Brief Reports

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Supergauge independence of Witten's gravitational energy expression

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Witten's proof of positive gravitational energy involved an apparent choice of supergauge, through a constraint on the basic spinor parameter. We show that the value of the energy is, as it must be on supergravity grounds, independent of this choice.

In his classic proof of gravitational energy positivity, Witten[1] required his fundamental spinor to satisfy the Dirac equation $\not{D}\epsilon \equiv \vec{\gamma}\cdot\vec{D}\epsilon = 0$ on the initial-value surface. In terms of the underlying classical supergravity,[2-5] this requirement corresponds to a natural, "Coulomb gauge," choice for the spin-$\frac{3}{2}$ field. However, the energy's value must be independent of supergauge (as was the original quantum supergravity proof[6]), and its form should reflect this fact: a demonstration is the main purpose of our note.

We begin with the energy expression[1] in a general gauge, in presence of an external $T_{\mu\nu}$ obeying the usual dominant energy condition:

$$\bar{\epsilon}_0\gamma_\mu\epsilon_0 P^\mu = 2\int d^3x\sqrt{g}\,(|D_i\epsilon|^2 - |\not{D}\epsilon|^2 + \epsilon^* S\epsilon)\ ,$$
$$S \equiv \tfrac{1}{2}G_{0\mu}\gamma_0\gamma^\mu = \tfrac{1}{2}T_{0\mu}\gamma_0\gamma^\mu \geqslant 0\ , \tag{1}$$

where ϵ_0 is the constant value of $\epsilon(x)$ at spatial infinity; $\bar{\epsilon}_0\gamma_\mu\epsilon_0$ is necessarily timelike. Our units are $8\pi G = 1$, signature $(-+++)$ and the Einstein constraints have been used to reexpress S in (1). If we take $\not{D}\epsilon = 0$, then the integral is manifestly non-negative. To see that this gauge choice is superfluous, let us decompose ϵ as follows:

$$\epsilon = \epsilon_1 + \epsilon_2,\ \ \not{D}\epsilon_1 = 0,\ \ \epsilon_1(x) \underset{\infty}{\sim} \epsilon_0,\ \ \epsilon_2(x) \underset{\infty}{\sim} 0\ . \tag{2}$$

This represents an arbitrary gauge, $\not{D}\epsilon = \not{D}\epsilon_2 \neq 0$. In an obvious notation, the right side of (1) reads

$$I(\epsilon^*\epsilon) = 2\int d^3x\sqrt{g}\,(|D_i\epsilon_1|^2 + \epsilon_1^* S\epsilon_1) + I(\epsilon_1^*\epsilon_2) + I(\epsilon_2^*\epsilon_1) + I(\epsilon_2^*\epsilon_2)\ . \tag{3}$$

We now show that only the positive first integral in (3) fails to vanish. The relevant identity is[7]

$$W \equiv -D_i^\dagger D^i + \not{D}\not{D} + S = 0\ , \tag{4}$$

where $D_i^\dagger$ is the formal adjoint of D_i in that

$$\int \alpha^* D_i^\dagger D^i\beta = -\int D_i\alpha^* D^i\beta$$

in our spatial integration. Thus, integration by parts in the unwanted terms of (3) produces precisely the identically vanishing forms

$$\epsilon_1^* W\epsilon_2,\ \ \epsilon_2^* W\epsilon_1,\ \ \epsilon_2^* W\epsilon_2\ ,$$

with no surface terms left over since ϵ_2 vanishes at infinity. This exhibits Witten's formula (1) expressed solely in terms of ϵ_1 and its asymptotic value ϵ_0:

$$(\bar{\epsilon}_0\gamma_\mu\epsilon_0)P^\mu = 2\int d^3x\sqrt{g}\,(|D_i\epsilon_1|^2 + \epsilon_1^* S\epsilon_1)\ . \tag{5}$$

Thus, in any gauge (ϵ arbitrary), the energy is manifestly positive and ϵ_2 independent. The supergravity basis for this is discussed in Ref. 3, where it is shown that only the $\vec{\gamma}$-transverse part of the generic gauge function $D_i\epsilon$ in $\delta\psi_i = D_i\epsilon$ survives in the supersymmetry generator $Q(\epsilon)$, and therefore also in the energy expression which is its square. The present demonstration merely underlines this fact directly without reference to the underlying classical supergravity. There, Q depends only on the dynamical, $\vec{\gamma}$- and $\vec{D}$-transverse, component ψ_i^t of the ψ_μ field; essentially, $Q \sim \int d^3x\sqrt{g}\,\bar{\vec{\psi}}^t\cdot\vec{D}\epsilon$. Orthogonality then implies that only the projections of ϵ obeying the conditions $\vec{\gamma}\cdot\vec{D}\epsilon = 0$, $\vec{D}^\dagger\cdot\vec{D}\epsilon = 0$ survive in Q and hence in the basic relation

$$\bar{\epsilon}_0\gamma_\mu\epsilon_0 P^\mu = \tfrac{1}{2}\{Q,Q\}_{\rm PB}\ .$$

These two conditions are manifestly compatible in the supersymmetric case where external sources are absent and S vanishes in (4); the same ϵ_2 independence is nevertheless valid also for $T_{\mu\nu} \neq 0$, as we have seen in the present context. Of course, the proofs[1,8,9] that $\not{D}\epsilon_1 = 0$, $\epsilon_1(x) \underset{\infty}{\sim} \epsilon_0$ has a unique solution are still essential in either method; it is basically the *global* supersymmetry gauge choice ϵ_0 which parametrizes the energy expression, independent of its interior continuation.

