})$ such that τ is a factor of c in $R \cdot M^k_{m,k}(\mathbb{F})$.*

Observe that this gives immediately the Graham-Leeb-Rothschild Theorem—Theorem 5.1—as every k-dimensional subspace of $\mathbb{F}^n$ can be represented as the linear span of the columns of a matrix in RCEF. The proof of Theorem 5.4 is a direct consequence of the DRT and the next propositions. In the following, we fix an ordering $<$ on the finite field $\mathbb{F}$ such that $0 < 1$ are the first two elements of $\mathbb{F}$. We let $\mathbb{F}^n$ be endowed with the corresponding antilexicographic order $<_{\mathrm{alex}}$. Let $\Phi_{n,k} : \mathrm{Epi}(n, \mathbb{F}^k) \to M^k_{n,k}$ be the function assigning to each rigid surjection f the matrix whose rows are $f(j)$ for $j < n$.

Lemma 5.5. *$\Phi_{n,k}(f)$ is a full rank matrix in RCEF.*

Proof. It is clear that $\Phi_{n,k}(f)$ is a full rank matrix. We prove that it is in RCEF. For each $i \in k$, let $j_i := \min\{j < n \ : \ A \cdot u_j = u_i\}$. The sequence $(j_i)_{i<m}$ is strictly increasing, since f is a rigid surjection. If $j < j_i$, then $A \cdot u_j <_{\mathrm{alex}} u_i$, by the minimality of j_i and the fact that f is a rigid surjection. Therefore $A \cdot u_j \in \langle u_l \rangle_{l<i}$. This shows that $(j_i)_{i<k}$ witnesses that the transpose A of $\Phi_{n,k}(f)$ is in RREF. □

Proposition 5.6. *For $A \in M^k_{k,n}(\mathbb{F})$ the following are equivalent.*

(1) A is in RREF.

(2) The linear map $T : \mathbb{F}^n \to \mathbb{F}^k$ represented by A in the corresponding unit bases is a rigid surjection and for every $i < k$ there is a column of A equal to u_i.

Proof. *(1)⇒(2)* Suppose that A is in RREF. We will prove that the linear operator $T : \mathbb{F}^n \to \mathbb{F}^k$, $T(u_i) := A \cdot u_i$ ($i < n$) is a rigid surjection from $\mathbb{F}^n$ to $\mathbb{F}^k$ endowed with the antilexicographical ordered $<_{\mathrm{alex}}$ explained before. Let $(j_i)_{i<k}$ be the strictly increasing sequence of in n witnessing that A is in RREF. By linearity, $T(0) = 0$. Fix now $w \in \mathbb{F}^k$.

Claim 5.6.1. $\min_{<_{\mathrm{alex}}} T^{-1}(w) = I_A \cdot w$.

From this, since $I_A : \mathbb{F}^k \to \mathbb{F}^n$ is clearly $<_{\mathrm{alex}}$-increasing, we can conclude that T is a rigid surjection.

Proof of Claim: So suppose that $(v_j)_{j<n} = \bar{v} = \min_{<_{\mathrm{alex}}}\{v \in \mathbb{F}^n \ : \ A \cdot v = w\}$. Set $z = (z_j)_j := I_B(w)$. We prove by induction on $i < k$ that $v_j = z_j$ for every $j \ge j_{k-i-1}$. Suppose that $k = 0$. Since for every $j > j_{k-1}$ one has that $z_j = 0$, we obtain that $v_j = 0$, by $<_{\mathrm{alex}}$-minimality of $\bar{v}$. Let $(A)_{k-1}$ be the $(k-1)^{\mathrm{th}}$-row of A. It follows that $(A)_{k-1} = u_{k-1} + y$, where $y \in \langle u_j \rangle_{j > j_{k-1}}$. Hence,

$$z_{j_{k-1}} = w_{k-1} = (A)_{k-1} \cdot \bar{v} = v_{j_{k-1}}.$$

Suppose that the conclusion holds for i, that is, $v_j = z_j$ for every $j \ge j_{k-i-1}$. We will prove that it also holds for $i+1$. Since $v \le_{\mathrm{alex}} z$, and $z_j = 0$ for every $j_{k-i-2} < j < j_{k-i-1}$, we obtain that $v_j = 0$ for such j's. Then the $(k-i-2)^{\mathrm{th}}$ row of A is of the form $(A)_{k-i-2} = u_{j_{k-i-2}} + y$ with y in the linear span of $\{u_j \ : \ j > j_{k-i-2},\ j \neq j_p \text{ for all } p\}$. Since $\bar{v} = x_0 + v_{j_{k-i-2}} + x_1$ with $x_0 \in \langle u_j \rangle_{j < j_{k-i-2}}$ and $x_1 \in \langle u_{j_l} \rangle_{k-i-2<l<k}$, it follows that

$$z_{j_{k-i-2}} = w_{k-i-2} = (A)_{k-i-2} \cdot \bar{v} = v_{j_{k-i-2}}.$$ □

(2)⇒(1) Now suppose that T is an rigid surjection from $\mathbb{F}^n$ to $\mathbb{F}^k$ with respect to the antilexicographical orderings, and that for every $i < k$ a column of A is u_i. For each $i < k$, let j_i be the first such column of A. We prove that $(j_i)_{i<k}$ witnesses that A is in RREF, that is:

Claim 5.6.2. $T\langle u_j \rangle_{j<j_i} = \langle u_l \rangle_{l<i}$ for every $i < k$.

Proof of Claim: The proof is by induction on i. If $i = 0$, then $T\langle u_j \rangle_{j<j_0} = 0$ because u_0 is the second element of $\mathbb{F}^n$ in the antilex ordering. Suppose the result is true for i, and let us extend it to $i+1$. In particular, we know that $j_{i+1} > j_i$, and it is clear that $\langle u_l \rangle_{l \le i} \subseteq T\langle u_j \rangle_{j \le j_i} \subseteq \langle u_j \rangle_{j < j_{i+1}}$. Suppose towards a contradiction that there exists j such that $j_i < j < j_{i+1}$ and $T(u_j) \notin \langle u_l \rangle_{l \le i}$. Denote by ξ the least such j. Thus, $u_{i+1} \le_{\mathrm{alex}} T(u_\xi)$, hence there is some $x \le_{\mathrm{alex}} u_\xi$ such that $T(x) = u_{i+1}$. This means, by the minimality of ξ, that $T(u_\xi) = y + u_{i+1}$ with $y \in \langle u_l \rangle_{l \le i}$. We know that $y \neq 0$ by the minimality of j_{i+1}; so $u_{i+1} <_{\mathrm{alex}} y + u_{i+1}$. Hence,

$$\min T^{-1}(u_{i+1}) <_{\mathrm{alex}} \min T^{-1}(y + u_{i+1}) = u_\xi.$$

In other words, there must be $x \in \langle u_j \rangle_{j<\xi}$ such that $T(x) = u_{i+1}$, which is impossible by the minimality of ξ. $\square$ $\square$

Proof of Theorem 5.4. Fix all parameters. We consider $\mathbb{F}^k$ and $\mathbb{F}^m$ antilexicographically ordered $<_{\mathrm{alex}}$ as explained before. Let n be obtained from the linear orderings $(\mathbb{F}^k, <_{\mathrm{alex}})$, $(\mathbb{F}^m, <_{\mathrm{alex}})$ and the number of colors r^λ, where $\lambda = \prod_{i=0}^{k-1}(p^k - p^i)$ is the order of the group $\mathrm{GL}(\mathbb{F}^k)$, by applying the Dual Ramsey Theorem for rigid surjections (Theorem 1.2). We claim that n satisfies the desired conclusions. Fix a coloring $c : M_{n,k}^k(\mathbb{F}) \to r$. Let $c_0 : \mathrm{Epi}(n, \mathbb{F}^k) \to r^{\mathrm{GL}(\mathbb{F}^k)}$ be the coloring defined by $c_0(\sigma) := (c(\Phi_{k,n}(\sigma) \cdot \Gamma^{-1}))_{\Gamma \in \mathrm{GL}(\mathbb{F}^k)}$ for $\sigma \in \mathrm{Epi}(n, \mathbb{F}^k)$. By the choice of n, there exists $\varrho \in \mathrm{Epi}(n, \mathbb{F}^m)$ such that c_0 is constant on $\mathrm{Epi}(\mathbb{F}^m, \mathbb{F}^k) \circ \varrho$ with constant value $\widehat{c} \in r^{\mathrm{GL}(\mathbb{F}^k)}$. Let $R := \Phi_{n,m}(\varrho)$. We claim that R and $\widehat{c}$ satisfy the conclusion of the statement in the theorem. It follows from Proposition 5.5 that $R \in \mathcal{E}_{n,m}$. Now let $A \in M_{m,k}^k(\mathbb{F})$. We have to prove that $c(R \cdot A) = \widehat{c}(\tau(R \cdot A))$. First, note that $\tau(R \cdot A) = \tau(A)$, because R is in RCEF. Let B be the transpose of $\mathrm{red}_c(A)$ (i.e. B is the RREF of the transpose of A), and let $T : \mathbb{F}^m \to \mathbb{F}^k$ be the linear operator defined by B in the canonical bases. We know by Proposition 5.6 that $T \in \mathrm{Epi}(\mathbb{F}^m, \mathbb{F}^k)$.

Claim 5.6.3. $\Phi_{n,k}(T \circ \varrho) = R \cdot \mathrm{red}_c(A)$.

Proof of Claim: Fix $j < m$. Then the j^{th}-row $(\Phi_{n,k}(T \circ \varrho))_j$ of $\Phi_{n,k}(T \circ \varrho)$ is the row vector $T(\varrho(j))$. Hence,

$$(\Phi_{n,k}(T \circ \varrho))_j = T(\varrho(j)) = ((\mathrm{red}_c(A))^{\mathrm{t}} \cdot ((R)_j)^{\mathrm{t}})^{\mathrm{t}} = (R)_j \cdot \mathrm{red}_c(A) = (R \cdot \mathrm{red}(A))_j. \qquad \square$$

So, given $\Gamma \in \mathrm{GL}_k(\mathbb{F})$ we have that

$$c(R \cdot A) = c(R \cdot \mathrm{red}_c A \cdot \tau(A)^{-1}) = (c_0(R \cdot \mathrm{red}_c A))(\tau(A)) = \widehat{c}(\tau(A)) = \widehat{c}(\tau(R \cdot A)). \qquad \square$$

5.1.1. *Square matrices of rank k.* We conclude our study of matrices over a finite field by presenting the Ramsey factorization result of square matrices.

Definition 5.7. Given k and n, let $\tau^{(2)} : M_{n,n}^k \to \mathrm{GL}(\mathbb{F}^k)$ be the mapping uniquely defined by the relation $A = A_0 \cdot \tau^{(2)}(A) \cdot A_1^{\mathrm{t}}$ for some $A_0, A_1 \in \mathcal{E}_{n,k}$.

It is routine to see that $\tau^{(2)}$ is well defined.

Theorem 5.8 (Factorization of square matrices)**.** *For every positive integers k, m and r there is n such that every $c : M_{n,n}^k(\mathbb{F}) \to r$ there are $R_0, R_1 \in \mathcal{E}_{n,m}$ such that $\tau^{(2)}$ is a factor of c in $R_0 \cdot M_{m,m}^k(\mathbb{F}) \cdot R_1^{\mathrm{t}}$, i.e., the c-coloring of $R_0 \cdot A \cdot R_1^{\mathrm{t}}$ only depends on $\tau^{(2)}(A)$.*

Proof. Let $n_0 := n_{\mathbb{F}}(k, m, r^{\mathrm{GL}(\mathbb{F}^k)})$, and let $n := n_{\mathbb{F}}(k, n_0, r^{\mathcal{M}_{n_0,k}^k})$. We claim that n works. Let $f : M_{n,n}^k \to r$. Let $P \in \mathcal{E}_{n,n_0}$ be fixed. We define the coloring $c : M_{n,k}^k \to r^{M_{n_0,k}^k}$ by

$$c(A) := (f(A \cdot B^{\mathrm{t}} \cdot P^{\mathrm{t}}))_{B \in \mathcal{M}_{n_0,k}^k}.$$

The coloring c is well defined because $A \cdot B^{\mathrm{t}} \cdot P^{\mathrm{t}}$ has always rank k. Let $R \in \mathcal{E}_{n,n_0}$ and $c_0 : \mathrm{GL}(\mathbb{F}^k) \to r^{M_{n_0,k}^k}$ be such that $c(R \cdot A) = c_0(\tau(A))$ for every $A \in M_{n_0,k}^k$. Now let $d : M_{n_0,k}^k \to r^{\mathrm{GL}(\mathbb{F}^k)}$ be defined by

$$d(B) := (c_0(\Gamma)(B))_{\Gamma \in \mathrm{GL}(\mathbb{F}^k)}.$$

Let $S \in \mathcal{E}_{n_1,m}$ and $d_0 : \mathrm{GL}(\mathbb{F}^k) \to r^{\mathrm{GL}(\mathbb{F}^k)}$ be such that $d(S \cdot B) = d_0(\tau(B))$ for every $B \in M_{m,k}^k$. Set $R_0 = R \cdot Q$ and $R_1 := P \cdot S$, where $Q \in \mathcal{E}_{n_0,m}$ is arbitrary. Finally, let $g : \mathrm{GL}(\mathbb{F}^k) \to r$ be defined by $g(\Gamma) = d_0(\Gamma_0)(\Gamma_1)$, where $\Gamma = \Gamma_0 \cdot \Gamma_1^{\mathrm{t}}$ are arbitrary. Notice that if $\Gamma = \Gamma_1 \cdot \Gamma_0^{\mathrm{t}}$, then it follows that

$$\begin{aligned} d_0(\Gamma_0)(\Gamma_1) =& d(S \cdot P_0 \cdot \Gamma_0)(\Gamma_1) = c_0(\Gamma_1)(S \cdot P_0 \cdot \Gamma_0) = c(R \cdot P_1 \cdot \Gamma_1)(S \cdot P_0 \cdot \Gamma_0) = \\ =& f(R \cdot P_1 \cdot \Gamma_1 \cdot \Gamma_0^{\mathrm{t}} \cdot P_0^{\mathrm{t}} \cdot R_1^{\mathrm{t}}) = f(R \cdot P_1 \cdot \Gamma \cdot P_0^{\mathrm{t}} \cdot R_1^{\mathrm{t}}) \end{aligned}$$

where $P_0 \in \mathcal{E}_{m,k}$ and $P_1 \in \mathcal{E}_{n_0,k}$ are arbitrary. This means that g does not depend on the chosen decomposition $\Gamma = \Gamma_1 \cdot \Gamma_0^{\mathrm{t}}$. Similarly one proves that $g(\tau^{(2)}(A)) = f(R_0 \cdot A \cdot R_1^{\mathrm{t}})$ for every $A \in M_{m,m}^k$. $\square$

6. Matrices and grassmannians over $\mathbb{R}, \mathbb{C}$, and normed spaces

We present now factorization results of compact colorings of matrices and Grassmannians over the fields $\mathbb{F} = \mathbb{R}, \mathbb{C}$. There are many such results, depending on the chosen metric. These factorizations are approximate, but apply to arbitrary colorings given by Lipschitz mappings with values in a compact metric space. The collection of matrices $M_{n,m}$ can be naturally turned into a metric space by fixing two norms M and N on $\mathbb{F}^m$ and $\mathbb{F}^n$, respectively, and then by identifying a matrix $A \in M_{n,m}$ with the linear operator $T_A : \mathbb{F}^m \to \mathbb{F}^n$, $T_A(x) := A \cdot x$. This allows to define the distance $d_{M,N}(A, B) := \|T_A - T_B\|_{(\mathbb{F}^m,M),(\mathbb{F}^n,N)}$. In this way, each full rank $n \times m$-matrix A defines a norm $\nu(A)$ on $\mathbb{F}^m$, $\nu(A)(x) := N(Ax)$. Roughly speaking, the factorization states that when M and N are, for example, the p-norms, for $p \in [1, +\infty]$ other than an even integer strictly larger than 2, every coloring of full rank matrices is determined, approximately, by ν. Geometrically, this means that the coloring of a full-rank matrix A is determined by the centered section of $\operatorname{Im} A$ with the unit ball of $(\mathbb{F}^n, N)$. Similarly, each k-dimensional subspace V of $\mathbb{F}^n$ determines a member of the *Banach-Mazur* compactum $\mathcal{B}_k$, that is, the isometry class $\tau_N(V)$ of all k-dimensional normed spaces isometric to (V, N). We prove that for appropriate choices of norms N, any coloring of the k-Grassmannians is determined by the centered section of V with the unit ball of $(\mathbb{F}^n, N)$, up to a k-linear transformation. In general, we obtain factorizations of this kind for a sequence of norms $(N_k)_k$ on each $\mathbb{F}^k$ such that $\mathcal{A} = \{(\mathbb{F}^k, N_k)\}_k$ is a stable Fraïssé class such that the limit $\operatorname{FLim} \mathcal{A}$ is approximately ultrahomogeneous.

We will also give in Subsection 6.3 a quantitative proof, with estimates on the corresponding Ramsey numbers, of the approximate Ramsey property of the finite-dimensional normed spaces using injective envelopes. Finally, by using Lipschitz free spaces associated to finite metric spaces, we give another proof of the approximate Ramsey property of finite metric spaces, proved first indirectly by Pestov in [62]. In fact, finite metric spaces also satisfy the (exact) Ramsey property itself, as later shown by Nešetřil in [56]. In this section $\mathbb{F}$ will be either $\mathbb{R}$ or $\mathbb{C}$. We start by introducing the notion of approximate factor.

Definition 6.1. Let (M, d_M), (N, d_N) and (P, d_P) be metric spaces, $\varepsilon > 0$, and $c : (M, d_M) \to (N, d_N)$ and $\pi : (M, d_M) \to (P, d_P)$ be metric colorings, i.e. 1-Lipschitz maps. We say that π is an *ε-approximate factor* (or simply *ε-factor*) of c if there is some coloring $\widehat{c} : (P, d_P) \to (N, d_N)$ such that

$$\sup_{x \in M} d_N(c(x), \widehat{c}(\pi(x))) \leq \varepsilon.$$

That is, "up to ε" $c = \widehat{c} \circ \pi$. Given $M_0 \subseteq M$ we say that π is an *ε-factor of c in M_0* if $\pi \restriction_{M_0} : M_0 \to N$ is an ε-factor of $c \restriction M_0$, i.e., there is some coloring $\widehat{c} : P \to N$ such that $\sup_{x \in M_0} d_N(c(x), \widehat{c}(\pi(x))) \leq \varepsilon$.

