Quantum Energy Teleportation in Two-dimensional Conformal Field Theories

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Abstract

We construct a set of quasi-local measurement operators in 2D CFT, and then use them to proceed the quantum energy teleportation (QET) protocol and show it is viable. These measurement operators are constructed out of the projectors constructed from shadow operators, but further acting on the product of two spatially separated primary fields. They are equivalently the OPE blocks in the large central charge limit up to some UV-cutoff dependent normalization. However, we show that the probabilities of their measurement outcomes are UV-cutoff independent. Furthermore, we also demonstrate that despite of their quasi-locality, the OPE blocks do satisfy the causality constraints. To show the essence of quantum entanglement for the viability of QET, we prove a no-go theorem if taking the long time limit $T \to \infty$, where T is the time duration between Alice's and Bob's operation. Beyond this limit we find the QET can be successfully realized. In contrast, we find that these measurement operators cannot violate CHSH inequality.

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1 Introduction

Quantum entanglement has been studied intensively in the past few years in quantum field theory (QFT) and many-body systems, partly inspired by the Ryu-Takayanagi formula of the holographic entanglement entropy [1,2], partly inspired by the new quantum order in the many-body condensed matter systems [3,4], and moreover by the connection of these twos [5–8]. There are usually two ways to characterize the quantum entanglement. One is to evaluate the entanglement entropy or Rényi entropies of the reduced density matrix of a quantum state. The more is the entanglement entropy, the higher the quantum state is entangled. This has been done quite extensively recently to understand the entangled nature of the quantum state such as the area-law for the ground state entanglement entropy, see for example [9]. The other way is to treat the entanglement of quantum state as the resources for some quantum information tasks, which will help to enhance the efficiency of the similar tasks in the classical computation and communication, and to reduce the complexity. There are many classic examples in the earlier development of quantum information sciences, such as quantum teleportation [10], dense coding [11] and so on. However, most of these examples are performed for the few-qubit systems, and seldom for the QFT or many-body systems.

Take the quantum teleportation as an example. Alice and Bob share a 2-qubit Bell state, and Alice would like to use the Bell state as the resource to send her unknown 1-qubit state to Bob by LOCC (local operation and classical communication) without actually sending her physical qubit. If we would like to formulate the similar task in the interacting QFT, e.g., for free QFT see [12] for some discussions , we will face many problems, some of which will be discussed in this paper. The most obvious problem is how to formulate the qubit-like degrees of freedom in QFT, which can encode some quantum information for the further manipulation. The other problem is how to formulate the quantum measurements in the sense of positive operator-valued measure (POVM).

On the other hand, a similar task of exploiting the quantum entanglement as the resources is the quantum energy teleportation (QET) [17–20], for which Alice will send the energy (not the quantum state) to Bob by LOCC. Since this task requires no specific qubit-like information, it is more suitable to be used to characterize the quantum entanglement in the QFT context. In this paper we will study the QET in the two-dimensional (2D) conformal field theory (CFT). In fact, in a previous paper [21] one of the authors and his collaborators have proposed a tentative QET protocol in the context of AdS/CFT correspondence. In this scenario Bob's local operation is formulated by deforming the UV hypersurface of AdS space by following the so-called surface/state correspondence [22, 23], which states that each (space-like) hypersurface in AdS space corresponds to a quantum state in the dual CFT, and all the hypersurface states are related by surface deformations or the local unitary transformations (of dual CFT). Even the tentative holographic QET protocol in [21] lacks a well-defined quantum measurement performed by Alice, it was shown that the tentative holographic QET protocol is a successful task so that Alice can teleport energy to Bob by local operation.

In this wok, instead of adopting the holographic principle, we tackle this problem directly in the context of 2D CFTs. Since the QET protocol involves two distant regions for which live Alice and Bob, respectively, we need to be able to define the local measurements and local operations in the sense of QFT. One way to define the local measurements is to smear the local operators for a finite region. The smearing is used to alleviate the UV divergence of the point-operators. However, in the relativistic sense, we need to ensure the smearing operators obey the causality constraint, i.e., two space-like separated operators commute, see [29] for discussions of this issue on CFTs. This will then impose some constraints on the smearing functions. Besides, for a set of operators to be qualified as the set for quantum measurement, we also require them to form a complete set, i.e., sum to identity operator.

In general, it is hard to find a set of smearing operators satisfying both requirements discussed above. In this work we propose that the OPE blocks formulated in [14] can be used as a set of local quantum measurements in the weak sense, i.e., just holds for ground state but not in the operator sense. The OPE blocks can be shown to be equivalent to be the projector operators P_k's with k labelling the outcomes, which are constructed in the shadow formalism [13], acting on the product of two separated local primary operators, i.e., $O_i(x_1)O_j(x_2)$. The projectors P_k's are not local but smear over the entire spacetime. However, in 2D CFTs they can be reduced to quasi-local ones over the causal diamond subtended by the interval $[x_1, x_2]$. The reason of the weak sense is that the set of OPE blocks cannot be complete. This is easy to see by the fact that the set of projectors constructed by shadow formalism is by construction complete, so that their quasi-locally reduced versions, i.e., OPE blocks, cannot be. Despite that, this is good enough to adopt them for the QET protocol by either in the weak sense or adopting the view of acting P_k's on the excited state $O_i(x_1)O_j(x_2)|0\rangle$ initially prepared by Alice for QET, where $|0\rangle$ is CFT's ground state. Moreover, we will also show that the OPE blocks are causal operators in the weak sense, i.e., when evaluating the vev of its commutators with primary operators.

By adopting the OPE blocks as the quasi-local quantum measurements, we can then proceed the QET protocol in 2D CFTs. First, Alice performs the quantum measurement on her interval $[x_1, x_2]$ at time $t = 0$, and send her outcome by classical communication to a distant Bob so that Bob can perform local operation on his interval $[x_3, x_4]$ according to Alice's message at time T ($T \geq |x_3 - x_2|$). We then evaluate the energetics for each step and then deduce how much energy can Bob locally extract from the new CFT state after the distant Alice perform her quasi-local measurements. We then find that the QET protocol cannot be successful, i.e., Bob cannot extract energy in the infinite time limit, i.e., $T \to \infty$. Otherwise, the QET works by appropriately tuning Bob's local operations given Alice's measurement outcome. This then justifies the QET protocol in 2D CFTs and the intuition that the quantum entanglement of CFT state as the QET resources will be ruined if we take the infinite time limit.

In the following the paper is organized as follows. In section 2 we will review the issues of POVM in QFT, and then propose the OPE blocks as the set of quasi-local quantum measurements in 2D CFTs. In section 3 we adopt the OPE blocks as the quantum measurements for the QET protocol and calculate the energetics at each step. We first show that the QET will fail in the infinite time limit, and then show that the sub-leading correction beyond this limit will then yield QET by appropriate quantum feedback control. Finally, we give a toy example for demonstration of viable QET in CFT. We then conclude our paper in section 4 and end with a discussion on Bell inequality of the OPE blocks. In the Appendix A we give the details of deriving the 3-point functions and conformal blocks from the OPE blocks, as the precursors for 3. In Appendix B we check the consistency of OPE blocks with the causality constraints. In the Appendix C we give a brief introduction for the $i\epsilon$ prescription of Lorentzian commutator. Appendix D just contains some technical details of section 3.2.

2 Projection Measurements in CFT

A quantum measurement process can be described by a set of positive operators $\{E_k\}$ whose sum is the identity operator, i.e.,

$$\sum_k E_k = \mathbb{I} \,. \tag{1}$$

Then, the probability of obtaining the outcome k when measuring the state $|\psi\rangle$ is

$$\mathrm{p}_k = \langle\psi|E_k|\psi\rangle \,. \tag{2}$$

This is known as the positive operator-valued measure (POVM). A special case is when the positive operators E_k's are all projection operators, i.e., $E_k^\dagger E_j = \delta_{k,j} E_k$, then the normalized post-measurement state of outcome k is

$$|\psi_k\rangle = \frac{E_k|\psi\rangle}{\sqrt{\langle\psi|E_k|\psi\rangle}} \ . \tag{3}$$

This is the so-called projective-valued measure (PVM).

Moreover, the POVM can also be constructed by introducing the auxiliary probe coupled to the state $|\psi\rangle$, so that the operator E_k can be obtained as follows: acting on the total system by the time evolution operator $U(t)$, and then projecting it onto the probe's eigenstate $|k\rangle_p$, i.e.,

$$E_k := M_k^\dagger M_k \tag{4}$$

with

$$M_k := \ {}_p\langle k|U(t)|0\rangle_p, \tag{5}$$

where the subscript p denotes "probe". It is easy to see that (1) is satisfied by $U^\dagger U = 1$.

Based on the above procedure, one may construct the POVM in quantum field theory (QFT) and then implement them on some quantum tasks, see for example [20] on constructing POVM of free QFT for quantum energy teleportation. However, in practical the construction of POVM for interacting QFT is not so straightforward due to nontrivial operator mixings.

