A POLYNOMIAL METHOD APPROACH TO ZERO-SUM SUBSETS IN $\mathbb{F}_p^2$

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ABSTRACT. In this paper we prove that every subset of $\mathbb{F}_p^2$ meeting all $p+1$ lines passing through the origin has a zero-sum subset. This is motivated by a result of Gao, Ruzsa and Thangadurai which states that $OL(\mathbb{F}_p^2) = p + OL(\mathbb{F}_p) - 1$, for sufficiently large primes p . Here $OL(G)$ denotes the so-called Olson constant of the additive group G and represents the smallest integer such that no subset of cardinality $OL(G)$ is zero-sum-free. Our proof is in the spirit of the Combinatorial Nullstellensatz.

1. INTRODUCTION

For a set A in an additive group G , denote

$$\Sigma(A) = \left\{ \sum_{x \in B} x \mid B \subset A \right\}, \quad \Sigma^*(A) = \left\{ \sum_{x \in B} x \mid B \subset A, B \neq \emptyset \right\}.$$

Define the *Olson constant* $OL(G)$ of a finite additive group G to be the minimal d such that every subset $A \subset G$ of cardinality d satisfies $0 \in \Sigma^*(A)$.

It turns out this is a difficult quantity to compute for general groups. First, Erdős and Heilbronn initially proved in [5] that there exists an absolute constant c such that $OL(\mathbb{F}_p) \leq c\sqrt{p}$, where p is a prime. They also conjectured that this bound extends to arbitrary additive groups G , and years later Szemerédi showed in [12] that $OL(G) \leq k\sqrt{|G|}$ for some $k > 0$. However, both results were far from being optimal. Olson [11] also came into the picture later and proved that $OL(G) \leq 2\sqrt{|G|}$, which has been subsequently improved by Hamidoune and Zemor [7], who proved that

$$OL(G) \leq \sqrt{2|G|} + O(|G|^{1/3} \log |G|),$$

conjecturing that $c = \sqrt{2}$ is perhaps optimal.

The story is not much better for groups of the form $\mathbb{F}_p^d$, where p is an odd prime. For $d = 1$, Hamidoune and Zemor were indeed correct; in [8], Nguyen, Szemerédi and Vu improved upon the Erdős-Heilbronn theorem and showed that for sufficiently large primes p , $OL(\mathbb{F}_p) \leq \sqrt{2p}$. This was subsequently also shown by Balandraud in [2] for all primes p , who in fact actually proved that for $A \subset \mathbb{F}_p$, with $|A| = d$ and such that $A \cap (-A) = \emptyset$, we have that

$$|\Sigma(A)| \geq \min \{p, d(d+1)/2 + 1\} \quad \text{and} \quad |\Sigma^*(A)| \geq \min \{p, d(d+1)/2\}.$$

Balandraud's original proof made use of some impressive recursions for binomial determinants, but it has been recently recasted by the same author in [3] as a natural

consequence of the Combinatorial Nullstellensatz due to Alon [2]. For $d = 2$, Gao, Ruzsa and Thangadurai showed in [7] that

$$OL(\mathbb{F}_p^2) = p + OL(\mathbb{F}_p) - 1$$

holds for $p > 4.67 \times 10^{34}$. Their argument was subsequently updated by Bhowmik and Schlage-Puchta to show that the equality holds for all primes > 6000 ; thus, given the result for $\mathbb{F}_p$, the Olson constant of $\mathbb{F}_p^2$ is also determined for large primes. For higher dimensions, Gao, Ruzsa and Thangadurai conjectured that

$$OL(\mathbb{F}_p^d) = p + OL(\mathbb{F}_p^{d-1}) - 1$$

should hold in general, but little is known in this case. We refer to [9] for an account of the state of the art.

In this paper we shall prove the following

Theorem 1. *Let p be an odd prime and let $\{x_1, \dots, x_n\} \subset \mathbb{F}_p^2 \setminus \{0, 0\}$ be a set of vectors spanning all $p + 1$ nonzero directions from 0 in $\mathbb{F}_p^2$. We prove that there exists a subset $I \subset \{1, \dots, n\}$ such that*

$$\sum_{i \in I} x_i = 0.$$

This is meant to complement the Gao-Ruzsa-Thangadurai-Bhowmik-Schlage-Puchta theorem. Here, we shall say that two vectors $(a, b), (c, d) \in \mathbb{F}_p^2$ have the same direction from zero if there is some $t \in \mathbb{F}_p \setminus \{0\}$ such that $(a, b) = (tc, td)$. The above result thus says that a set $A \subset \mathbb{F}_p^2$, with $(0, 0) \notin A$ and with the property that it meets all $p + 1$ lines of the form $ax + by = 0$, must contain a non-empty subset $B \subset A$ with zero sum.