We have seen in Section 1 how to make $M_{n,m}$ a metric space by considering $m \times n$-matrices as particular representations of a linear operator between normed spaces $(\mathbb{F}^n, N)$ and $(\mathbb{F}^m, M)$, and then considering the operator norm. The factorization results of colorings of matrices will be then reformulations of the corresponding results for colorings of collections of operators. We recall some basic notions concerning linear operators on Banach spaces.

Definition 6.2. Let X, Y be two Banach spaces, k be a positive integer, and $\lambda \geq 1$. Then

- $\operatorname{Ball}(X) = \{x \in X : \|x\| \leq 1\}$ denotes the unit ball of X;
- $\mathcal{L}(X, Y)$ is the Banach space of all bounded linear operators $T : X \to Y$, endowed with the operator norm $\|T\| := \sup_{\|x\|_X = 1} \|Tx\|_Y$, and the corresponding norm distance $d_{X,Y}(T, U) := \|T - U\|_{X,Y}$.
- When TX is finite-dimensional , let

  $$\|T^{-1}\| := \min\{a \geq 0 : \operatorname{Ball}(TX) \subseteq aT(\operatorname{Ball}(X))\}.$$

  Let $\mathcal{L}_\lambda(X, Y)$ be the set of all $T \in \mathcal{L}(X, Y)$ such that $\lambda^{-1} \cdot \operatorname{Ball}(TX) \subseteq T(\operatorname{Ball}(X)) \subseteq \lambda \cdot \operatorname{Ball}(Y)$, where $\operatorname{Im} T = (\operatorname{Im} T, \|\cdot\|_Y)$.
- Let $\mathcal{L}^k(X, Y)$ be the set of all $T \in \mathcal{L}(X, Y)$ such that $\dim \operatorname{Im} T = k$, and let $\mathcal{L}_r^k(X, Y) = \mathcal{L}_r(X, Y) \cap \mathcal{L}^k(X, Y)$.
- $\mathcal{L}^{k,\mathrm{w}^*}(X^*, X)$ is the metric space of all the w*-to-norm continuous linear operators from X^* to X of rank k, endowed with the operator metric; let $\mathcal{L}_\lambda^{k,\mathrm{w}^*}(X^*, X) := \mathcal{L}^{k,\mathrm{w}^*}(X^*, X) \cap \mathcal{L}_\lambda(X^*, X)$;
- the k-Grassmannian $\operatorname{Gr}(k, X)$ of X is naturally a topological space, as it can be identified with the quotient of X^k by the relation $(x_i)_{i<k} \sim (y_i)_{i<k}$ iff $\langle x_i \rangle_{i<k} = \langle y_i \rangle_{i<k}$. This turns $\operatorname{Gr}(k, X)$ into a Polish space. A natural compatible metric is the *gap (or opening) metric*, $\Lambda_X(V, W)$ defined as the Hausdorff distance, with respect to the norm metric in X, between the unit balls $\operatorname{Ball}(V, \|\cdot\|_X)$ and $\operatorname{Ball}(W, \|\cdot\|_X)$,

that is,

$$\Lambda_X(V,W) := \max\{\max_{v\in V, \|v\|_X\le 1}\min_{w\in W,\|w\|_X\le 1}\|v-w\|_X, \max_{w\in W,\|w\|_X\le 1}\min_{v\in V,\|v\|_X\le 1}\|v-w\|_X\}.$$

When V is a finite-dimensional vector space and E is a Banach space, we will write $\mathcal{L}(V,E)$ to denote the set of all linear maps from V to E. This is consistent with the above notation since any linear map from V to E is bounded with respect to any given norm on V. The choice of the notation $\|T^{-1}\|$ is justified by the fact that when $T: X\to Y$ is invertible, $\|T^{-1}\|$ is the norm of the inverse operator $T^{-1}: Y\to X$. In general, $\|T^{-1}\|$ is the norm of the inverse of the operator $T: X\to \operatorname{Im} T$. Observe that $T: X\to Y$ has rank k if and only if $T^*: Y^*\to X^*$ has rank k. In this case, $\|T^*\| = \|T\|$ and $\|(T^*)^{-1}\| = \|T^{-1}\|$. The special case when $T: X\to Y$ is 1-1 and $r=1$ corresponds to T being an isometric embedding. The collection of such maps is denoted by $\operatorname{Emb}(X,Y)$. As for the rank decomposition mentioned above, it is easy to see that an operator $T\in\mathcal{L}^k(E^*,E)$ is w*-to-norm continuous if and only if $T = T_0\circ T_1^*$ for some $T_0\in\mathcal{L}^k(\mathbb{F}^k,E)$ and $T_1\in\mathcal{L}^k((\mathbb{F}^k)^*,E)$.

Definition 6.3 (Canonical actions)**.** Let X,E be two Banach spaces. Recall that $\operatorname{Aut}(E)$ is the topological group of all surjective linear isometries on E, endowed with its strong operator topology. Then,

- $\operatorname{Aut}(E)\curvearrowright\mathcal{L}(X,E)$ is the canonical action by isometries $g\cdot T := g^{-1}\circ T$.
- $\operatorname{Aut}(E)^2\curvearrowright\mathcal{L}^{k,\mathrm{w}^*}(E^*,E)$ is the canonical action by isometries $(g,h)\cdot T := g\circ T\circ h^*$ for $(g,h)\in\operatorname{Iso}(E)^2$ and $T\in\mathcal{L}^{k,\mathrm{w}^*}(E^*,E)$.
- $\operatorname{Aut}(E)\curvearrowright\operatorname{Gr}(k,E)$ is the canonical action by isometries $g\cdot V := g(V)$.

Note that $\mathcal{L}^k(X,E)$ and $\mathcal{L}_r(X,E)$ are $\operatorname{Iso}(E)$-closed, while $\mathcal{L}_\lambda^{k,\mathrm{w}^*}(X^*,X)$ is $\operatorname{Iso}(E)^2$-invariant.

Lemma 6.4. *Suppose that X,E are Banach spaces such that X is finite-dimensional and $G := \operatorname{Aut}(E)$ is extremely amenable. Let $k\in\mathbb{N}^*$, $\lambda\ge 1$, $\varepsilon>0$, let Y be a finite-dimensional normed space that can be isometrically embedded into E, and let (K,d_K) be a compact metric space. Then,*

(1) for every K-coloring c of $(\mathcal{L}_\lambda^k(X,E), d_{X,E})$ there is $R\in\operatorname{Emb}(Y,E)$ such that the quotient map $\pi: \mathcal{L}_\lambda^k(X,E)\to\mathcal{L}_\lambda^k(X,E)/\!\!/G$ is an ε-factor of c in $R\circ\mathcal{L}_\lambda^k(X,Y)$;

(2) for every K-coloring c of $(\mathcal{L}_\lambda^{k,\mathrm{w}^*}(E^*,E), d_{E^*,E})$ there are $R_0,R_1\in\operatorname{Emb}(Y,E)$ such that the quotient map $\pi: \mathcal{L}_\lambda^{k,\mathrm{w}^*}(E^*,E)\to\mathcal{L}_\lambda^{k,\mathrm{w}^*}(E^*,E)/\!\!/G^2$ is an ε-factor of c in $R_0\circ\mathcal{L}_\lambda^{k,\mathrm{w}^*}(Y^*,Y)\circ R_1^*$;

(3) for every K-coloring c of $(\operatorname{Gr}(k,E),\Lambda_E)$ there exists $V\in\operatorname{Gr}(\dim Y,E)$ such that $(V,\|\cdot\|_E)$ is isometric to Y and such that the quotient map $\pi:\operatorname{Gr}(k,E)\to\operatorname{Gr}(k,E)/\!\!/G$ is an ε-factor of c in $\operatorname{Gr}(k,V)$.

Proof. This is a direct consequence of Proposition 1.1 applied, respectively, to the canonical actions $G\curvearrowright\mathcal{L}_r^k(X,E)$, $G\curvearrowright\operatorname{Gr}(k,E)$, and $G^2\curvearrowright\mathcal{L}^{k,\mathrm{w}^*}(E^*,E)$ introduced in Definition 6.3, together with the fact that $\mathcal{L}_\lambda^k(X,Y)$, $\mathcal{L}_\lambda^{k,\mathrm{w}^*}(Y^*,Y)$ and $\operatorname{Gr}(k,Y)$ are compact. □

The next notion will allow us to obtain an asymptotic version of Lemma 6.4. Recall that a Banach space E is approximately ultrahomogeneous when for every finite-dimensional subspace $X\subseteq E$, every isometric embedding $\phi: X\to M$, and every $\varepsilon>0$ there is $\alpha\in\operatorname{Aut}(E)$ such that $\|\alpha\restriction X-\phi\|\le\varepsilon$.

Definition 6.5. Let E be a separable approximately ultrahomogeneous Banach space. We will say that an increasing sequence $(E_n)_n$ of subspaces of E is *adequate* when

(i) $\dim E_n$ of dimension n.
(ii) $\bigcup_n E_n$ is dense in E.
(iii) For every m and every $g\in\operatorname{Aut}(E)$ there is n and $\gamma\in\operatorname{Emb}(E_m,E_n)$ such that $\|g\restriction_{E_m}-\gamma\|<\varepsilon$.

It is not difficult to see that whenever a class $\mathcal{A}$ is a Fraïssé class, its limit $\operatorname{FLim}\mathcal{A}$ has an adequate sequence consisting of a cofinal family $(E_n)_n$ in $\mathcal{A}$. (This means that for every X in $\mathcal{A}$ and $\delta>0$ there exists $n\in\mathbb{N}$ such that X admits a δ-embedding into E_n.) In particular, the spaces $L_p[0,1]$, $1\le p<\infty$, and $\mathbb{G}$ have an adequate sequence $(E_n)_n$ such that each E_n is isometric to ℓ_p^n and ℓ_∞^n, respectively. The following is the asymptotic version of Lemma 6.4.

Corollary 6.6. *Suppose that E,X are Banach spaces such that X is finite-dimensional and $G := \operatorname{Aut}(E)$ is extremely amenable, and suppose that $(E_n)_n$ is an adequate sequence of E. Let k be a positive integer, $\lambda\ge 1$, $\varepsilon>0$, let Y be a finite-dimensional space that can be isometrically embedded into some E_n, and let (K,d_K) be a compact metric space. Set $\bar\lambda := \max\{1,(1-\varepsilon)\lambda\}$. Then,*

(1) there is $n\in\mathbb{N}$ such that for every K-coloring c of $(\mathcal{L}_\lambda^k(X,E_n), d_{X,E_n})$ there is $R\in\operatorname{Emb}(Y,E_n)$ such that the quotient map $\pi:\mathcal{L}_\lambda^k(X,E)\to\mathcal{L}_\lambda^k(X,E)/\!\!/G$ is an ε-factor of c in $R\circ\mathcal{L}_\lambda^k(X,Y)$;

(2) there is $n \in \mathbb{N}$ such that for every K-coloring c of $(\mathcal{L}_\lambda^{k,\mathrm{w}^*}(E_n^*, E_n), d_{E_n^*, E_n})$ there are $R_0, R_1 \in \mathrm{Emb}(Y, E_n)$ such that the quotient map $\pi : \mathcal{L}_\lambda^{k,\mathrm{w}^*}(E^*, E) \to \mathcal{L}_\lambda^{k,\mathrm{w}^*}(E^*, E) /\!\!/ G^2$ is an ε-factor of c in $R_0 \circ \mathcal{L}_\lambda^{k,\mathrm{w}^*}(Y^*, Y) \circ R_1^*$;
(3) there is $n \in \mathbb{N}$ such that for every K-coloring c of $(\mathrm{Gr}(k, E_n), \Lambda_{E_n})$ there $V \in \mathrm{Gr}(\dim Y, E_n)$ such that $(V, \|\cdot\|_{E_n})$ is isometric to Y and such that the quotient map $\pi : \mathrm{Gr}(k, E) \to \mathrm{Gr}(k, E) /\!\!/ G$ is an ε-factor of c in $\mathrm{Gr}(k, V)$.

Proof. The proofs of *(1)*, *(2)*, and *(3)* are similar, and reminiscent of the proof of the implication *(2)* $\Rightarrow$ *(5)* in Proposition 1.15. We only give the details for *(1)*. Without loss of generality we may assume that $Y = E_m$ for some large m. Suppose by contradiction that, for some compact space (K, d_K), that there is no such n. Therefore for each n there exists a coloring $c_n : \mathcal{L}_\lambda^k(X, E_n) \to (K, d_K)$ providing a counterexample. Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. Set $\bar{\varepsilon} = \varepsilon/4$. For each $T \in \mathcal{L}_\lambda^k(X, E)$, let $c(T) \in K$ be defined as follows. Choose a sequence $(T_n)_{n\in\mathbb{N}}$ converging to T with $T_n \in \mathcal{L}_\lambda^k(X, E_n)$ for every $n \in \mathbb{N}$. Let $c(T) := \mathcal{U} - \lim c_n(T_n)$. This limit exists because K is compact. It is easy to see that c defines a coloring. Let $\pi : \mathcal{L}_\lambda^k(X, E) \to \mathcal{L}_\lambda^k(X, E) /\!\!/ G$ be the quotient mapping. By Lemma 6.4 there exist $S \in \mathrm{Emb}(Y, E)$ and a coloring $\widehat{c} : (\mathcal{L}_\lambda^k(X, E) /\!\!/ G, \widehat{d}) \to (K, d_K)$ such that $d_K(c(S \circ T), \widehat{c}(\pi(T)) \leq \varepsilon$ for every $T \in \mathcal{L}_\lambda^k(X, Y)$.

Let $(S_n)_n$ be a sequence converging to S with $S_n \in \mathrm{Emb}(Y, E_n)$ for every $n \in \mathbb{N}$. This sequence exists since $(E_n)_n$ is adequate (recall that we are supposing $Y = E_m$). It follows that for every $T \in \mathcal{L}_\lambda^k(X, Y)$ we have that $c(S \circ T) = \mathcal{U} - \lim c_n(S_n \circ T)$. Let D be a finite $\bar{\varepsilon}$-dense subset of $\mathcal{L}_\lambda^k(X, Y)$. Choose n such that

$$\max_{T\in D} d_K(c_n(S_n \circ T), c(S \circ T)) < \bar{\varepsilon}.$$

We claim that n, $R := S_n$ and $\widehat{c}$ contradict the assumption that c_n is a counterexample. Fix $T \in \mathcal{L}_\lambda^k(X, Y)$. Let $U \in D$ be such that $\|T - U\| \leq \bar{\varepsilon}$. It follows that

$$\begin{aligned} d_K(\widehat{c}(\pi(T)), c_n(S_n \circ T)) =& d_K(\widehat{c}(\pi(S_n \circ T)), c_n(S_n \circ T)) \leq d_K(\widehat{c}(\pi(S \circ T)), c(S \circ T)) + \\ &+ d_K(c_n(S_n \circ T), c(S \circ T)) \leq \bar{\varepsilon} + d_K(c_n(S_n \circ U), c(S \circ U)) + \\ &+ d_K(c_n(S_n \circ T), c_n(S_n \circ U)) + d_K(c(S \circ T), c(S \circ U)) \leq \varepsilon. \end{aligned}$$ □

6.1. **Orbit spaces for approximately ultrahomogeneous spaces.** In general, there is no known explicit description of the orbit spaces considered in Corollary 6.6. However, in the case of approximately ultrahomogeneous Banach spaces, one can give an explicit description of such orbit spaces, as we explain in this subsection.

Definition 6.7 (Canonical orbit spaces)**.** Let V be a finite-dimensional vector space, E be a Banach space, and k be a positive integer.

- $\mathcal{N}_V$ is the set of all norms on V, endowed with the topology of pointwise convergence;
- $\omega(N, P) := \log \max\{\|\mathrm{Id}\|_{N,P}, \|\mathrm{Id}\|_{P,N}\}$ is the *intrinsic metric* on $\mathcal{N}_V$, which is compatible with the topology of pointwise convergence;
- let $\mathrm{GL}(V) \curvearrowright \mathcal{N}_V \times \mathcal{N}_{V^*}$ be the action $\Delta \cdot (N_0, N_1) := (\Delta \cdot N_0, \Delta \cdot N_1)$, where $(\Delta \cdot N_0)(x) := N_0(\Delta^{-1}(x))$ for $x \in V$ and $(\Delta \cdot N_1)(f) := N_1(\Delta^*(f))$ for $f \in V^*$;
- Let $\mathcal{D}_V$ and $\mathcal{D}_V(E)$ be the orbit spaces $(\mathcal{N}_V \times \mathcal{N}_{V^*}) /\!\!/ \mathrm{GL}(V)$ and $(\mathcal{N}_V(E) \times \mathcal{N}_{V^*}(E)) /\!\!/ \mathrm{GL}(V)$, respectively;
- $\omega_2((N_0, N_1), (P_0, P_1)) := \omega(N_0, P_0) + \omega(N_1, P_1)$, and its corresponding quotient metric $\widehat{\omega}_2$ are compatible $\mathrm{GL}(V)$-invariant metrics on $\mathcal{N}_V \times \mathcal{N}_{V^*}$ and on $\mathcal{D}_V$, respectively;
- The *Banach-Mazur compactum* $\mathcal{B}_k$ is the orbit space $\mathcal{N}_k /\!\!/ \mathrm{GL}(\mathbb{F}^k)$;
- the quotient metric $d_{\mathrm{BM}}(N, P)$ on $\mathcal{B}_k$ is defined by
$$d_{\mathrm{BM}}(N, P) := \log \inf_{\Delta\in\mathrm{GL}(\mathbb{F}^k)} \|\Delta\|_{N,P} \cdot \|\Delta^{-1}\|_{P,N}.$$

The equivalence class of a given norm $N \in \mathcal{N}_k$ will be denoted by $[N]_{\mathrm{BM}}$. It is well known that the metric space $(\mathcal{B}_k, d_{\mathrm{BM}})$ is compact, and its diameter is at most $\log k$; see for example [1, Theorem 12.1.4]. Usually, the compatible distance considered in $\mathcal{B}_k$ is not the quotient metric $\widehat{\omega}$, but the 2-Lipschitz equivalent *Banach-Mazur metric*, $d([M]_{\mathrm{BM}}, [N]_{\mathrm{BM}}) = \inf_{\Delta\in\mathrm{GL}(\mathbb{F}^k)} \|\Delta\|_{(\mathbb{F}^k,M),(\mathbb{F}^k,N)} \cdot \|\Delta^{-1}\|_{(\mathbb{F}^k,N),(\mathbb{F}^k,M)}$.

