CP^1-Fermion Correspondence in Three Dimensions

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(Received 27 July 1988)

We show that in three dimensions the effective electromagnetic actions of the CP^1 model and of a charged fermion coincide in the vacuum and one-soliton sectors, to lowest order in inverse mass. This implies equivalence between the fermion and CP^1 current operators. In the vacuum sector, the first-order fermion-current–current self-interaction correction, required for equivalence at next-to-leading order, is also obtained. In the soliton phase, the two next to leading order terms appear with coefficients differing by simple numerical factors.

PACS numbers: 11.10.Lm, 11.30.Er, 75.10.Jm

The transmutation of spins and statistics in the one-soliton sector of the CP^1 model in three spacetime dimensions, and the central role played by the Hopf-Chern-Simons term have been known for some time.[1] Recently, Polyakov[2,3] has suggested that there is a more detailed fermion-boson equivalence to leading order in the inverse mass in this sector. We discuss here a related ed aspect of this equivalence: We shall compare the effective electromagnetic actions of the CP^1 and charged-fermion theories in both the vacuum and one-particle sectors, and establish their equality to leading order. This means that the ordinary Fermi current $j^\mu(\psi)\equiv\bar\psi\gamma^\mu\psi$ is equivalent to the CP^1 current

$$J^\mu\equiv-(\theta/4\pi^2)\,{}^*F^\mu(A)\equiv-(\theta/4\pi^2)\epsilon^{\mu\nu\alpha}\partial_\nu A_\alpha$$

for the particular value of the θ parameter in which the spin transmutation was originally noted: Current correlation functions are identical to lowest order in (∂/m) in the vacuum and one-particle sectors. In the vacuum sector, we can also calculate the lowest-order fermion self-interaction $\sim m^{-1}j^\mu j_\mu$ which is required to obtain agreement of the next terms in the effective actions. In the soliton sector, we find the same structure for the next two terms, but there are factors of 2 and 3 differences between their respective coefficients, although these may be affected by corrections we have not included.

It is well known[4,5] that a charged massive fermion in $D=3$ in an external field V_μ,

$$\mathcal{L}(\psi)=\bar\psi(\not{D}\pm m)\psi,\quad D_\mu\equiv i\,\partial_\mu+V_\mu, \tag{1}$$

leads to an effective action in the ψ-vacuum sector which is

$$W_\psi[V]=-i\ln\det(\not{D}(V)\pm m)$$
$$=\pm\pi W_{\rm CS}[V]+\frac{1}{24\pi|m|}\int F^2_{\mu\nu}(V)+O\left(\frac{\partial^2}{m^2}\right), \tag{2}$$

with $W_{\rm CS}[V]$ the Chern-Simons action

$$W_{\rm CS}[V]=\frac{1}{8\pi^2}\int\epsilon^{\mu\nu\alpha}V_\mu\partial_\nu V_\alpha. \tag{3}$$

In $D=3$, the result (2) is finite and cutoff independent, so the coefficients are precise. The sign of the Chern-Simons term is correlated to that of the parity-nonconserving fermion mass term, while that of $F^2_{\mu\nu}(V)$ is fixed (and opposite to that of the Maxwell action); the next term is of order $m^{-2}\epsilon^{\mu\nu\alpha}V_\mu\Box\partial_\nu V_\alpha$.

Consider now the CP^1 model,

$$\mathcal{L}(z,A)=\frac{2}{g_0^2}D^*_\mu(A)z^\dagger_\alpha D^\mu(A)z_\alpha-\frac{2}{g_0^2}\sigma(z^\dagger_\alpha z_\alpha-1)+\frac{\theta}{8\pi^2}\epsilon^{\mu\nu\alpha}A_\mu\partial_\nu A_\alpha, \tag{4}$$