2. PRELIMINARIES

Although not implicitly used in our proof, we first recall the statement of the Combinatorial Nullstellensatz.

Theorem 2. *Let $\mathbb{F}$ be an arbitrary field, and let $f = f(x_1, \dots, x_n)$ be a polynomial in $F[x_1, \dots, x_n]$. Suppose the degree $\deg(f)$ of f is $\sum_{i=1}^n t_i$, where each t_i is a nonnegative integer, and suppose the coefficient of $\prod_{i=1}^n x_i^{t_i}$ in f is nonzero. Then, if $S_1, \dots, S_n$ are the subsets of F with $|S_i| > t_i$, there are $s_1 \dots s_n \in S_1 \dots S_n$ so that*

$$f(s_1, \dots, s_n) \neq 0.$$

This is usually proven in general by induction on the number of variables (see for instance [1]), but whenever $S_i = \{0, 1\}$, $t_i = 1$ for each $i \in \{1, \dots, n\}$, there's a very neat way of thinking about this, which will be relevant to our proof of Theorem 1. One can simply consider the following “integral” of f on $\{0, 1\}^n$:

$$\mathbb{I}_f := \sum_{(x_1, \dots, x_n) \in \{0, 1\}^n} (-1)^{x_1 + \dots + x_n} f(x_1, \dots, x_n).$$

If f vanishes everywhere on $S_1 \times \dots \times S_n$, then clearly $\mathbb{I}_f = 0$.

On the other hand, if c is the nonzero coefficient of $\prod_{i=1}^n x_i$ in f , then

$$g(x_1, \dots, x_n) := f(x_1, \dots, x_n) - c \prod_{i=1}^n x_i$$

is a polynomial of degree less than n in $\mathbb{F}[x_1, \dots, x_n]$. In particular, every monomial appearing in $g \in \mathbb{F}[x_1, \dots, x_n]$ with nonzero coefficient has to be of the form $x_1^{i_1} \dots x_n^{i_n}$, with $i_j \in \{0, 1\}$ not all 1. For such a monomial, where say $i_K = 0$ for some $K \in \{1, \dots, n\}$, we then note that

$$\begin{aligned} & \sum_{(x_1, \dots, x_n) \in \{0, 1\}^n} (-1)^{x_1 + \dots + x_n} x_1^{i_1} \dots x_n^{i_n} \\ &= \sum_{x_K \in \{0, 1\}} (-1)^{x_K} \sum_{x_j \in \{0, 1\}, j \neq K} (-1)^{\sum_{j \neq K} x_j} x_1^{i_1} \dots x_n^{i_n} \\ &= 0. \end{aligned}$$

Therefore, $\mathbb{I}_g = 0$, which yields $\mathbb{I}_f = \mathbb{I}_c \prod_{i=1}^n x_i = \pm c \neq 0$, a contradiction.

We will consider such an integral in our proof of Theorem 1 as well. Before doing so however, let us mention first that one of the first certificates of the usefulness of the Combinatorial Nullstellensatz in [1] was a proof of the following theorem due to Chevalley and Warning.

Theorem 3. *Let p be a prime and let $\mathbb{F}$ be a finite field. Let*

$$P_1 = P_1(x_1, \dots, x_n), \dots, P_m = P_m(x_1, \dots, x_n)$$

be m polynomials in $\mathbb{F}[x_1, \dots, x_n]$ such that

$$n > \sum_{j=1}^m d_j,$$

where d_j is the total degree of f_j . Then, if the polynomials P_i have a common zero $(c_1, \dots, c_n)$, they must have another common zero.

We mention this because the one can use the Theorem 3 as a “black-box” to show something in the spirit of Theorem 1.