2.1 OPE block in CFT

Instead, in d-dimensional CFTs there is a set of projection operators constructed by the shadow operator formalism [13], and explicitly they are given by

$$P_k = \frac{\Gamma(\Delta_k)\Gamma(d-\Delta_k)}{\pi^d\Gamma(\Delta_k - \frac{d}{2})\Gamma(\frac{d}{2}-\Delta_k)} \int D^d X \ \mathcal{O}_k(X)|0\rangle\langle 0|\tilde{\mathcal{O}}_k(X), \tag{6}$$

where $\Gamma(x)$ is the Gammas function and Δ_k is the conformal dimension of $\mathcal{O}_k$. These projectors are complete if k runs over all primaries, i.e.,

$$\sum_{k\in \text{all primaries}} P_k = \mathbb{I}_{CFT} \ . \tag{7}$$

We have introduced the shadow operator [1] [2]

$$\tilde{\mathcal{O}}_k(X) := \int D^dY \; \frac{1}{(-2X\cdot Y)^{d-\Delta_k}}\mathcal{O}_k(Y), \tag{8}$$

so that it can be used to show that

$$P_i P_j = \delta_{i,j} P_i \;. \tag{9}$$

In the above, we adopt the notation of embedding space for the coordinate X, i.e., for CFT in d-dimensions, the "embedding space" is $\mathbb{R}^{d,2}$. The dimensional space is obtained by quotienting the null cone $X^2 = 0$ and by the rescaling $X \sim \lambda X$, $\lambda \in \mathbb{R}$. In particular, we can choose the Poincare section such that $X := (X^+, X^-, X^\mu) = (1, x^\mu x_\mu, x^\mu)$ such that

$$-2X_1 \cdot X_2 = (x_1 - x_2)^2 \;.$$

Even though P_j's are projection operators, however, it is not local and thus we cannot use them to implement local quantum measurements which are required in many quantum information tasks such as quantum (energy) teleportation. Fortunately, for 2D CFTs the P_j becomes a quasi-local operator when acting on the following states

$$O_1(x_1)O_2(x_2)|0\rangle, \tag{10}$$

where $|0\rangle$ is the ground state of CFT. In this case, the integration in (6) and (8) is over the casual diamond $\mathcal{D}_{\mathbf{A}}$ subtended by the interval $[x_1, x_2]$, i.e., $x_1 < x_2$ w.l.o.g.. For simplicity, we will only consider the case with $O_1 = O_2 := O_i$ the primary operator of conformal dimension $(h_i, \bar{h}_i)$. We can then view the state (10) as some quasi-local excitation prepared by Alice, and then she further performs a local projection measurement within her causal domain for some quantum information task.

Indeed, the post-measurement state is related to the OPE block defined in [14], i.e.,

$$P_k\, \mathcal{O}_i(x_1)\mathcal{O}_i(x_2)|0\rangle = x_{12}^{-2h_i-2\bar{h}_i} c_{iik}\, \mathcal{B}_k(x_1,x_2)|0\rangle \;. \tag{11}$$

where c_{iik} is the OPE coefficient and $x_{mn} := x_m - x_n$. By this definition, it is straightforward to relate the conformal block $g_k(u,v)$ and the two-point correlator

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[2] The "conformal integral" in (8) is defined by [13]

$$\int D^dX\; f(X) = \frac{1}{\text{Vol GL}(1,\mathbb{R})^+}\int_{X^+ + X^- \geq 0} d^{d+2}X\; \delta(X^2)\; f(X) \;.$$

of the OPE blocks, i.e.,

$$\begin{aligned} g_k(u,v) &:= (x_{12}x_{34})^{2h_i+2\bar{h}_i} c_{iik}^{-2} \langle 0|\mathcal{O}_i(x_3)\mathcal{O}_i(x_4)\ P_k\ \mathcal{O}_i(x_1)\mathcal{O}_i(x_2)|0\rangle \\ &= \langle 0|\mathcal{B}_k^\dagger(x_4,x_3)\mathcal{B}_k(x_1,x_2)|0\rangle, \end{aligned} \tag{12}$$

where the cross ratio u is

$$u := \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\ , v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\ . \tag{13}$$

Note that the second equality is arrived by the definition (11) of the OPE block.

In 2D Minkowski spacetime the OPE block $\mathcal{B}_k(x_1,x_2)$ can be expressed in terms of a smearing operator over the causal diamond $\mathcal{D}_{\mathbf{A}}$, i.e.,

$$\mathcal{B}_k(x_1,x_2) = \int_{\mathcal{D}_{\mathbf{A}}} d^2w\ G_k(w,\bar{w};x_1,x_2)\mathcal{O}_k(w,\bar{w}), \tag{14}$$

where O_k is a primary operator of conformal dimension $(h_k,\bar{h}_k)$. The smearing function $G_k(w,\bar{w};x_1,x_2)$ is the propagator constructed in the framework of the integral geometry [14], and takes the following form in the large central charge limit [3]:

$$G_k(w_0,\bar{w}_0;x_1,x_2) = n_k\bar{n}_k\Big(\frac{w_{01}w_{20}}{x_{21}}\Big)^{h_k-1}\Big(\frac{\bar{w}_{01}\bar{w}_{20}}{x_{21}}\Big)^{\bar{h}_k-1}, \tag{15}$$

where the overall factors n_k and $\bar{n}_k$ are

$$n_k := \frac{\Gamma(2h_k)}{\Gamma(h_k)^2}\ , \qquad \bar{n}_k := \frac{\Gamma(2\bar{h}_k)}{\Gamma(\bar{h}_k)^2}\ . \tag{16}$$

In the above and hereafter, we denote the lightcone coordinates of the spacetime point (t,x) by

$$w := x - t\ , \qquad \bar{w} := x + t\ . \tag{17}$$

Especially, $w = \bar{w} = x$ on $t = 0$ slice. We also introduce the short-handed notation: $w_{ij} := w_i - w_j$ and $\bar{w}_{ij} := \bar{w}_i - \bar{w}_j$.

2.2 OPE block as POVM

The form of (12) is similar as the definition of the probability of POVM (2) if we take the limit $x_1 \to x_4$ and $x_2 \to x_3$. This motives us to construct the POVM operators by OPE block with some suitable normalization and regularization.
From (2) the probability for the outcome k is formally given by

$$\mathrm{p}_k = \frac{c_{iik}^2 g_k(1,0)}{\sum_j c_{iij}^2 g_j(1,0)}\ , \qquad \text{with}\ \sum_k p_k = 1 \tag{18}$$

[3]In Appendix B we show the essence of the special form of (15) in its consistency with the causality constraint.

where $g_k(u,v)$ is the conformal block defined by (12). After the measurement, the outcome state becomes

$$|\psi_k\rangle := \frac{\mathcal{B}_k(x_1,x_2)|0\rangle}{\sqrt{g_k(1,0)}} \ . \tag{19}$$

Note that p_k is independent of the value of x_1 and x_2 though $|\psi_k\rangle$ does.

Using (14) and (15) we can evaluate the universal global conformal blocks for 2D CFTs, and the results are

$$g_k(u,v) = z^{h_k}\bar{z}^{\bar{h}_k}\ {}_2F_1(h_k,h_k,2h_k,z)\ {}_2F_1(\bar{h}_k,\bar{h}_k,2\bar{h}_k,\bar{z}) \ , \tag{20}$$

where $z = w_{12}w_{34}/(w_{13}w_{24})$ and $\bar{z} = \bar{w}_{12}\bar{w}_{34}/(\bar{w}_{13}\bar{w}_{24})$ are the cross ratios, and $u = z\bar{z}, v = (1-z)(1-\bar{z})$. The details of derivation can be found in Appendix A.

In the limit $x_1 \to x_4, x_2 \to x_3$, $z, \bar{z} \to 1$. Notice that for $h_k, \bar{h}_k > 0$

$$g_k(u \to 1, v \to 0) \to\ {}_2F_1(h_k,h_k,2h_k,1)\ {}_2F_1(\bar{h}_k,\bar{h}_k,2\bar{h}_k,1) \ , \tag{21}$$

which is formally divergent and needs some regularization. Moreover, it is easy to see that $g_k(1,0)$ should be dimensionless, and thus the divergence is log divergence. In fact, by definition of (19), this regularization can be understood as the wavefuntion renormalization. At this moment we only formally treat $g_{k\neq 0}(1,0)$ as a regularized function of running energy scale μ in the form of $\log\frac{\Lambda}{\mu}$ where Λ is some UV cutoff energy scale. We will discuss more details on regularization in next subsection. Obviously, the particular smearing function (15) helps to avoid the more serious divergence such as the power-law ones.

On the other hand, for the vacuum/identity global conformal block denoted by $k=0$ with $h_0 = \bar{h}_0 = 0$ we can check

$$g_0(1,0) = 1 \ . \tag{22}$$

For $k, k' \neq 0$, the ratio $g_k(u,v)/g_{k'}(u,v)$ is finite in the limit $u \to 1, v \to 0$, i.e.,

$$\lim_{u\to 1, v\to 0} \frac{g_k(u,v)}{g_{k'}(u,v)} = \frac{n_k\bar{n}_k}{n_{k'}\bar{n}_{k'}}, \tag{23}$$

which is only related to the conformal dimensions according to the definition (16) of n_k and $\bar{n}_k$.

From the above we can conclude that

$$\mathrm{p}_0 \simeq 0, \qquad \mathrm{p}_{k\neq 0} = \frac{c_{iik}^2 n_k\bar{n}_k}{\sum_{j\neq 0} c_{iij}^2 n_j\bar{n}_j} \tag{24}$$

where $\simeq$ means that the equality holds exactly if the UV cutoff is taken to infinity. Since $\Gamma(a>0) > 0$, thus p_k should be positive and finite for all $h_k, \bar{h}_k > 0$, as expected. It is interesting to see $\mathrm{p}_0 \simeq 0$ so that the identity channel is excluded as a physical outcome.

