Hamiltonian formulation of supergravity*

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(Received 29 April 1977)

The action of supergravity, including mass and cosmological terms, is cast into Hamiltonian form in terms of the graviton and fermion degrees of freedom, and the constraints corresponding to the coordinate and supersymmetry invariances are exhibited. Absence of contact terms in first-order form is an immediate consequence of the latter. Source interactions due to lowest-order exchange of a supergraviton are given in the massive and massless cases. The gravitational zero-mass discontinuity persists in the fermion sector. Invariances of the spin-3/2 field under chiral and dual transformations are analyzed.

I. INTRODUCTION

Supergravity[1,2] consists of two very particularly coupled gauge fields, the Einstein and massless Rarita-Schwinger systems. We propose here to analyze the dynamical properties of supergravity through reduction of the coupled system to Hamiltonian form. The action will be a sum of kinetic terms corresponding to its degrees of freedom together with the Lagrange-multiplier-constraint structure characteristic of the generators of the gauge coordinate and supersymmetry transformations. In the process, it will become clear, from the Hamiltonian viewpoint, why only minimal coupling in first-order form, with no explicit contact terms, can admit supersymmetry.

Before performing the canonical reduction, we shall explicitly analyze the free massless and massive Rarita-Schwinger actions, where a complete reduction to the unconstrained dynamical variables can be performed. When the field is coupled to a prescribed spinor current, the supermatter interactions resulting from single-fermion exchange have the form required by supersymmetry to complement the one-graviton exchange interaction between stress-tensor sources. Consequently the well-known[3] discontinuity between strictly massless and the $m=0$ limit of massive graviton exchange is to be expected in the fermion sector also, and will be displayed explicitly.

Massless Rarita-Schwinger theory exhibits two apparently separate formal invariances—the first is the usual chiral invariance associated with a massless spinor, which seems to play an important role in extended supergravity models.[4] The other is a duality invariance, quite similar to the electromagnetic one,[5] associated with the vector aspect of the field. We shall see that both transformations are indeed implementable in terms of conserved generators constructed from the canonical variables, and that they are in fact identical. Physically, helicity rotations on the spin-$\frac{1}{2}$ or spin-1 aspects of the field are equivalent as expected. We shall also comment on these transformations in the coupled case.

Hamiltonian analysis of full supergravity will involve a number of interesting features. We shall see that the natural dynamical variables of the fermion field are its components with respect to local frames rather than the usual world indices. It will also emerge that even when the system is massive, the characteristic Lagrange-multiplier form of gauge fields is still preserved. This unusual feature foreshadows the supersymmetry preserved when suitable matched mass and cosmological terms are present.[6] In its final Hamiltonian form, the action will have precisely the structure of a doubly gauge invariant system in the "vanishing" total Hamiltonian. Finally, we shall comment on the relation between this structure and the recent work of Teitelboim[7] on the algebra of the constraints.

II. FREE MASSLESS THEORY

The free massless Rarita-Schwinger system can be entirely reduced to unconstrained Hamiltonian form in terms of its two helicity-$\frac{3}{2}$ components. The analysis of the linear constraint corresponding to the Abelian supersymmetry will be instructive when we come to the full theory. In the notation of Ref. 2, the Lagrangian reads

$$\mathcal{L}_{3/2} = -\tfrac{1}{2} i \epsilon^{\rho\mu\nu\alpha} \bar{\psi}_\rho \gamma_5 \gamma_\mu \partial_\nu \psi_\alpha \,. \tag{2.1}$$

It is manifestly invariant under $\delta\psi_\beta = \partial_\beta \alpha(x)$. Performing the space-time decomposition, we have (up to a divergence)

$$\begin{aligned}\mathcal{L}_{3/2} &= -\tfrac{1}{2} i \epsilon^{0ijk} [\bar{\psi}_i \gamma_5 (\gamma_j \partial_0 - \gamma_0 \partial_j) \psi_k + 2 \bar{\psi}_0 \gamma_5 \gamma_i \partial_j \psi_k] \\ &= \mathcal{L}_0 + \mathcal{L}_k + \mathcal{L}_c \,. \end{aligned} \tag{2.2}$$

Clearly, ψ_0 is a Lagrange multiplier which enforces the constraint

$$\sigma_{ij}\partial_i\psi_j = 0\,, \tag{2.3}$$

where we have used $\epsilon^{0ijk}\gamma_5\gamma_i = 2\gamma^0\sigma^{jk}$. Let us first note that only the two gauge-invariant transverse parts ψ_i^T ($\partial_i\psi_i^T \equiv 0$) of the spatial components remain in the action by virtue of the constraint. The latter is needed in the time derivative ($\mathcal{L}_0$) term, while the kinetic ($\mathcal{L}_k$) term is manifestly transverse. However, the $\psi_i^T \equiv (\delta_{ij} - \nabla^{-2}\partial^2{}_{ij})\psi_j \equiv \rho_{ij}\psi_j$ are not all independent, since (2.3) is equivalent to

$$\vec{\gamma}\cdot\vec{\nabla}(\vec{\gamma}\cdot\vec{\psi}^T) = 0\,, \tag{2.4}$$

whose solution is clearly $\chi \equiv \vec{\gamma}\cdot\vec{\psi}^T = 0$. This means the canonical variables are the "doubly transverse" components $\vec{\psi}^{TT}$ of $\vec{\psi}$ satisfying

$$\vec{\nabla}\cdot\vec{\psi}^{TT} = 0 = \vec{\gamma}\cdot\vec{\psi}^{TT}\,. \tag{2.5}$$