Theorem 4. *Let p be a prime and let n be integers such that $n > 2(p-1)$. Then, among any n elements $v_1, \dots, v_n$ of $\mathbb{F}_p^2$ there exists a nonempty subsequence with a zero-sum.*

In fact, it is a result of Olson [10] that if G is an abelian p -group of the form $G = \mathbb{Z}_{p^{\alpha_1}} \oplus \dots \oplus \mathbb{Z}_{p^{\alpha_k}}$, and n is any integer such that

$$n > 1 + \sum_{i=1}^k (p^{\alpha_i} - 1),$$

then among every sequence of elements of G with length n there is a nonempty subsequence with a zero sum. We sketch the proof for the scenario described in Theorem 4.

Proof of Theorem 4. Assume without loss of generality that $n = 2p - 1$, and furthermore let $v_i = (a_{i1}, a_{i2})$, for each $1 \leq i \leq n$. Define

$$f_j(x_1, \dots, x_n) := \sum_{i=1}^n a_{ij} x_i^{p-1}$$

for each $j \in \{1, 2\}$. Note that these polynomials fulfil the hypothesis of the Chevalley-Warning theorem: the sum of their degrees is $2(p-1)$, which is less than $n = 2p - 1$, the number of variables. Also, note that the system $f_1 = f_2 = 0$ has the trivial solution $x_1 = \dots = x_n = 0$. By the Chevalley-Warning theorem, it thus has another one, say $(\alpha_1, \dots, \alpha_n)$. Setting $I := \{i \mid \alpha_i \neq 0\}$, this is non-empty and it is easy to check that

$$\sum_{i \in I} v_i = 0,$$

as desired. □

3. PROOF OF THEOREM 1

Let A be a set of vectors $\{x_1, \dots, x_n\} \subset \mathbb{F}_p^2 \setminus \{0, 0\}$ spanning all $p+1$ nonzero directions from 0 in $\mathbb{F}_p^2$. Recall that we want to prove that there exists a subset $I \subset \{1, \dots, n\}$ such that

$$\sum_{i \in I} x_i = 0.$$

Without loss of generality, assume that $n = p+1$ and that

$$A = \{(a_1, a_1), \dots, (a_{p-1}, (p-1)a_{p-1}), (a_p, 0), (0, a_{p+1})\}$$

where $a_1, \dots, a_{p+1}$ are non-zero elements in $\mathbb{F}_p$. We claim that there exists a non-empty subset of A with sum of elements equal to zero.

We will first prove a crucial Lemma.

Lemma 5. *The polynomial $f \in \mathbb{F}_p[x_1, \dots, x_{p+1}]$ defined by*

$$f(x_1, \dots, x_{p+1}) = (x_1 + \dots + x_p)^{p-1} (x_1 + 2x_2 + \dots + px_p + x_{p+1})^{p-1}$$

contains no monomial where variables $x_1, \dots, x_{p+1}$ all have positive exponents.

Proof of Lemma 5. Assume that the coefficient of some monomial $x_1^{c_1} \dots x_{p+1}^{c_{p+1}}$ appearing in the expansion of f is nonzero modulo p , where $c_i > 0$ for each $i \in \{1, \dots, p+1\}$. First, note that each c_i is also less than p , since otherwise the monomial would have degree at least $2p > 2p-2 = \deg f$. Moreover, the coefficient $[x_1^{c_1} \dots x_{p+1}^{c_{p+1}}](f)$ of $x_1^{c_1} \dots x_{p+1}^{c_{p+1}}$ in f satisfies

$$[x_1^{c_1} \dots x_{p+1}^{c_{p+1}}](f) = (c_1 \dots c_{p+1})^{-1} [x_1^{c_1-1} \dots x_{p+1}^{c_{p+1}-1}](g),$$

where the latter denotes the coefficient of $x_1^{c_1-1} \dots x_{p+1}^{c_{p+1}-1}$ in the derivative

$$g(x_1, \dots, x_{p+1}) := \frac{\partial^n f(x_1, \dots, x_{p+1})}{\partial x_1 \dots \partial x_{p+1}}.$$

Since $p > c_i > 0$ for each i , this coefficient is also nonzero modulo p .