Finally, based on all the above, we can write down the density matrix $\rho_{\mathbf{A}}$ for the resultant state after Alice's quasi-local measurement on the state (10), i.e.,

$$\rho_{\mathbf{A}} = \sum_k \mathrm{p}_k|\psi_k\rangle\langle\psi_k| = \sum_{k\neq 0} \frac{c_{iik}^2\mathcal{B}_k(x_1,x_2)|0\rangle\langle 0|\mathcal{B}_k^\dagger(x_1,x_2)}{\sum_{j\neq 0} c_{iij}^2 g_j(1,0)} \ . \tag{25}$$

We can also introduce the POVM-like measurement operator

$$M_k(\mathbf{A}) := \sqrt{\frac{\mathrm{p}_k}{g_k(1,0)}}\, \mathcal{B}_k(x_1, x_2) \tag{26}$$

so that $\rho_{\mathbf{A}}$ can be expressed as

$$\rho_{\mathbf{A}} := \sum_{k\neq 0} M_k(\mathbf{A})|0\rangle\langle 0|M_k^\dagger(\mathbf{A}) \ . \tag{27}$$

Thus,

$$\mathrm{p}_k = \langle 0|M_k^\dagger \mathbf{A})M_k(\mathbf{A})|0\rangle \ . \tag{28}$$

The above form of $\rho_{\mathbf{A}}$ suggests that we can also think that all the non-vacuum OPE blocks, i.e., $M_{k\neq 0}(\mathbf{A})$, form a complete set of non-trivial measurement operators when acting on the CFT ground state. After the measurement, the CFT ground state then get quasi-locally excited so that the vacuum OPE block is excluded, i.e., $\mathrm{p}_0 = 0$ but $\mathrm{p}_k \neq 0$. However, the set $\{M_{k\neq 0}(\mathbf{A})\}$ cannot be complete if it does not act on the CFT ground state just because in the operator sense

$$\sum_{k\neq 0} M_k^\dagger(\mathbf{A})M_k(\mathbf{A}) = \sum_{k\neq 0} \frac{\mathrm{p}_k}{g_k(1,0)}\, O_i(x_2)O_i(x_1)P_k O_i(x_1)O_i(x_2) \neq \mathbb{I} \ , \tag{29}$$

i.e., the completeness does not hold for arbitrary states.

2.3 On regularization

As mentioned in last subsection we meet with a divergent quantity $g_{k\neq 0}(1,0)$. Indeed when dealing with quantum field theory one often meets with the UV-divergence. But we expect we could obtain some physical quantities which are independent of UV cut-off by suitable regularization, such as the $S-$matrix in a scattering process. Here our definition of probability is similar.

Our starting point is the state (10), which is a local state in the sense that the energy density of this state is divergent at point x_1 and x_2. This state is not normalizable. One could regulate it by moving the operator slightly into Euclidean time, i.e, at $t = i\delta$, where δ is a small positive number. We could define a new state

$$e^{-\delta H} O_1(x_1)O_2(x_2)|0\rangle, \tag{30}$$

where H is the Hamiltonian of CFT. This method is used in [24] to discuss quantum quench by a boundary state, also in [25] [26] to discuss the local quench by a primary operator. Therefore we could define the post-measurement state (19) as

$$|\psi_k\rangle = N_k(\delta)^{-1} e^{-\delta H} \mathcal{B}_k(x_1, x_2)|0\rangle \ , \tag{31}$$

the normalization constant is related to the parameter δ, which is regarded as a UV cut-off. By using the result in A.2 we could obtain

$$N_k^2(\delta) = {}_2F_1(h_k, h_k, 2h_k, 1 - \frac{\delta^2}{L^2})\ {}_2F_1(\bar{h}_k, \bar{h}_k, 2\bar{h}_k, 1 - \frac{\delta^2}{L^2}) \ , \tag{32}$$

where $L := |x_{12}|$. Note that $N_k \sim \log \delta/L$ if $\delta \ll L$. This process actually regularize $g_k(1,0)$ by N_k^2. Therefore we get the regularized POVM-like operator

$$M_k(\mathbf{A}) := \frac{\sqrt{\mathrm{p}_k}}{N_k(\delta)}\, e^{-\delta H}\mathcal{B}_k(x_1, x_2). \tag{33}$$

The regularization make the post-measurement state (31) be a normalizable state. But the probability is still not dependent upon the UV cut-off as long as $\delta/L \ll 1$.

3 QET protocol in 2D CFTs

Based on the above construction of the measurement process for 2D CFTs, we are now ready to consider a corresponding QET protocol. The protocol goes as follows. First, Alice performs the projection measurement $\{P_k\}$ of (6) on the quasi-local excited state (10), and send her measurement outcome to distant Bob via classical communication (CC). According to the outcome, Bob then perform the following quasi-local unitary operation (LO) on the interval $[x_3, x_4]$ which is far from the interval $[x_1, x_2]$:

$$U_{\mathbf{B}} = e^{i\beta_k \theta G_{\mathbf{B}}} \tag{34}$$

where β_k is the feedback-control parameter associated with outcome k, θ labels the angle for unitary transformation, and we choose

$$G_{\mathbf{B}} = \int_{x_4}^{x_3} dx\; f(x)\; O_h(t_0, x), \tag{35}$$

with t_0 some constant time, O_h being some primary operator of conformal weight $(h, \bar{h})$, and $f(x)$ a smooth smearing real function. Thus $G_{\mathbf{B}}$ is hermitian so that $U_{\mathbf{B}}$ is unitary. In QET, one can tune β_k to help Bob extract energy by local operations.

Based on the above QET protocol with a many-body entangled state as the resource of the task, and with the help of LOCC, we will perform the energetic analysis for each step in the following.

We start with Alice's post-measurement state, which is already given by (27). Thus, the energy injected by the projection measurement is

$$E_{\mathbf{A}} = \mathrm{tr}(\rho_{\mathbf{A}} H_{CFT}) = \int dx \sum_{k\neq 0} \langle 0|M_k^\dagger(\mathbf{A}) T_{00}(x) M_k(\mathbf{A})|\mathbf{0}\rangle\ . \tag{36}$$

where $H_{CFT} := \int dx\; T_{00}(x)$ is the Hamiltonian of CFT. We can manipulate (36) further to get some explicit form of $E_{\mathbf{A}}$. However, the calculation is difficult to carry out to the end due to the complication of triple integration $\int dx \int d^2w \int d^2w' \cdots$, and the final form is irrelevant to the viability of QET as shown below. Thus, we will not pursue this further.

Assume the time elapses T before Bob performs the local operation, then in the Heisenberg's picture the measurement operator $M_k(\mathbf{A})$ evolves into $M_k(\mathbf{C}) := e^{-iH_{CFT}T}M_k(\mathbf{A})e^{iH_{CFT}T}$, or more explicitly,

$$M_k(\mathbf{C}) = \frac{\sqrt{p_k}}{N_k(\delta)}\int_{\mathcal{D}_\mathbf{A}} d^2w\ G_k(w,\bar{w};x_1,x_2)e^{-\delta H}O_k(w-T,\bar{w}+T)\ . \tag{37}$$

After Bob's local unitary operation the total state of CFT becomes

$$\rho_{QET} = \sum_{k\neq 0} U_\mathbf{B} M_k(\mathbf{C})|0\rangle\langle 0|M_k^\dagger(\mathbf{C})U_\mathbf{B}^\dagger\ . \tag{38}$$

We can then evaluate the amount of energy teleported from Alice to Bob as following:

$$E_\mathbf{B} = E_\mathbf{A} - \sum_{\kappa\neq 0}\langle 0|M_k^\dagger(\mathbf{C})U_\mathbf{B}^\dagger H_{CFT}U_\mathbf{B}M_k(\mathbf{C})|0\rangle. \tag{39}$$

If we assume θ is small, then we can express $E_\mathbf{B}$ in in terms of θ expansion, i.e.,

$$\begin{aligned} E_\mathbf{B} &= i\theta\sum_{k\neq 0}\beta_k\langle 0|M_k^\dagger(\mathbf{C})[H_{CFT},G_\mathbf{B}]M_k(\mathbf{C})|0\rangle \\ &\quad -\frac{\theta^2}{2}\sum_{k\neq 0}\beta_k^2\langle 0|M_k^\dagger(\mathbf{C})[[H_{CFT},G_\mathbf{B}],G_\mathbf{B}]M_k(\mathbf{C})|0\rangle + \cdots \end{aligned} \tag{40}$$

where we have used the fact that

$$\sum_{k\neq 0}\langle 0|M_k^\dagger(\mathbf{C})H_{CFT}M_k(\mathbf{C})|0\rangle = \sum_{k\neq 0}\langle 0|M_k^\dagger(\mathbf{A})e^{iH_{CFT}T}H_{CFT}e^{-iH_{CFT}T}M_k(\mathbf{A})|0\rangle = E_\mathbf{A}\ .$$

The commutator with H_{CFT} in the above can be reduced to time-derivative by Heisenberg equation, i.e.,

$$[H_{CFT},O_h(\xi,\bar{\xi})] = -i(\partial_\xi - \partial_{\bar{\xi}})O_h(\xi,\bar{\xi}) \tag{41}$$

3.1 No-go in the infinite time limit

In this subsection, we will show the impossibility of having QET energy gain in the infinite time limit, i.e., $T\to\infty$. To calculate the first order term of (40), denoted by $E_\mathbf{B}|_\theta$, for each $k\neq 0$ term we need to deal with

$$\begin{aligned} E_k^{(1)} &:= i\beta_k\theta\langle 0|M_k^\dagger(\mathbf{C})[H_{CFT},G_\mathbf{B}]M_k(\mathbf{C})|0\rangle \\ &= i\beta_k\theta\int_{x_3}^{x_4}dx\ f(x)\ \langle 0|M_k^\dagger(\mathbf{C})[H_{CFT},O_h(\xi,\bar{\xi})]M_k(\mathbf{C})|0\rangle \end{aligned} \tag{42}$$

where we have introduced the lightcone coordinates $\xi := x - t_0$ and $\bar{\xi} := x + t_0$. By using (37) and (41) we have

$$\langle 0|M_k^\dagger(\mathbf{C})[H_{CFT},O_h(\xi,\bar{\xi})]M_k(\mathbf{C})|0\rangle$$

$$= -i(\partial_\xi - \partial_{\bar{\xi}})\langle 0|M_k^\dagger(\mathbf{C})O_h(\xi,\bar{\xi})M_k(\mathbf{C})|0\rangle \tag{43}$$

$$= \frac{-ip_k}{N_k^2(\delta)}\int_{\mathcal{D}_\mathbf{A}}\int_{\mathcal{D}_\mathbf{A}} d^2wd^2w'\ G_k(w,\bar{w};x_1,x_2)\ G_k(w',\bar{w}';x_1,x_2) \tag{44}$$

$$\times(\partial_\xi - \partial_{\bar{\xi}})\langle O_k(w-T-i\delta,\bar{w}+T+i\delta)O_h(\xi,\bar{\xi})O_k(w'-T+i\delta,\bar{w}'+T-i\delta)\rangle,$$

which is related to the three point correlation function $\langle O_k(w-T,\bar{w}+T)O_h(\xi,\bar{\xi})O_k(w'-T,\bar{w}'+T)\rangle$,

$$\begin{aligned}
&\langle O_k(w-T-i\delta,\bar{w}+T+i\delta)O_h(\xi,\bar{\xi})O_k(w'-T+i\delta,\bar{w}'+T-i\delta)\rangle\\
&\propto \frac{1}{(\xi-w+T-i\delta)^h(\xi-w'+T+i\delta)^h(w-w'-2i\delta)^{2h_k-h}}\\
&\quad\times\frac{1}{(\bar{\xi}-\bar{w}-T+i\delta)^{\bar{h}}(\bar{\xi}-\bar{w}'-T-i\delta)^{\bar{h}}(\bar{w}-\bar{w}'+2i\delta)^{2\bar{h}_k-\bar{h}}},
\end{aligned} \tag{45}$$

which vanishes in the infinite time limit if $h\neq 0$. In short, this implies that

$$\lim_{T\to\infty}\langle 0|M_k^\dagger(\mathbf{C})O_{h\neq 0}(\xi,\bar{\xi})M_k(\mathbf{C})|0\rangle = 0\ . \tag{46}$$

Thus, $E^{(1)}_{k\neq 0}$ vanishes in the infinite time limit if $h\neq 0$. This yields the fact that there is no QET energy gain or loss at the first order of θ expansion if taking the limit $T\to\infty$.