These two conditions imply that $\vec{\psi}^{TT}$ is a single Majorana spinor, describing two helicity degrees of freedom. The general solution of (2.5) may be expressed through projection operators:

$$\psi_i^{TT} \equiv \tfrac{1}{2}\rho_{ik}\gamma_l\gamma_k\rho_{lj}\psi_j = \psi_i^T - \tfrac{1}{2}(\gamma_i\vec{\gamma}\cdot\vec{\psi}^T)^T\,. \tag{2.6}$$

We shall not need to insert the projector explicitly in the action, because the operators sandwiched by the $\vec{\psi}^{TT}$'s automatically respect their properties. (Note also that, while $\vec{\psi}^T$ and $\vec{\psi}^{TT}$ are gauge invariant, one can *pick* a gauge such that $\vec{\psi} = \vec{\psi}^{TT}$ by proper choice[8] of α just as in electrodynamics for A_i^T.) The canonical form of the action in terms of the unconstrained degrees of freedom $\vec{\psi}^{TT}$ is then

$$\mathcal{L}_{3/2} = -\tfrac{1}{2}i\epsilon^{0ijk}\bar{\psi}_i^{TT}\gamma_5(\gamma_j\partial_0 - \gamma_0\partial_j)\psi_k^{TT}\,, \tag{2.7}$$

which is equivalent to the form

$$\mathcal{L}_{3/2} = -\tfrac{1}{2}i\bar{\psi}_i^{TT}(\gamma^0\partial_0 + \not{\nabla})\psi_i^{TT} \tag{2.8}$$

appropriate to a real massless spinor field. In terms of the field strengths

$$E_i \equiv -\partial_0\psi_i^{TT}\,, \quad B^i \equiv \tfrac{1}{2}\epsilon^{ijk}\partial_j\psi_k^{TT} \equiv (\nabla\times\psi^{TT})^i\,, \tag{2.9}$$

we also have

$$\mathcal{L}_{3/2} = -\tfrac{1}{2}i\bar{\vec{\psi}}^{TT}\gamma_5\cdot(\vec{\gamma}\times\vec{E} - \gamma_0\vec{B})\,. \tag{2.10}$$

The canonical anticommutation relations may be variously expressed in terms of conjugate pairs of variables; for example, the chiral combination $\vec{\psi}_{L,R}^{TT} = 2^{-1/2}(1 \mp i\gamma_5)\vec{\psi}^{TT}$. In terms of these we have

$$\{\psi_L^{TT*}(\vec{r}), \vec{\psi}_L^{TT}(\vec{0})\} = \vec{\delta}_L^{TT}(\vec{r})\,, \tag{2.11}$$

where the $\vec{\delta}$ is the TT projection of the combined spin-plus-space unit operator, and similarly for $\vec{\psi}_R$. The reduced field equations obtained by varying $\vec{\psi}^{TT}$ are clearly

$$\vec{E} + \gamma_5\vec{B} = 0\,. \tag{2.12}$$

III. CHIRAL AND DUALITY INVARIANCES

There are two apparently disjoint formal invariances of the ψ_μ field in supergravity. In terms of the field strengths

$$f_{\mu\nu} \equiv D_\mu\psi_\nu - D_\nu\psi_\mu\,, \quad {}^*f^{\mu\nu} \equiv \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}f_{\alpha\beta}\,, \tag{3.1}$$

the spin-$\frac{3}{2}$ field equations have the equivalent forms

$$R^\nu \equiv \gamma_\mu {}^*f^{\mu\nu} = 0\,, \tag{3.2a}$$

$$\gamma_\mu f^{\mu\nu} = 0\,, \tag{3.2b}$$

$$f_{\mu\nu} + \gamma_5 {}^*f_{\mu\nu} = 0\,. \tag{3.2c}$$

The last equation is the covariant equivalent of the canonical equation (2.12). Clearly the duality rotation

$$\delta f_{\mu\nu} = {}^*f_{\mu\nu}, \quad \delta\,{}^*f_{\mu\nu} = -f_{\mu\nu} \tag{3.3}$$

transforms the equivalent forms (3.2a) and (3.2b) into each other, while (3.2c) rotates into γ_5 times itself. This is an indication of the equivalence of (3.3) and the chiral rotation

$$\delta\psi_\mu = \gamma_5\psi_\mu\,, \tag{3.4}$$

which has the same effect on (3.2c), since γ_5 commutes with the covariant derivative. A few words of caution about the formal character of the transformations must be said, however. First, equations (3.2) are only part of the full supergravity system, and the ψ field also enters in the Einstein and torsion equations. Second, it is not at all clear that there exist transformations on the basic fields ψ_μ which lead to (3.3) in general. Indeed, this is not the case for Yang-Mills fields[5] because the potential and field strength are not linearly related, and there is an implicit nonlinearity here as well, due to the torsion. Finally, it is clear that at the canonical level the transformation cannot be implemented on all components; ψ_0, for example, is not even a dynamical variable. We now investigate the situation for the free field, but note first that the chirality transformation (3.4) does leave the full supergravity action (and all field equations) invariant in either first- or second-order form, as is clear from simple Dirac algebra.