On the other hand, the polynomial g is identically zero. Indeed, in the computation of the multiple derivative, we differentiate $(x_1 + x_2 + \dots + x_p)^{p-1}$ with respect to some k variables ($2 \leq k \leq p-1$), including x_p but excluding x_{p+1} , and we differentiate $(x_1 + 2x_2 + \dots + px_p + x_{p+1})^{p-1}$ with respect to other $p+1-k$ variables (including x_{p+1} but excluding x_p). It is easy to check that for each summand appearing in g , a multiple of the $(p-k)$ -th elementary symmetric polynomials in the variables $1, 2, \dots, p$ arises for some $k \in \{2, \dots, p-1\}$. By Vieta's formulas, these are all equal to 0 over $\mathbb{F}_p$, as $(t-1)\dots(t-p) = t^p - t$. We have therefore arrived at a contradiction. $\square$

Now, suppose that A contains no non-empty subset with sum of elements equal to zero. Consider the following polynomial in $\mathbb{F}_p[x_1, \dots, x_{p+1}]$:

$$P(x_1, \dots, x_{p+1}) = \left[1 - \left(\sum_{i=1}^p a_i x_i \right)^{p-1} \right] \left[1 - \left(\sum_{j=1}^{p+1} j a_j x_j \right)^{p-1} \right].$$

Note that $P(x_1, \dots, x_{p+1}) = 1$ precisely when $\sum_{i=1}^p a_i x_i = \sum_{j=1}^{p+1} j a_j x_j = 0$ and is 0 otherwise; therefore, when restricted on $\{0, 1\}^{p+1}$, the polynomial P identifies zero-sum subsets of A . In particular, given our zero-sum-subset free assumption on A , we have that

$$P(0, \dots, 0) = 1 \quad \text{and} \quad P(x_1, \dots, x_{p+1}) = 0,$$

for every $(p+1)$ -tuple $(x_1, \dots, x_{p+1}) \in \{0, 1\}^{p+1} \setminus \{0, \dots, 0\}$. In particular,

$$\sum_{(x_1, \dots, x_{p+1}) \in \{0, 1\}^{p+1}} (-1)^{x_1 + \dots + x_{p+1}} P(x_1, \dots, x_{p+1}) = 1.$$

If $Q \in \mathbb{F}_p[x_1, \dots, x_{p+1}]$ is defined by

$$Q(x_1, \dots, x_{p+1}) = \left(\sum_{i=1}^p a_i x_i \right)^{p-1} \left(\sum_{j=1}^{p+1} j a_j x_j \right)^{p-1},$$

then $P - Q$ is a polynomial of degree $p-1$ in $\mathbb{F}_p[x_1, \dots, x_{p+1}]$. In particular, there is no monomial in $P - Q$ which contains all variables $x_1, \dots, x_{p+1}$, each with positive exponent. For each monomial $x_1^{i_1} \dots x_{p+1}^{i_{p+1}}$, for which $i_K = 0$ for some $K \in \{1, \dots, p+1\}$, we then note that

$$\begin{aligned} & \sum_{(x_1, \dots, x_{p+1}) \in \{0, 1\}^{p+1}} (-1)^{x_1 + \dots + x_{p+1}} x_1^{i_1} \dots x_{p+1}^{i_{p+1}} \\ &= \sum_{x_K \in \{0, 1\}} (-1)^{x_K} \sum_{x_j \in \{0, 1\}, j \neq K} (-1)^{x_1 + \dots + x_{p+1}} x_1^{i_1} \dots x_{p+1}^{i_{p+1}} \\ &= 0. \end{aligned}$$

Since this holds for every monomial appearing in $P - Q$, we get that

$$\sum_{(x_1, \dots, x_{p+1}) \in \{0, 1\}^{p+1}} (-1)^{x_1 + \dots + x_{p+1}} (P(x_1, \dots, x_{p+1}) - Q(x_1, \dots, x_{p+1})) = 0.$$

Consequently,

$$\sum_{(x_1, \dots, x_{p+1}) \in \{0,1\}^{p+1}} (-1)^{x_1 + \dots + x_{p+1}} Q(x_1, \dots, x_{p+1}) = 1.$$

But this gives a contradiction with the structure of A . Lemma 5 says that each monomial appearing in Q must have at least one variable absent; by performing the same trick as above, we can thus argue that for each monomial $x_1^{j_1} \dots x_{p+1}^{j_{p+1}}$ appearing in Q , we have that

$$\sum_{(x_1, \dots, x_{p+1}) \in \{0,1\}^{p+1}} (-1)^{x_1 + \dots + x_{p+1}} x_1^{j_1} \dots x_{p+1}^{j_{p+1}} = 0,$$

and therefore

$$\sum_{(x_1, \dots, x_{p+1}) \in \{0,1\}^{p+1}} (-1)^{x_1 + \dots + x_{p+1}} Q(x_1, \dots, x_{p+1}) = 0,$$

which is a contradiction. □

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