Supplemental Material
Time-quasiperiodic topological superconductors with Majorana multiplexing

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FLOQUET ANSATZ FOR TIME-QUASIPERIODIC SYSTEMS

In this section, we will prove the validity of Floquet ansatz in a time-quasiperiodic system. Namely, the solution $\Psi(t)$ to a time-dependent Schrödinger equation in a time-quasiperiodic system can be written as $\Psi(t) = e^{-i\epsilon t}\Phi(t)$ with quasienergy ϵ and time-quasiperiodic $\Phi(t)$.

A time-dependent Hamiltonian $H(t)$ is time-quasiperiodic with d frequencies if $H(t) = h(\omega_1 t, \ldots, \omega_d t)$, where $h(\theta_1, \ldots, \theta_d)$ is a function of with d 2π-periodic arguments $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_d)$ living on a d-dimensional torus $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$. The frequencies $\boldsymbol{\omega} = (\omega_1, \cdots, \omega_d)$ are assumed to be mutually irrational, namely

$$\sum_{j=1}^{d} n_j \omega_j \neq 0, \quad \forall \mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d. \tag{S1}$$

Consider the time evolution of an arbitrary state $\Psi(t)$ which obeys the time-dependent Schrödinger equation (SEQ)

$$i\dot{\Psi}(t) = H(t)\Psi(t), \quad \dot{\Psi} = \partial_t \Psi. \tag{S2}$$

If we write $\Psi(t) = \psi(\boldsymbol{\theta})$, with $\boldsymbol{\theta} = \boldsymbol{\omega} t$, the above equation can be rewritten as

$$i\boldsymbol{\omega} \cdot \boldsymbol{\nabla}_{\boldsymbol{\theta}} \psi(\boldsymbol{\theta}) = h(\boldsymbol{\theta})\psi(\boldsymbol{\theta}). \tag{S3}$$

Let us formally divide $\boldsymbol{\theta}$ into two parts as $\boldsymbol{\theta} = (\boldsymbol{\theta}_\perp, \theta_k)$, where $\boldsymbol{\theta}_\perp$ is a vector consisting of θ_js with $j = 1, \ldots, d, j \neq k$. Similarly, we write $\boldsymbol{\omega} = (\boldsymbol{\omega}_\perp, \omega_k)$.

Thus, we obtain a new SEQ

$$i\omega_k \partial_{\theta_k} \psi(\boldsymbol{\theta}_\perp, \theta_k) = \left[h(\boldsymbol{\theta}_\perp, \theta_k) - i\boldsymbol{\omega}_\perp \cdot \boldsymbol{\nabla}_{\theta_\perp}\right] \psi(\boldsymbol{\theta}_\perp, \theta_k). \tag{S4}$$

By Floquet theorem, the solutions to this SEQ can be written as

$$\psi(\boldsymbol{\theta}_\perp, \theta_k) = \exp(-i\epsilon_k \theta_k/\omega_k)\phi_d(\boldsymbol{\theta}_\perp, \theta_k) \tag{S5}$$

with $\phi_k(\boldsymbol{\theta}_\perp, \theta_k) = \phi_k(\boldsymbol{\theta}_\perp, \theta_k + 2\pi)$. Hence, $\psi(\boldsymbol{\theta}) \exp(i\epsilon_k \theta_k/\omega_k)$ is 2π-periodic in its kth argument θ_k. Since k is an arbitrary number from 1 to d,

$$\phi(\boldsymbol{\theta}) = \psi(\boldsymbol{\theta}) \exp(i\sum_{j=1}^{d} \epsilon_j \theta_j/\omega_j) \tag{S6}$$

will be 2π-periodic in all θ_js with proper chosen ϵ_js.

As a result, a quasiperiodic function $\Phi(t) = \phi(\boldsymbol{\theta})$ can be constructed by setting $\boldsymbol{\theta} = \boldsymbol{\omega} t$. We thus obtain a factorization

$$\Psi(t) = \Phi(t) \exp(-i\epsilon t), \quad \epsilon = \sum_{j=1}^{d} \epsilon_j, \tag{S7}$$

with $\Phi(t)$ time-quasiperiodic in the same frequencies. Moreover, this function satisfies

$$\left[H(t) - i\partial_t\right] \Phi(t) = \epsilon\Phi(t), \tag{S8}$$

which is Eq. (6) in the main text.

WANNIER-STARK LOCALIZATION OF FLOQUET MAJORANAS

Let us consider the time-periodic Kitaev chain introduced in the main text, with Hamiltonian $H(t) = H_K + M(\omega t)$. The static part is

$$H_K = -\mu \sum_{j=1}^{N} c_j^\dagger c_j - \sum_{j=1}^{N-1} [(J c_j^\dagger c_{j+1} + i\Delta c_j c_{j+1}) + h.c.], \tag{S9}$$

and the time-periodic part is

$$M(\omega t) = -i\Delta' \sum_{j=1}^{N-1} (e^{-i\omega t} c_j c_{j+1} - e^{i\omega t} c_{j+1}^\dagger c_j^\dagger). \tag{S10}$$

Introducing Nambu spinor $C_j^\dagger = (c_j^\dagger, c_j)$, we obtain the corresponding Bogoliubov–de Gennes Hamiltonian up to a constant term

$$H_{BdG} = H_{BdG,K} + M_{BdG}(\omega t) \tag{S11}$$

$$H_{BdG,K} = -\frac{\mu}{2} \sum_{j=1}^{N} C_j^\dagger \tau_z C_j - \frac{1}{2} \sum_{j=1}^{N-1} \left[C_j^\dagger \left(J\tau_z + i\Delta\tau_x \right) C_{j+1} + h.c. \right] \tag{S12}$$

$$M_{BdG}(\omega t) = -\frac{i\Delta'}{2} \sum_{j=1}^{N-1} \left[C_j^\dagger e^{i\omega t \tau_z} \tau_x C_{j+1} + h.c. \right]. \tag{S13}$$

If we rather consider a periodic boundary condition and take the Fourier expansion $C_j = \sum_{j=1}^{N} \Psi(k) e^{ikj}/\sqrt{N}$, we obtain the Bloch Hamiltonian given the in main text.

