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Equality of the density of states in a wide class of tight-binding Lorentzian random models

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We prove directly the equality of the density of states in a wide class of tight-binding Lorentzian random models, including the Lloyd model, the $\tan(2\pi\alpha n+\theta)$ model of Grempel, Fishman, and Prange, and a model with potential $\sum_i \psi_i \tan(2\pi\alpha_i n)$, where $\sum_i \psi_i = 1$ and the α_i are rationally independent.

In a recent paper,[1] Grempel *et al.* found exact solutions for the class of tight-binding Schrödinger operators $[(H_0 u)(n) = u(n+1) + u(n-1)]$.

$$(Hu)(n) = (H_0 u)(n) + \lambda \tan(2\pi\alpha n+\theta) u(n) \quad , \tag{1}$$

at least when α is an irrational not well approximated by rationals. They noted that the density of states for the model was identical to that for the Lloyd model,[2] that is, a model of the form

$$H = H_0 + \lambda V \quad , \tag{2}$$

where $V(n)$ are independent identically distributed random variables with distribution

$$\frac{1}{\pi}(1+x^2)^{-1}\,dx \quad . \tag{3}$$

Our goal in this Brief Report is to give a new proof of this equality which, at the same time, shows that other objects such as the off-diagonal—averaged Green's function and the finite-volume—averaged density of states are equal. More interestingly, we will show a large number of other models having the same density of states, e.g., (2) with $V(n) = \sum_{i=1}^{k} \psi_i \tan(2\pi\alpha_i n)$, where $\sum_{i=1}^{k} \psi_i = \lambda$ and the α_i are rationally independent, or the model of Ref. 3 where $V(n) = \lambda \tan(\alpha n^2+\theta)$ so long as θ is averaged over. Our results are not restricted to one dimension.

The key indications of the general phenomena found here are twofold: (a) The probability distribution of $x = \tan\theta$ if θ is uniformly distributed on $[0,\pi)$ is given by Eq. (3), suggesting the equality in Ref. 1 is not coincidental. (b) The solution of the Lloyd model can be expressed by saying that its density of states is the same as that in a third model, namely, the average of the density of states over the ensemble of operators where V is a constant λc, with c a random variable with distribution given by Eq. (3).[4] This striking fact about the original Lloyd model, that constant Lorentzian and completely independent Lorentzian have the same density of states, suggests a special property of Lorentzians is at work. This is seen in the following:

Lemma: Fix arbitrary reals $\alpha_1, \ldots, \alpha_k$ and positive numbers $\psi_1, \ldots, \psi_k$ with $\sum_1^k \psi_j = 1$. Let

$$x(\theta) = \sum_{j=1}^{k} \psi_j \tan(\alpha_j+\theta) \quad .$$

Then

$$\frac{1}{2\pi}\int_0^{2\pi} e^{itx(\theta)}\,d\theta = e^{-|t|} \quad .$$

Proof: By changing the sign of α_i and θ, we can suppose $t>0$. Since $\operatorname{Im}\tan(z) > 0$ if $\operatorname{Im} z > 0$, we see that[5]

$$\lim_{\epsilon\downarrow 0} \frac{1}{2\pi}\int_0^{2\pi} e^{itx(\theta+i\epsilon)}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{itx(\theta)}\,d\theta \quad .$$

In the integral, we change variables to $z = e^{i\theta}$ and observe that $x(\theta+i\epsilon)$ has no singularities in $|z| \le 1$, so the entire integral comes from the pole at $z=0$ due to the change of variables $d\theta = dz/(iz)^{-1}$. At $z=0$, x is i, which proves the result.

The α's (and for that matter the ψ's, so long as $\sum_{j=1}^{k} \psi_j = 1$) can be random variables and the expectation value of e^{itx} is still $e^{-|t|}$ so long as θ is uniformly distributed. Therefore, for large classes of random V's, including the Lloyd model,[2] the Maryland model,[1] the above random-constant model, and the $\sum_i \psi_i \tan(2\pi\alpha_i n+\theta_i)$ model discussed above,[6] one has it that any positive[7] combination of the potential at different sites, $\sum_k a_k V(k)$, has a Lorentz distribution with half-width $\sum_k a_k$.

Next, suppose that M_0 is a finite matrix and $V(n)$ is a random diagonal matrix and we want to evaluate

$$\operatorname{Exp}[e^{it(m_0+V)}(n,m)] \equiv P_{nm}(t) \quad ,$$

where Exp is the expectation value over the ensemble of V's. Since M_0 and V are finite matrices, we can expand (and obtain a convergent series)

$$e^{it(M_0+V)}=e^{itV}+i\int_0^t e^{isV}M_0e^{i(t-s)V}+\cdots .$$

Taking expectations, we see that $P_{nm}(t)$ is a sum (over j and intermediate sites $n_1,\ldots,n_j$) and an integral (over s variables) of

$$\mathrm{Exp}(e^{is_1V(n_1)}\cdots e^{is_jV(n_j)})$$

with all $s_j>0$ and $\sum_i s_i=t$. Thus $P_{nm}(t)$ only depends on the distribution of convex combinations of V.[8] In particular, if we now specialize to a situation where for $s_j\geq 0$, $\sum_j s_j=t$,

$$\mathrm{Exp}\left[\exp\left(i\sum_{k=1}^{j}s_kV(k)\right)\right]=e^{-\lambda|t|} , \qquad (4)$$

and we see that

$$P_{nm}(t)=e^{-\lambda|t|}P_{nm}^{(0)}(t) , \qquad (5)$$

where $P^{(0)}$ is the value of P when $V\equiv 0$, i.e., $e^{itM_0}(n,m)$.