The free-field duality rotation among electric and magnetic components (2.9) states

$$\delta\vec{\nabla}\times\vec{\psi}^{TT} = (-\dot{\vec{\psi}}^{TT})\,, \quad \delta(-\dot{\vec{\psi}}) = \vec{\nabla}\times\vec{\psi}^{TT}\,, \tag{3.5}$$

or equivalently

$$\delta\vec{\psi}^{TT} = -\nabla^{-2}\vec{\nabla}\times\dot{\vec{\psi}}^{TT}\,, \quad \delta\dot{\vec{\psi}} = -\vec{\nabla}\times\vec{\psi}^{TT}\,. \tag{3.6}$$

However, since this is a first-order system, $\dot{\vec{\psi}}^{TT}$ is determined by (2.12) to be $\gamma_5\vec{\nabla}\times\vec{\psi}^{TT}$, so that on shell the duality transformation reduces to a chiral one, $\delta\vec{\psi}^{TT} = \gamma_5\vec{\psi}^{TT}$. The corresponding generator is

$$G = \tfrac{1}{2}i\int d^3r\,\bar{\vec{\psi}}^{TT}\cdot\gamma_5\gamma_0\vec{\psi}^{TT}\,, \tag{3.7}$$

which may be written in covariant form

$$G=\tfrac{1}{2}i\int d\sigma_\mu \epsilon^{\mu\nu\alpha\beta}\bar{\psi}_\nu\gamma_\alpha\psi_\beta=\int d\sigma_\mu j^\mu . \quad (3.8)$$

The axial-vector current j^μ, which is the dual of the spin density $S^{\nu\alpha\beta}$, is clearly conserved, since

$$\partial_\mu j^\mu=\bar{\psi}_\lambda\gamma_5 R^\lambda=0 ,$$

where R^λ is the Rarita-Schwinger operator of (3.2a). Just as the dual transformation parallels the spin-1 case, the chiral one follows the spin-$\frac{1}{2}$ pattern, where the generator of $\delta\lambda=\gamma_5\lambda$ is just

$$G=\int d\sigma_\mu j^\mu, \quad j^\mu=\bar{\lambda}\gamma_5\gamma^\mu\lambda ,$$

and j^μ is the dual of the spin density $\epsilon^{\mu\nu\alpha\beta}j_\beta$. Thus, there is a degeneracy (for the free field) between the vector-like dual and spinor-like chiral rotations to which the vector-spinor ψ_μ can be subjected, since both affect the helicity in the same way.

One further important point deserves mention. It has been shown that, in the extension of supergravity with appropriate mass and cosmological terms,[7] the fermion still behaves as a massless field.[9] It still has only 2 degrees of freedom, when analyzed in the de Sitter background which corresponds to Minkowski space when a cosmological term is present. One would therefore expect some generalized version of our invariances to hold here, as a manifestation of "masslessness." The field equations in de Sitter space are exactly as in (3.2), but in terms of a modified $f_{\mu\nu}$ defined as follows:

$$\tilde{f}_{\mu\nu}\equiv\mathfrak{D}_\mu\psi_\nu-\mathfrak{D}_\nu\psi_\mu , \quad \mathfrak{D}_\mu\equiv D_\mu+\tfrac{1}{2}m\gamma_\mu , \quad (3.9)$$

where D_μ is the usual metric covariant derivative on a spin-$\frac{1}{2}$ field. In de Sitter space the quantity $\mathfrak{D}_\mu$ has the property that

$$[\mathfrak{D}_\mu,\mathfrak{D}_\nu]\alpha=0 \quad (3.10)$$

for any spin-$\frac{1}{2}$ quantity $\alpha(x)$. Clearly, γ_5 invariance no longer holds, because as usual with mass terms

$$\mathfrak{D}_\mu(m)\gamma_5=\gamma_5\mathfrak{D}_\mu(-m) . \quad (3.11)$$

But dual invariance can be generalized: The formal transformation $\delta\tilde{f}_{\mu\nu}={}^*\tilde{f}_{\mu\nu}$ is an invariance of (3.2) as a function of $\tilde{f}_{\mu\nu}$. But is it a permitted transformation? The quantity ${}^*\tilde{f}^{\mu\nu}$ satisfies the identity $\mathfrak{D}_\mu {}^*\tilde{f}^{\mu\nu}\equiv 0$ in the de Sitter background, as it does in flat space. Therefore we must have $\mathfrak{D}_\mu\delta\tilde{f}^{\mu\nu}=0$, which is compatible with the on-shell equation[10] $\mathfrak{D}_\mu\tilde{f}^{\mu\nu}=0$. It therefore appears that there remains duality, but not γ_5, invariance. The duality-transformed version of (3.2c) is still

$$\delta(\tilde{f}+\gamma_5{}^*\tilde{f})=-\gamma_5(\tilde{f}+\gamma_5{}^*\tilde{f}) , \quad (3.12)$$

but the latter form is not what is obtained when we set $\delta\psi_\mu=\gamma_5\psi_\mu$. Only the spin-1 rotation remains as a corroboration of the effective masslessness of the ψ field. This is perhaps not surprising since the spin-1 content is unaffected by the conformally flat de Sitter metric, while a massive spin-$\frac{1}{2}$ field does not enjoy γ_5 invariance whatever the value of the cosmological constant.

IV. FREE MASSIVE THEORY

The massive Rarita-Schwinger Lagrangian has the form

$$\mathfrak{L}_m=\mathfrak{L}_{3/2}+im\bar{\psi}_\mu\sigma^{\mu\nu}\psi_\nu , \quad (4.1)$$

from which it follows that both $\gamma^\mu\psi_\mu=0$ and $\partial_\mu\psi^\mu=0$, the surviving components satisfying the Dirac equation with mass m. The space-time decomposition now leads to the form

$$\mathfrak{L}_m=-\tfrac{1}{2}i\epsilon^{0ijk}\bar{\psi}_i\gamma_5(\gamma_j\partial_0-\gamma_0\partial_j)\psi_k+im\bar{\psi}_i\sigma^{ij}\psi_j - i\bar{\psi}_0(2\gamma^0\sigma^{jk}\partial_j\psi_k-m\gamma^0\gamma^j\gamma_j) . \quad (4.2)$$