where z_α is a complex two-component field, $D_\mu\equiv\partial_\mu+iA_\mu$, and A_μ is to be varied independently; the θ term is the Hopf-Chern-Simons invariant, while the Lagrange multiplier σ enforces the constraint $z^\dagger z=1$. Now in $D=2$ this theory is renormalizable and it is well known that the one-loop corrections lead to dimensional transmutation in which the z field acquires a mass [we shall follow Coleman's approach (Ref. 6)]. For our vacuum sector discussion, we treat z as a field (rather than consider its soliton aspects) and show that, despite nonrenormalizability for $D=3$, a similar phenomenon takes place. That is, we calculate the effective potential $V(\sigma)$ at one-loop order and show that σ acquires a constant value which acts like a z-field mass to lowest order. In terms of a cutoff Λ, we find that

$$-\frac{1}{2}V(\sigma)=\frac{\sigma}{g_0^2}-\int d^3p[\ln(p^2+\sigma)-\ln p^2]$$
$$=\frac{\sigma}{g_0^2}-\frac{1}{2\pi^2}\left[\Lambda\sigma-\frac{\pi}{3}\sigma^{3/2}\right]+O(1/\Lambda), \tag{5}$$

including in $V(\sigma)$ the linear σ term in (4) and rescaling $z \rightarrow z(g_0/\sqrt{2})$. Renormalizing the coupling constant g_0^2 by

$$\frac{1}{g^2} \equiv -\frac{1}{2}\frac{dV}{d\sigma}\bigg|_{\sigma=M^2} = \frac{1}{g_0^2} - \frac{\Lambda}{2\pi^2} + \frac{\sqrt{M^2}}{4\pi} \quad (6)$$

leads to

$$-\frac{1}{2}V(\sigma) = \frac{\sigma}{g^2} - \frac{M\sigma}{4\pi} + \frac{\sigma^{3/2}}{6\pi} = \frac{\sigma^{3/2}}{6\pi} - \frac{\sqrt{\sigma_0}\sigma}{4\pi}, \quad (7)$$

where σ_0 is the value of σ at the stationary point of $V(\sigma)$. [We require $\sqrt{\sigma_0} = M - 4\pi/g^2 > 0$, but $\sqrt{\sigma_0} < 0$ might be the signal of a phase transition in the (2+1)-dimensional CP^1 model.] Note that Λ has disappeared in favor of the renormalized g^2 and the z field has become effectively massive, with mass $\sqrt{\sigma_0}$. To get the effective action in our external V_μ, we couple the model to V_μ through a current which we take to be

$$J^\mu(A) = \frac{2\theta}{8\pi^2}\epsilon^{\mu\nu\alpha}\partial_\nu A_\alpha = \frac{2\theta}{8\pi^2}\,{}^*F^\mu(A). \quad (8)$$

Thus our Lagrangian becomes (in the approximation where σ contributes only a z mass)

$$\mathcal{L}(z,A;V) = D_\mu^*(A)z^\dagger D^\mu(A)z - \sigma_0 z^\dagger z + (\theta/8\pi^2)\epsilon^{\mu\nu\alpha}(A_\mu\partial_\nu A_\alpha + 2V_\mu\partial_\nu A_\alpha). \quad (9)$$

The change of variables $A_\mu = C_\mu - V_\mu$ leads to

$$\mathcal{L}(z,C-V;V) = \mathcal{L}'(z,C,V) - \frac{\theta}{8\pi^2}\epsilon^{\mu\nu\alpha}V_\mu\partial_\nu V_\alpha,$$
$$\mathcal{L}'(z,C,V) = D_\mu^*(C-V)z^\dagger D^\mu(C-V)z - \sigma_0 z^\dagger z + \frac{\theta}{8\pi^2}\epsilon^{\mu\nu\alpha}C_\mu\partial_\mu C_\alpha; \quad (10)$$

thus the effective action (choosing N so that $W_{CP}[0]=0$) is

$$e^{iW_{CP}[V]} = N^{-1}\exp(-i\theta W_{\rm CS}[V])\int(Dz^\dagger)(Dz)(dC)e^{I'[z,C,V]}. \quad (11)$$