This time-periodic Hamiltonian can be mapped to a 1D synthetic lattice with an additional electric field, giving rise to a Wannier-Stark ladder. The on-site Hamiltonian at the nth rung is $h_0 - n\omega \mathbb{I}_{2N}$, with

$$h_0 = -\frac{1}{2} \begin{pmatrix} \mu\tau_z & J\tau_z + i\Delta\tau_x & & & \\ J\tau_z - i\Delta\tau_x & \mu\tau_z & J\tau_z + i\Delta\tau_x & & \\ & J\tau_z - i\Delta\tau_x & \ddots & \ddots & \\ & & \ddots & \mu\tau_z & J\tau_z + i\Delta\tau_x \\ & & & J\tau_z - i\Delta\tau_x & \mu\tau_z \end{pmatrix} \tag{S14}$$

a $2N \times 2N$ matrix describing a finite Kitaev chain of length N (in unit of lattice constant), and $\mathbb{I}_{2N}$ is the identity matrix of the same size. The nearest-neighbor hopping matrix along the ladder (from the nth to the $(n+1)$th rung of the ladder) is

$$h_{-1} = -\frac{i\Delta'}{2} \begin{pmatrix} 0 & \tau_+ & & & \\ -\tau_+ & 0 & \tau_+ & & \\ & -\tau_+ & \ddots & \ddots & \\ & & \ddots & 0 & \tau_+ \\ & & & -\tau_+ & 0 \end{pmatrix}, \tag{S15}$$

with $\tau_\pm = (\tau_x \pm i\tau_y)/2$. Hopping in the opposite direction is given by the matrix $h_1 = h_{-1}^\dagger$.

In Fig. S1, we numerically calculate the Floquet Majorana wave function $\phi_n(j) = (\phi_{n,e}(j), \phi_{n,h}(j))$ at quasienergies 0 and $\omega/2$ in a Kitaev chain of $N = 100$ sites. We take 30 rungs of the Wannier-Stark ladder in our numerical simulation. We see that both Majoranas are perfectly localized in both physical space and the synthetic lattice.

LOCALIZATION IN A QUASIPERIODIC LADDER

When the d-time-quasiperiodic system is mapped to a d dimensional synthetic lattice, the presence of the electric field $\boldsymbol{\omega}$ naturally cuts the lattice into a layers of quasicrystals living in one dimension lower. These quasicrystals are

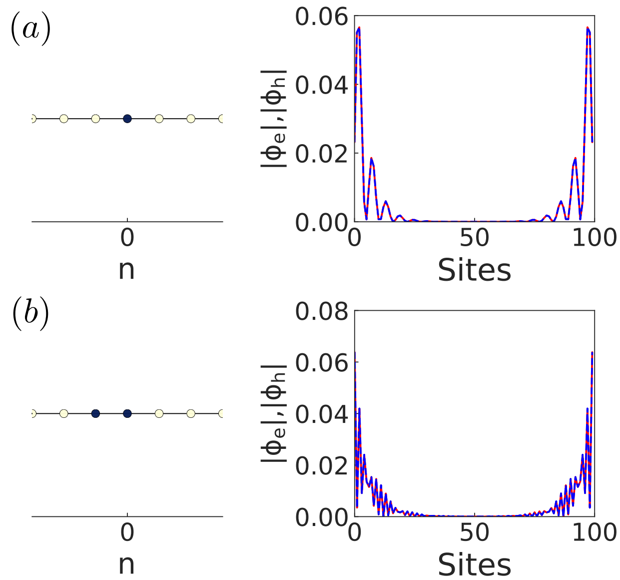

Figure S1. Numerical results for the Floquet Majorana wave functions in a Wannier-Stark ladder of 30 rungs, for a time-periodic Kitaev chain of $N = 100$ sites. The left panels are the magnitude of $|\phi_n|$) (summed over electron and hole components), where darker color corresponds to larger magnitude. The right panels are the absolute value of the corresponding Majorana wave function, summed over the 1D synthetic lattice. The electron and hole components ϕ_e and ϕ_h are plotted as red solid and blue dashed curves. (a) and (b) are for Majoranas at quasienergies 0 and $\omega/2$, respectively. The other parameters are $J/\omega = 0.51$, $\mu/\omega = 0.87$, and $\Delta/\omega = 0.051$.

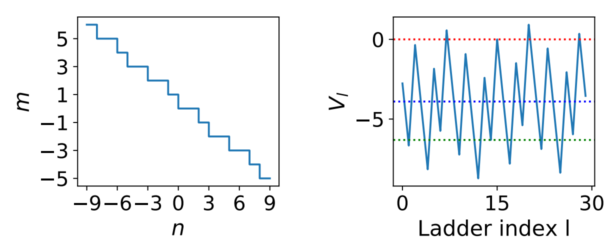

Figure S2. Left: 1D quasiperiodic ladder obtained by cutting the 2D synthetic lattice with equipotential surfaces. Right: Onsite potential V_l as a function of the ladder index l. We indicate the energy at 0, $-\omega_1$, $-\omega_2$ by the red, blue and green dotted lines for reference.

constructed by taking all the lattice points in between two equipotential surfaces perpendicular to the electric field, as described in the main text. Hence, the on-site potentials of the quasicrystal stays close to the average potential of the two surfaces. On the other hand, the on-site potential within the quasicrystal varies from site to site. For two neighboring sites, the potential difference is one of ω_js for $j = 1, \ldots d$. Thus, this quasiperiodic structure can be viewed as a mixuture of d Wannier-Stark ladders, which stays at a constant height in average. When the potential difference is larger compared to the coupling strength between neighboring rungs in this mixed ladder, the eigenstates of the system are localized.

The time-quasiperiodic Kitaev chain introduced in the main text can be mapped to a 2D synthetic lattice with an additional electric field. Perpendicular to the field, we have a quasiperiodic ladder climbing up or down by either ω_1 or ω_2 between two rungs, depending on whether these two rungs are connected horizontally or vertically in the original 2D synthetic lattice. In Fig. S2 we show a quasiperiodic ladder of length 30 obtained in a 2D lattice, and its on-site potential V_l as a function the ladder index l.