Definition 6.8. Let V be a finite-dimensional vector space, E be a Banach space, k be a positive integer, and $\lambda \geq 1$. We define

- given a norm N on V, let $B_\omega(N; \lambda)$, $B_\omega[N; \lambda]$, be the open, respectively closed, ω-ball of $\mathcal{N}_V$ centered in N and of radius $\log \lambda$;
- the dual norm N^* of N is a norm in V^* defined for $f \in V^* = \mathcal{L}(V, \mathbb{F})$ by $N^*(f) := \max_{N(v)=1} f(v)$;

- let $\mathcal{N}_V(E)$ be the collection of all norms N on V such that (V,N) can be isometrically embedded into E, and let $B_\omega^E(N;\lambda)$ and let $B_\omega^E[N;\lambda]$ by the relative versions of ω-balls to $(\mathcal{N}_V(E),\omega)$;
- let $\mathcal{D}_V^\lambda := \{([N_0,N_1] \in \mathcal{N}_V \times \mathcal{N}_{V^*} \,:\, \lambda^{-1}N_1^*(x) \le N_0(x) \le \lambda N_1^*(x)$ for all $x\in V\}$, and let $\mathcal{D}_V^\lambda(E) := \mathcal{D}_V(E)\cap \mathcal{D}_V^\lambda$;
- let $\mathcal{B}_k(E)$ be the collection of the classes $[N]_{\mathrm{BM}}$, where N ranges in $\mathcal{N}_k(E)$.

In general, $\mathcal{N}_V(E)$, $\mathcal{D}_V^\lambda(E)$ or $\mathcal{B}_k(E)$ are not closed, but they are closed when E is approximately ultrahomogeneous; see Proposition 6.12.

Definition 6.9 (Interpretation of the quotient mappings)**.** Let V be a vector space with $\dim V = k$ and let E be a Banach space. We define

- $\nu_{V,E} : \mathcal{L}^k(V,E) \to \mathcal{N}_V(E)$ is the mapping $T\in \mathcal{L}^k(V,E)\mapsto \nu_{V,E}(T)$, $\nu_{V,E}(T)(x) := \|T(x)\|_E$ for every $x\in V$;
- $\nu_{k,E}^2 : \mathcal{L}^{k,\mathrm{w}^*}(E^*,E) \to \mathcal{D}_k(E)$ assigns to $T\in \mathcal{L}^{k,\mathrm{w}^*}(E^*,E)$ the $\mathrm{GL}(\mathbb{F}^k)$-orbit $[(\nu_{\mathbb{F}^k,E}(T_0),\nu_{(\mathbb{F}^k)^*,E}(T_1))]$ where $T_0\in \mathcal{L}^k(\mathbb{F}^k,E)$, $T_1\in\mathcal{L}^k((\mathbb{F}^k)^*,E)$ are such that $T = T_0\circ T_1^*$;
- $\tau_{k,E} : \mathrm{Gr}(k,E)\to \mathcal{B}_k(E)$ is defined for each $V\in \mathrm{Gr}(k,E)$ by $\tau_{k,E}(V) = [\nu_{k,E}(T)]_{\mathrm{BM}}$ for some $T\in \mathcal{L}(\mathbb{F}^k,E)$ such that $\operatorname{Im} T = V$.

For convenience, we isolate some useful facts, to be used in the following.

Proposition 6.10. *Let V be a vector space of dimension k, and let E be a Banach space.*
(a) $(\nu_{V,E}(T))^*(f) = \min\{\|g\|_{E^*} \,:\, T^*(g) = f\}$ for every $T\in\mathcal{L}^k(X,E)$ and $f\in X^*$;
(b) $\omega(N,P) = \omega(N^*,P^*)$;
(c) the canonical action $\mathrm{GL}(V)\curvearrowright (\mathcal{N}_V,\omega)$, $(\Delta\cdot N)(x) := N(\Delta^{-1}(x))$, is by isometries;
(d) the dual action $\mathrm{GL}(V)\curvearrowright(\mathcal{N}_{V^*},\omega)$, $(\Delta\cdot N)(f) := N(\Delta^*(f))$, is by isometries and $(\Delta\cdot N)^* = \Delta^*\cdot N^*$.

Proof. (a): Set $N := \nu_{V,E}(T)$. Then $T:(V,N)\to E$ is by definition an isometry, so the desired inequality follows from the fact that $\|T^*\| = \|(T^*)^{-1}\|$. The rest is not difficult to prove; we leave the details to the reader. □

Proposition 6.11. *Suppose that $X = (X,N)$ is a normed space with $\dim X = k$, E is a Banach space and $\lambda\ge 1$. Then*
(1) The mappings $\nu_{X,E}$, $\nu_{k,E}^2$ and $\tau_{k,E}$ are well defined and continuous;
(2) $\nu_{X,E}(\mathcal{L}_\lambda^k(X,E)) = B_\omega^E[N;\lambda]$ and $\nu_{k,E}^2(\mathcal{L}_\lambda^{k,\mathrm{w}^*}(E^*,E)) = \mathcal{D}_k^\lambda(E)$.

Proof. (1): The continuity of $\nu_{X,E}$ follows from the fact that convergence in norm implies pointwise converge. Now we show that $\nu_{k,E}^2$ is well defined. Suppose that $T_0\circ T_1^* = U_0\circ U_1^*$. Then there is $\Delta\in\mathrm{GL}(\mathbb{F}^k)$ such that $U_0 = T_1\circ\Delta^{-1}$ and $U_1 = T_1\circ\Delta^*$. It follows that $\nu_{\mathbb{F}^k,E}(U_0) = \Delta\cdot\nu_{\mathbb{F}^k,E}(T_0)$ and $\nu_{(\mathbb{F}^k)^*,E}(U_1) = \nu_{(\mathbb{F}^k)^*,E}(T_1\circ\Delta^*) = \Delta\cdot\nu_{(\mathbb{F}^k)^*,E}(T_1)$, so $[(\nu_{\mathbb{F}^k,E}(T_0),\nu_{(\mathbb{F}^k)^*,E}(T_1))] = [(\nu_{\mathbb{F}^k,E}(U_0),\nu_{(\mathbb{F}^k)^*,E}(U_1))]$. We check now the continuity of $\nu_{k,E}^2$. Suppose that $T_n\to T$ in norm for $n\to\infty$. It follows that $\mathrm{Im}(T_n)\to\operatorname{Im}T$ in the opening distance Λ_E. Let $(e_i)_{i<d}$ be a basis of $\operatorname{Im}T$, and let $(x_i)_{i<k}$ be a linearly independent sequence in E such that $T = T_0\circ T_1^*$, where $T_0:\mathbb{F}^k\to E$ is linearly defined by $T_0(u_i) = e_i$ and $T_1 : (\mathbb{F}^k)^*\to E$ is defined by $T_1(u_i^*) = x_i$. For large enough n choose a basis $\{e_i^n\}_{i<k}$ of $\operatorname{Im}T_n$ such that $e_i^n\to e_i$ for $n\to\infty$ in norm for every $i<k$. Similarly than for T we define $T_0^n:\mathbb{F}^k\to E$, $T_0^n(u_i) := e_i^n$, and let $T_1^n : (\mathbb{F}^k)^*\to E$ be such that $T_n = T_0^n\circ(T_1^n)^*$. It follows that $T_0^n\to_n T_0$, so by continuity of $\nu_{\mathbb{F}^k,E}$, it follows that $\nu_{\mathbb{F}^k,E}(T_0^n)\to\nu_{\mathbb{F}^k,E}(T_0)$ for $n\to\infty$. On the other hand, T_0, T_0^n are 1-1, so

$$\|(T_1^n)^* - (T_1)^*\| \le \|T_0^{-1}\|\cdot(\|T_0^n\circ(T_1^n)^* - T_0\circ T_1^*\| + \|T_0^n - T_0\|\|(T_0^n)^{-1}\|\cdot\|T\|).$$

Hence $T_1^n\to T_1$, and $\nu_{(\mathbb{F}^k)^*,E}(T_1^n)\to\nu_{(\mathbb{F}^k)^*,E}(T_1)$ for $n\to\infty$.

Finally, we show that $\tau_{k,E}$ is well defined and continuous. Suppose that $\operatorname{Im}T = \operatorname{Im}U$; then there is $\Delta\in\mathrm{GL}(\mathbb{F}^k)$ such that $U = T\circ\Delta$. So, $\nu_{k,E}(U) = \nu_{k,E}(T\circ\Delta) = \Delta^{-1}\cdot\nu_{k,E}(T)$. To show the continuity, suppose that $V_n\to V$ for $n\to\infty$ in $\mathrm{Gr}(k,E)$ in the opening metric Λ_E. Let $T\in\mathcal{L}^k(\mathbb{F}^k,E)$ be such that $\operatorname{Im}T = V$, and for each $i<k$ and n choose $x_i^n\in B_{(V_n,\|\cdot\|_E}$ such that $\|x_i^n - T(u_i)\|_E\to 0$ for $n\to\infty$. It is clear that from some point on $(T(x_i^n))_{i<k}$ are linearly independent, so the mapping $T_n:\mathbb{F}^k\to E$, $T_n(u_i) = x_i^n$, belongs to $\mathcal{L}^k(\mathbb{F}^k,E)$ and satisfies that $d_{X,E}(T_n - T)\to 0$ for $n\to\infty$, where $X = (\mathbb{F}^k,\nu_{\mathbb{F}^k,E}(T))$. It follows from the continuity of $\nu_{X,E}$ that $\nu_{X,E}(T_n)\to\nu_{X,E}(T)$, so $\tau_{k,E}(V_n)\to_n\tau_{k,E}(V)$ for $n\to\infty$.

(2): $\nu_{X,E}(\mathcal{L}_\lambda^k(X,E)) = B_\omega^E[N;\lambda]$ follows from the fact that given $T\in\mathcal{L}^k(X,E)$, setting $P := \nu_{X,E}(T)$, we have that $\exp(\omega(N,P)) = \max\{\|T\|,\|T^{-1}\|\}$. We show now that $\nu_{k,E}^2(\mathcal{L}_\lambda^{k,\mathrm{w}^*}(E^*,E)) = \mathcal{D}_k^\lambda(E)$. Let $[\mathbf{N}]\in\mathcal{D}_k(E)$

be defined by $\mathbf{N} = (N_0, N_1)$. Choose $T_0 \in \mathcal{L}^k(\mathbb{F}^k, E)$ and $T_1 \in \mathcal{L}^k((\mathbb{F}^k)^*, E)$ such that $\nu_{\mathbb{F}^k,E}(T_0) = N_0$ and $\nu_{(\mathbb{F}^k)^*,E}(T_1) = N_1$. We claim that $T_0 \circ T_1^* \in \mathcal{L}_\lambda^{k,\mathrm{w}^*}(E^*, E)$. Let $\|g\|_{E^*} = 1$. Then

$$\|T_0(T_1^*(g))\|_E = N_0(T_1^*(g)) \leq \lambda N_1^*(T_1^*(g)) \leq \lambda \|g\|_{E^*},$$

where the last inequality holds by Proposition 6.10 (a). Now suppose that $\|T_0(T_1^*(g))\|_E \leq \lambda^{-1}$. It follows that $N_0(T_1^*(g)) \leq \lambda^{-1}$, so $N_1^*(T_1^*(g)) \leq 1$. Hence, by Proposition 6.10 (a), there is $h \in E^*$ such that $T_1^*(h) = T_1^*(g)$ and $\|h\|_{E^*} \leq 1$. This implies that $\mathrm{Ball}(\mathrm{Im}(T_0 \circ T_1^*)) \subseteq \lambda \cdot (T_0 \circ T_1^*)(\mathrm{Ball}(E^*))$. □

Proposition 6.12. *Suppose that $X = (X, N)$ is a normed space with $\dim X = k$, E is an approximately ultrahomogeneous Banach space and $\lambda \geq 1$. Then*

(1) $\nu_{X,E}(T) = \nu_{X,E}(U)$ if and only if $[T] = [U]$, $\widehat{\nu}_{X,E} : \mathcal{L}^k(X, E) /\!\!/ \mathrm{Aut}(E) \to \mathcal{N}_X(E)$ is a homeomorphism and $\mathcal{N}_X(E) \subseteq \mathcal{N}_X$ is closed;

(2) $\nu_{k,E}^2(T) = \nu_{k,E}^2(U)$ if and only if $[T] = [U]$, $\widehat{\nu}_{k,E}^2 : \mathcal{L}^{k,\mathrm{w}^*}(E^*, E) /\!\!/ (\mathrm{Iso}(E))^2 \to \mathcal{D}_k$ is a homeomorphism and $\mathcal{D}_k(E) \subseteq \mathcal{D}_k$ is closed;

(3) $\tau_{k,E}(V) = \tau_{k,E}(W)$ if and only if $[V] = [W]$, $\widehat{\tau}_{k,E} : \mathrm{Gr}(k, E) /\!\!/ \mathrm{Iso}(E) \to \mathcal{B}_k(E)$ is a homeomorphism and $\mathcal{B}_k(E) \subseteq \mathcal{B}_k$ is closed.

Proof. (1): Suppose that $\nu_{X,E}(T) = \nu_{X,E}(U)$. Then we can find, for each $\varepsilon > 0$ an isometry $g \in \mathrm{Aut}(E)$ such that $d_{X,E}(g \circ T, U) < \varepsilon$, and hence $U \in [T]$. We now show that $\widehat{\nu}_{X,E}$ is a homeomorphism. Suppose that $(P_i)_i$ is a converging sequence in $\mathcal{N}_X(E)$ with limit $P \in \mathcal{N}_X$. It follows by the approximate ultrahomogeneity of E that there is a *Cauchy* sequence $U_i \in \mathcal{L}^k(X, E)$ such that $\nu_{X,E}(U_i) = P_i$ for every i. Since $P_i \to P$ for $i \to \infty$, from some point on the sequence $(U_i)_i$ is in $\mathcal{L}_\lambda^k(X, E)$ for some $\lambda > 1$. Thus $(U_i)_i$ converges to $U \in \mathcal{L}^k(X, E)$, that is, $([U_i])_i$ converges to $[U]$. This proves that $\widehat{\nu}_{X,E}$ is open and that $\mathcal{N}_X(E)$ is closed in $\mathcal{N}_X$.

(2): Suppose that $\nu_{k,E}^2(T) = \nu_{k,E}^2(U)$. Let $T = T_0 \circ T_1^*$, $U = U_0 \circ U_1^*$ be such that $\nu_{\mathbb{F}^k,E}(T_0) = \nu_{\mathbb{F}^k,E}(U_0)$ and $\nu_{(\mathbb{F}^k)^*,E}(T_1) = \nu_{(\mathbb{F}^k)^*,E}(U_1)$. By the approximate ultrahomogeneity of E we can find $g, h \in \mathrm{Iso}(E)$ such that $\|g \circ T_0 - U_0\| \leq \varepsilon/(2\|T_1\|)$ and such that $\|h \circ T_1 - U_1\| \leq \varepsilon/(2\|U_0\|)$. Hence,

$$\|g \circ T \circ h^* - U\| \leq \|g \circ T_0 - U_0\| \cdot \|T_1\| + \|h \circ T_1 - U_1\| \cdot \|U_0\| \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $[U] = [T]$. We see now that $\widehat{\nu}_{k,E}^2$ is a homeomorphism. Suppose that $\widehat{\nu}_{k,E}^2([T_n]) \to \widehat{\nu}_{k,E}^2([T]) = [\mathbf{N}]$ for $n \to \infty$. Our goal is to find a subsequence of $([T_n])_n$ that converges to $[T]$: We first choose $T = T_0 \circ T_1^*$, a subsequence $(T_{n_m})_m$ and decompositions $T_{n_m} = T_0^m \circ T_1^m$ in a way that both $\omega(\nu_{\mathbb{F}^k,E}(T_0^m), \nu_{\mathbb{F}^k,E}(T_0)) < m^{-1}$ and $\omega(\nu_{(\mathbb{F}^k)^*,E}(T_1^m), \nu_{(\mathbb{F}^k)^*,E}(T_1)) < m^{-1}$ for every $m \in \mathbb{N}$. It follows from *(1)* that $[T_0^m] \to [T_0]$ and $[T_1^m] \to [T_1]$ for $m \to \infty$. This easily implies that $[T_{n_m}] \to [T]$ for $m \to \infty$.

(3): Suppose that $\tau_{k,E}(V) = \tau_{k,E}(W)$. By the approximately ultrahomogeneity of E, for a given $\varepsilon > 0$ we can find an isometry $g \in \mathrm{Aut}(E)$ such that $\Lambda_E(V, g \cdot W) < \varepsilon$, and hence $V \in [W]$. The fact that $\widehat{\tau}_{k,E}$ is a homeomorphism follows from *(1)*. □

We have then the following.

Corollary 6.13. *Suppose that E is approximately ultrahomogeneous, $G := \mathrm{Aut}(E)$. Then,*

(1) $\partial_{X,E}(N, P) := \widehat{d}_{X,E}([T], [U])$, where $T, U \in \mathcal{L}(X, E)$ satisfy $\nu_{X,E}(T) = N$ and $\nu_{X,E}(U) = P$, defines a compatible metric on $\mathcal{N}_X(E)$ such that $\widehat{\nu}_{X,E} : (\mathcal{L}^k(X, E) /\!\!/ G, \widehat{d}_{X,E}) \to (\mathcal{N}_X(E), \partial_{X,E})$ is an isometry;

(2) $\partial_{E^*,E,k}([\mathbf{N}], [\mathbf{P}]) := \widehat{d}_{E^*,E}([T], [U])$, where $\nu_{k,E}^2[T] = [\mathbf{N}]$ and $\nu_{k,E}^2[U] = [\mathbf{P}]$, defines a compatible metric on $\mathcal{D}_k(E)$ such that $\widehat{\nu}_{k,E}^2 : (\mathcal{L}^{k,\mathrm{w}^*}(E^*, E) /\!\!/ G^2, \widehat{d}_{E^*,E}) \to (\mathcal{D}_k(E), \partial_{E^*,E,k})$ is an isometry;

*(3) the E-*Kadets* metric $\gamma_{E,k}$ on $\mathcal{B}_k(E)$, $\gamma_{E,k}([N], [P]) := \widehat{\Lambda}_E([V], [W])$ for V, W such that $\tau_{k,E}(V) = N$ and $\tau_{k,E}(W) = P$, is a compatible metric and $\widehat{\tau}_{k,E} : (\mathrm{Gr}(k, E) /\!\!/ G \to (\mathcal{B}_k(E), \gamma_E)$ is an isometry.*

In the literature the Kadets metric γ corresponds to the metric $\gamma_{\mathbb{G}}$ for Grassmannians of the Gurarij space $\mathbb{G}$; see [35].