While (5) shows that the energy's *value* depends only on the zero-mode part of a general ϵ, its *form* can be altered by making different gauge choices (a standard property of gauge-invariant quantities). For example, we may remove the explicit source dependence in (1) by arranging for the last two integrals there to cancel (for positive S). This can be accomplished by imposing the condition $\not{D}\epsilon = \sqrt{S}\,\epsilon$ (where $\sqrt{S}$ is the matrix square root), which still maintains[10] $\epsilon \underset{\infty}{\sim} \epsilon_0$ provided S falls off as r^{-4}. The resulting form,

$$\bar{\epsilon}_0 \gamma_\mu \epsilon_0 P^\mu = 2\int d^3x\sqrt{g}\,|D_l\epsilon|^2$$

reminds us in the present context that energy density is not a meaningful concept in relativity.

Finally, we mention that Witten's method can also be formulated entirely in terms of quantities intrinsic to the initial surface[11] to provide a weaker but simple and instructive form of the energy theorem. The identity corresponding to (4) becomes (here[7] $\nabla^\dagger \equiv \nabla$)

$$\overline{W} \equiv -\nabla^i\nabla_i + \not{\nabla}\not{\nabla} + \bar{S} = 0, \quad \bar{S} \equiv \tfrac{1}{4}\,{}^3R = \tfrac{1}{2}(\tau + T_{00}) \ ,$$
$$\tau \equiv \tfrac{1}{2}(K^i_j K^j_i - K^2) \ , \qquad (6)$$

where $\vec{\nabla}$ is the surface-intrinsic covariant gradient, and ${}^3R$ the intrinsic curvature scalar. We have used the Einstein constraint $G_{00} = T_{00}$ in $\bar{S}$, thereby introducing the second fundamental form (extrinsic curvature) K_{ij} ($K \equiv K^i_i$) of the initial surface in the combination τ, which may be thought of as the gravitational field's kinetic energy density. Integrating $\epsilon^* \overline{W} \epsilon = 0$ here yields

$$\int \epsilon^* \vec{\nabla}\epsilon \cdot d\vec{S} = |\epsilon_0|^2 E = 2\int d^3x\sqrt{g}\,(|\vec{\nabla}\epsilon|^2 - |\not{\nabla}\epsilon|^2 + \epsilon^*\bar{S}\epsilon) \ . \qquad (7)$$

This is an expression for the energy alone, rather than four-momentum, in terms of the external T_{00} alone and the gravitational kinetic density τ. The latter is locally positive only on a minimal surface, $K = 0$. In that case, we may proceed as before to show that only the corresponding ϵ_1 contributes in (7), or we may cancel the last two terms there against each other by gauge choice. It is also easy to show that vanishing energy implies flat spacetime. Since $E = 0$ implies $\vec{\nabla}\epsilon_1 = 0 = K_{ij} = T_{00}$, and $\vec{\nabla}\epsilon = 0$ immediately tells us three-space is flat, the initial data are those of the vacuum: vanishing intrinsic and extrinsic curvatures and matter excitations, from which spacetime flatness follows.[12] It is amusing that the energy theorem is so easily established on a minimal surface.

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[2]G. Horowitz and A. Strominger, Phys. Rev. D **27**, 2793 (1983).

[3]S. Deser, Phys. Rev. D **27**, 2805 (1983).

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[5]C. Teitelboim, Phys. Rev. D **29**, 2763 (1984).

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[7]While the derivatives D_i are the spatial components of the full D_μ and so fail to commute with intrinsic three-quantities such as γ^j, this is exactly cancelled by their abnormal behavior under integration by parts as shown in detail in Ref. 8, where Eq. (4) is derived. An elementary derivation of the cancellation is also given by L. D. Faddeev, Usp. Fiz. Nauk **136**, 435 (1982) [Sov. Phys. Usp. **25**, 130 (1982)]. See further the "note added" to Ref. 1, where the identical process is used. In the three-dimensional approach below, these complications never arise.

[8]T. Parker and C. H. Taubes, Commun. Math. Phys. **84**, 223 (1982).

[9]O. Reula, J. Math. Phys. **23**, 810 (1982).

[10]T. Parker has now rigorously established the existence of this gauge choice (private communication).

[11]The results (6) and (7) have been derived independently by T. Parker (private communication); a somewhat different intrinsic approach was used in Ref. 9.

[12]R. Arnowitt and S. Deser, Ann. Phys. (N.Y.) **23**, 318 (1963).