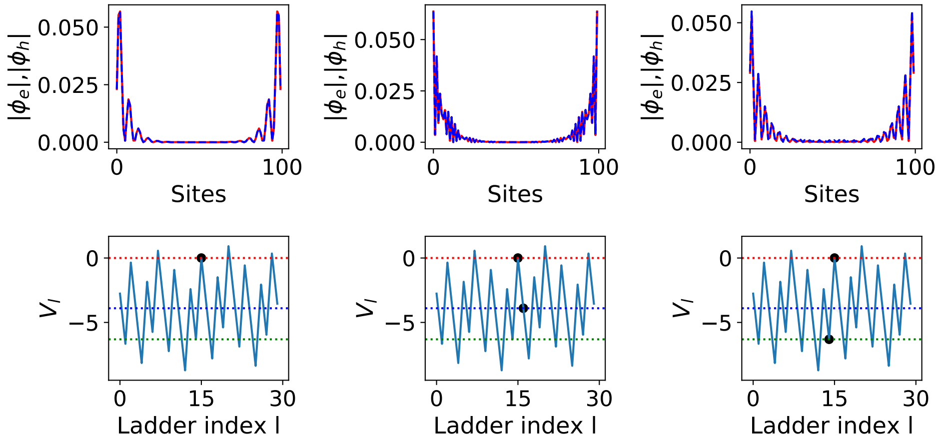

Figure S3. Majorana wave functions in a quasiperiodic ladder of 30 rungs, where each rung contains a Kitaev chain of $N = 100$ sites. Upper: the absolute value of the Majorana wave function, summed over the quasiperiodic ladder. The electron and hole components ϕ_e and ϕ_h are plotted as red solid and blue dashed curves. Lower: the magnitude of $|\phi_j|$) (summed over electron and hole components) plotted on top of the on-site potential as a function of the ladder index, where darker color corresponds to larger magnitude. The left, middle, and right panels are for Majoranas at 0, $-\omega_1/2$ and $-\omega_2/2$ energies, respectively. The other parameters are $\omega_1 = 3.9$, $\omega_2 = \omega_1 \times (\sqrt{5}+1)/2$, $J = 2$, $\mu = 3.4$, $\Delta = 0.2$, and $\Delta' = 0.15$.

Moreover, the hopping matrix (from the nth to the $(n+1)$th rung of the ladder) is

$$h_{-1,0} = -\frac{i\Delta'}{2}\begin{pmatrix} 0 & \tau_+ & & & \\ -\tau_+ & 0 & \tau_+ & & \\ & -\tau_+ & \ddots & \ddots & \\ & & \ddots & 0 & \tau_+ \\ & & & -\tau_+ & 0 \end{pmatrix} \tag{S16}$$

for a horizontal hopping, or

$$h_{0,1} = -\frac{i\Delta'}{2}\begin{pmatrix} 0 & \tau_- & & & \\ -\tau_- & 0 & \tau_- & & \\ & -\tau_- & \ddots & \ddots & \\ & & \ddots & 0 & \tau_- \\ & & & -\tau_- & 0 \end{pmatrix} \tag{S17}$$

for a vertical hopping.

In Fig. S3, we numerically calculate the Majorana wave function $\phi_l(j) = (\phi_{l,e}(j), \phi_{l,h}(j))$ (l is the ladder index) at quasienergies 0, $-\omega_1/2$, and $-\omega_2/2$ in a Kitaev chain of $N = 100$ sites. We take 30 rungs of the quasiperiodic ladder in our numerical simulation. We see that both Majoranas are perfectly localized in both physical space and the quasiperiodic ladder.

Indeed, combining the two localization mechanisms, i.e., the Wannier-Stark localization and quasiperiodic localization, time-quasiperiodic Majoranas can be localized in the synthetic dimensions as discussed in the main text.

PARTICLE-HOLE SYMMETRY OF TIME-QUASIPERIODIC MAJORANAS

In this section, we numerically show that the time-quasiperiodic Majorana wave functions are particle-hole symmetric at all times.

The Majorana wave function $\psi_j(x,t)$ at position x and time t can be written as

$$\psi_j(x,t) = e^{-i\epsilon_j t}\sum_{n_1,n_2} e^{-i(n_1\omega_1+n_2\omega_2)t}\phi_j(x;n_1,n_2), \quad j = 0,1,2, \tag{S18}$$

with $\epsilon_0 = 0$ and $\epsilon_j = \omega_j/2$ for $j = 1, 2$. Here $\psi_j(x,t)$ is a two-component wave function consisting electron and hole components $\psi_j^e(x,t)$ and $\psi_j^h(x,t)$. To show the particle-symmetry of $\psi_j(x,t)$ at any time t, one can compute the difference

$$\chi(x,t) = \left|\psi_j^e(x,t)\right|^2 - \left|\psi_j^h(h,t)\right|^2 \tag{S19}$$

and show it vanishes at all x and t.

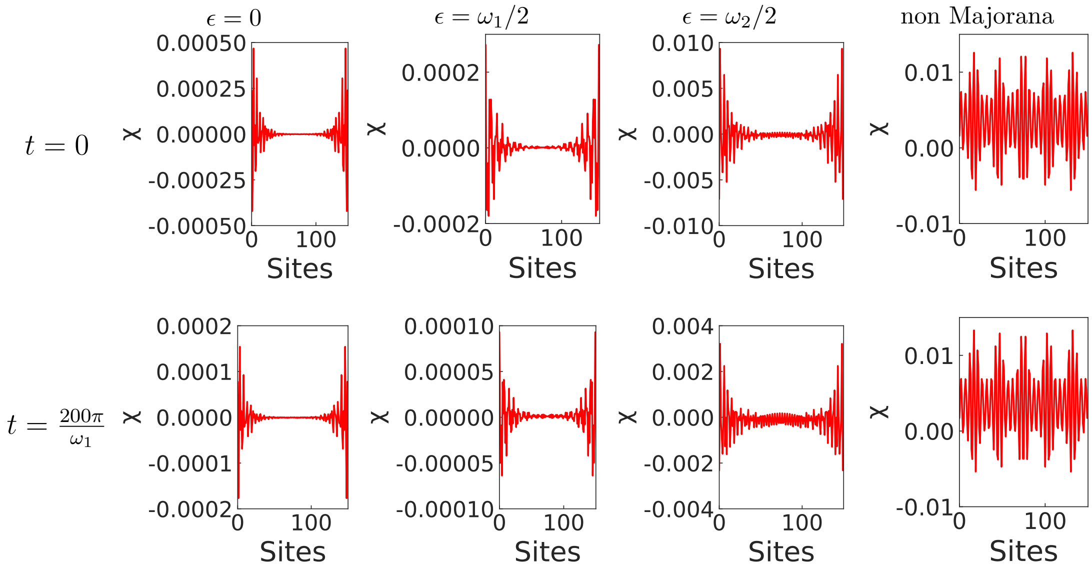

Figure S4. Difference in electron and hole components χ of the time-quasiperiodic Majoranas with energy ϵ (first three columns) as well as a generic non Majorana state (last column), at times $t = 0$ and $t = 100 \times 2\pi/\omega_1$ (two rows). The parameters are: $\omega_2/\omega_1 = (\sqrt{5}+1)/2$, $J/\omega_1 = 0.51$, $\mu/\omega_1 = 0.87$, $\Delta/\omega_1 = 0.051$ and $\Delta'/\omega_1 = 0.038$ for a chain of $L = 150$ sites.