6.2. **Factorizing colorings of matrices and grassmannians over $\mathbb{R}, \mathbb{C}$.** We restate Corollary 6.6 in terms of matrices, using the canonical identification of the orbit spaces for the spaces E_p, where $1 \leq p \leq \infty$ is not an even integer strictly larger than 2.

The set of matrices $M_{n,m}$ over $\mathbb{F}$ becomes a Polish space when identified with the product space $\mathbb{F}^{n \cdot m}$. Each matrix $A \in M_{n,m}$ represents a linear operator $T_A : \mathbb{F}^m \to \mathbb{F}^n$, $T_A(x) := A \cdot x$. In this way one naturally transfers all the concepts for operators introduced above to matrices.

Definition 6.14. Let $1 \le p \le \infty$, and let p^* be the *conjugate* of p, defined by $p^{-1} + (p^*)^{-1} = 1$, $m, n, k \in \mathbb{N}^*$ and $\lambda \ge 1$. Let also $E_p := L_p[0,1]$ if $1 \le p < \infty$ and $E_\infty := \mathbb{G}$. We define

- d_p is the compatible metrics on $M_{n,m}$, $d_p(A,B) := \|T_A\|_{\ell_p^m, \ell_p^n} = \max_{\|x\|_p = 1} \|(A - B) \cdot x\|_p$. Let
  - $M_{n,m}(p,\lambda) := \{A \in M_{n,m} \, : \, T_A \in \mathcal{L}_\lambda(\ell_p^m, \ell_p^n)\}$,
  - $M_{n,m}^k$ to be the collection of $n \times m$ matrices of rank k, $M_n^k := M_{n,n}^k$,
  - $M_{n,m}^k(p,\lambda) := M_{n,m}^k \cap M_{n,m}(p,\lambda)$,
  - $\mathcal{E}_{n,k}^p := M_{n\times k}^k(p,1)$.
- d_p^* is the metric on M_n^k defined by $d_p^*(A,B) := \|T_A - T_B\|_{\ell_{p^*}^n, \ell_p^n} = \max_{\|x\|_{p^*}=1} \|(A-B)\cdot x\|_p$. Let
  - $M_n(p^*, p, \lambda) := \{A \in M_n \, : \, T_A \in \mathcal{L}_\lambda(\ell_{p^*}^n, \ell_p^n)\}$,
  - $M_n^k(p^*, p, \lambda) := M_n(p^*, p, \lambda) \cap M_n^k$.
- $\mathcal{N}_m^p := \mathcal{N}_{\mathbb{F}^m}(E_p)$, $\partial_{p,k} := \partial_{\mathbb{F}^k, E_p}$ and $\mathcal{N}_m^p(\lambda) := B_\omega[\|\cdot\|_{\ell_p^m}, \lambda]$. Let $\nu_p : M_{n,k}^k \to \mathcal{N}_k^p$, $\nu_p(A) := \nu_{\mathbb{F}^n, E_p}(T_A)$; that is, ν_p assigns to each full ranked $n \times k$-matrix A the norm $\nu_p(A) : \mathbb{F}^k \to \mathbb{R}$ defined for $x \in \mathbb{F}^k$ by $(\nu_p(A))(x) := \|A \cdot x\|_p$.
- $\mathcal{D}_k^p := \mathcal{D}_k(E_p)$, $\partial_{p,k}^* := \partial_{E_p^*, E_p, k}$, $\nu_p^2 : M_n^k \to \mathcal{D}_k^p$ is the mapping that assigns to an n-squared matrix A of rank k, the class $\mathrm{GL}(\mathbb{F}^k)$-orbit of the pair $(\nu_p(B), \nu_p(C))$ where $B, C \in M_{n,k}^k$ are such that $A = B \cdot C^*$.
- $\Lambda_{k,p}$ is the opening distance $\Lambda_{\ell_p^k}$ on $\mathrm{Gr}(k, E_p)$, $\gamma_{p,k}$ is the E_p-Kadets distance $\gamma_{E_p,k}$ on $\mathcal{B}_k(E_p)$, and $\tau_{k,p}$ denotes and τ_{k,E_p}.

The following is the factorization of compact colorings of matrices and grassmannians over $\mathbb{R}, \mathbb{C}$. In the next, let $1 \le p \le \infty$ p be different from an even integer strictly larger than 2.

Theorem 6.15 (Factorization of colorings of matrices and grassmannians)**.** *Fix $k, m \in \mathbb{N}$, real numbers $\varepsilon > 0$ and $\lambda \ge 1$. Set $\bar{\lambda} = \max\{1, (1-\varepsilon)\lambda\}$. Let (K, d_K) be a compact metric space.*

(1) there is $n \in \mathbb{N}$ such that for every K-coloring c of $(M_{n,k}^k(p,\lambda), d_p)$ there is $R \in \mathcal{E}_{n,m}^p$ such that ν_p is an ε-factor of c in $R \cdot M_{m,k}^k(p, \bar{\lambda})$; that is, there is a coloring $\widehat{c} : (\mathcal{N}_k^p(\bar{\lambda}), \partial_p) \to (K, d_K)$ such that $d_K(c(R \cdot A), \widehat{c}(\nu_p(A))) \le \varepsilon$ for every $A \in M_{m,k}^k(p, \bar{\lambda})$;

(2) there is $n \in \mathbb{N}$ such that for every coloring $c : (M_n^k(p^*, p, \lambda), d_p^*) \to (K, d_K)$ there are $R_0, R_1 \in \mathcal{E}_{n\times m}^p$ such that ν_p^2 is an ε-factor of c in $R_0 \cdot M_m^k(p^*, p, \bar{\lambda}) \cdot R_1^*$;

(3) there is $n \ge k$ such that for every coloring $c : (\mathrm{Gr}(k, \mathbb{F}^n), \Lambda_{k,p}) \to (K, d_K)$ there is $V \in \mathrm{Gr}(m, \mathbb{F}^n)$ with $(V, \|\cdot\|_{\ell_p^n})$ isometric to ℓ_p^m such that $\tau_{k,p}$ is an ε-factor of c in $\mathrm{Gr}(k, V)$.

Observe that when $\lambda = 1$, the set $\mathcal{N}_k^p(1)$ consist of a single element $\{\|\cdot\|_p\}$. Therefore in this case Theorem 6.15 *(1)* is a reformulation of the compact (ARP) of the collection $\{\ell_p^n\}_{n\in\mathbb{N}}$.

Proof of Theorem 6.15. Under the hypotheses of the theorem, the space E_p is approximately ultrahomogeous. Therefore the theorem is a consequence of Corollary 6.6 and Corollary 6.13 when applied to the spaces E_p. □

The extreme amenability of $\mathbb{G}$ is proved here in Theorem 2.8, the fact that $\{\ell_\infty^n\}$ is a stable Fraïssé class can be found in [39, 46], while the approximate ultrahomogeneity of $\mathbb{G}$ is proved by Gurarij in [32]. The extreme amenability of $\mathrm{Aut}(L_2)$ was proved by Gromov and Milman in [30]. The fact that $\{\ell_2^n\}_n$ is a stable Fraïssé class, and that L_2 is approximately ultrahomogeneous can be easily proved using the polar decomposition of a bounded operator on the Hilbert space. Finally, if $1 \le p \ne 2 < \infty$, the extreme amenability of $\mathrm{Aut}(L_p[0,1])$ was proved by Giordano and Pestov in [21] (notice that for these p's all these groups $\mathrm{Aut}(L_p[0,1])$ are topologically isomorphic). The fact that $\{\ell_p^n\}_n$ is a stable Fraïssé class is obtained in [19], as an application of the result of Schechtman from [68] stating that δ-embeddings between ℓ_p^n's are uniformly closed to embeddings. The approximate ultrahomogeneity of $L_p[0,1]$ for p different from an even integer strictly larger than 2 was proved by Lusky in [50]. We do not know if there exist other approximately ultrahomogeneous Banach spaces other than the E_p's for $1 \le p \le \infty$, p different from an even integer strictly larger than 2.

The following statement can be considered as a version of the Graham-Leeb-Rothschild Theorem for the fields $\mathbb{R}, \mathbb{C}$.

Corollary 6.16 (Graham-Leeb-Rothschild for $\mathbb{R}$, $\mathbb{C}$)**.** *For every $1 \le p \le \infty$, p different from an even integer strictly larger than 2, every k, m, every $\varepsilon > 0$ and every compact metric space (K, d_K) there is $n \ge k$ such that every coloring $c : (\mathrm{Gr}(k, \mathbb{F}^n), \Lambda_{\ell_p^n}) \to (X, d_X)$ ε-stabilizes on $\mathrm{Gr}(k, W)$ for some $W \in \mathrm{Gr}(m, \mathbb{F}^n)$.*

Proof. This is direct consequence of Theorem 6.15 *(3)* and the facts that for large r the space ℓ_p^r contains almost isometric copies of ℓ_2^m and that $\mathcal{B}_k(\ell_2^m)$ consists of a point. □

Recall that Dvoretzky's Theorem asserts than any finite-dimensional normed space X of dimension r contains almost isometric copies of ℓ_2^m with m proportional to $\log(r)$, and uniformly on the dimension r (see [1, Theorem 12.3.6]). In the previous argument, by replacing the fact ℓ_p^r contains almost isometric copies of ℓ_2^m with Dvoretzky's theorem, one obtains a similar result for any sequence of norms $(M_n)_n$, each on $\mathbb{F}^n$.

6.2.1. *Explicit metrics for $p = \infty$.* We give explicit formulas to compute the metrics on the orbit spaces for the case $p = \infty$. The proofs rely on the fact that the Gurarij space is universal for separable Banach spaces.

Proposition 6.17. *For every finite-dimensional normed space $X = (X, N)$ and every $P, Q \in \mathcal{N}_X$ one has that*

$$\partial_{X,\mathbb{G}}(P, Q) = \alpha_X(P, Q) := d_{\mathcal{H}}^{N^*}(\mathrm{Ball}((V, P)^*), \mathrm{Ball}((V, Q)^*)),$$

where $d_{\mathcal{H}}^{N^*}(K, L)$ is the Hausdorff distance (with respect to the norm distance induced by N^*) between K and L, is a compatible distance in $\mathcal{N}_X$.

Proof. We first show that $\alpha_X(P, Q) \le \partial_{X,E}(P, Q)$ for every Banach space E and $P, Q \in \mathcal{N}_X(E)$: Let $T, U \in \mathcal{L}(X, E)$ be such that $\nu_{X,E}(T) = P$ and $\nu(X, E)(U) = Q$ and $\partial_{X,E}(P, Q) = \|T - U\|$. Let $f \in \mathrm{Ball}((X, P)^*)$, and let $g \in \mathrm{Ball}(E^*)$ be such that $T^*(g) = f$. It follows that $d^{X^*}(f, \mathrm{Ball}((X, Q)^*)) \le \|f - U^*(g)\|_{X^*} \le \|T^* - U^*\| = \partial_{X,E}(P, Q)$. Finally, we show that $\partial_{X,\mathbb{G}}(P, Q) \le \alpha_X(P, Q)$: By the universality property of $\mathbb{G}$, $\mathcal{N}_X = \mathcal{N}_X(\mathbb{G})$. Let $P, Q \in \mathcal{N}_X$, and let $T, U \in \mathcal{L}(V, \mathbb{G})$ be such that $P = \nu_{X,\mathbb{G}}(T)$ and $Q = \nu_{X,\mathbb{G}}(U)$. Fix $\varepsilon > 0$. Find $Y \subseteq \mathbb{G}$ a subspace isometric of ℓ_∞^m and $T_0, U_0 \in \mathrm{Emb}(X, Y)$ such that $\|T - T_0\|_{X,\mathbb{G}}, \|U - U_0\|_{X,\mathbb{G}} < \varepsilon$. Set $P_0 := \nu_{X,\mathbb{G}}(T_0)$, $Q_0 := \nu_{X,\mathbb{G}}(U_0)$. Fix a sequence $(v_i)_{i=1}^m$ in Y that is 1-equivalent to the unit basis of ℓ_∞^n. For every $1 \le i \le n$ choose $x_i, y_i \in \mathrm{Ball}(Y^*)$ such that

$$\|T_0^*(v_i^*) - U_0^*(y_i)\|_{X^*} = d^{X^*}(T_0^*(v_i^*), B_{(X,N_0)^*}) \text{ and } \qquad \|U_0^*(v_i^*) - T_0^*(x_i)\|_{X^*} = d^{X^*}(U_0^*(v_i^*), B_{(X,P_0)^*}).$$

Since Y^* is isometric to ℓ_1^n, and since $(v_i^*)_{i=1}^n$ is 1-equivalent to the unit basis of ℓ_1^n it follows that

$$\|T_0 - U_0\|_{X,\mathbb{G}} = \|T_0^* - U_0^*\|_{Y^*,X^*} = \max_{1\le i\le n} \max\{\|T_0^*(v_i^*) - U_0^*(y_i)\|_{X^*}, \|U_0^*(v_i^*) - T_0^*(x_i)\|_{X^*}\} = \alpha_X(P_0, Q_0).$$

Let $Y \subseteq Z \subseteq \mathbb{G}$ be a subspace isometric to ℓ_∞^{2n}, and let $(w_i)_{i=1}^{2n}$ be a sequence in Z that is 1-equivalent to the unit basis of ℓ_∞^{2n}. Let $\xi, \eta \in \mathrm{Emb}(Y, Z)$ be defined dually by $\xi^*(w_i^*) := v_i^*$, $\xi^*(w_{n+i}^*) := x_i^*$, $\eta^*(w_i^*) := y_i$ and $\eta^*(w_{n+i}^*) = v_i^*$ for $1 \le i \le n$. Finally, let $g, h \in \mathrm{Aut}(\mathbb{G})$ be such that $\|g \restriction_Y - \xi\|_{Y,\mathbb{G}}, \|h \restriction_Y - \eta\|_{Y,\mathbb{G}} < \varepsilon$. It follows that

$$\begin{aligned}\partial_{X,\mathbb{G}}(P, Q) \le &\|g \circ T - h \circ U\|_{X,\mathbb{G}} \le \|\xi \circ T_0 - \eta \circ U_0\|_{X,Z} + 4\varepsilon = \alpha_X(P_0, Q_0) + 4\varepsilon \le \alpha_X(P, Q) + \alpha_X(P, P_0) + \\ &+ \alpha_X(Q, Q_0) + 4\varepsilon \le \alpha_X(P, Q) + \partial_{X,\mathbb{G}}(P, P_0) + \partial_{X,\mathbb{G}}(Q, Q_0) + 4\varepsilon \le \alpha_X(P, Q) + 6\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the proof is concluded. □

We see now that $\gamma_{\mathbb{G}}$ is Lipschitz equivalent to the Banach-Mazur metric on $\mathcal{B}_k$. The following result is a slight modification of [58, Proposition 6.2]. It will be later used when we present a direct proof of the approximate Ramsey property of the finite-dimensional normed spaces.

Proposition 6.18. *Suppose that X and Y are two finite-dimensional normed spaces, and let $T : X \to Y$ be a 1-1 linear operator. Then there is a normed space Z with $\dim Z \le \dim X \cdot \dim Y$ and isometric embeddings $I : X \to Z$ and $J : Y \to Z$ such that:*
(a) if $1 \le \|T\|, \|T^{-1}\|$, then $\|I - J \circ T\| \le \|T\| \cdot \|T^{-1}\| - 1$;
(b) if $\dim X = \dim Y$ and $\|T\| = 1$, then $\Lambda_Z(I(X), J(Y)) \le \|T^{-1}\| - 1$.

Proof. Fix a 1-1 linear operator $T : X \to Y$. Define on the direct sum $X \oplus Y$ the seminorm:

$$Q(x, y) := \max\left\{\|\frac{Tx}{\|T\|} + y\|_Y, \max_{g\in D}\left|\frac{g}{\|T\|}(y) + \frac{(T^*g)(x)}{\|T^*g\|_{X^*}}\right|\right\},$$

where $(T^*)^{-1}(\|T^{-1}\|^{-1} \cdot \partial_e(\mathrm{Ball}(X^*))) \subseteq D \subseteq \mathrm{Ball}(X)$ Let Z by the quotient of $X \oplus Y$ by the kernel of Q. Let $I : X \to Z$, $J : Y \to Z$ be the two canonical injections $I(x) := [(x, 0)]$, $J(y) := [(0, y)]$. It is routine to check that I, J and Z has the desired properties. □

Corollary 6.19. d_{BM} *and $\gamma = \gamma_{\mathbb{G}}$ are Lipschitz equivalent. In fact,*

$$\frac{1}{4k \log k} d_{\mathrm{BM}}([N], [P]) \le \gamma([N], [P]) \le (\log k) d_{\mathrm{BM}}([N], [P])$$

for every $N, P \in \mathcal{N}_k$. Consequently, $\gamma_{\mathbb{G}}$ is Lipschitz equivalent to τ_{BM}.

Proof. We start with the following.

Claim 6.19.1. $d_{\mathrm{BM}}([P],[Q]) \leq 4k \log k \gamma_E([P],[Q])$ for every Banach space E and every $P, Q \in \mathcal{N}_k(E)$.