Since $E_{\mathbf{B}}|_\theta = 0$ in the infinite time limit, we now go to evaluate the second order term of $E_{\mathbf{B}}$, denoted as $E_{\mathbf{B}}|_{\theta^2}$, i.e.,

$$E_{\mathbf{B}}|_{\theta^2} := \frac{\theta^2}{2}\sum_{k\neq 0}E_k^{(2)} = -\frac{\theta^2}{2}\sum_{k\neq 0}\beta_k^2\ \langle 0|M_k^\dagger(\mathbf{C})[[H_{CFT},G_{\mathbf{B}}],G_{\mathbf{B}}]M_k(\mathbf{C})|0\rangle\ . \tag{47}$$

Similar to simplification for the $E_k^{(1)}$, by using (41) we can express $E_k^{(2)}$ as following:

$$E_k^{(2)} = i\beta_k^2\int_{x_3}^{x_4}dy_1 f(y_1)\int_{x_3}^{x_4}dy_2 f(y_2)\ (\partial_{\xi_1}-\partial_{\bar{\xi}_1})\langle 0|M_k^\dagger(\mathbf{C})[O_h(\xi_1,\bar{\xi}_1),O_h(\xi_2,\bar{\xi}_2)]M_k(\mathbf{C})|0\rangle$$

where we have introduced $\xi_i := y_i - t_0$ and $\bar{\xi}_i := y_i + t_0$ for $i=1,2$. Using the definition of $M_k(\mathbf{C})$, the correlator inside the above double integral can be further expressed in terms of four-point function $\langle O(w-T,\bar{w}+T)O_h(x)O_h(y)O(w'-T,\bar{w}'+T)\rangle$. We can further reduce this into the sum of three-point functions by the OPE of $O_h(x)O_h(y)$. Using the fact of (46) in the long time limit and also (28), we can arrive

$$\lim_{T\to\infty}\langle 0|M_k^\dagger(\mathbf{C})O_h(\xi_1,\bar{\xi}_1)O_h(\xi_2,\bar{\xi}_2)M_k(\mathbf{C})|0\rangle = \frac{c_{hh0}\ p_k}{(\xi_1-\xi_2)^{2h}(\bar{\xi}_1-\bar{\xi}_2)^{2\bar{h}}} \tag{48}$$

where c_{hh0} is the OPE coefficient for the identity channel. If we recognize the above power-law factor obtained from OPE as the two-point function $\langle 0|O_h(\xi_1,\bar{\xi}_1)O_h(\xi_2,\bar{\xi}_2)|0\rangle$, then we find that the 4-point function is cluster-decomposed in the infinite time limit. This then implies that

$$\begin{aligned}
&\lim_{T\to\infty}(\partial_{\xi_1}-\partial_{\bar{\xi}_1})\langle 0|M_k^\dagger(\mathbf{C})[O_h(\xi_1,\bar{\xi}_1),O_h(\xi_2,\bar{\xi}_2)]M_k(\mathbf{C})|0\rangle\\
&\qquad = c_{hh0}\ p_k\ (\partial_{\xi_1}-\partial_{\bar{\xi}_1})\langle 0|[O_h(\xi_1,\bar{\xi}_1),O_h(\xi_2,\bar{\xi}_2)]|0\rangle\ .
\end{aligned} \tag{49}$$

As $O_h(\xi_1,\bar{\xi}_1)$ and $O_h(\xi_2,\bar{\xi}_2)$ are operators on time slice $t=t_0$, naively the commutator seems to be zero except $\xi_1=\xi_2$. However, due to the overall derivative on the commutator, one should shift the coordinate away for the slice $t = t_0$ to make it well-defined. So generally this would be a non-zero result.

To see the sign of $E_k^{(2)}$ let's turn to Fourier space. To carry out the calculations, we assume the x-direction to be periodic with $x \sim x + L$. By a coordinate transformation in the Euclidean space

$$z^E = e^{-2\pi i \xi^E / L}, \tag{50}$$

the cylinder is mapped to infinite plane, on which the Euclidean correlator is

$$\langle O(z_1^E, \bar{z}_1^E) O(z_1^E, \bar{z}_2^E) \rangle = \frac{1}{(z_1^E - z_2^E)^{2h} (\bar{z}_1^E - \bar{z}_2^E)^{2\bar{h}}}. \tag{51}$$

The correlator on the cylinder can then be obtained as

$$\langle O(\xi_1^E, \bar{\xi}_1^E) O(\xi_2^E, \bar{\xi}_1^E) \rangle = (\frac{2\pi}{L})^{2h} (\frac{2\pi}{L})^{2\bar{h}} \frac{e^{2\pi i h(\xi_1^E - \xi_2^E)/L} e^{-2\pi i h(\bar{\xi}_1^E - \bar{\xi}_2^E)/L}}{(1 - e^{2\pi i(\xi_1^E - \xi_2^E)/L})^{2h} (1 - e^{-2\pi i(\bar{\xi}_1^E - \bar{\xi}_2^E)/L})^{2\bar{h}}}. \tag{52}$$

By the $i\epsilon$ prescription as briefly introduced in Appendix C, the corresponding Minkowski correlators are

$$\langle O(\xi_1, \bar{\xi}_1) O(\xi_2, \bar{\xi}_2) \rangle = (\frac{2\pi}{L})^{2h} (\frac{2\pi}{L})^{2\bar{h}} \frac{e^{2\pi i h(\xi_1 - \xi_2)/L} e^{-2\pi i h(\bar{\xi}_1 - \bar{\xi}_2)/L}}{(1 - e^{2\pi i(\xi_1 - \xi_2)/L - \epsilon})^{2h} (1 - e^{-2\pi i(\bar{\xi}_1 - \bar{\xi}_2)/L - \epsilon})^{2\bar{h}}}, \tag{53}$$

and

$$\langle O(\xi_2, \bar{\xi}_2) O(\xi_1, \bar{\xi}_1) \rangle = (\frac{2\pi}{L})^{2h} (\frac{2\pi}{L})^{2\bar{h}} \frac{e^{2\pi i h(\xi_1 - \xi_2)/L} e^{-2\pi i h(\bar{\xi}_1 - \bar{\xi}_2)/L}}{(1 - e^{2\pi i(\xi_1 - \xi_2)/L + \epsilon})^{2h} (1 - e^{-2\pi i(\bar{\xi}_1 - \bar{\xi}_2)/L + \epsilon})^{2\bar{h}}}, \tag{54}$$

where ϵ is a small positive number. Thus we could expand the above expressions as

$$\langle O(\xi_1, \bar{\xi}_1) O(\xi_2, \bar{\xi}_2) \rangle = (\frac{2\pi}{L})^{2h} (\frac{2\pi}{L})^{2\bar{h}} \sum_{n \geq 0} \sum_{m \geq 0} F_n(h) F_m(h) e^{2\pi i(\xi_1 - \xi_2)(n+h)/L} e^{-2\pi i(\bar{\xi}_1 - \bar{\xi}_2)(m+\bar{h})/L}, \tag{55}$$

and

$$\langle O(\xi_2, \bar{\xi}_2) O(\xi_1, \bar{\xi}_1) \rangle = (\frac{2\pi}{L})^{2h} (\frac{2\pi}{L})^{2\bar{h}} \sum_{n \geq 0} \sum_{m \geq 0} F_n(h) F_m(h) e^{-2\pi i(\xi_1 - \xi_2)(n+h)/L} e^{2\pi i(\bar{\xi}_1 - \bar{\xi}_2)(m+\bar{h})/L}, \tag{56}$$

where F_n are the coefficients of the Tayor series of $(1 - x)^{-2h}$, which are positive definite. Now we make the coordinates $(\xi_1, \bar{\xi}_1)$ and $(\xi_2, \bar{\xi}_2)$ approach to the time slice $t = t_0$, then $E_k^{(2)}$ becomes

$$E_k^{(2)} = -\beta_k^2 \, c_{hh0} \, p_k \, (\frac{4\pi}{L})(\frac{2\pi}{L})^{2h} (\frac{2\pi}{L})^{2\bar{h}} \sum_{n \geq 0} \sum_{m \geq 0} F_n(h) F_m(h)(m+n+h+\bar{h}) |f_{n-m}(h)|^2 \,, \tag{57}$$

where we have defined

$$f_{n-m} = \int_{x_3}^{x_4} dy\, f(y)\, e^{2\pi i(n-m+h-\bar{h})} \, .$$

From (57) we could see $E_k^{(2)}$ is negative definite, and thus $E_{\mathbf{B}}|_{\theta^2}$ is always negative. This means that Bob cannot extract energy in the infinite time limit up to second order. This is consistent with the passivity of the QFT's vacuum state [27], and the quantum interest conjecture [28]. [4]

3.2 Sub-leading correction beyond long time limit

We now try to consider the sub-leading correction of $E_{\mathbf{B}}|_\theta$ in the large T expansion. We will see that there are nonzero energy changes for each channel so that we can manipulate the feedback control parameters to obtain QET. Thus, the no-go theorem is lifted beyond the long time limit.