To summarize, if V obeys (4), then (5) will hold, and by the lemma, (4) holds for many distinct choices of V, in particular, both for the Lloyd model and for the Maryland model, Eq. (1) averaged over θ.

Once (5) holds in finite volume, it remains true for infinite volume (if $P^{(0)}$ has a limit as it does if $M_0=H_0$). $P_{00}(t)$ is just the Fourier transform of the averaged[9] density of states, so the density of states only depends on λ and is just the convolution of the density of states for M_0 and the Lorentz distribution $\pi^{-1}(x^2+\lambda^2)^{-1}(\lambda\, dx)$. The same argument works for the Fourier transform of $P_{nm}(t)$, which is the averaged Green's function which one can thereby see decays exponentially when $\lambda>0$ and $M_0=H_0$. In one dimension, once the density of states only depends on V through (4), the same is true of the Lyaponov exponent by the Thouless formula.[10]

We close with a series of remarks about further examples. (a) While the α independence of the density of states in (1) was only proven in Ref. 1 for α, which are not well approximated by rationals, our argument does not depend on Diophantine properties. If α is a Liouville number, one can show that the spectrum is purely singular continuous,[11] so we have examples of dense point and singular continuous spectrum with the same density of states. This illustrates once again[12] that the density of states and the spectrum are often crude indications of the true physics. (b) The method works in any dimension: Thus the ν-dimensional Lloyd model has the same density of states as $\sum_i\psi_i\tan(2\pi\vec{\alpha}_i\cdot\vec{n})$, where the $\vec{\alpha}$ are rationally independent ν-component vectors. (c) M_0 need not be translation invariant. Thus, if V is any potential for which H_0+V has a density of states $d\tilde{k}$, then the average dk of the density of states of $H_0+V+\epsilon\tan(\alpha n+\theta)$ over θ is just the convolution of $d\tilde{k}$ and $\pi^{-1}\epsilon\, dx/[x^2+\epsilon^2]$. In particular, dk is supported everywhere and γ, the average over θ of the Lyaponov exponent, is strictly positive for all E. In the case where the density of states is independent of θ, we see the perturbed operator has spectrum $(-\infty,\infty)$ with *no* absolutely continuous component.[13] In particular, for *any* almost periodic function V, if α is rationally independent of the frequencies of V and ϵ is *any* nonzero number, $H_0+V+\epsilon\tan(\alpha n)$ has these properties.

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[1]D. Grempel, S. Fishman, and R. Prange, Phys. Rev. Lett. 49, 833 (1982).

[2]P. Lloyd, J. Phys. C 2, 1717 (1969).

[3]S. Fishman, D. Grempel, and R. Prange, Phys. Rev. Lett. 49, 509 (1982).

[4]That is, the density of states is just the convolution of the free density of states and the distribution of λc.

[5]This is somewhat subtle, e.g., it is false if $t<0$ (where we have to replace $\epsilon\downarrow 0$ by $\epsilon\uparrow 0$).

[6]Since the α_i are rationally independent, limits of translates of V are precisely the class of potentials with θ_i arbitrary and independently distributed.

[7]Of course, other combinations have a distribution which is dependent on which V is chosen. Interesting physics like localization which is model dependent must then depend on more than such positive combinations.

[8]This realization and the argument in this paragraph were arrived at in discussions with Tom Spencer.

[9]In many cases, e.g., rationally independent α, one can show that the density of states is equal for almost all potentials in the class, so that averaging is not necessary.

[10]D. Herbert and R. Jones, J. Phys. C 4, 1145 (1971); D. Thouless, *ibid.* 5, 77 (1972).

[11]One does this by using the analysis of J. Avron and B. Simon, Bull. Am. Math. Soc. 6, 81 (1982). One can show that for almost all θ, the argument of A. Gordon, Usp. Mat. Nauk 31, 257 (1976), applies to show there can be no normalizable eigenfunctions if α is a Liouville number.

[12]Aubry duality for the density of states [see G. Andre and S. Aubry, Ann. Israel Phys. Soc. 3, 133 (1980)] shows that in the $\cos(2\pi\alpha n)$ model the density of states are up to a scale factor the same in the extended and localized region.

[13]That a strictly positive Lyaponov exponent implies that there is no absolutely continuous spectrum is a result of L. Pastur, Commun. Math. Phys. 75, 179 (1980).