Proof of Claim: Suppose first that $\gamma_E([P],[Q]) < (3k)^{-1}$. Let $V, W \in \mathrm{Gr}(k, E)$ be such that $\tau_{k,E}(V) = [P]$, $\tau_{k,E}([Q])$, and $\gamma_E([P],[Q]) = \Lambda_E(V, W)$. Let $(x_i)_{i<k}$ be an Auerbach basis of $(V, \|\cdot\|_E)$; see [10, Chapter 4, Theorem 13]. For each $i < k$, let $y_i \in \mathrm{Ball}((V, \|\cdot\|_E))$ be such that $\|x_i - y_i\|_E \leq \Lambda_E(V, W)$. Notice that

$$\begin{aligned}\|\sum_{i<k} \lambda_i y_i\|_E \geq & \|\sum_{i<k} \lambda_i x_i\|_E - \|\sum_{i<k} \lambda_i (x_i - y_i)\|_E \geq \|\sum_{i<k} \lambda_i x_i\|_E - k\Lambda_E(V,W) \max_{i<k} |(\lambda_i| \geq \\ \geq & (1 - k\Lambda_E(V,W))\|\sum_{i<k} \lambda_i x_i\|_E > 0. \end{aligned} \tag{1}$$

It follows that $(y_i)_{i<k}$ is a basis of W. Let $T : V \to W$ be the invertible operator sending x_i to $T(x_i) := y_i$. It follows from (1) that $\|T^{-1}\|_{(W,\|\cdot\|_E),(V,\|\cdot\|_E)} \leq (1-k\Lambda_E(V,W))^{-1}$. Similarly one can show that $\|T\|_{(V,\|\cdot\|_E),(W,\|\cdot\|_E)} \leq 1 + k\Lambda_E(V, W)$. Since $(1 + x)/(1 - x) \leq \exp(9x/4)$ for every $0 \leq x \leq 1/3$, it follows that

$$d_{\mathrm{BM}}(\tau_{k,E}(V), \tau_{k,E}(W)) \leq \frac{9}{4} k\Lambda_E(V, W).$$

Now suppose that $\Lambda_E(V, W) \geq (3k)^{-1}$. Since the diameter of $\mathcal{B}_k$ is at most $\log(k)$, we obtain that

$$d_{\mathrm{BM}}(\tau_{k,E}(V), \tau_{k,E}(W)) \leq \log(k) \leq 3k \log(k)\Lambda_E(V, W).$$

In any case, $d_{\mathrm{BM}}(\tau_{k,E}(V), \tau_{k,E}(W)) \leq 4k \log(k)\Lambda_E(V, W)$. □

Fix two norms $N, P \in \mathcal{N}_k$, and set $X := (\mathbb{F}^k, N)$ and $Y := (\mathbb{F}^k, P)$. Let $T : X \to Y$ be such that $\|T\| \cdot \|T^{-1}\| = \exp(d_{\mathrm{BM}}([N], [P]))$, and without loss of generality we assume that $\|T\| = 1$. We apply Proposition 6.18 to T to obtain a normed space Z and isometric embeddings $I : X \to Z$ and $J : Y \to Z$ such that (b) holds, that is, $\Lambda_Z(I(X), J(Y)) \leq \exp(d_{\mathrm{BM}}([N], [P])) - 1$. Since $d_{BM}([N], [P]) \leq \log(k)$, it follows that $\exp(d_{\mathrm{BM}}([N], [P])) - 1 \leq \log k \cdot d_{\mathrm{BM}}([N], [P])$. Thus $\gamma([N], [P]) \leq \Lambda_Z(X, Y) \leq \log k \cdot d_{\mathrm{BM}}([N], [P])$. □

We conclude by proving that $\partial_{\mathbb{G}^*,\mathbb{G}}$ and $\widehat{\omega}_2$—see Definition 6.7—are Lipschitz equivalent on $\mathcal{D}_k^\lambda$.

Proposition 6.20. *For every* $[\mathbf{N}], [\mathbf{P}] \in \mathcal{D}_k^\lambda$ *one has that*

$$\frac{1}{2\log(\lambda k)\lambda\sqrt{k}} \widehat{\omega}_2([\mathbf{N}], [\mathbf{P}]) \leq \partial_{\mathbb{G}^*,\mathbb{G}}([\mathbf{N}], [\mathbf{P}]) \leq k^2\lambda^3 \widehat{\omega}_2([\mathbf{N}], [\mathbf{P}]).$$

We will use the following.

Lemma 6.21. *Let* $P, Q \in \mathcal{N}_V$ *and suppose that* $P, Q \in B_\omega[N; \lambda]$. *Then* $\lambda^{-1} \cdot \omega(P, Q) \leq \alpha_X(P, Q) \leq \lambda \cdot \omega(P, Q)$.

Proof. Since $\omega(P, Q) = \omega(P^*, Q^*)$, it suffices to prove that $\lambda^{-1} \cdot \omega(P, Q) \leq d^N_{\mathcal{H}}(B_P, B_Q) \leq \lambda \cdot \omega(P, Q)$ provided that $\lambda^{-1}N(x) \leq P(x), Q(x) \leq \lambda N(x)$ for every $x \in X$. Let us show the first inequality. Without of generality we assume that there is $x \in X$ with $P(x) = 1$ such that $d^N_{\mathcal{H}}(B_P, B_Q) = d^M(x, B_Q)$. We also assume that $d^N_{\mathcal{H}}(P, Q) > 0$. It follows that $Q(x) > 1$ and $d^N_{\mathcal{H}}(P, Q) \leq N(x - x/N(x)) \leq \lambda|1 - Q(x)^{-1}| = \lambda(1 - Q(x)^{-1}) \leq \lambda(1-\exp(-\omega(P, Q))) \leq \lambda\omega(P, Q)$. Suppose now that $P(x) = 1$. Let $y \in X$ be such that $Q(y) \leq 1$ and $N(x-y) \leq d^N_{\mathcal{H}}(P, Q)$. It follows that $Q(x) \leq Q(y) + Q(x - y) \leq 1 + \lambda N(x - y) \leq 1 + \lambda d^N_{\mathcal{H}}(P, Q) \leq \exp(\lambda \cdot d^N_{\mathcal{H}}(P, Q))$. Consequently, $\omega(P, Q) \leq \lambda d^N_{\mathcal{H}}(P, Q)$. □

Lemma 6.22. *For every* $[\mathbf{N}], [\mathbf{P}] \in \mathcal{D}_V^\lambda$ *one has that*

$$\widehat{\omega}_2([\mathbf{N}], [\mathbf{P}]) \leq 2(\log \lambda + \min\{d_{\mathrm{BM}}([N_0], [P_0]), d_{\mathrm{BM}}([N_1], [P_1])\}).$$

Consequently, $\mathrm{diam}(\mathcal{D}_V^\lambda) \leq 2\log(\lambda \dim V)$.

Proof. Fix $[\mathbf{N}], [\mathbf{P}] \in \mathcal{D}_V^\lambda$. It follows that $\omega(N_0, N_1^*), \omega(P_0, P_1^*) \leq \log(\lambda)$. Let $\Delta \in \mathrm{GL}(V)$ be such that $d_{\mathrm{BM}}([N_0], [P_0]) = \omega(N_0, \Delta \cdot P_0)$. It follows that $\omega(N_1, \Delta \cdot P_1) \leq \omega(N_1, N_0^*) + \omega(N_0^*, \Delta \cdot P_0^*) + \omega(\Delta \cdot P_0^*, \Delta \cdot P_1) \leq 2\log \lambda + d_{\mathrm{BM}}(N_0, P_0)$. Hence, $\omega_2([\mathbf{N}], [\mathbf{P}]) \leq 2(\log \lambda + \min\{d_{\mathrm{BM}}([N_0], [P_0]), d_{\mathrm{BM}}([N_1], [P_1])\})$ □

Lemma 6.23. *Let* E *be a Banach space. For every* $T, U \in \mathcal{L}_\lambda^{k,\mathrm{w}^*}(E^*, E)$ *such that* $\lambda\sqrt{k}\|T - U\| < 1$ *one has that* $\widehat{\omega_2}(\tau^2_{k,E}(T), \tau^2_{k,E}(U)) \leq \lambda\sqrt{k}\|T - U\|$.

Proof. Let $T_0, U_0 \in \mathcal{L}^k(\mathbb{F}^k, E)$ and $T_1, U_1 \in \mathcal{L}^k((\mathbb{F}^k)^*, E)$ be such that $T = T_0 \circ T_1^*$ and $U = U_0 \circ U_1^*$. Define $N_0 := \nu_{\mathbb{F}^k,E}(T_0)$, $N_1 := \nu_{(\mathbb{F}^k)^*,E}(T_1)$, and $P_0 := \nu_{\mathbb{F}^k,E}(U_0)$, $P_1 := \nu_{(\mathbb{F}^k)^*,E}(U_1)$. We use the *Kadets-Snobar Theorem*—see for example [1, Theorem 12.1.6]—to fix a projection $P : E \to \mathrm{Im}(T_1)$ of norm at most $\sqrt{k}$. Observe that if $i : \mathrm{Im}\, T_1 \to E$, then $i \circ P \circ T_1 = T_1$, so $T_1^* = T_1^* \circ P^* \circ r$, where $r : E^* \to (\mathrm{Im}\, T_1)^*$ is the restriction map $r(g) := g \restriction \mathrm{Im}\, T_1$. In particular, the rank of $T_1^* \restriction \mathrm{Im}\, P^*$ is k.

Claim 6.23.1. (a) $T_1^* : \operatorname{Im} P^* \to (\mathbb{F}^k, N_0)$ is an isomorphism such that $(\lambda\sqrt{k})^{-1}\|g\|_{E^*} \le N_0(T_1^*(g)) \le \lambda\|g\|_{E^*}$;
(b) $U_1^* : \operatorname{Im} P^* \to (\mathbb{F}^k, P_0)$ is an isomorphism;
(c) $(U_1^* \circ (T_1^*)^{-1}) \cdot N_1 = P_1$;
(d) $\omega(N_0, (U_1^* \circ (T_1^*)^{-1})^{-1} \cdot P_0) \le \lambda\sqrt{k}\|T - U\|$.

Proof of Claim: (a): Let $g \in \operatorname{Im} P^*$, $\|g\|_{E^*} = 1$. Then, $N_0(T_1^*(g)) = \|Tg\|_E \le \lambda$. Now suppose that $N_0(T_1^*(g)) = 1$, that is, $\|T(g)\| = 1$. There is then $h \in E^*$ with $\|h\|_{E^*} \le \lambda$ such that $T^*(h) = T^*(g)$, so $T_1^*(h) = T_1^*(g)$. It follows that $T_1^*(h) = T_1^*(P^*(h \restriction_{\operatorname{Im} T_1}))$. Hence, $T_1^*(g) = T_1^*(g')$ with $\|g'\|_{E^*} \le \|P^*\|\|h\|_{E^*} \le \lambda\sqrt{k}$. But since $T_1^* \restriction_{\operatorname{Im} P^*}$ has dimension k, it must be injective. Thus $g = g'$, and hence, $\|g\|_{E^*} \le \lambda\sqrt{k}$, as required.
(b): We know that $\|U_1^*\|_{E^*,P_0} = \|U\|_{E^*,E} \le \lambda$. On the other hand, suppose that $g \in \operatorname{Im} P^*$ is such that $\|g\|_{E^*} = 1$. Then, $|P_0(U_1^*(g)) - N_0(T_1^*(g))| = |\|Tg\|_E - \|U(g)\|_E| \le \|T - U\| < 1/(\lambda\sqrt{k})$. So, $P_0(U_1^*(g)) > 0$, hence $U_1^*(g) \ne 0$, and since $\dim \operatorname{Im} P^* = k$, it follows that U_1^* is an isomorphism.
(c): Set $\Delta := U_1^* \circ (T_1^*)^{-1} \in \mathrm{GL}(\mathbb{F}^k)$. Let $f \in (\mathbb{F}^k)^*$. Then $(\Delta \cdot N_1)(f) = N_1(\Delta^*(f)) = \|T_1(T_1^{-1}(U_1(f)))\|_E = \|U_1(f)\|_E = P_1(f)$.
(d): Set $\Delta := U_1^* \circ (T_1^*)^{-1}$. Fix $x \in \mathbb{F}^k$ such that $N_0(x) = 1$, and set $g := (T_1^*)^{-1}(x)$. Notice that $\|g\|_{E^*} \le \lambda\sqrt{k}$. Then $|N_0(x) - P_0(\Delta(x))| = |\|T_0(T_1^*(g))\|_E - \|U_0(U_1^*(g)\|_E| \le \|T - U\|_{E^*,E}\|g\|_{E^*} \le \lambda \cdot \sqrt{k} \cdot \|T - U\|_{E^*,E}$. So, $P_0(\Delta(x)) \le (\lambda \cdot \sqrt{k} \cdot \|T - U\| + 1)N_0(x)$. Similarly, $N_0(x) \le (\lambda \cdot \sqrt{k} \cdot \|T - U\| + 1)(\Delta \cdot P_0)(x)$. It follows that

$$\omega(N_0, \Delta^{-1} \cdot P_0) \le \log(\lambda \cdot \sqrt{k} \cdot \|T - U\| + 1) \le \lambda \cdot \sqrt{k}\|T - U\|. \qquad \square$$

Proof of Proposition 6.20. Fix $k \ge 1$.

Claim 6.20.1. If E is approximately ultrahomogeneous, then $\widehat{\omega}_2([\mathbf{N}], [\mathbf{P}]) \le 2\log(\lambda k)\lambda\sqrt{k}\partial_{E^*,E}([\mathbf{N}], [\mathbf{P}])$ for every $[\mathbf{N}], [\mathbf{P}] \in \mathcal{D}_k^\lambda(E)$.

Proof of Claim: This follows from Lemma 6.23 and the fact that the diameter of $\mathcal{D}_V^\lambda$ is at most $2\log(\lambda k)$. $\square$

Claim 6.20.2. For every $[\mathbf{N}], [\mathbf{P}] \in \mathcal{D}_k^\lambda$ one has that $\partial_{\mathbb{G}^*\mathbb{G}}([\mathbf{N}], [\mathbf{P}]) \le k^2\lambda^3\widehat{\omega}_2([\mathbf{N}], [\mathbf{P}])$.

Proof of Claim: Let $\Delta \in \mathrm{GL}(\mathbb{F}^k)$ be such that $\widehat{\omega}_2([\mathbf{N}], [\mathbf{P}]) = \omega(N_0, \Delta \cdot P_0) + \omega(N_1, \Delta \cdot P_1)$. We use Proposition 6.17 to find $T_0, U_0 \in \mathcal{L}(\mathbb{F}^k, \mathbb{G})$ and $T_1, U_1 \in \mathcal{L}((\mathbb{F}^k)^*, \mathbb{G})$ be such that:
(a) $N_0 = \nu_{\mathbb{F}^k,\mathbb{G}}(T_0)$, $\Delta \cdot P_0 = \nu_{\mathbb{F}^k,\mathbb{G}}(U_0)$, $N_1 = \nu_{(\mathbb{F}^k)^*,\mathbb{G}}(T_1)$ and $\Delta \cdot P_1 = \nu_{(\mathbb{F}^k)^*,\mathbb{G}}(U_1)$;
(b) $\alpha_{(\mathbb{F}^k,N_1^*)}(N_0, \Delta \cdot P_0) = \|T_0 - U_0\|_{(\mathbb{F}^k,N_1^*),\mathbb{G}}$ and $\alpha_{((\mathbb{F}^k)^*,N_0^*)}(N_1, \Delta \cdot P_1) = \|T_1 - U_1\|_{((\mathbb{F}^k)^*,N_0^*),\mathbb{G}}$.
Let $T := T_0 \circ T_1^*$, $U := U_0 \circ U_1^*$. It follows that $\nu^2([T]) = [\mathbf{N}]$, $\nu^2([U]) = [\mathbf{P}]$, and $\partial_{\mathbb{G}^*,\mathbb{G}}([\mathbf{N}], [\mathbf{P}]) \le \|T - U\|_{\mathbb{G}^*,\mathbb{G}}$. Now let $g \in \mathbb{G}^*$ be such that $\|g\|_{\mathbb{G}^*} = 1$. Then,

$$\begin{aligned}\|(T-U)(g)\|_{\mathbb{G}} \le &\|T_0(T_1^* - U_1^*)(g)\|_{\mathbb{G}} + \|(T_0 - U_0)(U_1^*(g))\|_{\mathbb{G}} \le \\ \le &\|T_0\|_{(\mathbb{F}^k,N_0),\mathbb{G}}\|T_1^* - U_1^*\|_{\mathbb{G}^*,(\mathbb{F}^k,N_0)} + \|T_0 - U_0\|_{(\mathbb{F}^k,N_1^*),\mathbb{G}}\|U_1^*\|_{\mathbb{G}^*,(\mathbb{F}^k,N_1)^*} = \\ = &\alpha_{((\mathbb{F}^k)^*,N_0^*)}(N_1, \Delta \cdot P_1) + \alpha_{(\mathbb{F}^k,N_1^*)}(N_0, \Delta \cdot P_0)\end{aligned} \tag{2}$$

On the other hand, by Lemma 6.22,

$$\max\{\|\mathrm{Id}\|_{\Delta\cdot P_1,N_0^*}, \|\mathrm{Id}\|_{\Delta\cdot P_0,N_1^*}\} \le \max\{\|\mathrm{Id}\|_{\Delta\cdot P_1,N_1}\|\mathrm{Id}\|_{N_1,N_0^*}, \|\mathrm{Id}\|_{\Delta\cdot P_0,N_0}\|\mathrm{Id}\|_{N_0,N_1^*}\} \le \lambda e^{\widehat{\omega}_2([N],[P])} \le k^2\lambda^3.$$

It follows from the inequality in (2) and Lemma 6.21 that

$$\partial_{\mathbb{G}^*,\mathbb{G}}([N], [P]) \le \|(T - U)\|_{\mathbb{G}^*,\mathbb{G}} \le k^2\lambda^3(\omega(N_1, \Delta \cdot P_1) + \omega(N_0, \Delta \cdot P_0)) \le k^2\lambda^3\widehat{\omega}_2([\mathbf{N}], [\mathbf{P}]). \qquad \square \quad \square$$

We do not know if there is a similar explicit description of the metrics on the canonical quotient spaces for values of p other than ∞.

6.3. **Bounds of Ramsey numbers for finite-dimensional normed spaces.** The goal of this subsection is to give a quantitative proof of the approximate Ramsey property of the finite-dimensional normed $\mathbb{R}$-spaces, which yields and explicit upper bound for the corresponding Ramsey numbers. We also present a rather simple proof of a result by Gowers on discretizations of ℓ_∞^n, and a new proof of the approximate Ramsey property for finite metric spaces. Racall that $(u_i)_{i<d}$ is the canonical unit basis of ℓ_∞^d.