Without taking the long time limit, we should deal with $E_k^{(1)}$ in the following form:

$$E_k^{(1)} = \beta_k \theta \int_{x_3}^{x_4} dx\, f(x)(\partial_\xi - \partial_{\bar{\xi}})\langle 0|M_k^\dagger(\mathbf{C}) O_h(\xi,\bar{\xi}) M_k(\mathbf{C})|0\rangle \, . \tag{58}$$

For simplicity, hereafter we will set $f(x) = 1$.

We can either calculate the 3-point function in (58) directly, or we can just consider the sub-leading contribution in the large T expansion. Both yields the similar results, and for simplicity we will just consider the latter [5],

$$E_k^{(1)} \simeq \mathcal{N}_{\theta,k,T} \int_{\mathcal{D}_{\mathbf{A}}} d^2 w_0 \int_{\mathcal{D}_{\mathbf{A}}} dw_3^2\, G_k(w_0,\bar{w}_0;x_2,x_1) G_k(w_3,\bar{w}_3;x_1,x_2) \frac{1}{w_{03}^{2h_k-h}\bar{w}_{03}^{2\bar{h}-\bar{h}_k}} \tag{59}$$

with the overall factor

$$\mathcal{N}_{\theta,k,T} := -\beta_k\, \theta\, \frac{\mathrm{p}_k}{N_k^2(\delta)} \frac{2\Delta_h}{T^{2\Delta_h+1}}\, w_{21} \tag{60}$$

where $\Delta_h := h + \bar{h}$.

The integral (59) is similar as the one to compute the conformal block in Appendix A. Plugging (15) into (59)

$$E_k^{(1)} \simeq \mathcal{N}_{\theta,k,T} n_k \bar{n}_k I_{03} \bar{I}_{03}, \tag{61}$$

[4] The quantum interest conjecture says that a negative energy pulse should always be followed by a larger positive energy pulse, that is, one should pay the interest when temporarily harvesting the energy of the vacuum by the negative energy pulse. Interestingly, the time-reversal statement of quantum interest conjecture implies the passivity of quantum state, i.e., Bob cannot extract more energy (by a negative energy pulse) than what Alice has injected earlier by her local measurement (with a positive energy pulse). In this sense, the no-go theorem just discussed is consistent with passivity and quantum interest conjecture. In particular, when Alice did not inject any energy pulse, there should have no followup negative energy pulse for Bob to extract.

[5] As we can see from (45) the integration should also depend on the cut-off δ, here we only consider the leading contributions, so ignore δ in the integration.

where we define

$$I_{03} = \int_{w_1}^{w_2} dw_0 \int_{w_1}^{w_2} dw_3 \, (w_{01}w_{20})^{h_k-1}(w_{31}w_{23})^{h_k-1}w_{03}^{h-2h_k}, \tag{62}$$

and $\bar{I}_{03}$ is the anti-holomorphic part. If $h-2h_k$ is not an integer, the integral will have branch cut. To simplify the calculation we assume $h-2h_k$ is an integer, and constrain $h-2h_k > -1$. In this case we could obtain an analytic result of the integral I_{03}. It is given by

$$I_{03} = w_{21}^{h+2h_k-2}(1+(-1)^{h-2h_k})\frac{\sqrt{\pi}}{2^h}\frac{\Gamma(h_k)^2\Gamma(h-2h_k+1)\Gamma(\frac{h}{2})}{\Gamma(\frac{h+1}{2})\Gamma(1+\frac{h}{2}-h_k)\Gamma(\frac{h}{2}+h_k)}, \tag{63}$$

which is positive or zero in the region $h-2h_k > -1$. One could find the details of the calculation in Appendix D. $\bar{I}_{03}$ can be obtained by $h \to \bar{h}$ and $h_k \to \bar{h}_k$ in expression (63). Plugging them into (61) we get the final result of $E_k^{(1)}$, which could be positive by tuning parameter β_k. The first non-vanishing contribution to Bob's extraction energy beyond the infinite time limit is then

$$E_{\mathbf{B}}^{(1)} := \sum_{k\neq 0} E_k^{(1)}, \tag{64}$$

where $E_k^{(1)}$ is given by (61).

Finally, we remark about requiring $h-2h_k$ to be some integer in the above discussion. This is assumed to avoid the branch cut for the integral related to the integral representation of hypergeometric function. Supposed that we do not restrict to the integer values of $h-2h_k$, then we need to perform suitable analytic continuation to carry out the integration, and it may result in a complex-valued energy, i.e., complex $E_k^{(1)}$. Physically, the complex energy is expected as the quasi-local states such as $M_k(\mathbf{C})|0\rangle$ or $U_{\mathbf{B}}M_k(\mathbf{C})|0\rangle$ are not energy eigenstates, i.e., the states may not be stable under evolution. This then implies that these states with non-integer $h-2h_k$ may be quasi-normal states with the imaginary part of the energy as their decay width.

3.3 QET in a toy 2D CFT model

In this subsection we will use a toy example to show our previous abstract discussion of viable QET. Assume in this model the OPE of $\mathcal{O}_i(x_1)\mathcal{O}_i(x_2)$ only has two channels except the identity channel, i.e.,

$$\mathcal{O}_i(x_1)\mathcal{O}_i(x_2) = x_{12}^{-2h_i-2\bar{h}_i} \sum_{k\in\{0,1,2\}} c_{iik}\,\mathcal{B}_k(x_1,x_2)\,, \tag{65}$$

where $k=0$ refers to the identity, $1,2$ are two others. Without loss of generality, we normalize $c_{ii1}=1$, denote $c_{ii2}=c$. According to (24) we have

$$p_0 = 0, \qquad p_1 = \frac{n_1\bar{n}_1}{n_1\bar{n}_1 + cn_2\bar{n}_2}, \qquad p_2 = \frac{cn_2\bar{n}_2}{n_1\bar{n}_1 + cn_2\bar{n}_2}, \tag{66}$$

where $n_i = \frac{\Gamma(2h_i)}{\Gamma(h_i)^2}$ and $\bar{n}_i = \frac{\Gamma(2\bar{h}_i)}{\Gamma(\bar{h}_i)^2}(i = 1, 2)$. By using (64), we obtain the energy Bob can extract $E_{\mathbf{B}}^{(1)}$,

$$E_{\mathbf{B}}^{(1)} = -C\frac{1}{T^{2(h+\bar{h})+1}}\sum_{k=1,2}\beta_k p_k M_k \bar{M}_k, \tag{67}$$

where C is positive constant unrelated to k, and

$$\begin{aligned} M_k &:= (1 + (-1)^{h-2h_k})\frac{\Gamma(2h_k)\Gamma(h - 2h_k + 1)}{\Gamma(1 + \frac{h}{2} - h_k)\Gamma(\frac{h}{2} + h_k)}, \\ \bar{M}_k &:= (1 + (-1)^{\bar{h}-2\bar{h}_k})\frac{\Gamma(2\bar{h}_k)\Gamma(\bar{h} - 2\bar{h}_k + 1)}{\Gamma(1 + \frac{\bar{h}}{2} - \bar{h}_k)\Gamma(\frac{\bar{h}}{2} + \bar{h}_k)}. \end{aligned} \tag{68}$$

Let's do some numerical calculation. Assume $h_1 = \bar{h}_1 = 1$ and $h_2 = \bar{h}_2 = \frac{3}{2}$, we need $h > 2$ since the constraint $h - 2h_k > -1$. When $h = 2n$ ($n \geq 2$ and $n \in \mathcal{Z}$), we have

$$E_{\mathbf{B}}^{(1)} = -C\frac{1}{T^{4n+1}}\beta_1\frac{1}{1 + c\frac{64}{\pi^2}}\Big(\frac{\Gamma(2n-1)}{\Gamma(n)\Gamma(n+1)}\Big)^2. \tag{69}$$

Therefore, in this case as long as taking $\beta_1 < 0$, we will have $E_{\mathbf{B}}^{(1)} > 0$, which means that Bob can extract energy by the unitary operation. We also notice that the result is proportional to $1/T^{2(h+\bar{h}+1)}$, so we should not use too large h to ensure the energy $E_{\mathbf{B}}^{(1)}$ will not decay too fast. Also note that the decay behavior of T is independent of h_k.

4 Discussion and Conclusion

In comparison with the system of finite degrees of freedom, defining the quantum measurement process in QFT is a challenging problem, especially for the ones with non-trivial interactions. An obvious difficulty is the UV-divergence when the measurement operators contact with each others at the same spacetime point. But we usually expect that some suitable regularization methods could deal with this and help us to define physical quantities, which should be independent with UV-cutoff.

In this paper we make a modest step towards this problem. Our starting point is the shadow operator P_k given in (6), which was once used to study the conformal blocks in Euclidean CFT. These operators P_k, which are complete, can be taken as projection measurements in CFT. When these operators work on a local state $\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|0\rangle$, we could obtains a set of quasi-local states in 2D CFT. These state are obtained by the respective smearing operators, the so-called OPE blocks $\mathcal{B}_k(x_1, x_2)$ over the casual diamond $\mathcal{D}$ of the interval $[x_1, x_2]$.

We shows that the obtained smearing operators satisfy several interesting properties. Firstly, when considering the OPE of $\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)$ they appears as OPE blocks. Thus their normalization is related to the conformal blocks $g_k(u)$. However, these are global conformal blocks, which are the leading contribution of the so-called Virasoro conformal block in the large c limit, since we have not considered all the descendants contributions. By suitable normalization we define the POVM-like measurement operators $M_k(x_1, x_2)$, which satisfy the expected properties of POVM operators in the weak sense, i.e., hold only when acting on the vacuum state. A remarkable consequence is that the defined probabilities are independent with UV-cutoff, thus physical.