Note that this is quite different from massive electrodynamics, where introduction of $m^2A_\mu{}^2$ gives rise to a term in $m^2A_0{}^2$ and A_0 is no longer a Lagrange multiplier. Thus, we still have a gauge-like constraint even though there is no gauge invariance. The resolution of this paradox lies in the fact that while ψ_0 still disappears from the reduced action without "using up" the constraint equation, the latter no longer removes $\chi\equiv\vec{\gamma}\cdot\vec{\psi}^T$ but rather expresses the longitudinal part, $\vec{\gamma}\cdot\vec{\psi}^L$ (which appears in the $m\bar{\psi}_i\sigma^{ij}\psi_j$ term) as a function of χ. Explicitly, the constraint reads[11]

$$(\not{\nabla}-m)\chi-m\vec{\gamma}\cdot\vec{\psi}^L=0 , \quad (4.3)$$

and χ (or $m\vec{\gamma}\cdot\vec{\psi}^L$) remains to describe the helicity-($\pm\frac{1}{2}$) states. However, as noted earlier, the absence of $\psi_0{}^2$ terms corresponds to the existence of extended supergravity with mass and cosmological terms which is invariant under a slightly modified supersymmetry transformation. After some straightforward algebra, and use of the fact that $\bar{\chi}\gamma^0\not{\nabla}\dot{\chi}$ is a total divergence, one finds for L_m the form

$$\mathfrak{L}_m=-\tfrac{1}{2}i\bar{\vec{\psi}}^{TT}\cdot(\gamma\cdot\partial+m)\vec{\psi}^{TT}-\tfrac{1}{2}i(\tfrac{3}{2})\bar{\chi}(\gamma\cdot\partial-m)\chi . \quad (4.4)$$

Redefining $\chi'=(\frac{3}{2})^{1/2}\gamma_5\chi$ puts the helicity-$\frac{1}{2}$ term in the desired form, $-\frac{1}{2}i\bar{\chi}'(\gamma\cdot\partial+m)\chi'$; note that, as in electrodynamics, these "longitudinal" excitations do not vanish as $m\to 0$. We shall return to their coupling in the next section.

V. INTERACTION WITH SOURCES

Linearized gravitation coupled to a (necessarily conserved) prescribed stress tensor $T_{\mu\nu}$ is well known to lead to an interaction of the form

$$I_2(m=0) \sim \int d^4x d^4x' \left[T_{\mu\nu} D(x-x') T'_{\mu\nu} - \tfrac{1}{2} T_\mu{}^\mu D(x-x') T'^{\nu}_{\nu}\right], \quad (5.1)$$

whereas the massless limit of massive spin 2 leads to the physically different result[3]

$$I_2(m\to 0) \sim \int d^4x d^4x' \left[T_{\mu\nu} D(x-x') T'_{\mu\nu} - \tfrac{1}{3} T_\mu{}^\mu D(x-x') T'^{\nu}_{\nu}\right] \quad (5.2)$$

because the helicity-zero mode does not decouple from $T_\mu{}^\mu$ even in the limit. Is this characteristic of higher-spin Bose fields repeated for spin $\frac{3}{2}$? We shall investigate this in two ways, from the structure of the numerators of the covariant propagator and from the canonical point of view. We may read off the equivalent forms to (5.1) and (5.2) from the corresponding Rarita-Schwinger propagators coupled to a conserved prescribed spinor current[12] j_μ and discover immediately that the corresponding interactions read

$$I_{3/2}(m=0) \sim i \int d^4x d^4x' \left[\bar{j}_\mu S(x-x') j'_\mu - \tfrac{1}{2} \overline{\gamma\cdot j}\, S(x-x') \gamma\cdot j'\right], \quad (5.3)$$

$$I_{3/2}(m\to 0) \sim i \int d^4x d^4x' \left[\bar{j}_\mu S(x-x') j'_\mu - \tfrac{1}{3} \overline{\gamma\cdot j}\, S(x-x') \gamma\cdot j'\right], \quad (5.4)$$

where $S(x)$ is the Dirac propagator. Indeed, there is a gratifying supersymmetry basis for these facts, since it is known that for any system

$$\int \left(t_{\mu\nu}{}^2 + i\bar{j}_\mu \not{\partial} j_\mu - \tfrac{3}{2} c_\mu \Box c_\mu\right) \quad (5.5)$$

is a global invariant,[13] where c_μ is the conserved axial-vector current, and this is still the case when we insert D between all the terms to get

$$\int \left(t_{\mu\nu} D t_{\mu\nu} + i\bar{j}_\mu S j_\mu - \tfrac{3}{2} c_\mu c_\mu\right), \quad (5.6)$$

where $c_\mu{}^2$ is now a contact term. Likewise, it may be shown that the following quantity is also a global invariant:

$$\int \left(t_\mu{}^\mu D t_\nu{}^\nu - i\overline{\gamma\cdot j} S \gamma\cdot j\right), \quad (5.7)$$

Thus, we expect (apart from the irrelevant contact term) that linearized supergravity leads respectively to the sums of (5.1) and (5.3) or (5.2) and (5.4), given in rigid combinations of the invariants (5.6) and (5.7). In this sense, the $m\to 0$ discontinuity persists in both sectors. Of course, just as there are sources with $T_\mu{}^\mu = 0$ in gravity, there are also supermultiplets such as the $(1,\frac{1}{2})$, for which both $T_\mu{}^\mu$ and $\gamma\cdot j$ vanish. On the other hand since there is also nonconformal supermatter, e.g., $(\frac{1}{2},0)$, the difference in coupling of the trace parts gives rise to the same discontinuity for the current-current interaction as arose in the stress-tensor interactions.