Definition 6.24. Given $d, m, r \in \mathbb{N}$ and $\varepsilon > 0$, let $\mathbf{n}_\infty(d, m, r, \varepsilon)$ be the least $n \in \mathbb{N}$ such that every r-coloring of $\mathrm{Emb}(\ell_\infty^k, \ell_\infty^n)$ has an ε-monochromatic set of the form $\gamma \circ \mathrm{Emb}(\ell_\infty^k, \ell_\infty^m)$ for some $\gamma \in \mathrm{Emb}(\ell_\infty^m, \ell_\infty^n)$.

The existence of such n is proved in Lemma 2.4. In fact, the case $d = 1$ was first proved by Gowers [27], indirectly, as it follows easily via a compactness argument from the oscillation stability of the space c_0. We start by presenting a proof of this result for *positive* embeddings. Given integers k and n, let $\mathrm{FIN}_k(n)$ be the collection of all mappings from n into $k+1 = \{0, 1, \dots, k-1, k\}$ such that k is in its range. Let $T : \mathrm{FIN}_k(n) \to \mathrm{FIN}_{k-1}(n)$ be the *tetris* operation defined pointwise for $f \in \mathrm{FIN}_k(n)$ by $T(f)(i) := \max\{f(i) - 1, 0\}$. Given disjointly supported $f_0, \dots, f_{l-1}$ in $\mathrm{FIN}_j(n)$, the combinatorial space $\langle f_i \rangle_{i<l}$ is the collection of all combinations $\sum_{i<l} T^{k-j_i}(f_i)$ where $(j_i)_{i<l} \in \mathrm{FIN}_k(l)$.

Proposition 6.25 (Gowers)**.** *For every k, m and every r there is some n such that every r-coloring of $\mathrm{FIN}_k(n)$ has a monochromatic set of the form $\langle f_i \rangle_{i<m}$ for some disjointly supported sequence $(f_i)_{i<m}$ in $\mathrm{FIN}_k(n)$.*

Let $\mathbf{GR}(d, m, r)$ be the minimal n so that the Dual Ramsey Theorem holds for the parameters d, m and r.

Proof of Proposition 6.25. Fix k, m and r. We claim that $n = \mathbf{GR}(k+1, km+1, r)$ works. Fix an r-coloring c of $\mathrm{FIN}_k(n)$. We consider $k+1$, $mk+1$, and n canonically ordered. For a subset A of n, we let $\mathbb{1}_A$ be the indicator function of A. Let $\Phi : \mathrm{Epi}(n, k+1) \to \mathrm{FIN}_k(n)$ be defined by $\Phi(\sigma) := \sum_{i \le k} i \cdot \mathbb{1}_{\sigma^{-1}(i)}$. By the Ramsey property of n there is some rigid surjection $\varrho : n \to mk+1$ such that $c \circ \Phi$ is constant on $\mathrm{Epi}(mk+1, k+1) \circ \varrho$ with value $\widehat{r}$. For each $j < m$, let $f_j := \sum_{1 \le i \le k} i \cdot \mathbb{1}_{\varrho^{-1}(jk+i)}$. Then c is constant on $\langle f_j \rangle_{j<m}$. To see this, given $f = \sum_{l<m} T^{k-j_l} f_l \in \langle f_j \rangle_{j<m}$ we define $\sigma : mk+1 \to k+1$ by $\sigma(0) = 0$ and $\sigma(lk+i) := \max\{i-k+j_l, 0\}$ for $l < m$ and $1 \le i \le k$. Then for $0 < i_0$ one has that $\min \sigma^{-1}(i_0) = kl_0 + (i_0 + k - j_{l_0})$ where $l_0 = \min\{l < m : i_0 \le j_l\}$, so σ is a rigid surjection. It is not difficult to see that $\Phi(\sigma \circ \varrho) = f$, so $c(f) = \widehat{r}$. □

We will need the following simple estimate of the cardinality of an ε-separated subset of the unit ball of an k-dimensional normed space which can also be found in [55].

Proposition 6.26. *Let X be a k-dimensional normed space, $0 < \varepsilon \le 1$ and $r > 0$.*

(a) For every $A \subseteq B \subseteq r\mathrm{Ball}(X)$ such that A is ε-separated there is some $A \subseteq D \subseteq B$ which is ε-dense in B and such that $\#D \le (1 + 2r/\varepsilon)^k \le \exp(k \log((2r+1)/\varepsilon))$.

(b) Suppose that $A \subseteq \mathrm{Sph}(X) = \{x \in X : \|x\| = 1\}$ is ε-separated. Then there is $\{0\} \cup A \subseteq D \subseteq \mathrm{Ball}(X)$ which is ε-dense in $\mathrm{Ball}(X)$ such that

(b.1) $\#D \le (1 + 4\varepsilon^{-1})^k \le \exp(k \log(5\varepsilon^{-1}))$, and

(b.2) For every $x \in \mathrm{Ball}(X) \setminus \{0\}$ there is some $y \in D$ such that $\|x - y\|_X < \varepsilon$ and $\|y\|_X < \|x\|_X$.

Proof. Fix all data. Let $A \subseteq D \subseteq B \subseteq r\mathrm{Ball}(X)$ be a maximal ε-separated subset of B. It follows that $\{B(x, \varepsilon/2)\}_{x \in E}$ is a pairwise disjoint family of balls, all contained in $B(0, r + \varepsilon/2) \setminus B(0, r - \varepsilon/2)$. Comparing volumes (the volume induced by declaring that a ball of radius r has r^k volume), one obtains that $\#D \cdot (\varepsilon/2)^k \le (r + \varepsilon/2)^k - (r - \varepsilon/2)^k$, which gives the estimate on $\#D$ in *(a)*. To prove *(b)*, we use the same idea and we find for each $i = 0, \dots, m$, $m := \lfloor \varepsilon^{-1} \rfloor - 1$, we can find ε-dense subsets $D_i \subseteq (1 - i\varepsilon) \cdot \mathrm{Sph}(X)$ with $\#D_i \cdot (\varepsilon/2)^k \le ((1 - i\varepsilon) + \varepsilon/2)^k - ((1 - i\varepsilon) - \varepsilon/2)^k$. Summing up,

$$(\frac{\varepsilon}{2})^k \sum_{i=0}^{m} \#D_i \le (1 + \frac{2}{\varepsilon})^k - (1 - \varepsilon m - \frac{2}{\varepsilon})^k \le (\frac{\varepsilon}{2})^k,$$

where the last inequality follows from the choice of m. Now if we set $D_\varepsilon := \bigcup_{i \le m} D_i \cup \{0\}$, then D is a ε-dense subset of $\mathrm{Ball}(X)$ of cardinality $\le (1 + 2\varepsilon^{-1})^k$ such that for every $x \in \mathrm{Ball}(X) \setminus \{0\}$ there is $y \in D_\varepsilon$ such that $\|y\|_X < \|x\|_X$ and $\|y - x\|_X < 2\varepsilon$. So the corresponding set $D_{\varepsilon/2}$ satisfies *(b.1)* and *(b.2)*. □

Theorem 6.27. $\mathbf{n}_\infty(d, m, r, \varepsilon) \le \mathbf{GR}(\lfloor (1 + 4\varepsilon^{-1})^d \rfloor, \lfloor (1 + 4\varepsilon^{-1})^d \rfloor 2^d m!/(m-d)!, r)$.

Proof. We adapt the proof of Lemma 2.5. Fix d, m, r and $\varepsilon > 0$. Let $\mathcal{D}$ be a finite ε-dense subset of $\mathrm{Ball}(\ell_1^d)$ obtained as in Proposition 6.26 from $A = \{0\} \cup \{u_i\}_{i<d}$. Let $\prec$ be any linear ordering of $\mathcal{D}$ such that if $\|x\|_1 < \|y\|_1$ then $x \prec y$. Let $\mathrm{emb}(d, m)$ be the collection of all 1-1 mappings $f : d \to m$. For each $(f, \theta) \in \mathrm{emb}(d, m) \times \{-1, 1\}^d$, let $h_{f,\theta} : \ell_\infty^d \to \ell_\infty^m$ be the linear map obtained by setting $h_{f,\theta}(u_i) := \theta_i \cdot u_{f(i)}$. Then $h_{f,\theta} \in \mathrm{Emb}(\ell_\infty^d, \ell_\infty^m)$ and for any other $T \in \mathrm{Emb}(\ell_\infty^d, \ell_\infty^m)$ there is a pair $(f, \theta) \in \mathrm{emb}(d, m) \times \{-1, 1\}^d$ such that $T^* \circ h_{f,\theta} = \mathrm{Id}_{\ell_\infty^d}$.

Let $\Delta := \mathcal{D} \times \mathrm{emb}(d, m) \times \{-1, 1\}^d$ be ordered by its lexicographical ordering from $\mathcal{N}$ ordered by $\prec$, and both $\mathrm{emb}(d, m)$ and $\{-1, 1\}^d$ being lexicographically ordered. Notice that $\#\mathrm{emb}(d, m) = m!/(m-d)!$, hence $\#\Delta = \lfloor (1 + 4\varepsilon^{-1})^d 2^d m!/(m-d)!$. We claim that $n := \mathbf{GR}(\lfloor (1 + 4\varepsilon^{-1})^d, \lfloor (1 + 4\varepsilon^{-1})^d 2^d m!/(m-d)!, r)$ works. Indeed, let c be an r coloring of $\mathrm{Emb}(\ell_\infty^d, \ell_\infty^n)$. We define an injection $\Phi : \mathrm{Epi}(n, \mathcal{D}) \to \mathrm{Emb}(\ell_\infty^d, \ell_\infty^n)$ by assigning to each $\sigma \in \mathrm{Epi}(n, \mathcal{D})$ the operator $T := \Phi(\sigma) : \ell_\infty^d \to \ell_\infty^n$ such that, for each $\xi < n$, the ξ^{th}-row vector of the matrix corresponding to $\Phi(\sigma)$ in the respective unit bases is $\sigma(\xi)$. Equivalently, $T^*(u_\xi) := \sigma(\xi)$ for every $\xi < n$. It is easily verified that T is always an isometric embedding. It follows by the Dual Ramsey Theorem applied to the coloring $\widehat{c} = c \circ \Phi$ that there is $\gamma_0 \in \mathrm{Epi}(n, \Delta)$ such that

$$\widehat{c} \text{ is constant on } \mathrm{Epi}(\Delta, \mathcal{D}) \circ \gamma_0 \text{ with value } r_0.$$

Let $R \in \mathrm{Emb}(\ell_\infty^m, \ell_\infty^n)$ be the isometric embedding such that, for every $\xi < n$, one has that $R^*(u_\xi) = h_{f,\theta}(v)$, where $(v, f, \theta) = \gamma_0(\xi)$. The proof is finished once we establish the following.

Claim 6.27.1. For every $T \in \mathrm{Emb}(\ell_\infty^d, \ell_\infty^m)$ there exists $\phi \in \mathrm{Epi}(\Delta, \mathcal{D})$ such that

$$\|\Phi(\phi \circ \gamma_0) - R \circ T\| \le \varepsilon.$$

Proof of Claim: Fix $T \in \mathrm{Emb}(\ell_\infty^d, \ell_\infty^m)$, and let $(\bar{f}, \bar{\theta}) \in \mathrm{emb}(d, m) \times \{-1, 1\}^d$ such that $T^* \circ h_{\bar{f},\bar{\theta}} = \mathrm{Id}_{\ell_\infty^d}$. Now we define the rigid surjection $\phi : \Delta \to \mathcal{D}$ as follows. we set $\phi(0, f, \theta) := 0$ for every f, θ. Suppose that $v \neq 0$, and define $\phi(v, f, \theta) := v$ if $(f, \theta) = (\bar{f}, \bar{\theta})$, and $\phi(v, f, \theta) := w \in \mathcal{D}$ such that $\|T^* h_{f,\theta}(v) - w\|_{\ell_1} \le \varepsilon$, and such that $\|w\|_1 < \|v\|_1$. Such a w exists because $\|T^* h_{f,\theta}(v)\|_{\ell_1^d} \le \|T^*\| \|v\|_{\ell_1} = \|v\|_{\ell_1}$, and by the choice of $\mathcal{D}$. It is not difficult to see that $\min \phi^{-1}(v) = (v, \bar{f}, \bar{\theta})$, so ϕ is a rigid surjection. We compute $\|(\Phi(\phi \circ \gamma_0))^* - (R \circ T)^*\|_{\ell_1^n, \ell_1^d} = \max_{\xi < n} \|(\Phi(\phi \circ \gamma_0))^*(u_\xi) - T^*(R^*(u_\xi))\|_{\ell_1^d}$. So, fix $\xi < n$, and suppose that $\gamma_0(\xi) = (v, f, \theta)$. Then $\Phi(\phi \circ \gamma_0)^*(u_\xi) = \phi(\gamma_0(\xi))$, and $T^*(R^*(u_\xi)) = T^*(h_{f,\theta}(v))$. So if $v = 0$, it follows that $\phi(0, f, \theta)$ and $T^*(R^*(u_\xi)) = T^*(h_{f,\theta}(0)) = 0$. Suppose that $v \neq 0$. If $(f, \theta) = (\bar{f}, \bar{\theta})$, then $T^*(R^*(u_\xi)) = T^*(h_{f,\theta}(v)) = v$ and $\phi(v, f, \theta) = v$. Finally, if $(f, \theta) \neq (\bar{f}, \bar{\theta})$, then $\phi(v, f, \theta) = w$ is chosen such that $\|T^*(h_{f,\theta}(v)) - w\|_{\ell_1} < \varepsilon$, so in any case $\|(\Phi(\phi \circ \gamma_0))^*(u_\xi) - T^*(R^*(u_\xi))\|_{\ell_1^d} \le \varepsilon$. □ □

6.3.1. *Approximate Ramsey property of Polyhedral spaces.* We give now an explicit proof of the approximate Ramsey property of finite-dimensional polyhedral spaces, spaces whose unit balls have finitely many extreme points. This is done by using injective envelopes of polyhedral spaces, and in this way by reducing colorings of polyhedral spaces to colorings of ℓ_∞^n-spaces. In this way, knowing the number of extreme points of the dual unit ball of given spaces, we can estimate upper bounds of the corresponding Ramsey numbers.

Definition 6.28. Recall that a finite-dimensional space F is called *polyhedral* when its unit ball $\mathrm{Ball}(F)$ is a polyhedron, i.e., when the set $\partial_e(B_F)$ of *extreme* points of its unit ball B_F is finite.

The spaces ℓ_∞^n and ℓ_1^n are polyhedral. In fact, a finite-dimensional space is polyhedral if and only if its dual ball is polyhedral. It follows from this, a separation argument, and the Milman Theorem asserting that a finite-dimensional space F is polyhedral if and only if there is a finite set $A \subseteq \mathrm{Sph}(F^*)$ such that $\|x\| = \max_{f \in A} f(x)$ for every $x \in F$. Also, every subspace of a polyhedral space is also polyhedral, and every finite-dimensional polyhedral space embeds into ℓ_∞^n for some $n \in \mathbb{N}$.

Definition 6.29. (*Ramsey number for polyhedral spaces*) Given an integer d, let Pol_d be the class of all polyhedral spaces F such that $\#\partial_e(B_{F^*}) = 2d$. Given $d, m \in \mathbb{N}$, $r \in \mathbb{N}$ and $\varepsilon > 0$, let $\mathbf{n_{pol}}(d, m, r, \varepsilon)$ be the minimal integer $n \ge m$ such that for every $F \in \mathrm{Pol}_d$ and $G \in \mathrm{Pol}_m$, every r-coloring of $\mathrm{Emb}(F, \ell_\infty^n)$ has an ε-monochromatic set of the form $T \circ \mathrm{Emb}(F, G)$ for some $T \in \mathrm{Emb}(G, \ell_\infty^n)$.

Definition 6.30 (injective envelope of a polyhedral space)**.** The injective envelope of a polyhedral space F is a pair (d_F, T_F), where d_F is an integer and $\Psi_F \in \mathrm{Emb}(F, \ell_\infty^{d_F})$ such that for every other isometric embedding $T : F \to \ell_\infty^n$ there is an isometric embedding $U : \ell_\infty^{n_F} \to \ell_\infty$ such that $T = U \circ \Psi_F$.

Proposition 6.31. $\mathbf{n_{pol}}(d, m, r, \varepsilon) = \mathbf{n}_\infty(d, m, r, \varepsilon)$.

Proof of Proposition 6.31. First of all, $\ell_\infty^k \in \mathrm{Pol}_k$, so $\mathbf{n_{pol}}(d, m, r, \varepsilon) \ge \mathbf{n}_\infty(d, m, r, \varepsilon)$. Fix now an r coloring c of $\mathrm{Emb}(F, \ell_\infty^n)$. Let $\widehat{c} : \mathrm{Emb}(\ell_\infty^d, \ell_\infty^n) \to r$ be defined for $U \in \mathrm{Emb}(\ell_\infty^d, \ell_\infty^n)$ by $\widehat{c}(U) := c(U \circ \Psi_F)$. Let $\widehat{T} \in \mathrm{Emb}(\ell_\infty^m, \ell_\infty^n)$ and $\widehat{r} < r$ be such that

$$\widehat{T} \circ \mathrm{Emb}(\ell_\infty^d, \ell_\infty^m) \subseteq (\widehat{c}^{-1}\{\widehat{r}\})_\varepsilon. \tag{3}$$

Let $T := \widehat{T} \circ \Psi_G$. We claim that $T \circ \mathrm{Emb}(F, G) \subseteq (c^{-1}\{\widehat{r}\})_\varepsilon$. Let $U \in \mathrm{Emb}(F, G)$, and let $W \in \mathrm{Emb}(\ell_\infty^d, \ell_\infty^m)$ be such that $\Psi_G \circ U = W \circ \Psi_F$. From the inclusion in (3) there exists $V \in \mathrm{Emb}(\ell_\infty^d, \ell_\infty^m)$ such that $\widehat{c}(V) = \widehat{r}$ and $\|V - \widehat{T} \circ W\| < \varepsilon$. Let $\widehat{V} := V \circ \Psi_F$. Then $c(V \circ \Psi_F) = \widehat{c}(V) = \widehat{r}$, while

$$\|\widehat{V} - T \circ U\| = \|V \circ \Psi_F - \widehat{T} \circ \Psi_G \circ U\| = \|V \circ \Psi_F - \widehat{T} \circ W \circ \Psi_F\| \le \|V - \widehat{T} \circ W\| < \varepsilon. \qquad \square$$

6.3.2. *Approximate Ramsey property for finite-dimensional normed spaces.* We give an explicit, constructive proof of approximate Ramsey property arbitrary finite-dimensional normed spaces. The proof is based on the approximate Ramsey property of polyhedral spaces and the well known fact that the finite-dimensional polyhedral spaces are dense in the class of finite-dimensional normed spaces with respect to the Banach-Mazur distance. In fact, we have the following.