Secondly, the operators $M_k(x_1, x_2)$ are quasi-local. Since they are operators smearing over the causal diamond of interval $[x_1, x_2]$, for two spacelike intervals the corresponding POVM-like operators are also commutative. This permits us to discuss some information tasks. We check the smearing form of OPE in three different ways, which are shown in Appendix A.

Finally, we use this operators to proceed the QET protocol in 2D CFT. We prove a no-go theorem if taking the long time limit. We calculate the energy that Bob could gain up to second order of θ, and find it impossible for Bob to gain any energy. This result is physical. As we know the entanglement of the state shared by Alice and Bob is the key for QET protocol's success. Our result implies that the infinite time evolution will somehow destroy the entanglement resources shared between Alice and Bob for successful QET. This is also consistent with or due to the observed cluster decomposition in the infinite time limit. By considering the correction in the finite time duration we successfully realize the QET protocol. The energy Bob can extract depends on the UV-cutoff, which is different from the probabilities. This is reasonable because the input energy by Alice is expected to be dependent with UV-cutoff.

Before we close our paper in this section, we will comment on the issue of checking Bell inequality by using our weak-sense POVM-like operators. Bell inequality formulated in the CHSH form needs two pairs of Hermitian operators, say A_1, A_2 for Alice and B_1, B_2 for Bob, the norms of which are required to be smaller than one. We assume $A_1 := M_{k_1}(\mathbf{A})$, $A_2 := M_{k_2}(\mathbf{A})$, $B_1 := M_{k_2}(\mathbf{B})$, and $B_2 := M_{k_1}(\mathbf{B})$ with $k_1 \neq k_2$. Note that the norms of these operators in the vacuum state are all smaller than one as we can see from (28). CHSH inequality is

$$\gamma := |\langle A_1(B_1 + B_2) + A_2(B_1 - B_2)\rangle| \leq 2. \tag{70}$$

If existing some operators such that the inequality is violated, we could claim the state has quantum entanglement. Let us see whether our measurement operators could make this. According to the definition of M_k we have

$$\langle 0|M_{k_1}(\mathbf{A})M_{k_2}(\mathbf{B})|\mathbf{0}\rangle = \mathbf{0}. \tag{71}$$

Therefore we arrive

$$\gamma = \langle 0|M_{k_1}(\mathbf{A})M_{k_1}(\mathbf{B})|0\rangle + \langle 0|M_{k_2}(\mathbf{A})M_{k_2}(\mathbf{B})|\mathbf{0}\rangle. \tag{72}$$

If $x_3 - x_2 = L \neq 0$, i.e., the interval $[x_1, x_2]$ of Alice and the $[x_3, x_4]$ of Bob are separate, then $\langle 0|M_{k_1}(\mathbf{A})M_{k_1}(\mathbf{B})|0\rangle$ will vanish if taking the UV-cutoff to zero. This can be seen as follows. Since $M_k = \frac{\sqrt{p_k}}{N_k(\delta)}\mathcal{B}_k$ so that

$$\langle 0|M_{k_1}(\mathbf{A})M_{k_1}(\mathbf{B})|0\rangle = \frac{p_{k_1}}{N_k^2(\delta)}\langle 0|\mathcal{B}_{k_1}(\mathbf{A})\mathcal{B}_{k_1}(\mathbf{B})|0\rangle = p_{k_1}\frac{g_{k_1}(u,v)}{N_k^2(\delta)}, \tag{73}$$

where $u < 1$ is the cross ratio. As $g_{k_1}(u,v)$ is finite, thus γ will approach to zero. On the other hand, if $L = 0$ we will have $u \to 1$, then $g_{k_1}(u,v) \to N_k^2(\delta)$, $\gamma \to p_{k_1} + p_{k_2} \leq 2$. In conclusion, the measurement operators M_k's adopted here cannot violate the Bell inequality.

Overall, our study implies that QET is viable in CFTs and can be used to detect the entanglement of the underlying quantum state even the corresponding Bell inequality using the same set of weak-sense measurement operators is not violated. Besides, we also point out many subtle issues regarding the quantum measurements and causality constraints in CFTs, which should deserve further investigations.

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A Three-point function and conformal block from OPE block

As a consistency check we will reproduce below the three-point function and global conformal block from the OPE block given in (14) and (15).

A.1 Three-point function

Let's first show the three-point function can be reproduced from the OPE block with the smearing function (15). We would like to consider the three-point correlation function whose form is fixed by the conformal symmetry as follows:

$$\begin{aligned} & c_{iik}^{-1} \langle O_i(x_1) O_i(x_2) O_k(w_3, \bar{w}_3) \rangle \\ & = \frac{1}{x_{21}^{2\Delta_i - \Delta_k} [(w_{13} + i\epsilon)(w_{23} + i\epsilon)]^{h_k} [(\bar{w}_{13} - i\epsilon)(\bar{w}_{23} - i\epsilon)]^{\bar{h}_k}} \end{aligned} \tag{74}$$

where $\Delta_j = h_j + \bar{h}_j$, and we use the notation $w_{ij} := w_i - w_j$ and $\bar{w}_{ij} := \bar{w}_i - \bar{w}_j$. Note also here $w_i = \bar{w}_i = x_i$ for $i = 1, 2$.

On the other hand, we can use (14) to evaluate the three-point function, i.e.,

$$c_{iik}^{-1} \langle O_i(x_1) O_i(x_2) O_k(w_3, \bar{w}_3) \rangle = x_{21}^{-2h_i - 2\bar{h}_i} \int_{\mathcal{D}_{12}} G_k(w, \bar{w}; x_1, x_2) \langle O_k(w, \bar{w}) O_k(w_3, \bar{w}_3) \rangle \tag{75}$$

where we use the fact that $\langle O_j O_k\rangle = 0$ if $j \neq k$. Using (15) we can further simplify, for simplicity we reduce the integration over $\mathcal{D}_{12}$ into the integration over interval $[x_1, x_2]$, and also only deal with the holomorphic part, i.e.,

$$\begin{aligned}
&c_{iik}^{-1} \langle O_i(x_1)O_i(x_2)O_k(w_3)\rangle \\
&= x_{21}^{-2h_i} n_k \int_{x_1}^{x_2} dw_0 \left(\frac{w_{01}w_{20}}{x_{21}}\right)^{h_k-1} \frac{1}{(w_{03}+i\epsilon)^{2h_k}} \\
&= x_{21}^{-2h_i+h_k}(w_{13}+i\epsilon)^{-2h_k} n_k \int_0^1 dy\, [y(1-y)]^{h_k-1}\Big(1-y(\frac{x_{12}}{w_{13}+i\epsilon})\Big)^{-2h_k} \\
&= x_{21}^{-2h_i+h_k}(w_{13}+i\epsilon)^{-2h_k}\ {}_2F_1(2h_k, h_k, 2h_k; \frac{x_{12}}{w_{13}+i\epsilon}) \\
&= x_{21}^{-2h_i+h_k}[(w_{13}+i\epsilon)(w_{23}+i\epsilon)]^{-h_k}
\end{aligned} \tag{76}$$

where n_k is defined in (16). The anti-holomorphic part can be obtained similarly, and the combined result yields the three-point function as expected.

A.2 Global conformal block

We now show that the global conformal block can be obtained as the two-point function of OPE blocks. For simplicity, we consider the conformal block with its arguments x_i's lying on the $t = 0$ slice with $x_1 < x_2 < x_3 < x_4$ WLOG. Start with (12) or (12) for the definition of the conformal block, and using (14) we have

$$g_k(u) = \int_{\mathcal{D}_\mathbf{A}}\int_{\mathcal{D}_\mathbf{B}} d^2w d^2w' G_k(w,\bar{w};x_1,x_2)G_k(w',\bar{w}';x_3,x_4)\langle O_k(w,\bar{w})O_k(w',\bar{w}')\rangle \tag{77}$$

where $\mathcal{D}_\mathbf{A}$ and $\mathcal{D}_\mathbf{B}$ are the causal diamonds subtended by the intervals $[x_1, x_2]$ and $[x_3, x_4]$, respectively. However, for simplicity reduce the integration to the one over only the intervals $[x_1, x_2]$ and $[x_3, x_4]$.

Since the holomorphic and anti-holomorphic parts factorize in the conformal block, we first deal with the holomorphic part and the anti-holomorphic part can be done in the similar way. Using (15) we can further simplify the holomprhic part of (77) as follows

$$g_k(u)|_{holo} = (x_{21}x_{43})^{1-h_k} n_k^2\, I_k(x_i) \tag{78}$$

where the cross ratio $u := \frac{x_{12}x_{34}}{x_{13}x_{24}}$ as defined before. and

$$I_k(x_i) := \int_{x_1}^{x_2} dw_0\, (w_{01}w_{20})^{h_k-1} \int_{x_3}^{x_4} dw_5\, [w_{53}w_{45}]^{h_k-1}\, w_{05}^{-2h_k}\ .$$

Note that here we have $w_i = \bar{w}_i = x_i$ for $i = 1, 2, 3, 4$.