The above derivation assumed conservation of both $T_{\mu\nu}$ and j_μ. Strictly speaking this is no longer required when the gauge fields acquire mass. However, as in the gravitational case, one can see that the $m\to 0$ limit is meaningful only for conserved sources. Specifically, the field equation

$$-\gamma_5 \gamma_\nu {}^*f^{\mu\nu} + 2m\sigma^{\mu\nu}\psi_\nu = j^\mu \quad (5.8)$$

has divergence

$$m\sigma^{\mu\nu} f_{\mu\nu} = \partial_\mu j^\mu, \quad (5.9)$$

and, unless $\partial\cdot j = 0$, one cannot recover the massless equations. Contracting γ_μ into (5.8) gives the constraint

$$3m\gamma_\mu \psi^\mu = \gamma_\mu j^\mu. \quad (5.10)$$

This is the analog of the trace of the linearized massive Einstein equation

$$3m^2 h^\alpha{}_\alpha = T^\alpha{}_\alpha. \quad (5.11)$$

In both cases, these equations determine the nondynamical variables $\gamma\cdot\psi$ and $h_{\alpha\alpha}$ in terms of the sources, and their apparently singular dependence on m^{-1} is in fact acceptable when all appropriate factors are included, in contrast to (5.9), which involves the true dynamical variables.

Let us now approach the coupling to sources from the canonical point of view by analyzing the massive action (4.1) in the presence of a source term with

$$\mathcal{L}_J = -i\bar{\psi}_\mu j^\mu = -i(\bar{\psi}_0 j^0 + \bar{\psi}_i^T j_i^T + \bar{\psi}_i^L j_i^L). \quad (5.12)$$

Carrying through the reduction as before, we find that the net coupling has the form

$$-i\bar{\psi}_i^{TT} j_i^{TT} - i\bar{\chi}\gamma_\mu j^\mu + \tfrac{3}{2} i\bar{\chi}\gamma_i j_i - (i/m)\bar{\chi}\partial_\mu j^\mu. \quad (5.13)$$

We see a nonvanishing coupling of the helicity-$\frac{1}{2}$ field χ to $\vec{\gamma}\cdot\vec{j}$, and there is more than just the expected TT coupling even as $m\to 0$. We see again from the last term why the current must be conserved for the limit to exist at all. Had we begun with the strictly massless theory, with the same term (5.12), we would have found the coupling to have the form

$$-i\overline{\psi}_i^{TT} j_i^{TT} - \frac{i}{4}\overline{j}^0 \frac{1}{\nabla} j^0 , \tag{5.14}$$

with the characteristic instantaneous term in addition to the expected TT coupling. The interactions (5.13) and (5.14) correspond respectively to the interactions (5.4) and (5.3).

VI. HAMILTONIAN FORM OF SUPERGRAVITY

The general Hamiltonian structure of supergravity is dictated by its local gauge invariances and by the geometrically appropriate choices of initial-value data for the two fields. A suitable set of degrees of freedom could be taken from spatial components of the vierbein $e_{\mu\alpha}$ and of the fermion field ψ_μ. As we shall see, the appropriate fermionic variables are in fact the local components $\psi_\alpha = e^\mu{}_\alpha \psi_\mu$, rather than ψ_μ itself.[13a] In addition to the "$p\dot{q}$" kinetic terms in these variables, there must appear the usual four general covariance constraints with the $g_{0\mu}$ as Lagrange multipliers, six constraints corresponding to freedom of local vierbein rotations,[14] and finally a fermionic constraint due to local supersymmetry, whose Lagrange multiplier will be proportional to the time component of ψ_α. The standard count of variables eliminated by the constraints then gives the correct number of variables for the two graviton (helicity-2) and fermion (helicity-$\frac{3}{2}$) components.[15] We therefore expect to find that the supergravity Lagrangian

$$\mathfrak{L}_{SG}(e,\omega,\psi) = -\tfrac{1}{2}eR(e,\omega) - \tfrac{1}{2}i\epsilon^{\lambda\mu\nu\rho}\overline{\psi}_\lambda\gamma_5\gamma_\mu D_\nu\psi_\rho$$

will take the Hamiltonian form of a gauge system[16]

$$\mathfrak{L}_{SG} = p^i{}_\alpha \dot{e}_i{}^\alpha - \tfrac{1}{2}i\epsilon^{ijk}\overline{\phi}_i\gamma_5\gamma_j\dot{\phi}_k - N_\mu\mathcal{H}^\mu(p,e,\phi) - \lambda^{\alpha\beta}J_{\alpha\beta}(p,e,\phi) - \overline{\eta}S(p,e,\phi), \tag{6.1}$$

where $(p^{i\alpha}, \phi_i)$ are some suitable set of vierbein momenta and fermion initial data and the Hamiltonian is a sum of constraints. Since even the reduction of spin-$\frac{1}{2}$ fields coupled to gravity is complicated enough in specially chosen local vierbein gauge,[17] and we are not especially interested in keeping generality in this sector, we will immediately dispense with the freedom of local orientations and work in the usual time gauge defined by the six conditions consisting of $e^0{}_a = 0 = e_i{}^0$, together with vanishing of antisymmetric space components, $e_{ia} - e_{ai} = 0$. In this gauge, $e_{ia}e^i{}_b = \delta_{ab}$, and the determinants are related according to ${}^3e/{}^4e = e^0{}_0$. [Fixing the local vierbein gauge eliminates the λJ constraint term in (6.1). Spatial vierbein rotation freedom could easily be kept, at the price of three J_{ab} constraint terms in the final result.]