Proposition 6.32. *Suppose that* $\dim X = k$. *For every* $0 < \varepsilon < 1$ *there is a polyhedral space* $X_0 \in \mathrm{Pol}_d$ *such that* $d_{\mathrm{BM}}(X, X_0) \leq \varepsilon$, *where* $d \leq ((2+3\varepsilon)/\varepsilon)^k$.

Proof. Let $\delta := \varepsilon(1+\varepsilon)^{-1}$. Let $D \subseteq \mathrm{Sph}(X^*)$ be a finite δ-dense subset of S_{X^*} of cardinality $\leq (1+2\delta^{-1})^k = ((2+3\varepsilon)/\varepsilon)^k$. On X we define the polyhedral norm $N(x) := \max_{f\in D} |f(x)|$. It follows that $X_0 := (X, N) \in \mathrm{Pol}_d$ with $d \leq \#D$, and $d_{\mathrm{BM}}(X, X_0) \leq \varepsilon$. □

Theorem 6.33. *For every finite-dimensional normed spaces* F *and* G, *every* $r \in \mathbb{N}$ *and every* $\varepsilon > 0$ *there is a finite-dimensional space* H *such that* F *isometrically embeds into* H *and such that every* r*-coloring of* $\mathrm{Emb}(F, H)$ *has a* ε*-monochromatic set of the form* $\gamma \circ \mathrm{Emb}(F, G)$ *for some* $\gamma \in \mathrm{Emb}(G, H)$. *In fact,* H *can be found such that*

$$\dim H \leq (\dim F)^{(1+\frac{8(5+\varepsilon)}{\varepsilon})^{n\cdot \dim F}} \cdot (\dim G)^{(1+\frac{8(5+\varepsilon)}{\varepsilon})^{n\cdot\dim G}} \cdot n,$$

where

$$n = \mathbf{n}_{\mathrm{pol}}(\lfloor (\frac{10+3\varepsilon}{\varepsilon})^{\dim F} \rfloor, \lfloor (\frac{10+3\varepsilon}{\varepsilon})^{\dim G} + (\frac{10+3\varepsilon}{\varepsilon})^{\dim F} \cdot (1+\frac{8(5+\varepsilon)}{\varepsilon})^{\dim F\cdot \dim G} \rfloor, r, \frac{\varepsilon}{4}).$$

The proof uses Proposition 6.32 and results on approximating almost isometric embeddings by isometric ones, and that is a consequence of the stable amalgamation property of the class $\langle \{\ell_\infty^n\}_n \rangle$, that is the class of finite dimensional normed spaces. In general, it is not true that an approximate isometry is close to an isometry: There are spaces with only $\pm\mathrm{Id}$ as isometries, yet with many approximate isometries far of being small perturbations of $\pm\mathrm{Id}$. For example, consider $\mathbb{R}^2$ with the euclidean norm $\|\cdot\|_2$. It is well know that for every $\varepsilon > 0$ there is another norm $\|\cdot\|$ such that $\|x\| \sim_{1+\varepsilon} \|x\|_2$ for every $x \in \mathbb{R}^2$ and such that $\pm\mathrm{Id}$ are the only isometries of $(\mathbb{R}^2, \|\cdot\|)$. It follows that the rotation on $\mathbb{R}^2$ by $\pi/2$ is a $1+\varepsilon$-isometry of $(\mathbb{R}^2, \|\cdot\|)$ which cannot be approximated by $\pm\mathrm{Id}$ for small enough ε. There is however the following fact, a slight generalization of a result by Kubis and Solecki in [39]. We introduce first some notions. The first one is a more convenient slight modification, $\mathcal{E}\mathrm{mb}_\theta(X, Y)$, of $\mathrm{Emb}_\theta(X, Y)$.

Definition 6.34. Given X of finite dimension, and $\theta \geq 1$, let $\mathcal{E}\mathrm{mb}_\theta(X, Y)$ be the collection of all 1-1 mappings $T : X \to Y$ such that $1 \leq \|T\|, \|T^{-1}\|$ and $\|T\| \cdot \|T^{-1}\| \leq \theta$.

Let $(X_i)_{i\leq n}$ be a sequence of Banach spaces. We say that a pair (Y, J) of a Banach space Y and $J \in \mathrm{Emb}(X_n, Y)$ is (θ,τ)*-correcting* for $\bar{X}$ $(1 < \theta < \tau)$ when every X_i isometrically embeds into Y, and for every $i < n$ and every $\gamma \in \mathcal{E}\mathrm{mb}_\theta(X_i, X_n)$ there exists $I_\gamma \in \mathrm{Emb}(X_i, Y)$ such that $\|J \circ \gamma - I_\gamma\| < \tau - 1$.

The next proposition can be seen as quantitative version of Proposition 1.12 for the class of finite-dimensional normed spaces.

Proposition 6.35. *Every finite sequence of finite-dimensional spaces* $(X_i)_{i\leq n}$ *and every* $1 < \theta < \tau$ *has a* (θ, τ)*-correcting pair* (Y, J) *such that*

$$\dim Y \leq (\prod_{i=0}^{n-1} (\dim X_i)^{l_i}) \dim X_n, \text{ where } l_i \leq (1+\frac{2\theta}{\tau-\theta})^{\dim X_i \cdot \dim X_n}.$$

When each X_i *is polyhedral,* $X_i \in \mathrm{Pol}_{d_i}$, *then* Y *can be taken polyhedral such that* $Y \in \mathrm{Pol}_d$ *with*

$$d \leq p_n + \sum_{i=0}^{n-1} p_i (1+\frac{2\theta}{\tau-\theta})^{\dim X_i\cdot \dim X_n}.$$

Proof. The proof is by induction on $n \geq 1$. Suppose first that $n = 1$.

Claim 6.35.1. Suppose that $\mathcal{N} \subseteq \mathcal{E}\mathrm{mb}_\theta(X_0, X_1)$ is finite. Then there exist a finite-dimensional space Y with $\dim Y \leq (\dim X_0)^{\#\mathcal{N}} \cdot \dim X_1$ and $\Theta \in \mathrm{Emb}(X_1, Y)$ such that for every $T \in \mathcal{N}$ there is $I \in \mathrm{Emb}(X_0, Y)$ such that $\|I - \Theta \circ T\| < \theta - 1$.

Proof of Claim: This is done by a simple induction on $\#\mathcal{N}$, and using that the case $\#\mathcal{N} = 1$ is done in Proposition 6.18. □

Let $\mathcal{N}$ be a finite $(\tau-\theta)$-net of $\mathcal{E}\mathrm{mb}_\theta(X_0, X_1)$ of cardinality $\le (1+2\theta/(\tau-\theta))^{\dim X_0 \cdot \dim X_1}$. In view of Proposition 6.26 *(a)*, this is possible because $\mathcal{E}\mathrm{mb}_\theta(X_0,X_1) \subseteq \theta \mathrm{Ball}(\mathcal{L}(X_0,X_1))$. Then the pair (X, I) obtained by applying Claim 6.35 to $\mathcal{N}$ is (θ,τ)-correcting for (X_0, X_1).

Now suppose that $n>1$. Find a (θ,τ)-correcting pair (Y_0,Θ_0) for $(X_i)_{i=1}^n$ with

$$\dim Y_0 \le (\prod_{i=1}^{n-1} (\dim X_i)^{l_i}) \dim X_n, \text{ where } l_i \le (1+\frac{2\theta}{\tau-\theta})^{\dim X_i \cdot \dim X_n}.$$

Let $\mathcal{N}$ be a finite $(\tau-\theta)$-net of $\mathcal{E}\mathrm{mb}_\theta(X_0,X_n)$ of cardinality $\le (1+2\theta/(\tau-\theta))^{\dim X_0 \cdot \dim X_n}$. Let (Y,Θ_1) be a pair obtained by applying Claim 6.35.1 to $\Theta_0 \circ \int N$. It can be easily verified that $(Y, \Theta_1 \circ \Theta_0)$ is a (θ,τ)-correcting pair for $(X_i)_{i\le n}$ with the desired estimate on the dimension $\dim Y$. □

Proof of Theorem 6.33. Fix F, G, $r \in \mathbb{N}$, $\varepsilon>0$ and $\theta \ge 1$, and set $\delta := \varepsilon/5$. Let $F_0 \in \mathrm{Pol}_d$, G_0 be polyhedral, and surjective isomorphisms $\Phi_F: F \to F_0$ and $\Phi_G : G \to G_0$ such that $\|\Phi_F\| = \|\Phi_G\| = 1$ and $\|\Phi_F^{-1}\|, \|\Phi_G^{-1}\| < 1+\varepsilon/5$. Notice that d can be taken such that $d \le ((10+3\varepsilon)/\varepsilon)^{\dim F}$. Let (H_0,Θ_0) be a $(1+\varepsilon/5, 1+\varepsilon/4)$-correcting pair for (F_0, G_0) with H_0 polyhedral, and let m be such that $H_0 \in \mathrm{Pol}_m$ with $m \le ((10+3\varepsilon)/\varepsilon)^{\dim G} + ((10+3\varepsilon)/\varepsilon)^{\dim F} \cdot (1+(8(5+\varepsilon))/\varepsilon)^{\dim F \cdot \dim G}$. Finally, let (H,Θ_1) be a $(1+\varepsilon/5, 1+\varepsilon/4)$-correcting pair for the triple (F, G, ℓ_∞^n) such that $\dim H \le (\dim F)^{(1+(8(5+\varepsilon))/\varepsilon)^{n\cdot \dim F}} \cdot (\dim G)^{(1+(8(5+\varepsilon))/\varepsilon)^{n \cdot \dim G}}$. n, where $n := \mathbf{n}_{\mathrm{pol}}(d,m,r,\varepsilon/4)$. We claim that H works. Fix $c: \mathrm{Emb}(F,H) \to r$. Let $\widehat{c}: \mathrm{Emb}(F_0, \ell_\infty^n) \to r$ be the induced coloring defined for $\gamma \in \mathrm{Emb}(F_0,\ell_\infty^n)$ by choosing $I_\gamma \in \mathrm{Emb}(F,H)$ such that $\|I_\gamma - \Theta_1 \circ \gamma \circ \Phi_F\| < \varepsilon/4$ and declaring $\widehat{c}(\gamma) := c(I_\gamma)$. By the Ramsey property of n, there exists $\varrho \in \mathrm{Emb}(H_0, \ell_\infty^n)$ and $\widehat{r} < r$ such that $\varrho \circ \mathrm{Emb}(F_0,H_0) \subseteq (\widehat{c}^{-1}\{\widehat{r}\})_{\varepsilon/4}$. Let $S \in \mathrm{Emb}(G,H)$ be such that

$$\|S - \Theta_1 \circ \varrho \circ \Theta_0 \circ \Phi_G\| < \frac{\varepsilon}{4}. \tag{4}$$

Claim 6.35.2. $S \circ \mathrm{Emb}(F,G) \subseteq (c^{-1}(\widehat{r}))_\varepsilon$.

Proof of Claim: Fix $T \in \mathrm{Emb}(F,G)$. Let $\tau \in \mathrm{Emb}(F_0,H_0)$ be such that $\|\tau - \Theta_0 \circ \Phi_G \circ T \circ \Phi_F^{-1}\| < \varepsilon/4$. This is possible because $\Phi_G \circ T \circ \Phi_F^{-1} \in \mathcal{E}\mathrm{mb}_{1+\varepsilon/5}(F_0,G_0)$. Let now $\gamma \in \mathrm{Emb}(F_0,\ell_\infty^n)$ be such that $\widehat{c}(\gamma) = \widehat{r}$ and $\|\gamma - \varrho \circ \tau\| < \varepsilon/4$. Then, $c(I_\gamma) = \widehat{r}$ and $\|S \circ T - I_\gamma\| < \varepsilon$. It follows from (4) and the fact that the operator T satisfies that $\|T\| = 1$, that

$$\|S \circ T - \Theta_1 \circ \varrho \circ \Theta_0 \circ \Phi_G \circ T\| < \frac{\varepsilon}{4}.$$

This is the diagram:

$$\begin{tikzcd}
F \arrow[r, "T"] \arrow[d, "\Phi_F"'] \arrow[rrrrd, bend right=70, "I_\gamma"'] & G \arrow[d, "\Phi_G"] \arrow[rrrd, bend left, "S"] & & & \\
F_0 \arrow[r, "t"'] \arrow[rr, bend right=50, "\tau"'] \arrow[rrr, bend right=60, "\gamma"'] & G_0 \arrow[r, "\Theta_0"] & H_0 \arrow[r, "\varrho"] & \ell_\infty^n \arrow[r, "\Theta_1"] & H
\end{tikzcd}$$

Consequently, $\|S \circ T - I_\gamma\| < \varepsilon$. □

6.4. **Finite metric spaces.** Recall that the Ramsey property for finite metric spaces, proved by Nešetřil [56], states that for every finite *ordered* metric spaces X and Y and every $r \in \mathbb{N}$ there exists a finite ordered metric space Z such that for every r-coloring of the set $\binom{Z}{X}_<$ of order isometric copies of X in Z there exist an order isometric copy Y_0 of Y in Z such that $\binom{Y_0}{X}_<$ is monochromatic. As a consequence, the isometry group of the *Urysohn* space $\mathbb{U}$ is extremely amenable, and since $\mathbb{U}$ is ultrahomogeneous, one obtains the following:

Theorem 6.36. *For every finite metric spaces M and N, $r \in \mathbb{N}$ and $\varepsilon > 0$ there exists a finite metric space P such that every r-coloring* $\mathrm{emb}(M,P)$ *has a ε-monochromatic set of the form* $\sigma \circ \mathrm{emb}(M,N)$ *for some* $\sigma \in \mathrm{emb}(N,P)$.

In the previous statement $\mathrm{emb}(M,P)$ is the set of all isometric embeddings from the metric space (M,d_M) into the metric space (N,d_N), endowed with its natural distance $d(\sigma,\tau) := \max_{x\in M} d_N(\sigma(x),\tau(x))$. Recall that the Uryshon space $\mathbb{U}$ is the unique (up to isometry) ultrahomogeneous separable complete metric space such that every isometric embedding from a finite metric space into $\mathbb{U}$ is the restriction of a surjective isometry of $\mathbb{U}$. Pestov proved in [62] that the group $\mathrm{Iso}(\mathbb{U})$ of surjective isometries of $\mathbb{U}$, and also that the ARP of finite metric spaces is equivalent to it, so proving Theorem 6.36. Later, Nešetřil established the (exact) Ramsey property of finite ordered metric spaces [56], thus gives another proof of the extreme amenability of $\mathrm{Iso}(\mathbb{U})$. We present here a third proof, which uses the approximate Ramsey property of the class of finite-dimensional polyhedral spaces.

Recall that a *pointed* metric space (X,d,p) is a metric space (X,d) with a distinguished point $p\in X$. Given two pointed metric spaces (M,p) and (N,q), let $\mathrm{emb}_0(M,N)$ be the set of pointed isometric embeddings, that is, all isometric embeddings from M into N sending p to q. Recall that when X and Y are normed spaces, we use $\mathrm{Emb}(X,Y)$ to denote *linear* isometric embeddings.

Definition 6.37. Given a pointed metric space (M,d,p) let $\mathrm{Lip}_0(M,p)$ be the Banach space of all Lipschitz maps $f:M\to\mathbb{R}$ such that $f(p)=0$ endowed with the *Lipschitz* norm,

$$\|f\| := \sup\left\{\frac{|f(x)-f(y)|}{d(x,y)} \ : \ x\neq y\in X\right\}.$$

Let $\mathcal{F}(M,p)$ be the *(Lipschitz) free space* over the pointed metric space (M,p) defined as the closed linear span of the *molecules* $\{\delta_x-\delta_p\}_{x\in M}$ in the dual space $\mathrm{Lip}_0(M,p)^*$, where δ_x for $x\in X$ denotes the *evaluation functional* at x. It turns out that $\mathcal{F}(M,p)^*$ is isometric to $\mathrm{Lip}_0(M,p)$. Since the space $\mathcal{F}(M,p)^*$ does not depend on the choice of p up to isometry, it is simply denoted by $\mathcal{F}(M)^*$. The same applies to $Lip_0(M,p)$.

The space $\mathcal{F}(M)$ is also known as the *Arens-Eells* space. More information on Arens–Eells spaces can be found in [71]. It is easy to see that the mapping $x\in M\mapsto \delta_x\in\mathcal{F}(M)$ is an isometric embedding. Given finite metric spaces M and N such that M isometrically embeds into N, let $M_\infty = M\cup\{p_\infty\}$, $N_\infty := N\cup\{q_\infty\}$ be one-point extensions of M and N with the distance $d(p_\infty,x)=d(q_\infty,y):=\min_{z\neq t\in N} d(z,t)$ for every $z\in M$, $y\in N$. Clearly M_∞ and N_∞ are metric spaces.

Proposition 6.38. *Suppose that M and N are metric spaces. Then every isometric embedding $\sigma: M\to N$ extends to a unique linear isometric embedding $T_\sigma:\mathcal{F}(M_\infty,p_\infty)\to\mathcal{F}(N_\infty,q_\infty)$.* □

The proof is a straightforward use of a standard duality argument, the McShane-Whitney extension Theorem for Lipschitz functions [71, Theorem 1.5.6], and the fact that $\delta_{p_\infty}=0$ in $\mathcal{F}(M_\infty,p_\infty)$ and $\delta_{q_\infty}=0$ in $\mathcal{F}(N_\infty,p_\infty)$.