We can further manipulate $I(x_i)$ as follows:

$$
\begin{aligned}
I_k(x_i) &= \int_{x_1}^{x_2} dw_0 (w_{01}w_{20})^{h_k-1} x_{43}^{2h_k-1} w_{03}^{-2h_k} \int_0^1 dy\ [y(1-y)]^{h_k-1} (1-y\frac{x_{43}}{w_{03}})^{-2h_k} \\
&= x_{43}^{2h_k-1}\ n_k^{-1} \int_{x_1}^{x_2} dw_0\ (w_{01}w_{20})^{h_k-1} w_{03}^{-2h_k}\ {}_2F_1(2h_k, h_k, 2h_k; \frac{x_{43}}{w-x_3}) \\
&= x_{43}^{2h_k-1}\ n_k^{-1} \int_{x_1}^{x_2} dw_0\ (w_{01}w_{20})^{h_k-1} (w_{03}w_{04})^{-h_k} \\
&= (x_{43}x_{21})^{2h_k-1} (x_{31}x_{41})^{-h_k}\ n_k^{-1} \int_0^1 dy\ [y(1-y)]^{h_k-1} [(1-\frac{x_{21}}{x_{31}}y)(1-\frac{x_{21}}{x_{41}}y)]^{-h_k}
\end{aligned}
$$

To further simplify, we need Feynman parametrization, i.e.,

$$
\frac{1}{A_1^{\alpha_1} A_2^{\alpha_2}} = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 dy_1 \int_0^1 dy_2\ \frac{\delta(y_1+y_2-1)\ y_1^{\alpha_1-1} y_2^{\alpha_2-1}}{(y_1 A_1 + y_2 A_2)^{\alpha_1+\alpha_2}} .
$$

Then,

$$
\begin{aligned}
I_k(x_i) &= (x_{43}x_{21})^{2h_k-1} (x_{31}x_{41})^{-h_k} \int_0^1 dy_1 \int_0^1 dy_2\ \delta(y_1+y_2-1)(y_1 y_2)^{h_k-1} \\
&\quad \times \int_0^1 dy\ [y(1-y)]^{h_k-1} \Big(1-(y_1\frac{x_{21}}{x_{31}} + y_2\frac{x_{21}}{x_{41}})y\Big)^{-2h_k} \\
&= (x_{43}x_{21})^{2h_k-1} (x_{31}x_{41})^{-h_k} \int_0^1 dy_1 \int_0^1 dy_2\ \delta(y_1+y_2-1)(y_1 y_2)^{h_k-1} \\
&\quad \times n_k^{-1} \Big(1-(y_1\frac{x_{21}}{x_{31}} + y_2\frac{x_{21}}{x_{41}})\Big)^{-h_k} \\
&= (x_{43}x_{21})^{2h_k-1} (x_{31}x_{42})^{-h_k} n_k^{-1} \int_0^1 dy_1\ [y_1(1-y_1)]^{h_k-1} (1-uy_1)^{-h_k} \\
&= (x_{43}x_{21})^{h_k-1}\ u^{h_k}\ n_k^{-2}\ {}_2F_1(h_k, h_k, 2h_k; u) .
\end{aligned}
$$

Plugging this into (78) we get

$$
g_k(u)|_{holo} = n_k^{-2} u^{h_k}\ {}_2F_1(h_k, h_k, 2h_k; u) . \tag{79}
$$

Combining with the anti-holomorphic part obtained in the similar way, we arrive (20) for global conformal block.

B Consistency of OPE blocks with causality constraint

The causality constraint should be the essential property for the physical observables or the quantum measurement operators. The usual local operators does satisfy the causality constraints. However, the product of local operators leads to nonlocal OPE blocks, it is then interesting to check the consistency of OPE blocks with the causality constraints. This will also show the essence of the special form (15) of the smearing function.

For operators $O_k(x_3)$, with the point (t_3, x_3)(or alternatively denoted as $(w_3, \bar{w}_3)$) being spacelike to the points x_1 and x_2 on $t = 0$ slice, we expect

$$[O_i(x_1)O_i(x_2), O_j(w_3, \bar{w}_3)] = 0 \; . \tag{80}$$

This can be easily seen by using the fact that $[AB, C] = [A, C]B + A[B, C]$.

On the other hand, according to (11) we can decompose $O_i(x_1)O_i(x_2)$ into sum of OPE blocks which are non-local smearing operators. In such a case, it is not so clear if the following holds true or not:

$$[B_k(x_1, x_2), O_j(w_3, \bar{w}_3)] = 0 \; . \tag{81}$$

Note that (81) implies (80) but not vice versa. However, we will show below that (81) holds true.

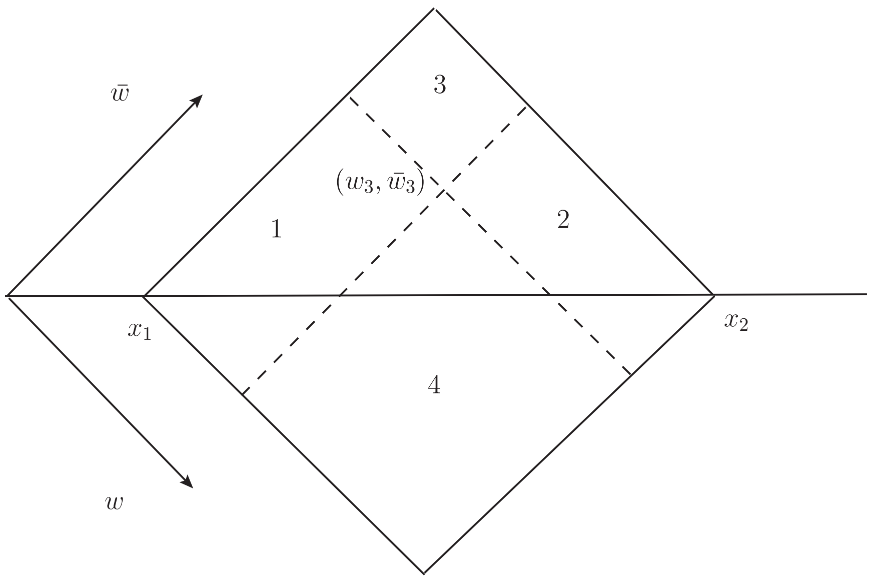

Figure 1: Dividing the causal diamond subtended by the interval (x_1, x_2) into four regions.

Since the OPE block defined by (14) is integrated over the causal diamond $\mathcal{D}_{\mathbf{A}}$, the points inside which are spacelike to x_1 and x_2 on $t = 0$ slice, we expect (81) should be satisfied if $(w_3, \bar{w}_3)$ lies inside $\mathcal{D}_{\mathbf{A}}$. Thus, we basically just need to verify

$$\int_{\mathcal{D}_{\mathbf{A}}} d^2w \; G_k(w, \bar{w}; x_1, x_2)[O_k(w, \bar{w}), O(w_3, \bar{w}_3)] = 0 \; . \tag{82}$$

We can divide the causal region $\mathcal{D}_{\mathbf{A}}$ into four regions divided by the lightcone passing through the point $(w_3, \bar{w}_3)$, as shown in Figure 1. However, the region 1 and region 2 are outside the lightcone of the point $(w_3, \bar{w}_3)$ so that $[O_k(w, \bar{w}), O_j(w_3, \bar{w}_3)] = 0$ in these two regions. Thus, we only need to check if the operator identity (82) for the sum of the region 3 and region 4 holds for arbitrary states. As for the purpose of this paper, we will only check it for the ground state and leave the proof for general states to the future works. That is, we will check

$$J_3 + J_4 = 0 \tag{83}$$

where

$$J_a := \int_{region\ a} d^2 w_0\ G_k(w_0, \bar{w}_0; x_1, x_2) \langle 0 | [O_k(w_0, \bar{w}_0), O_j(w_3, \bar{w}_3)] | 0 \rangle\ . \tag{84}$$

To start the check, we need the the vev of the commutator given by

$$\langle 0 | [O_k(w_0, \bar{w}_0), O_j(w_3, \bar{w}_3)] | 0 \rangle = \pm\ \Delta\phi_k \frac{\delta_{j,k}}{|w_{03}|^{2h_k} |\bar{w}_{03}|^{2\bar{h}_k}}, \tag{85}$$

where the "+" one is for region 3 and the "−" one for region 4 and

$$\Delta\phi_k := (e^{-2i\pi h_k} - e^{2i\pi \bar{h}_k})\ . \tag{86}$$

These Lorentzian correlators are obtained from the Euclidean 2-point function by $i\epsilon$ prescription [29]. We show above result in Appendix C.

Plugging (85) into (84), we arrive

$$J_3 = \delta_{j,k}\ \Delta\phi_k n_k \bar{n}_k w_{21}^{1-h_k} \bar{w}_{21}^{1-\bar{h}_k} \int_{w_1}^{w_3} dw_0 (w_{01} w_{20})^{h_k - 1} w_{30}^{-2h_k} \int_{\bar{w}_3}^{\bar{w}_2} d\bar{w}_0 (\bar{w}_{01} \bar{w}_{20})^{\bar{h}_k - 1} \bar{w}_{03}^{-2\bar{h}_k}\ , \tag{87}$$

and

$$J_4 = -\delta_{j,k}\ \Delta\phi_k n_k \bar{n}_k w_{21}^{1-h_k} \bar{w}_{21}^{1-\bar{h}_k} \int_{w_3}^{w_2} dw_0 (w_{01} w_{20})^{h_k - 1} w_{03}^{-2h_k} \int_{\bar{w}_1}^{\bar{w}_3} d\bar{w}_0 (\bar{w}_{01} \bar{w}_{20})^{\bar{h}_k - 1} \bar{w}_{30}^{-2\bar{h}_k}\ . \tag{88}$$

Note that the upper and lower limits of the integration are defined by using the lightcone coordinates for the corresponding region.