The z integration only produces a term of order $(1/\sqrt{\sigma_0})F_{\mu\nu}^2(C-V)$, and so the leading part of W_{CP} is

$$W_{CP} = -\theta W_{\rm CS}[V] + O(F_{\mu\nu}^2/\sqrt{\sigma_0}). \quad (12)$$

Comparing with the fermionic effective action (2), we see that $W_{CP} = W_\psi$ for (and only for) the value $\theta = \mp\pi$. Hence the current correlation functions are equal in the vacuum sector,

$$\langle j^\mu(\psi(x_1))\cdots j^\lambda(\psi(x_n))\rangle = \langle J^\mu(A(x_1))\cdots J^\lambda(A(x_n))\rangle, \quad (13)$$

where $J^\mu(A)$ is defined in (8).

It is actually possible to go further and obtain the corrections to the free charged-fermion action which will give the same nonleading $(1/\sqrt{\sigma_0})F^2[V]$ corrections to the lowest order $W_\psi[V]$ as those given by the CP^1 model, (9). On the fermion side, consider

$$I[\psi] = I_0 + I_{\rm int} = \int \bar\psi(\not{D} - m)\psi + (c/m)j_\mu(\psi)j^\mu(\psi). \quad (14)$$

Now

$$\int(D\bar\psi)(D\psi)e^{iI_\psi} \sim \left[1 - \frac{ic}{m}\int d^3x\frac{\delta^2}{\delta V_\mu(x)\delta V^\mu(x)}\right]\int(D\bar\psi)(D\psi)e^{iI_0}$$
$$\sim \left[1 - \frac{ic}{m}\int d^3x\frac{\delta^2}{\delta V_\mu(x)\delta V^\mu(x)}\right]\exp\left[\pm i\pi W_{\rm CS}[V] + \frac{i}{24\pi m}\int F_{\mu\nu}^2(V)\right]$$
$$\sim \left[1 + \frac{ic}{16\pi^2 m}\int F_{\mu\nu}^2(V)\right]\exp\left[\pm i\pi W_{\rm CS}[V] + \frac{i}{24\pi m}\int F_{\mu\nu}^2(V)\right]$$
$$\sim \exp\left[\pm\frac{i}{8\pi}\int\epsilon^{\mu\nu\alpha}V_\mu\partial_\nu V_\alpha + i\left[\frac{1}{24\pi m} + \frac{c}{16\pi^2 m}\right]\int F_{\mu\nu}^2(V)\right]. \quad (15)$$

We have made the obvious expansions in $1/m$ to obtain the final $W_\psi[V]$ form, and we have dropped a V-independent contribution. The calculation on the CP^1 side is straightforward: the vacuum polarization by the two charged scalar fields in (9) leads, upon integrating out the z fields and setting $\theta=\mp\pi$,

$$e^{iW_{CP}[V]}=N^{-1}e^{\pm i\pi W_{\rm CS}[V]}\int(DC)\exp\left[\mp i\pi W_{\rm CS}[C]-\frac{i}{24\pi\sqrt{\sigma_0}}\int F_{\mu\nu}^2(C-V)\right]$$
$$=N^{-1}e^{\pm i\pi W_{\rm CS}[V]}\int(DC)\exp(\mp i\pi W_{\rm CS}[C])\times\left[1-\frac{i}{24\pi\sqrt{\sigma_0}}\left(\int F_{\mu\nu}^2(C)-2F_{\mu\nu}(C)F^{\mu\nu}(V)+F_{\mu\nu}^2(V)\right)\right]$$
$$=N'^{-1}e^{\pm i\pi W_{\rm CS}[V]}\left[1-\frac{i}{24\pi\sqrt{\sigma_0}}\int F_{\mu\nu}^2(V)+O\left(\frac{1}{\sigma_0}\right)\right]. \quad (16)$$

Thus for $c=-(2\pi/3)(m/\sqrt{\sigma_0}+1)$ we have equivalence also to the next order, thereby establishing an extended correspondence between the models. Beyond this correction, nonrenormalizability renders the calculations unreliable, although it may not be so dangerous in a specific condensed-matter model application, where the lattice spacing (i.e., the cutoff) is established for some physical reason.