Proposition 6.39. *If M is a finite metric space, then $\mathcal{F}(M)$ is a finite-dimensional polyhedral space.*

Proof. Observe that for each $x\neq y$ in M, $\mu_{x,y} := (\delta_x-\delta_y)/d(x,y)$ has norm 1 in $\mathrm{Lip}_0(M)$ since clearly $\|\mu_{x,y}\|\le 1$, and the mapping $d_x(t):=d(x,t)$ for each $t\in M$ is 1-Lipschitz and $\mu_{x,y}(d_x)=1$. It follows from the definition of the Lipschitz norm that the convex hull of $\{\mu_{x,y}\}_{x\neq y \text{ in } M}$ is equal to $B_{\mathcal{F}(M)}$. □

Lemma 6.40. *Suppose that M and N are two finite metric spaces, suppose $r\in\mathbb{N}$, and that $\varepsilon>0$. Let $\varrho:=\mathrm{diam}(N)$. Then there exists $n\in\mathbb{N}$ such that every r-coloring of $\mathrm{emb}(M,\varrho\cdot B_{\ell_\infty^n})$ has an ε-monochromatic set of the form $\sigma\circ\mathrm{emb}(M,N)$ for some $\sigma\in\mathrm{emb}(N,\varrho\cdot B_{\ell_\infty^n})$.*

Proof. Fix finite pointed metric spaces (M,p), (N,q), r and $\varepsilon>0$. We assume that M isometrically embeds into N since otherwise the statement above is trivially true. Let d,m be such that $\mathcal{F}(M_\infty)\in\mathrm{Pol}_d$ and $\mathcal{F}(N_\infty)\in\mathrm{Pol}_m$. Then $n:=\mathbf{n}_{\mathrm{pol}}(d,m,r,\varepsilon_0)$, for $\varepsilon_0=\varepsilon/\mathrm{diam}(M)$ works. Fix a coloring $c:\mathrm{emb}(M,\varrho\cdot B_{\ell_\infty^n})\to r$. Define $\widehat{c}:\mathrm{Emb}(\mathcal{F}(M_\infty),\ell_\infty^n)\to r$ by $\widehat{c}(\gamma)=c(\sigma_\gamma)$, where $\sigma_\gamma: M\to\varrho\mathrm{Ball}(\ell_\infty^n)$ is defined by $\sigma_\gamma(x)=\gamma(\delta_x)$ for every $x\in M$. This is well defined since $\|\delta_x\|=\|\delta_x-\delta_p\|\le d(x,p)\le\mathrm{diam}(M)\le\mathrm{diam}(N)$, where the last inequality holds since $\mathrm{Emb}(M,N)\neq\emptyset$. Let $\bar{\alpha}\in\mathrm{Emb}(\mathcal{F}(N_\infty),\ell_\infty^n)$ and $\bar{r}<r$ be such that $\bar{\alpha}\circ\mathrm{Emb}(\mathcal{F}(M_\infty),\mathcal{F}(N_\infty))\subseteq(\widehat{c}^{-1}(\bar{r}))_{\varepsilon_0}$. Let $\bar{\tau}:N\to\varrho\mathrm{Ball}(\ell_\infty^n)$ be the embedding defined by $\bar{\tau}(x)=\bar{\alpha}(\delta_x)$. We claim that $\bar{\tau}$ works. In fact, $\bar{\tau}\circ\mathrm{emb}(M,N)\subseteq(c^{-1}(\bar{r}))_\varepsilon$. Let $\sigma\in\mathrm{emb}(M,N)$. Then there exists a unique extension $\gamma_\sigma\in\mathrm{Emb}(\mathcal{F}(M_\infty),\mathcal{F}(N_\infty))$. Let $\psi\in\mathrm{Emb}(\mathcal{F}(M_\infty),\ell_\infty^n)$ such that $\widehat{c}(\psi)=\bar{r}$ and $\|\psi-\bar{\alpha}\circ\gamma_\sigma\|<\varepsilon_0$. Then $\sigma_\psi(x)=\psi(\delta_x)$ for every $x\in M$ satisfies that $c(\sigma_\psi)=\bar{r}$ and

$$d(\sigma_\psi,\bar{\tau}\circ\sigma)=\max_{x\in M}\|\psi(\delta_x)-\bar{\alpha}(\delta_{\sigma(x)})\|_\infty=\|\psi(\delta_x)-\bar{\alpha}(\gamma_\sigma(\delta_x)\|_\infty<\varepsilon_0\cdot\mathrm{diam}(M)=\varepsilon.$$ □

Proof of Theorem 6.36. This is a consequence of Lemma 6.40, via a compactness argument. Fix M, N, r and $\varepsilon>0$. Let n be obtained from M, N, r and $\varepsilon/3$ by applying Lemma 6.40. Let $\varrho:=\mathrm{diam}(N)$. Since M

and N are finite and $\varrho\mathrm{Ball}(\ell_\infty^n)$ is compact, there exists $P \subseteq \varrho\mathrm{Ball}(\ell_\infty^n)$ finite such that

$$\mathrm{emb}(M, \varrho\mathrm{Ball}(\ell_\infty^n)) \subseteq (\mathrm{emb}(M, P))_{\frac{\varepsilon}{3}} \text{ and } \mathrm{emb}(N, \varrho\mathrm{Ball}(\ell_\infty^n)) \subseteq (\mathrm{emb}(N, P))_{\frac{\varepsilon}{3}}.$$

We claim that (P, d_∞) works. To this end, let $c : \mathrm{emb}(P, A) \to r$. Let $\widetilde{c} : \mathrm{emb}(M, \varrho\mathrm{Ball}(\ell_\infty^n)) \to r$ be defined by $\widetilde{c}(\gamma) = c(\sigma_\gamma)$ where $\sigma_\gamma \in \mathrm{emb}(M, A)$ is chosen such that $d(\gamma, \sigma_\gamma) < \varepsilon/3$. By the property of n, there is $\gamma \in \mathrm{emb}(N, \varrho\mathrm{Ball}(\ell_\infty^n))$ and $\bar{r} < r$ such that $\gamma \circ \mathrm{emb}(M, N) \subseteq (\widetilde{c}^{-1}(\bar{r}))_{\varepsilon/3}$. Let $\bar{\gamma} \in \mathrm{emb}(N, P)$ be such that $d(\gamma, \bar{\gamma}) < \varepsilon/3$. It takes a simple computation to see that $\bar{\gamma} \circ \mathrm{emb}(M, N) \subseteq (c^{-1}(r))_\varepsilon$. □

References

1. Fernando Albiac and Nigel J. Kalton, *Topics in Banach space theory*, Graduate Texts in Mathematics, vol. 233, Springer, New York, 2006.
2. Erik M. Alfsen, *Compact convex sets and boundary integrals*, Springer-Verlag, New York-Heidelberg, 1971.
3. Erik M. Alfsen and Edward G. Effros, *Structure in real Banach spaces. I*, Annals of Mathematics. Second Series **96** (1972), 98–128.
4. ______, *Structure in real Banach spaces. II*, Annals of Mathematics. Second Series **96** (1972), 129–173.
5. Itaï Ben Yaacov, *Fraïssé limits of metric structures*, Journal of Symbolic Logic **80** (2015), no. 1, 100–115.
6. Itaï Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, *Model theory for metric structures*, Model theory with applications to algebra and analysis. Vol. 2, London Mathematical Society Lecture Note Series, vol. 350, Cambridge University Press, 2008, pp. 315–427.
7. Itaï Ben Yaacov, Julien Melleray, and Todor Tsankov, *Metrizable universal minimal flows of polish groups have a comeagre orbit*, Geometric and Functional Analysis **27** (2017), no. 1, 67–77.
8. David P. Blecher and Christian Le Merdy, *Operator algebras and their modules—an operator space approach*, London Mathematical Society Monographs. New Series, vol. 30, Oxford University Press, Oxford, 2004.
9. David P. Blecher and Matthew Neal, *Metric characterizations of isometries and of unital operator spaces and systems*, Proceedings of the American Mathematical Society **139** (2011), no. 3, 985–998.
10. Béla Bollobás, *Linear analysis*, second ed., Cambridge University Press, Cambridge, 1999.
11. Man-Duen Choi and Edward G. Effros, *Injectivity and operator spaces*, Journal of Functional Analysis **24** (1977), no. 2, 156–209.
12. Man Duen Choi and Edward G. Effros, *Lifting problems and the cohomology of C*-algebras*, Canadian Journal of Mathematics **29** (1977), no. 5, 1092–1111.
13. Clinton Conley and Asger Törnquist, *A Fraïssé approach to the Poulsen simplex*, Sets and Computations, World Scientific, 2017, pp. 11–24.
14. Christopher J. Eagle, Ilijas Farah, Bradd Hart, Boris Kadets, Vladyslav Kalashnyk, and Martino Lupini, *Fraïssé limits of C*-algebras*, Journal of Symbolic Logic, in press.
15. Caleb Eckhardt, *Perturbations of completely positive maps and strong NF algebras*, Proceedings of the London Mathematical Society **101** (2010), no. 3, 795–820.
16. Edward G. Effros, *On a class of complex Banach spaces*, Illinois Journal of Mathematics **18** (1974), 48–59.
17. Edward G. Effros and Zhong-Jin Ruan, *Operator spaces*, London Mathematical Society Monographs. New Series, vol. 23, Oxford University Press, 2000.
18. Robert Ellis, *Universal minimal sets*, Proceedings of the American Mathematical Society **11** (1960), 540–543.
19. Valentin Ferenczi, Jordi Lopez-Abad, Brice Mbombo, and Stevo Todorcevic, *Amagamation and approximate Ramsey properties of ℓ_p-spaces*, in preparation.
20. Adam H. Fuller, Michael Hartz, and Martino Lupini, *Boundary representations of operator spaces, and compact rectangular matrix convex sets*, arXiv:1610.05828 (2016).
21. Thierry Giordano and Vladimir Pestov, *Some extremely amenable groups related to operator algebras and ergodic theory*, Journal of the Institute of Mathematics of Jussieu **6** (2007), no. 02, 279–315.
22. Eli Glasner and Benjamin Weiss, *Minimal actions of the group $\mathbb{S}(\mathbb{Z})$ of permutations of the integers*, Geometric and Functional Analysis **12** (2002), no. 5, 964–988.
23. ______, *The universal minimal system for the group of homeomorphisms of the Cantor set*, Fundamenta Mathematicae **176** (2003), no. 3, 277–289.
24. Shmuel Glasner, *Distal and semisimple affine flows*, American Journal of Mathematics **109** (1987), no. 1, 115–131.
25. Isaac Goldbring and Martino Lupini, *Model-theoretic aspects of the Gurarij operator space*, Israel Journal of Mathematics, in press.

26. Isaac Goldbring and Thomas Sinclair, *On Kirchberg's embedding problem*, Journal of Functional Analysis **269** (2015), no. 1, 155–198.
27. Timothy Gowers, *Lipschitz functions on classical spaces*, European Journal of Combinatorics **13**, no. 3, 141–151.
28. Ronald L. Graham, Klaus Leeb, and Bruce L. Rothschild, *Ramsey's theorem for a class of categories*, Proceedings of the National Academy of Sciences of the United States of America **69** (1972), 119–120.
29. Ronald L. Graham and Bruce L. Rothschild, *Ramsey's theorem for n-parameter sets*, Transactions of the American Mathematical Society **159** (1971).
30. Mikhael Gromov and Vitali D. Milman, *A topological application of the isoperimetric inequality*, American Journal of Mathematics **105** (1983), no. 4, 843–854.
31. Ved Prakash Gupta, Prabha Mandayam, and V. S. Sunder, *The functional analysis of quantum information theory*, Lecture Notes in Physics, vol. 902, Springer, Cham, 2015.
32. Vladimir I. Gurariĭ, *Spaces of universal placement, isotropic spaces and a problem of Mazur on rotations of Banach spaces*, Siberian Mathematical Journal **7** (1966), 1002–1013.
33. Yonatan Gutman and Hanfeng Li, *A new short proof for the uniqueness of the universal minimal space*, Proceedings of the American Mathematical Society **141** (2013), no. 1, 265–267.
34. Trevor Irwin and Sławomir Solecki, *Projective Fraïssé limits and the pseudo-arc*, Transactions of the American Mathematical Society **358** (2006), no. 7, 3077–3096.
35. Nigel J. Kalton and Mikhail I. Ostrovskii, *Distances between banach spaces*, Forum Mathematicum **11** (2008), no. 1, 17–48.
36. Ali S. Kavruk, Vern I. Paulsen, Ivan G. Todorov, and Mark Tomforde, *Quotients, exactness, and nuclearity in the operator system category*, Advances in Mathematics **235** (2013), 321–360.
37. Alexander S. Kechris, Vladimir Pestov, and Stevo Todorcevic, *Fraïssé limits, ramsey theory, and topological dynamics of automorphism groups*, Geometric and Functional Analysis **15** (2005), no. 1, 106–189.
38. Wiesław Kubiś and Aleksandra Kwiatkowska, *The Lelek fan and the Poulsen simplex as Fraïssé limits*, arXiv:1512.09252 (2015).
39. Wiesław Kubiś and Sławomir Solecki, *A proof of uniqueness of the Gurariĭ space*, Israel Journal of Mathematics **195** (2013), no. 1, 449–456.
40. Anselm Lambert, *Operatorfolgenräume*, Ph.D. thesis, Universität des Saarlandes, 2002.
41. Aldo J. Lazar, *The unit ball in conjugate L_1 spaces*, Duke Mathematical Journal **39** (1972), 1–8.
42. Aldo J. Lazar and Joram Lindenstrauss, *Banach spaces whose duals are L_1 spaces and their representing matrices*, Acta Mathematica **126** (1971), no. 1, 165–193.
43. Franz Lehner, *M_n-espaces, sommes d'unitaires et analyse harmonique sur le groupe libre*, Ph.D. thesis, Université de Paris 6, 1997.
44. Joram Lindenstrauss, Gunnar Olsen, and Yaki Sternfeld, *The Poulsen simplex*, Annales de l'Institut Fourier **28** (1978), no. 1, 91–114.
45. Martino Lupini, *Fraïssé theory and the Poulsen simplex*, Discrete Mathematics, to appear.
46. ———, *Fraïssé limits in functional analysis*, arXiv:1510.05188 (2015).
47. ———, *Operator space and operator system analogs of Kirchberg's nuclear embedding theorem*, Journal of Mathematical Analysis and Applications **431** (2015), no. 1, 47–56.
48. ———, *Uniqueness, universality, and homogeneity of the noncommutative Gurarij space*, Advances in Mathematics **298** (2016), 286–324.
49. Wolfgang Lusky, *The Gurarij spaces are unique*, Archiv der Mathematik **27** (1976), no. 6, 627–635.
50. ———, *Some consequences of W. Rudin's paper: "L_p-isometries and equimeasurability"*, Indiana University Mathematics Journal **27** (1978), no. 5, 859–866.
51. ———, *On a construction of Lindenstrauss and Wulbert*, Journal of Functional Analysis **31** (1979), no. 1, 42–51.
52. Julien Melleray, *A note on hjorth's oscillation theorem*, Journal of Symbolic Logic **75** (2010), no. 04, 1359–1365.
53. Julien Melleray, Lionel Nguyen Van Thé, and Todor Tsankov, *Polish groups with metrizable universal minimal flows*, International Mathematics Research Notices. (2016), no. 5, 1285–1307.
54. Julien Melleray and Todor Tsankov, *Extremely amenable groups via continuous logic*, arXiv:1404.4590 (2014).
55. Vitali D. Milman and Gideon Schechtman, *Asymptotic theory of finite-dimensional normed spaces*, Lecture Notes in Mathematics, vol. 1200, Springer-Verlag, Berlin, 1986, With an appendix by M. Gromov.
56. Jaroslav Nešetřil, *Metric spaces are Ramsey*, European Journal of Combinatorics **28** (2007), no. 1, 457–468.

57. Timur Oikhberg, *The non-commutative Gurarii space*, Archiv der Mathematik **86** (2006), no. 4, 356–364.
58. Mikhail I. Ostrovskiĭ, *Topologies on the set of all subspaces of a Banach space and related questions of Banach space geometry*, Quaestiones Math. **17** (1994), no. 3, 259–319.
59. Vern I. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics, vol. 78, Cambridge University Press, Cambridge, 2002.
60. Vladimir Pestov, *The isometry group of the Urysohn space as a Lévy group*, **154**, no. 10, 2173–2184.
61. ______, *On free actions, minimal flows, and a problem by ellis*, Transactions of the American Mathematical Society **350** (1998), no. 10, 4149–4165.
62. ______, *Ramsey-Milman phenomenon, Urysohn metric spaces, and extremely amenable groups*, Israel Journal of Mathematics **127** (2002), no. 1, 317–357.
63. ______, *Dynamics of infinite-dimensional groups*, University Lecture Series, vol. 40, American Mathematical Society, Providence, RI, 2006.
64. Gilles Pisier, *Exact operator spaces*, Astérisque (1995), no. 232, 159–186, Recent advances in operator algebras (Orléans, 1992).
65. ______, *The operator Hilbert space* OH*, complex interpolation and tensor norms*, Memoirs of the American Mathematical Society **122** (1996), no. 585, viii+103.
66. ______, *Introduction to operator space theory*, London Mathematical Society Lecture Note Series, vol. 294, Cambridge University Press, Cambridge, 2003.
67. Ebbe T. Poulsen, *A simplex with dense extreme points*, Annales de l'Institut Fourier **11** (1961), 83–87.
68. Gideon Schechtman, *Almost isometric L_p subspaces of $L_p(0,1)$*, The Journal of the London Mathematical Society **20** (1979), no. 3, 516–528.
69. Roger R. Smith, *Finite dimensional injective operator spaces*, Proceedings of the American Mathematical Society **128** (2000), no. 11, 3461–3462.
70. Vladimir V. Uspenskij, *On universal minimal compact G-spaces*, Topology Proceedings, vol. 25, 2000, pp. 301–308.
71. Nik Weaver, *Lipschitz algebras*, World Scientific Publishing Co., Inc., River Edge, NJ, 1999.
72. Corran Webster and Soren Winkler, *The Krein-Milman theorem in operator convexity*, Transactions of the American Mathematical Society **351** (1999), no. 1, 307–322.
73. Przemysław Wojtaszczyk, *Some remarks on the Gurarij space*, Studia Mathematica **41** (1972), 207–210.
74. Blerina Xhabli, *The super operator system structures and their applications in quantum entanglement theory*, Journal of Functional Analysis **262** (2012), no. 4, 1466–1497.

Department of Mathematical Sciences, Carnegie Mellon University, Pennsylvania, USA
E-mail address: dbartoso@andrew.cmu.edu

Departamento de Matemáticas Fundamentales, Facultad de Ciencias, UNED, 28040 Madrid, Spain
E-mail address: abad@mat.uned.es

Mathematics Department, California Institute of Technology, 1200 E. California Blvd, MC 253-37, Pasadena, CA 91125
E-mail address: lupini@caltech.edu
URL: http://www.lupini.org/

Department of Mathematics and Statistics, University of Ottawa, Ottawa, ON, K1N 6N5, Canada
E-mail address: bmbombod@uottawa.ca