Our choice of fermionic variables is made by requiring that their kinetic term be explicitly vierbein-independent. It is easy to see from the action (6.1) that a natural choice is

$$\begin{aligned} &\phi_a \equiv ({}^3e)^{1/2}\psi_a = ({}^3e)^{1/2}e^\mu{}_a\psi_\mu , \\ &\eta = e_0{}^0({}^3e)^{1/2}e^\mu{}_0\psi_\mu . \end{aligned} \tag{6.2}$$

The supergravity action written entirely in terms of the local indices (except those connected with explicit derivatives) reads

$$\mathfrak{L}_{SG} = e(-\tfrac{1}{2}\Omega_{\alpha\beta\gamma}\omega^{\gamma\alpha\beta} + \Omega_{\alpha\beta}{}^\beta\omega_\gamma{}^{\alpha\gamma} - \tfrac{1}{2}\omega_\alpha{}^\alpha{}_\beta\omega_\gamma{}^{\gamma\beta} + \tfrac{1}{2}\omega_\alpha{}^{\beta\gamma}\omega_\beta{}^\alpha{}_\gamma) - \tfrac{1}{2}i\epsilon^{\alpha\beta\gamma\delta}\overline{\psi}_\alpha\gamma_5\gamma_\beta(\tfrac{1}{2}\Omega_{\gamma\delta}{}^\lambda\psi_\lambda + e^\mu{}_\gamma\,\partial_\mu\psi_\delta - \tfrac{1}{2}\omega_{\gamma\lambda\kappa}\sigma^{\lambda\kappa}\psi_\delta), \tag{6.3}$$

where $\Omega_{\alpha\beta}{}^\gamma = e^\mu{}_\alpha e^\nu{}_\beta(\partial_\mu e_\nu{}^\gamma - \partial_\nu e_\mu{}^\gamma)$. Following Kibble's discussion of the spin-$\frac{1}{2}$ case, we group the explicit ω term in $\mathfrak{L}_{3/2}$ with the purely gravitational one, writing it as

$$\tfrac{1}{4}ie\overline{\psi}_\alpha\gamma^\delta\psi_\beta\omega_\delta{}^{\alpha\beta} + \tfrac{1}{2}ie\overline{\psi}_\alpha\gamma^\alpha\psi_\beta\omega_\delta{}^{\beta\delta}. \tag{6.4}$$

The algebraic constraints in the ω field equation

$$\omega_{\alpha\beta\gamma} = \tfrac{1}{2}(\Omega_{\beta\gamma\alpha} - \Omega_{\gamma\alpha\beta} + \Omega_{\beta\alpha\gamma}) - \tfrac{1}{4}i(\overline{\psi}_\beta\gamma_\alpha\psi_\gamma + \overline{\psi}_\alpha\gamma_\beta\psi_\gamma - \overline{\psi}_\alpha\gamma_\gamma\psi_\beta) \tag{6.5}$$

can be eliminated. They are just the components not involving time derivatives, i.e., ω_{abc}, $(\omega_{ab0} - \omega_{ba0})$, ω_{00b}. At this point, the gravitational Lagrangian has become

$$\begin{aligned}\mathfrak{L}_G = {}&({}^3e)e_0{}^0\Omega_{a0b}(\pi_{ab} - \tfrac{1}{4}ie^{-1}\overline{\phi}_a\gamma_0\phi_b) + e_0{}^0({}^3e)[-\tfrac{1}{2}(\pi_{ab}{}^2 - \tfrac{1}{2}\pi_{aa}{}^2) - \tfrac{1}{32}e^{-1}(\overline{\phi}_a\gamma_0\phi_b)^2] \\ &+ \overline{\eta}(\tfrac{1}{2}i\gamma_a\phi_b\pi_{ab} - \tfrac{1}{8}e^{-1}\gamma_a\phi_b\overline{\phi}_a\gamma_0\phi_b - \tfrac{1}{2}i\gamma_0\phi_a\omega_{bab}) + e_0{}^0[-\tfrac{1}{2}({}^3e)^3R(\omega) + \tfrac{1}{4}i\overline{\phi}_a\gamma_c\phi_b\omega_{cab} + \tfrac{1}{2}i\overline{\phi}_a\gamma_a\phi_b\omega_{cbc}]\end{aligned} \tag{6.6}$$

in terms of

$$\begin{aligned} &\pi_{ab} \equiv \omega_{(ab)0} - \delta_{ab}\omega_{cc0} , \\ &({}^3e)^3R(\omega) = -2\partial_i(e^i{}_a{}^3e\omega_{bab}) + {}^3e\Omega_{abc}\omega_{cab} - 2({}^3e)\Omega_{abb}\omega_{cac} + {}^3e\omega_{aab}\omega_{ccb} - {}^3e\omega_{abc}\omega_{bac} . \end{aligned} \tag{6.7}$$