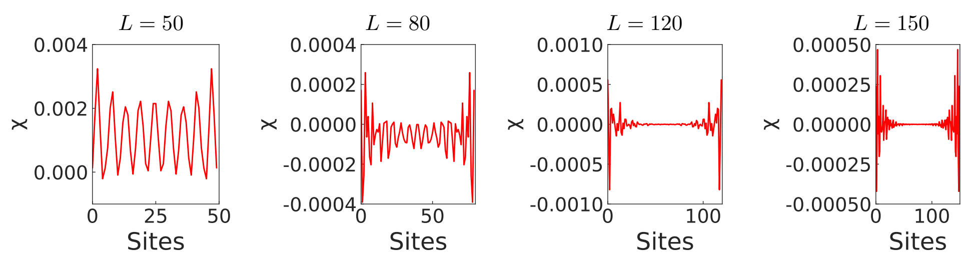

Figure S5. Difference in electron compoenent and hole component χ of the time-quasiperiodic Majoranas with energy $\epsilon = 0$ at time $t = 0$, computed with different chain lengths L. The parameters are the same as the ones in Fig. S4.

In Fig. S4, we show the difference in electron and hole components χ of the time-quasiperiodic Majoranas with energy ϵ in the first three columns, at times $t = 0$ and $t = 100 \times 2\pi/\omega_1$, in two rows. We see that χ for time-quasiperiodic Majoranas is very close to zero, compared to χ of a generic non Majorana state. In fact, the small deviation of χ from zero is due to finite size effect of the 1D chain we used in our numerical calculation. In Fig. S5, we show different χs computed with different chain lengths. We see that indeed as the number of sites L along the chain increases, χ approches zero.

MORE GENERAL DISCUSSION WITH TIME-DEPENDENT CHEMICAL POTENTIAL AND HOPPING

Floquet system and Wannier-Stark localization

Time dependent chemical potential and hopping term can be characterized by the time-dependent function $\xi_k(t) = \epsilon_k(t) - \mu(t)$, where k is the momentum when considering periodic boundary condition along the 1D superconductor,

$\epsilon_k(t)$ is the time-dependent band structure, and $\mu(t)$ is the time-dependent chemical potential. For simplicity, let us consider

$$\xi_k(t) = \xi_k + 2J' \cos \omega t. \tag{S20}$$

The time-periodic Kitaev chain in general can be written as

$$H_k(t) = \xi_k(t)\tau_z + \Delta_k(t)\tau_x, \tag{S21}$$

with $\Delta_k(t)$ a time-periodic pairing function. For each k, the time-dependent Hamiltonian can be mapped to a Wannier-Stark ladder of two level systems according to Eq. (1) in the main text. One can first neglect the superconducting pairing potential $\Delta_k(t)\tau_x$, and focus on the normal dispersion only. The Schrödinger equation corresponding to the mapped time-independent system can be written as

$$J'\tau_z(\phi(n+1) + \phi(n-1)) + (\xi_k\tau_z - n\omega)\phi(n) = E\phi(n), \tag{S22}$$

where $\phi(n)$ is the two component wave function amplitude at the nth rung at energy E.

If we define dimensionless quantity $\alpha = J'/\omega$, $\tilde{\xi_k} = \xi_k/\omega$, $\epsilon = E/\omega$, the above equation can be rewritten as

$$\phi(n+1) + \phi(n-1) = \frac{\epsilon\tau_z - \tilde{\xi_k} + n\tau_z}{\alpha}\phi(n). \tag{S23}$$

Recall the recurrence relation for Bessel function

$$Z_{n+1}(x) + Z_{n-1}(x) = \frac{2n}{x} Z_n(x) \tag{S24}$$

where $Z_n(x)$ can be the Bessel function of the first kind $J_n(x)$ or of the second kind $N_n(x)$. Hence, we require

$$\pm\,\epsilon - \tilde{\xi_k} \in \mathbb{Z}. \tag{S25}$$

We can label these energies as

$$\epsilon_m^+ = m + \tilde{\xi_k}, \quad \epsilon_l^- = l - \tilde{\xi_k}, \tag{S26}$$

with $l, m \in \mathbb{Z}$.

$$\phi_{n+1} + \phi_{n-1} = \frac{-l-n}{\alpha}\phi_n \tag{S27}$$

$$\tag{S28}$$

Since the wave function needs to be normalizable, we get two set of solutions

$$\phi_m^+(n) \equiv \langle n|\phi_m^+\rangle = \begin{pmatrix} J_{m+n}(2\alpha) \\ 0 \end{pmatrix} \quad \text{and} \quad \phi_l^-(n) \equiv \langle n|\phi_l^-\rangle = \begin{pmatrix} 0 \\ J_{-l-n}(2\alpha) \end{pmatrix}, \tag{S29}$$

corresponding states at energies ϵ_m^+ and ϵ_l^-. Given the fact that for small arguement $0 < z \ll \sqrt{\alpha+1}$

$$J_\alpha(z) \simeq \frac{1}{\alpha!}\left(\frac{z}{2}\right)^\alpha, \tag{S30}$$

we see that $\phi_m^\pm$ are localized at the $-m$th rung of the Wannier-Stark ladder (Wannier-Stark localization).