After some manipulations similar to the ones in Appendix A, we get

$$\int_{w_1}^{w_3} dw_0 (w_{01} w_{20})^{h_k - 1} w_{03}^{-2h_k} = w_{31}^{-h_k} w_{21}^{h_k - 1} m_k\ {}_2F_1(1 - h_k, h_k, 1 - h_k; \frac{w_{31}}{w_{21}})\ .$$

where $m_k := \frac{\Gamma(h_k)\Gamma(1-2h_k)}{\Gamma(1-h_k)}$. Using the identity of hypergeometric function ${}_2F_1(a, b, a; z) = (1 - z)^{-b}$, the above result can be further simplified to

$$\int_{w_1}^{w_3} dw_0 (w_{01} w_{20})^{h_k - 1} w_{03}^{-2h_k} = m_k (w_{31} w_{23})^{-h_k} w_{21}^{2h_k - 1}\ . \tag{89}$$

Similarly, we can obtain the following:

$$\int_{\bar{w}_3}^{\bar{w}_2} d\bar{w}_0 (\bar{w}_{01}\bar{w}_{20})^{\bar{h}_k-1}\bar{w}_{03}^{-2\bar{h}_k} = \bar{m}_k(\bar{w}_{13}\bar{w}_{32})^{-\bar{h}_k}\bar{w}_{21}^{2\bar{h}_k-1} \tag{90}$$

$$\int_{w_3}^{w_2} dw_0 (w_{01}w_{20})^{h_k-1}w_{03}^{-2h_k} = m_k(w_{31}w_{23})^{-h_k}w_{21}^{2h_k-1} \tag{91}$$

$$\int_{\bar{w}_1}^{\bar{w}_3} d\bar{w}_0 (\bar{w}_{01}\bar{w}_{20})^{\bar{h}_k-1}\bar{w}_{03}^{-2\bar{h}_k} = \bar{m}_k(\bar{w}_{31}\bar{w}_{23})^{-\bar{h}_k}\bar{w}_{21}^{2\bar{h}_k-1} \tag{92}$$

where $\bar{m}_k := \frac{\Gamma(\bar{h}_k)\Gamma(1-2\bar{h}_k)}{\Gamma(1-\bar{h}_k)}$. Plugging (89) to (92) into (87) and (88), we obtain

$$J_3 = -J_4 = \delta_{j,k}\,\Delta\phi_k n_k \bar{n}_k m_k \bar{m}_k \Big(\frac{w_{21}}{w_{31}w_{23}}\Big)^{h_k}\Big(\frac{\bar{w}_{21}}{\bar{w}_{31}\bar{w}_{23}}\Big)^{\bar{h}_k}. \tag{93}$$

This confirms the causality check of (81) for the OPE block, at least for the ground state.

From the above check, the form of the smearing function is quite essential to ensure the cancellation between J_3 and J_4 by the results of (89) to (92). Thus, not arbitrary smearing function can yield the causality result. This implies that OPE blocks are special quasi-local operators which can satisfy the causality constraint. This implication deserves further works for clarification or consolidation.

C Lorentzian commutator of primary fields

The $i\epsilon$ prescription is a way to compute the Loretzian correlators by analytically continuing Euclidean correlators. For any ordering Loretzian correlators,

$$\begin{aligned}&\langle O_1(w_1,\bar{w}_1)O_2(w_2,\bar{w}_2)...O_n(w_n,\bar{w}_n)\rangle\\ &\quad= \lim_{\epsilon_i\to 0}\langle O_1(w_1^E,\bar{w}_1^E)O_2(w_2^E,\bar{w}_2^E)...O_n(w_n^E,\bar{w}_n^E)\rangle\rangle_{\tau_i\to it+\epsilon_i},\end{aligned} \tag{94}$$

with

$$\epsilon_1 > \epsilon_2 > ... > \epsilon_n.$$

We would like to compute the commutator

$$\langle 0|[O_k(w_0,\bar{w}_0),O_j(w_3,\bar{w}_3)]|0\rangle = \langle 0|O_k(w_0,\bar{w}_0)O_j(w_3,\bar{w}_3)|0\rangle - \langle 0|O_j(w_3,\bar{w}_3)O_k(w_0,\bar{w}_0)|0\rangle. \tag{95}$$

By the $i\epsilon$ prescription we have

$$\langle 0|O_k(w_0,\bar{w}_0)O_j(w_3,\bar{w}_3)|0\rangle = \frac{\delta_{j,k}}{(w_0-w_3+i\epsilon)^{2h_k}(\bar{w}_0-\bar{w}_3-i\epsilon)^{2\bar{h}_k}}, \tag{96}$$

and

$$\langle 0|O_j(w_3,\bar{w}_3)O_k(w_0,\bar{w}_0)|0\rangle = \frac{\delta_{j,k}}{(w_0-w_3-i\epsilon)^{2h_k}(\bar{w}_0-\bar{w}_3+i\epsilon)^{2\bar{h}_k}}, \tag{97}$$

where ϵ is positive constant. Notice the sign before ϵ, which is important to keep the casual relation in Minkowski spacetime.
In the region 3 , $w_0 < w_3$ and $\bar{w}_0 > \bar{w}_3$, we have

$$\begin{aligned}&\langle 0|O_k(w_0,\bar{w}_0)O_j(w_3,\bar{w}_3)|0\rangle\\&=\frac{\delta_{j,k}}{(w_0-w_3+i\epsilon)^{2h_k}(\bar{w}_0-\bar{w}_3-i\epsilon)^{2\bar{h}_k}}\\&=\delta_{j,k}e^{-2h_k\log(w_0-w_3+i\epsilon)-2\bar{h}_k\log(\bar{w}_0-\bar{w}_3-i\epsilon)}\\&=\delta_{j,k}e^{-2h_k i\pi}\frac{1}{|w_{03}|^{2h_k}|\bar{w}_{03}|^{2\bar{h}_k}},\end{aligned}\tag{98}$$

where we have used the fact that $-a+i\epsilon=|a|e^{i(\pi-\epsilon)}$ for a positive a. Similarly,

$$\begin{aligned}&\langle 0|O_j(w_3,\bar{w}_3)O_k(w_0,\bar{w}_0)|0\rangle\\&=\frac{\delta_{j,k}}{(w_3-w_0+i\epsilon)^{2h_k}(\bar{w}_3-\bar{w}_0-i\epsilon)^{2\bar{h}_k}}\\&=\delta_{j,k}e^{-2h_k\log(w_3-w_0+i\epsilon)-2\bar{h}_k\log(\bar{w}_3-\bar{w}_0-i\epsilon)}\\&=\delta_{j,k}e^{2\pi i\bar{h}_k}\frac{1}{|w_{30}|^{2h}|\bar{w}_{30}|^{2\bar{h}_k}},\end{aligned}\tag{99}$$

where in the last step we use $-a-i\epsilon=|a|e^{i(-\pi+\epsilon)}$.
In the region 4, $w_0 > w_3$ and $\bar{w}_0 < \bar{w}_3$, by similar analysis we have

$$\begin{aligned}&\langle 0|O_k(w_0,\bar{w}_0)O_j(w_3,\bar{w}_3)|0\rangle\\&=\delta_{j,k}e^{2\bar{h}_k i\pi}\frac{1}{|w_{03}|^{2h_k}|\bar{w}_{03}|^{2\bar{h}_k}},\end{aligned}\tag{100}$$

and

$$\begin{aligned}&\langle 0|O_j(w_3,\bar{w}_3)O_k(w_0,\bar{w}_0)|0\rangle\\&=\delta_{j,k}e^{-2\pi i h_k}\frac{1}{|w_{30}|^{2h}|\bar{w}_{30}|^{2\bar{h}_k}}.\end{aligned}\tag{101}$$

Using above results one could obtain (85).

D An integral in section 3.2

Recall the integral (62)

$$\begin{aligned}I_{03}&=\int_{w_1}^{w_2}dw_0\int_{w_1}^{w_2}dw_3\ (w_{01}w_{20})^{h_k-1}(w_{31}w_{23})^{h_k-1}w_{03}^{h-2h_k},\\&=\int_{w_1}^{w_2}dw_0(w_{01}w_{20})^{h_k-1}I_0,\end{aligned}\tag{102}$$

where

$$I_0=\int_{w_1}^{w_2}dw_3(w_{31}w_{23})^{h_k-1}w_{03}^{h-2h_k}.$$

Let's divide I_0 into two parts,

$$
\begin{aligned}
I_0 &= \int_{w_1}^{w_0} dw_3 (w_{31})^{h_k-1}(w_{23})^{h_k-1}(w_{03})^{h-2h_k} + \int_{w_0}^{w_2} dw_3 (w_{31})^{h_k-1}(w_{23})^{h_k-1}(w_{03})^{h-2h_k} \\
&= w_{21}^{h_k-1}\frac{\Gamma(h_k)\Gamma(h-2h_k+1)}{\Gamma(h-h_k+1)}(w_{01})^{h-h_k}\ {}_2F_1(1-h_k,h_k,h-h_k+1,\frac{w_{01}}{w_{21}}) \\
&+(-1)^{h-2h_k} w_{21}^{h_k-1}\frac{\Gamma(h_k)\Gamma(h-2h_k+1)}{\Gamma(h-h_k+1)}(w_{20})^{h-h_k}\ {}_2F_1(1-h_k,h_k,h-h_k+1,\frac{w_{20}}{w_{21}}).
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
I_{03} &= w_{21}^{h_k-1}\frac{\Gamma(h_k)\Gamma(h-2h_k+1)}{\Gamma(h-h_k+1)}\Big(\int_{w_1}^{w_2} dw_0 w_{01}^{h-1} w_{20}^{h_k-1}\ {}_2F_1(1-h_k,h_k,h-h_k+1,\frac{w_{01}}{w_{21}}) \\
&+(-1)^{h-2h_k}\int_{w_1}^{w_2} dw_0 w_{01}^{h_k-1} w_{20}^{h-1}\ {}_2F_1(1-h_k,h_k,h-h_k+1,\frac{w_{20}}{w_{21}})\Big) \\
&= w_{21}^{h_k-1}\frac{\Gamma(h_k)\Gamma(h-2h_k+1)}{\Gamma(h-h_k+1)}(1+(-1)^{h-2h_k}) \\
&\times \int_{w_1}^{w_2} dw_0 w_{01}^{h_k-1} w_{20}^{h-1}\ {}_2F_1(1-h_k,h_k,h-h_k+1,\frac{w_{01}}{w_{21}}) \qquad (103) \\
&= w_{21}^{h+2h_k-2}(1+(-1)^{h-2h_k})\frac{\sqrt{\pi}}{2^h}\frac{\Gamma(h_k)^2\Gamma(h-2h_k+1)\Gamma(\frac{h}{2})}{\Gamma(\frac{h+1}{2})\Gamma(1+\frac{h}{2}-h_k)\Gamma(\frac{h}{2}+h_k)}.
\end{aligned}
$$

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