In arriving at the form (6.6) several important cancellations have occurred, in particular among terms quadratic in the constraint variable η in the spinorial terms naturally associated with $\mathfrak{L}_G$. Also, we can already note why a necessary condition for a supersymmetric theory is the absence of any ψ^4 (or higher) contact terms in first-order form (or equivalently the presence of just those contact terms corresponding to it in second-order form). Any quartic terms would necessarily have at least a quadratic dependence on ψ_0. [Terms such as $\epsilon^{\alpha\beta\gamma\delta}(\bar{\psi}_\alpha\gamma_\epsilon\psi_\beta)(\bar{\psi}_\gamma\gamma_\epsilon\psi_\delta)$, which would be linear in ψ_0, vanish identically.] Thus one of the surprising properties of supergravity lies in the possibility of having ψ_0 enter only as a linear Lagrange-multiplier coefficient.[18] Otherwise, there would be no gauge-constraint form. The ω-independent part of $\mathcal{L}_{3/2}$ decomposes into

$$\mathcal{L}_{3/2}=\tfrac{1}{2}i\epsilon^{abc}\bar{\phi}_a\gamma_5\gamma_b\dot{\phi}_c+\tfrac{1}{2}ie_0{}^0\epsilon^{abc}(\bar{\phi}_a\gamma_5\gamma_b e^i{}_0\partial_i\phi_c-\bar{\phi}_a\gamma_5\gamma_0 e^i{}_b\partial_i\phi_c-\tfrac{1}{2}\bar{\phi}_a\gamma_5\gamma_0\Omega_{bcf}\phi_f-\bar{\phi}_a\gamma_5\gamma_b\Omega_{c0f}\phi_f)$$
$$+\tfrac{1}{2}i\epsilon^{abc}\bar{\eta}\gamma_5\gamma_a[2e^i{}_b\partial_i\phi_c+(\partial_i e^i{}_b)\phi_c+\tfrac{1}{2}\Omega_{bcf}\phi_f]\,. \quad (6.8)$$

Defining the usual Lagrange multipliers

$$N\equiv e_0{}^0\,,\quad N^i=e_0{}^0e^i{}_0\,, \quad (6.9)$$

we have

$$\Omega_{a0}{}^b=-e^i{}_a e^0{}_0(\dot{e}_i{}^b+e_j{}^b\partial_i N^j+N^j\partial_j e_i{}^b)\,. \quad (6.10)$$

The presence of torsion, i.e., derivative coupling, means that the true gravitational momentum must be translated in (6.6) from its uncoupled form π_{ab} to include appropriate $\bar{\phi}\phi$ terms. In the case of spin $\frac{1}{2}$, this is a rather trivial (though formally complicated) process, because the coefficient S_{ab} of $e^i{}_b\dot{e}_{ia}$ is purely antisymmetric.[17] Thus, in a coordinate frame in which g_{ij} (and hence e_{ia}) is diagonal it would disappear. However, here we encounter both symmetric and antisymmetric terms in $\phi\phi$. In terms of

$$\Sigma_{ab}=-\tfrac{1}{4}(\epsilon^{fga}\bar{\phi}_f\gamma_5\gamma_g\phi_b+\epsilon^{fgb}\bar{\phi}_f\gamma_5\gamma_g\phi_a) \quad (6.11a)$$

and

$$\tilde{S}_{ab}\equiv-\tfrac{1}{4}i\bar{\phi}_a\gamma_0\phi_b-\tfrac{1}{4}i(\epsilon^{fga}\phi_f\gamma_5\gamma_g\phi_b-\epsilon^{fgb}\bar{\phi}_f\gamma_5\gamma_g\phi_a)\,, \quad (6.11b)$$

we define

$$p^i{}_a\equiv e^i{}_b[({}^3e)\pi_{ab}+\Sigma_{ab}+\tilde{S}_{ab}]\,. \quad (6.12)$$

The final form of the action is

$$\mathcal{L}=\tfrac{1}{2}i\epsilon^{abc}\bar{\phi}_a\gamma_5\gamma_b\dot{\phi}_c-p^i{}_a\dot{e}_i{}^a+N_a[D_i(e)p^{ia}+\tfrac{1}{2}i\epsilon^{bcd}\bar{\phi}_b\gamma_5\gamma_c e^i{}_a D_i(e)\phi_d]$$
$$-N\{\tfrac{1}{2}({}^3e)^3R(\omega)+\tfrac{1}{2}e^{-1}[p_{ia}{}^2-\tfrac{1}{2}(p^i{}_a e_i{}^a)^2]+e^{-1}p_{ij}(-\Sigma^{ij}-\tfrac{1}{2}\tilde{S}^{ij})+\tfrac{1}{2}i\epsilon^{abc}\bar{\phi}_a\gamma_5\gamma_0 e^i{}_b D_i(\omega)\phi_c+Q\}$$
$$+\bar{\eta}[i\epsilon^{abc}\gamma_5\gamma_a e^i{}_b D_i(\omega)\phi_c+\tfrac{1}{2}ie^{-1}\gamma_a\phi_b e_i{}^b p^{ia}-C]\,, \quad (6.13)$$

where

$$Q\equiv e^{-1}[\tfrac{1}{2}\Sigma_{ab}{}^2+\tfrac{1}{32}(\bar{\phi}_a\gamma_0\phi_b)^2-\tfrac{1}{8}i\epsilon^{abc}(\bar{\phi}_a\gamma_5\gamma_0\phi_d)(\bar{\phi}_b\gamma^d\phi_c)]\,,$$
$$C\equiv e^{-1}\{\epsilon^{abc}\gamma_5\gamma_a[\tfrac{1}{4}\phi_c(\bar{\phi}_d\gamma_d\phi_b)-\tfrac{1}{8}\phi_d(\bar{\phi}_b\gamma_d\phi_c)]+\tfrac{1}{4}\epsilon^{abc}\gamma_d\phi_c(\bar{\phi}_a\gamma_5\gamma_b\phi_d)\}\,,$$

and where the torsion covariant derivative is given by

$$D_i(\omega)\phi_c\equiv\partial_i\phi_c-\omega_{icd}\phi_d-\tfrac{1}{2}\omega_{iab}\sigma^{ab}\phi_c-\tfrac{1}{2}e^{-1}\partial_i(e)\phi_c\,,$$

while $D_i(e)$ is the usual metric three-covariant derivative. Note also that we have some torsion contributions left implicit by using ${}^3R(\omega)$ rather than expressing it in terms of ${}^3R(e)$ and spin density.