Let us now take into account the time-periodic pairing potential $\Delta_k(t)\tau_x$, which creates coupling between states $|\phi_m^\pm\rangle$in the mapped time-independent problem. Assuming

$$\Delta_k(t) = \sum_{n\in\mathbb{Z}} e^{-in\omega t}\Delta_k^{(n)}, \tag{S31}$$

then the only nonzero matrix elements are

$$\begin{aligned}\langle\phi_m^+|\,\hat{\Delta}_k\tau_x\,|\phi_l^-\rangle &= \sum_{nn'\in\mathbb{Z}}\langle\phi_m^+|n\rangle\,\langle n|\,\hat{\Delta}_k\tau_x\,|n'\rangle\,\langle n'|\phi_l^-\rangle\\ &= \sum_{nn'\in\mathbb{Z}} J_{m+n}(2\alpha)J_{-l-n'}(2\alpha)\Delta_k^{(n-n')}\\ &= \sum_{s\in\mathbb{Z}}\Delta_k^{(s)}\sum_{n\in\mathbb{Z}} J_{m+n}(2\alpha)J_{-l-n+s}(2\alpha)\\ &= \sum_{s\in\mathbb{Z}}\Delta_k^{(s)}\sum_{r\in\mathbb{Z}} J_r(2\alpha)J_{m-l+s-r}(2\alpha)\\ &= \sum_{s\in\mathbb{Z}}\Delta_k^{(s)} J_{m-l+s}(4\alpha) = \langle\phi_{m-l}^+|\,\hat{\Delta}_k\tau_x\,|\phi_0^-\rangle \equiv D_{m-l},\end{aligned} \tag{S32}$$

where we used the Bessel function addition theorem

$$\sum_{m\in\mathbb{Z}} J_{n-m}(x)J_m(y) = J_n(x+y). \tag{S33}$$

To create Majoranas at zero quasienergy, we need $\epsilon_0^{\pm}$ cross at some k, at which $D_0 \neq 0$. Similarly, to have Majoranas at $\omega/2$ quasienergy, we require, for example ϵ_0^+ crosses ϵ_1^- at some k when $D_{-1}\neq 0$. Even if we take static pairing $\Delta_k(t) = \Delta\sin k$, namely $\Delta_k^{(s)} = \delta_{s0}\Delta_k$ and $D_r = J_r(4\alpha)$, we can have both types of Majoranas, for example taking $J_0(4\alpha)\simeq J_1(4\alpha)\gg J_\nu(4\alpha)$ with $\nu\geq 2\in\mathbb{Z}$.

Localization in time-quasiperiodic system

We now generalization the previous Floquet superconductor to a two-frequency-time-quasiperiodic superconductor. By using two-dimensional Bessel functions [1] and performing similar analysis, we will show localization in the mapped two dimensional time-independent synthetic lattice. We will then construct time-quasiperiodic topological superconductor with Majorana multiplexing.

Consider time-dependent dispersion as a function of time t and momentum k

$$\xi_k(t) = \xi_k + 2J_1'\cos\omega_1 t + 2J_2'\cos\omega_2 t \tag{S34}$$

for simplicity. The time-quasiperiodic Kitaev chain we consider is

$$H_k(t) = \xi_k(t)\tau_z + \Delta_k(t)\tau_x, \tag{S35}$$

where $\Delta_k(t)$ is a time-quasiperiodic function at frequencies ω_1 and ω_2. It is helpful to first consider rational case with $\omega_1/\omega_2 = p/q$ with coprime intergers p and q. The time-quasiperiodic case with irrational ω_1/ω_2 can be regarded as the limiting procedure

$$\frac{\omega_1}{\omega_2} = \lim_{p_n,q_n\to\infty}\frac{p_n}{q_n}. \tag{S36}$$

For each k, the two-frequency-time-quasiperiodic system can be mapped to a two-dimensional synthetic lattice, in which each lattice site corresponding to a two level system. Let us denote the two component wave function amplitude at the site (n,m) as $\phi(n,m)$, the Schrödinger equation can then be written as

$$J_1'\tau_z(\phi(n+1,m)+\phi(n-1,m)) + J_2'\tau_z(\phi(n,m+1)+\phi(n,m-1)) + (\xi_k\tau_z - n\omega_1 - m\omega_2)\phi(n,m) = E\phi(n,m). \tag{S37}$$

We further more introduce dimensionless quantities

$$\alpha_1 = \frac{J_1'}{\omega_1},\quad \alpha_2 = \frac{J_2'}{\omega_2},\quad \epsilon = \frac{E}{\omega_1}p = \frac{E}{\omega_2}q,\quad \tilde{\xi}_k = \frac{\xi_k}{\omega_1}p = \frac{\xi_k}{\omega_2}q. \tag{S38}$$

We can rewrite the above equation as

$$p\alpha_1\tau_z(\phi(n+1,m)+\phi(n-1,m)) + q\alpha_2\tau_z(\phi(n,m+1)+\phi(n,m-1)) = (\epsilon - \tilde{\xi}_k\tau_z + pn + qm)\phi(n,m). \tag{S39}$$

We introduce the two dimensional Bessel function [1]

$$J_n^{p,q}(u,v) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dt\, e^{i(u\sin pt + v\sin qt - nt)} = \frac{1}{\pi}\int_0^{\pi} dt\, \cos\left(u\sin pt + v\sin qt - nt\right), \tag{S40}$$

which fulfill the following recurrence relation

$$pu(J_{\nu-p}^{p,q}(u,v) + J_{\nu+p}^{p,q}(u,v)) + qv(J_{\nu-q}^{p,q}(u,v) + J_{\nu+q}^{p,q}(u,v)) = 2\nu J_\nu^{p,q}(u,v). \tag{S41}$$

Comparing Eq.(S39) with (S41), we finds two sets of solutions

$$\phi_r^+(n,m) = \begin{pmatrix} J_{r+pn+qm}^{p,q}(2\alpha_1, 2\alpha_2) \\ 0 \end{pmatrix}, \quad \phi_s^-(n,m) = \begin{pmatrix} 0 \\ J_{-s-pn-qm}^{p,q}(2\alpha_1, 2\alpha_2) \end{pmatrix}, \quad r,s\in\mathbb{Z}, \tag{S42}$$

with eigenvalues

$$\epsilon_r^+ = r + \tilde{\xi}_k, \quad \epsilon_s^- = s - \tilde{\xi}_k. \tag{S43}$$

Let us take a closer look into the two dimensional Bessel function can be represented in terms of ordinary Bessel function [1]

$$J_\nu^{p,q}(u,v) = \sum_{(N,M)\in S_\nu} J_N(u)J_M(v), \tag{S44}$$

where the sum is over all pairs (N,M) in the set of solutions

$$S_\nu = \{(N,M)|pN + qM = \nu\} \tag{S45}$$

of the Diophantine equation

$$pN + qM = \nu. \tag{S46}$$

If (N_0, M_0) is a particular solution of the above equation, which can be found by the Euclidean algorithm, then all solutions can be written as $(N_0 - qw, M_0 + pw)$ with $w \in \mathbb{Z}$. By Eq.(S30), we see that $J_\nu^{pq}(u,v)$ is mainly contributed from $|N| \lesssim u$, $|M| \lesssim v$. In particular, when $q \gg u$, $p \gg v$ as we will consider the irrational limit, there is at most one solution (N_0, M_0) satisfying $|N_0| \lesssim u, |M_0| \lesssim v$, since $|N_0 - qw| \gg u$, $|M_0 + pw| \gg v$ for $w \neq 0$. When such a solution exist, we have $J_\nu^{p,q}(u,v) \simeq J_{N_0}(u)J_{M_0}(v)$; otherwise $J_\nu^{p,q}(u,v)$ is very small. In other words, as we change ν, like the ordinary Bessel function, $J_\nu^{p,q}(u,v)$ is localized around $\nu = pN_0 + qM_0$ with $(N_0, M_0) \in S_\nu$ with $|N_0| \lesssim u, |M_0| \lesssim v$.