Let us mention a few other aspects of our results so far. First, the vacuum sector is really not very sensitive (in leading order) to the detailed bosonic model (CP^1) that we have used, as long as there is a conserved current and hence a Hopf term, since it is the latter which has produced the $\epsilon V\partial V$ part of $W[V]$. That is, any charged-bosonic theory will only yield $(1/m)F^2$ corrections to $\epsilon V\partial V$; this has also been noted in other contexts.[7,8] Second, our choice (8) of the current J^μ in CP^1 is motivated by several considerations: (a) it is conserved (as is the equivalent quantity $z^\dagger\partial z$), (b) we know the unique fermionic form of W_ψ for which we must aim, and (c) J^μ even resembles the bosonic current $J^\mu=\epsilon^{\mu\nu}\partial_\nu\phi$ in the sine-Gordon–Thirring model for $D=2$. Indeed, this resemblance to a truly fermionizable model raises the question of whether Polyakov's[2] association $z_a(x)\sim z_a^0\exp(i\int^x A\cdot dx)\leftrightarrow\psi_a$ in the one-particle approximation to CP^1 resembles the $D=2$ case[8]: $\exp[i\int^x\dot\phi+C\phi(x,t)]:\leftrightarrow\psi$ (C is a constant given in Ref. 9). Perhaps a vortex picture[9] would supply an association between these phases. Finally, one might ask whether other spins could be associated with other values of θ in the vacuum sector of CP^1. An example is provided by replacing the fermion ψ by a charged vector field B_μ. In order to have a parity-nonconserving action, the B field must have an Abelian Chern-Simons mass term[4]

$$\mathcal{L}(B)=-\tfrac{1}{4}F_{\mu\nu}(B)F^{\mu\nu}(B)+m\epsilon^{\mu\nu\alpha}B_\mu^\dagger\overleftrightarrow{\partial}_\nu B_\alpha. \quad (17)$$

Minimal coupling to V_μ clearly yields a lowest-order effective action $W_B[V]=\theta'\epsilon^{\mu\nu\alpha}V_\mu\partial_\nu V_\alpha$ with a different value θ'. Although we have not calculated θ', power counting implies a unique value for it, and hence for the θ parameter needed to match it in the CP^1 theory {here the current $J^\mu[B]\sim m\epsilon^{\mu\nu\alpha}B_\nu^\dagger B_\alpha+O(B^\dagger\partial B)$}.

We come now to the one-soliton sector. In the spirit of Polyakov's scheme, we approximate this excitation by a point particle, so that in this sector [cf. Ref. (7)]

$$I_{CP}\to\int m(\dot X_\mu\dot X^\mu)^{1/2}+\int d^3y[J^\mu(y)A_\mu(y)+(\theta/8\pi^2)\epsilon^{\mu\nu\alpha}A_\mu\partial_\nu A_\alpha], \quad (18)$$
$$J^\mu(y)\equiv\int d\tau\dot X^\mu(\tau)\delta(y-X^\mu),$$

where we have dropped the seagull (A^2) term in CP^1 and $J\cdot A$ corresponds to the $z^\dagger d_\mu zA^\mu$ term in (4). We again couple the particle to an external V_μ field in the same spirit, with a term $\epsilon^{\mu\nu\alpha}V_\mu\partial_\nu A_\alpha$: Thus,

$$I_{CP}=m\int d\tau\sqrt{\dot X^2}+\int J^\mu(C-V)_\mu+(\theta/8\pi^2)(\epsilon^{\mu\nu\alpha}C_\mu\partial_\nu C_\alpha-\epsilon^{\mu\nu\alpha}V_\mu\partial_\nu V_\alpha).$$