In our spatially symmetric gauge, only the symmetric part of p^{ia} is the conjugate momentum, and its antisymmetric part is to be regarded as the bilinear expression in ϕ arising from the antisymmetric part of $\tilde{S}_{ia}\equiv e_i{}^b\tilde{S}_{ba}$. For example, the matter momentum density is partly contained in the $D_i(e)p^{[ia]}$ term.

While rather complicated algebraically, the structure of (6.13) becomes transparent when guided by our "prediction" in the fixed vierbein gauge.[19] In addition to the bilinear kinetic terms in $\phi\dot{\phi}$ and $p\dot{e}$, we have the four gravitational constraints $N^\mu\mathcal{H}_\mu(\phi,p,e)$ which involve the energy-momentum density of the graviton and fermion fields, and the supersymmetry constraint, $\bar{\eta}S$, whose linearized limit we recognize from the free-field discussion. The canonical form (6.13) shares, as it should, with that of pure gravity the vanishing of the total generator, in this case the sum of the "Hamiltonians" $\mathcal{H}_\mu$ and the "supercurrent" S. In principle, the coupled constraint equations can be solved iteratively for the constraint variables conjugate to the desired gauge components, in terms of the reduced dynamical variables, e.g., $\vec{\psi}^{TT}$ and the transversetraceless parts of the gravitational data.

The extension of supergravity with mass and cosmological terms also fits into the present framework. This is easily seen by expressing these terms in 3-dimensional form,

$$\begin{aligned}&\lambda({}^4e) = N(\lambda {}^3e)\,,\\ &m({}^4e)\bar{\psi}_\alpha \sigma^{\alpha\beta}\psi_\beta = N(m\bar{\phi}_a\sigma^{ab}\phi_b) + \bar{\eta}(m\gamma^0\gamma^a\phi_a)\,,\end{aligned} \tag{6.14}$$

and adding them to the appropriate constraints. Note that Lagrange-multiplier form is preserved. In particular, the special relation $\lambda = 3m^2$ characterizing the extension[6] of supergravity should in principle be observed from this point of view either as that value at which the local supersymmetry algebra (see below) holds, or equivalently as the value at which the S constraint has a flux-integral form, as discussed previously.

Our explicit results for the canonical form of the action may be compared to the very recent beautiful work of Teitelboim.[8] There, the structure of the local constraints $(S, \mathcal{H}_\mu)$ is obtained by purely Hamiltonian arguments, and they are found to obey a very natural *local* graded algebra. These considerations account for the presence of various otherwise surprising terms in our constraints, e.g., the dependence of S on the gravitational momenta, although we have not checked the details of the algebra from the fundamental commutators and anticommutators.[20] From Teitelboim's point of view the algebra follows just because S is constructed to be essentially the square root of the gravitational Hamiltonian constraint $\mathcal{H}_0$.

Finally, we discuss the question of integrated charges. Because they are gauge generators, $(\mathcal{H}_\mu, S)$ vanish on the mass shell. However, as for pure gravity, one must be careful in extracting the corresponding physical quantities. Thus the total energy-momentum P^μ of an asymptotically flat space-time does not vanish because the boundary conditions define preferred Minkowski frames at infinity. One divides the $\mathcal{H}^\mu$ into their linearized and nonlinear parts with respect to the asymptotic (flat) metric, $\mathcal{H}^\mu \equiv \mathcal{H}^\mu_L + \mathcal{H}^\mu_N = 0$. Now the $\mathcal{H}^\mu_L$ are effectively Laplacian operators on metric components, $\mathcal{H}^L_\mu \sim \nabla^2 g$, and P^μ is just given by

$$P^\mu \sim \int d^3r\, \mathcal{H}^\mu_L = \oint_\infty dS_i\, \partial_i \varphi^\mu = -\int d^3r\, \mathcal{H}^\mu_N\,. \tag{6.15}$$

The same can be done here for the generator S, and a total fermionic charge Q can be defined according to

$$Q = \int d^3r\, S_L = -\int d^3r\, S_N\,. \tag{6.16}$$

One would then expect the global supersymmetric relation $\{Q, Q\} = \gamma^\mu p_\mu$ to hold and in particular that

$$\mathrm{tr}\, QQ = P_0 \tag{6.17}$$

is valid for the total (asymptotically defined) "charges." If this can indeed be accomplished,[21] it may simultaneously provide an elegant solution to the long-standing problem of positive gravitational field energy, since the total energy of supergravity would be manifestly positive according to (6.17).

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*Work supported by National Science Foundation under Grant No. PHY-76-07299.

†Work supported in part by a National Science Foundation Fellowship.

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[10] Of course, on the mass shell, we may also replace $\mathfrak{D}_\mu f^{\mu\nu}$ by $D_\mu \tilde{f}^{\mu\nu}$ since $\gamma \cdot \tilde{f} = \gamma \cdot {}^*\tilde{f} = 0$.

[11] Note that the constraint (4.3) differs from a genuine gauge constraint in that it is not in flux-integral form, $\nabla \cdot v = 0$, unlike electrodynamics or gravity, but rather involves the finite range operator $(\not\nabla - m)$.

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[15] The six vierbein constraints are algebraic and reduce the sixteen $e_{\mu\alpha}$ to ten $g_{\mu\nu}$. The reduction from ten to two is then as in the metric formulation, where each gauge constraint accounts for two variables. For the

fermion, the single differential constraint eliminates two variables, leaving only one Majorana spinor since ψ_0 disappears, being a Lagrange multiplier.

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