By Eq.(S42), we see that the wave function amplitude are the same at sites (n,m) with constant $pn + qm$, which are sites along the direction $(-q,p)$ perpendicular to the field direction (p,q). Combining the properties of the two dimensional Bessel function discussed above, we know that $\phi_r^\pm(n,m)$ are actually localized around (n,m) when there exists

$$(N_0, M_0) \in S_r + (n,m) \tag{S47}$$

with $|N_0| \lesssim 2\alpha_1, |M_0| \lesssim 2\alpha_2$. Since the separation between peaks in the wave function amplitudes is $(-q,p)$ in the 2D synthetic lattice, in the irrational limit, we actually have true quasiperiodic localization along $(-q,p)$. Along the direction of the field (p,q), the states are also localized, which is understood as the Wannier-Stark localization.

Let us now take into account the time-periodic pairing potential $\Delta_k(t)\tau_x$, which creates coupling between states $|\phi_r^\pm\rangle$in the mapped time-independent problem. Assuming

$$\Delta_k(t) = \sum_{n_1\in\mathbb{Z}} e^{-i(n_1\omega_1 + n_2\omega_2)t}\Delta_k^{(n_1,n_2)}, \tag{S48}$$

then the only nonzero matrix elements are

$$\begin{aligned}
\langle\phi_r^+|\,\hat{\Delta}_k\tau_x\,|\phi_s^-\rangle &= \sum_{n_1,n_2,n_1',n_2'\in\mathbb{Z}} \langle\phi_r^+|n_1n_2\rangle\,\langle n_1n_2|\,\hat{\Delta}_k\tau_x\,|n_1'n_2'\rangle\,\langle n_1n_2|\phi_s^-\rangle \\
&= \sum_{n_1,n_2,n_1',n_2'\in\mathbb{Z}} J^{p,q}_{r+pn_1+qn_2}(2\alpha_1,2\alpha_2)J^{p,q}_{-s-pn_1'-qn_2'}(2\alpha_1,2\alpha_2)\Delta_k^{(n_1-n_1',n_2-n_2')} \\
&= \sum_{m_1,m_2\in\mathbb{Z}} \Delta_k^{(m_1,m_2)} \sum_{n_1,n_2\in\mathbb{Z}} J^{p,q}_{r+pn_1+qn_2}(2\alpha_1,2\alpha_2)J^{p,q}_{-s+pm_1+qm_2-pn_1-qn_2}(2\alpha_1,2\alpha_2) \\
&= \sum_{m_1,m_2\in\mathbb{Z}} \Delta_k^{(m_1,m_2)} \sum_{\nu\in\mathbb{Z}} J^{p,q}_{r+\nu}(2\alpha_1,2\alpha_2)J^{p,q}_{-s+pm_1+qm_2-\nu}(2\alpha_1,2\alpha_2) \\
&= \sum_{m_1,m_2\in\mathbb{Z}} \Delta_k^{(m_1,m_2)} J^{p,q}_{pm_1+qm_2+r-s}(4\alpha_1,4\alpha_2) \\
&= \langle\phi_{r-s}^+|\,\hat{\Delta}_k\tau_x\,|\phi_0^-\rangle \equiv F_{r-s},
\end{aligned} \tag{S49}$$

where we used the Bézout's identity [2]

$$\{pn_1+qn_2|n_1,n_2\in\mathbb{Z},\gcd(p,q)=1\}=\mathbb{Z} \tag{S50}$$

and the addition theorem for the two dimensional Bessel function [1]

$$\sum_{m\in\mathbb{Z}} J^{p,q}_{n-m}(u_1,v_1)J^{p,q}_m(u_2,v_2) = J^{p,q}_n(u_1+u_2,v_1+v_2). \tag{S51}$$

Note that if $\phi_r^\pm(n,m)$ is localized around (n_0,m_0), then $\phi_{r-p}^\pm(n,m)$ is localized around (n_0+1,m_0), and $\phi_{r-q}^\pm(n,m)$ is localized around (n_0,m_0+1).

To create Majoranas at zero quasienergy, we need $\epsilon_0^\pm$ cross at some k, at which $F_0\neq 0$. To have Majoranas at $\omega_1/2$ quasienergy, we require, for example ϵ_0^+ crosses ϵ_p^- at some k when $F_{-p}\neq 0$. Similarly, to have Majoranas at $\omega_2/2$ quasienergy, we require, for example ϵ_0^+ crosses ϵ_q^- at some k when $F_{-q}\neq 0$. Even if we take static pairing $\Delta_k(t)=\Delta\sin k$, namely $\Delta_k^{(s)}=\delta_{s0}\Delta_k$ and $D_r=J_r^{p,q}(4\alpha_1,4\alpha_2)$, we can have both types of Majoranas, for example taking nonvanishing $J_0^{p,q}(4\alpha_1,4\alpha_2)$, $J_p^{p,q}(4\alpha_1,4\alpha_2)$, $J_q^{p,q}(4\alpha_1,4\alpha_2)$.

SIGNATURES OF MAJORANA MULTIPLEXING IN CORRELATION FUNCTIONS

Majorana operators in second quantization

Before analyzing signature of Majorana multiplexing, it is helpful to first introduce Majorana operators in second-quantization. Let $|\Psi_\alpha(t)\rangle=\exp(-i\epsilon_\alpha t)\,|\Phi_\alpha(t)\rangle$ be a solution to the time-dependent Schrödinger equation

$$i\partial_t\,|\Psi_\alpha(t)\rangle = H(t)\,|\Psi_\alpha(t)\rangle\,, \tag{S52}$$

where $H(t)$ and $|\Phi_\alpha(t)\rangle$ are time-quasiperiodic with the same frequencies, and ϵ_α is the quasienergy. Creation (annihilation) operators $\psi_\alpha^\dagger(t)$ ($\psi_\alpha(t)=(\psi_\alpha^\dagger(t))^\dagger$) corresponding to $|\Psi_\alpha(t)\rangle$ can be defined as

$$\psi_\alpha^\dagger(t)=\sum_j C_j^\dagger\,\langle j|\Psi_\alpha(t)\rangle\,, \tag{S53}$$

where j is the real space index, $C_j^\dagger$ is the creation operator (may have multicomponents) at position j and $\langle j|\Psi_\alpha(t)\rangle$ is the real space wave function (with the same number of components as in C_j) of $|\Psi_\alpha(t)\rangle$.