Hence so we find, upon integrating out C_μ,

$$\int dC\int dXe^{iI_{CP}}=N_p^{-1}e^{-i\theta W_{\rm CS}[V]}\int dXe^{iI_X}, \quad (19)$$
$$I_X=m\int\sqrt{\dot X^2}+\frac{2\pi^2}{\theta}\int d^3k\,J_\mu(k)\epsilon^{\mu\nu\alpha}\frac{k_\nu}{k^2}J_\alpha(-k)-\int J^\mu V_\mu,$$

with N_p a normalization constant; in this sector we take boundary conditions $X(\tau_i)=x$ and $X(\tau_f)=y$ for the particle. This amplitude is to be compared to[2]

$$N_\psi\int(d\bar\psi)(d\psi)e^{i(I_\psi+\int j\cdot V)}{\rm tr}[\bar\psi(x)\psi(y)]=e^{i[-\pi W_{\rm CS}+O(\partial/m)]}\langle{\rm tr}\bar\psi(x)\psi(y)\rangle_V, \quad (20)$$

where

$$-\tfrac{1}{2}\int d(x-y)e^{-i(x-y)\cdot p}\langle \mathrm{tr}\bar{\psi}(x)\psi(y)\rangle_V = \mathrm{tr}\frac{-1}{\not p+m} - \int dk\, V_\mu(k)\mathrm{tr}\left[\frac{1}{\not p+m}\gamma^\mu\frac{1}{\not p+\not k+m}\right] + O(V^2)$$

$$= \frac{m}{p^2-m^2} + m\int dk\, V_\mu(k)\frac{(2p+k)^\mu + i\epsilon^{\mu\nu\alpha}p_\alpha k_\nu/m}{(p^2-m^2)[(p+k)^2-m^2]} + O(V^2),$$

for the fermions with normalization N_ψ.

On the particle side we find, after some calculation from (19) that

$$N_p^{-1}e^{-i\theta W_{\mathrm{CS}}}\int d(X)e^{ilx}e^{-i\theta W_{\mathrm{CS}}}G_V(x-y) \qquad (21)$$

where

$$\int d(x-y)e^{-i(x-y)p}G_V(x-y) = \frac{m}{p^2-m^2} + m\int dk V_\mu(k)\frac{\tfrac{1}{2}[(2p+k)^\mu + \pi i\epsilon^{\mu\nu\alpha}p_\alpha k_\nu/3\theta m + O(p^2/m^2)]}{(p^2-m^2)[(p+k)^2-m^2]} + O(V^2).$$

Thus agreement between (20) and (21) to lowest ∂/m order requires $\theta=\pi$ to match the W_{CS} coefficients, as in the vacuum sector (a factor $-m$ has been absorbed in N_p^{-1}). Note that this bosonic system has given rise to a term $\epsilon^{\mu\nu\alpha}p_\alpha k_\nu$ characteristic of the (parity violating) fermionic sector. However, this choice of θ gives a discrepancy for the two terms linear in V_μ. It is possible that a more careful treatment (e.g. including the CP^1 indices) will provide a factor of 2 for these terms in (21) which would still leave a factor 3 problem in the $\epsilon^{\mu\nu\alpha}p_\alpha k_\nu$ term. Perhaps a fermion jj interaction will also be required as in the vacuum sector. Thus, we see that in the one-soliton sector, to lowest order, the CP^1-fermion effective action correspondence persists, although we do find numerical discrepancies at next order (see also Ref. 11).

In summary, we have shown that there is an equivalence at the level of electromagnetic currents and effective actions between charged fermions and CP^1 fields both at the vacuum and the one-soliton sectors to leading ∂/m order. In addition, we have been able to obtain the self-coupling corrections to the fermion action required to maintain equivalence in the vacuum sector to next order in ∂/m. In the soliton sector, we found qualitative agreement to the next two orders, but with some (possibly curable) factor 2 and 3 discrepancies in their coefficients.

We thank M. Bowick, A. Das, T. Jacobson, D. Karabali, B. Marston, A. Schwimmer, and G. Semenoff for stimulating conversations. One of us (S.D.) thanks the Institute for Advanced Study for hospitality. This work was supported by National Science Foundation Grant No. 88-04561 and by the U.S. Department of Energy under Contract No. DE-AC02-76ER02220.

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