In the case of time-quasiperiodic Kitaev chain, we have

$$\psi_\alpha(t)=e^{-i\epsilon_\alpha t}\sum_{j=1}^N\left[c_j^\dagger\phi_{\alpha,e}(j,t)+c_j\phi_{\alpha,h}(j,t)\right], \tag{S54}$$

where $\phi_{\alpha,e}(j,t)$ and $\phi_{\alpha,h}(j,t)$ are the two components in the Nambu wave function $\langle j|\phi_\alpha(t)\rangle = (\phi_{\alpha,e}(j,t), \phi_{\alpha,h}(j,t))$. Due to time-quasiperiodicity, we have

$$\phi_{\alpha,e/h}(j,t) = \sum_{\mathbf{m}} \exp(-i\mathbf{m}\cdot\boldsymbol{\omega})\phi^{\alpha}_{\mathbf{m},e/h}(j), \tag{S55}$$

where $\phi_{\alpha,\mathbf{m},e/h}(j)$ are the solution of Eq. (1) in the main text represented in both real space and the synthetic lattice. For Majorana operators, in particular, we have $\psi_\alpha(t) = \psi^\dagger_\alpha(t)$ for all t. This restricts the quasienergy to be $\mathbf{n}\cdot\boldsymbol{\omega}/2$. The wave function at quasienergy $\epsilon_{\mathbf{n}} = \mathbf{n}\cdot\boldsymbol{\omega}/2$ is also restricted to satisfy $\phi_{\mathbf{m},h} = \phi^*_{-(\mathbf{m}+\mathbf{n}),e}$.

When there are two frequencies ω_1 and ω_2, the Majorana operators of the chain at quasienergies 0, $\omega_1/2$ and $\omega_2/2$ can be written as

$$\psi^\dagger_0(t) = \sum_{j=1}^{N}\sum_{n,m} e^{-i(n\omega_1+m\omega_2)t}\left[\phi_{n,m,e}(j)c^\dagger_j + \phi_{n,m,h}(j)c_j\right] \simeq \sum_{j=1}\left[\phi_{0,0,e}(j)c^\dagger_j + \phi_{0,0,h}(j)c_j\right], \tag{S56}$$

$$\psi^\dagger_1(t) = \sum_{j=1}^{N}\sum_{n,m} e^{-i(n\omega_1+m\omega_2)t}\left[e^{i\omega_1 t/2}\phi_{n-1,m,e}(j)c^\dagger_j + e^{-i\omega_1 t/2}\phi_{n,m,h}(j)c_j\right] \tag{S57}$$

$$\simeq \sum_{j=1}^{N}\left[e^{i\omega_1 t/2}\phi_{-1,0,e}(j)c^\dagger_j + e^{-i\omega_1 t/2}\phi_{0,0,h}(j)c_j\right], \tag{S58}$$

and

$$\psi^\dagger_2(t) = \sum_{j=1}^{N}\sum_{n,m} e^{-i(n\omega_1+m\omega_2)t}\left[e^{i\omega_2 t/2}\phi_{n,m,e}(j)c^\dagger_j + e^{-i\omega_2 t/2}\phi_{n,m-1,h}(j)c_j\right] \tag{S59}$$

$$\simeq \sum_{j=1}^{N}\left[e^{i\omega_2 t/2}\phi_{0,-1,e}(j)c^\dagger_j + e^{-i\omega_2 t/2}\phi_{0,0,h}(j)c_j\right] \tag{S60}$$

respectively. For Majoranas localized near the first site of the chain, the functions $\phi_{n,m,e/h}(j)$ appeared in the above expressions decays exponentially as j increases.

Correlation function

The presense of Majoranas of different types at the end of a time-quasiperiodic Kitaev chain can be detected using correlation functions of some local operators, such as the single particle Green's function. To be concrete, let us consider $\langle 0|\gamma_1(t)\gamma_1(0)|0\rangle$ where $\gamma_1 = (c_1 + c^\dagger_1)/\sqrt{2}$, and $|0\rangle$ represents the BCS vaccuum at $t=0$. The existence of Majoranas localized around the first site enables us to write

$$\gamma_1(t) = c_0\psi_0(t) + c_1\psi_1(t) + c_2\psi_2(t) + \ldots, \tag{S61}$$

where $\ldots$ includes other extended state which has less contribution compared to the Majoranas. Hence, we have $\langle 0|\gamma_1(t)\gamma_1(0)|0\rangle$ will oscillate at frequencies $\omega_1/2$ and $\omega_2/2$.

Temporal disorder

To explore the robustness of these Majoranas in the presense of temporal disorder, we consider exponential correlated Gaussian noise in the drive. We replace $\omega_i t$ by $\omega_i t + \delta_i(t)$ with

$$\langle\dot{\delta}_i(t)\dot{\delta}_j(t')\rangle = \delta_{ij}\sigma^2\exp(-|t-t'|/\tau_d), \tag{S62}$$

where τ_d is the the correlation time, and $\delta_i(t)$ is a Gaussian distributed random variable with zero mean and variance σ.

In Fig. S6, we show two numerical simulations of $\langle\gamma_1(t)\gamma_1(0)\rangle$ using the same parameters as the ones in the main text, with additional correlated Gaussian noise. We see that peaks at 0, $\omega_1/2$ and $\omega_2/2$ are robust against moderate disorder strength σ, and correlation time τ_d. As τ_d gets longer, these peaks get broader.

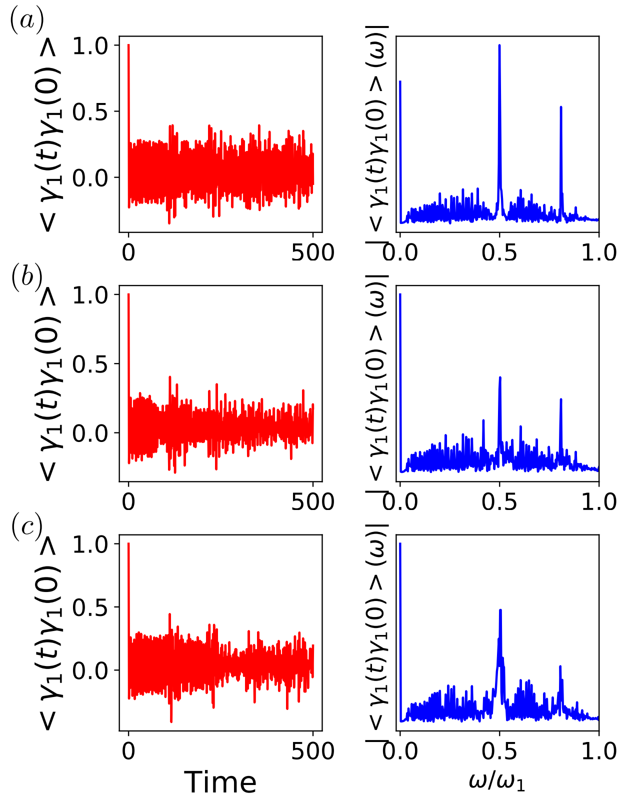

Figure S6.
Time evolution of $\langle\gamma_1(t)\gamma_1(0)\rangle$ (left panels) and its Fourier transform in the frequency domain (right panels), simulated on the time-quasiperiodic Kitaev chain, with addtional correlated Gaussian noise defined in Eq. (S62). The other parameters are the same as in Fig. 4 of the main text. The parameters for the noise are $\sigma = 0.1$; (a) $\omega_2\tau_d = 1$, (b) $\omega_2\tau_d = 20$, and (c) $\omega_2\tau_d = n100$.

Commensurate frequencies

Practically, the two frequencies ω_1 and ω_2 can hardly be mutually irrational. Let us assume $\omega_2/\omega_1 = p/q$, with $p, q \in \mathbb{Z}$. In the synthetic space, the system is still Wannier-Stark localized along the electric field, while perpendicular to the field it becomes periodic, with a large unit cell when p and q are large. In this case, the wave functions are still localized within the unit cell due to the large variation of on-site energies between different sites. We still have Majoranas from pariing within the same site or between neighboring sites.

Let approximate the golden ratio $(\sqrt{2}+1)/2$ by $5/3$, and take $\omega_2/\omega_1 = 5/3$ for the time-dependent Kitaev chain. Fig. S7 shows the wave function of the Majoranas in synthetic space and in real space. We find that the Majorana amplitudes are only localized with unit cells perpendicular to the direction of the electric field. In Fig. S8, we show the correlation $\langle\gamma_1(t)\gamma_1(0)\rangle$, and also find peaks at $\omega_1/2$ and $\omega_2/2$.

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[2] G. A. Jones and J. M. Jones, *Elementary number theory* (Springer Science & Business Media, 1998).

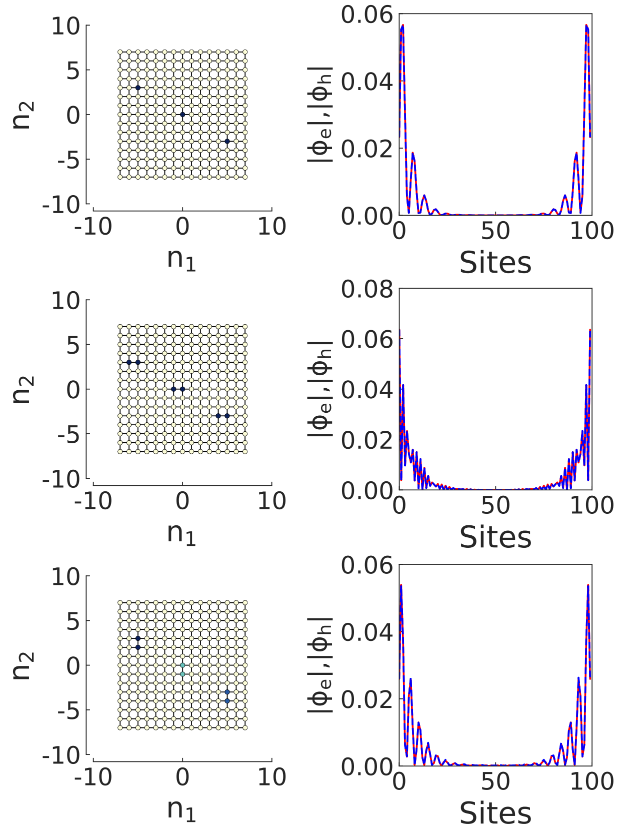

Figure S7. Numerical solution of the 0-frequency and time-quasiperiodic Majorana states on the 2D synthetic lattice of size 15×15. Each site of the lattice corresponding to a Kitaev chain of length $N = 100$. Left: $|\phi_{n_1,n_2}|^2$ for the 0, $\frac{\omega_1}{2}$, and $\frac{\omega_2}{2}$ Majoranas on the 2D synthetic lattice, where the darker color corresponds to a larger magnitude. Right: the absolute value of the corresponding Majorana wave function, summed over the 2D synthetic lattice. The electron and hole components ϕ_e, ϕ_h are plotted as red solid and blue dashed curves. The other parameters are $\omega_2/\omega_1 = 5/3$, $J/\omega_1 = 0.51$, $\mu/\omega_1 = 0.87$, $\Delta/\omega_1 = 0.051$, and $\Delta'/\omega_1 = 0.038$.

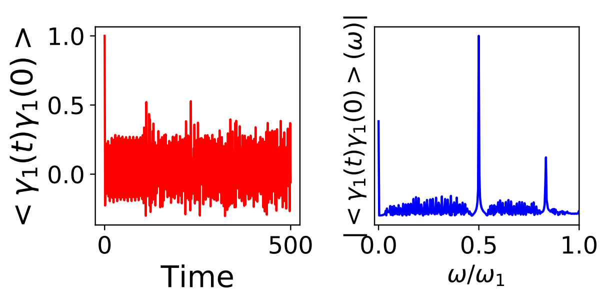

Figure S8. Time evolution of $\langle\gamma_1(t)\gamma_1(0)\rangle$ (left panels) and its Fourier transform in the frequency domain (right panels), with $\omega_2/\omega_1 = 5/3$, $J/\omega_1 = 0.51$, $\mu/\omega_1 = 0.87$, $\Delta/\omega_1 = 0.051$, and $\Delta'/\omega_1 = 0.038$.