[4] David Harker, "The Application of the Three-Dimensional Patterson Method and the Crystal Structures of Proustite, Ag_3AsS_3, and Pyrargyrite, Ag_3SbS_3," *Journ. Chem. Phys.*, **4**, 381–390 (1936).

[5] W. F. deJong, J. Bouman, and J. J. deLange, "X-Ray Photography of Zero-Order Reciprocal Net Planes of a Crystal," *Physika*, **5**, 188–192 (1938).

[6] W. F. deJong and J. Bouman, "Das Photographieren von reziproken Kristallnetzen mittels Röntgenstrahlen," *Zeit. Krist.* (A) **98**, 456–459 (1938).

[7] W. F. deJong and J. Bouman, "Das Photographieren von reziproken Netzebenen eines Kristalles mittels Röntgenstrahlen," *Physika*, **5**, 220–224 (1938).

[8] J. Bouman and W. F. deJong, "Die Intensitäten der Punkte einer photographierten reziproken Netzebene," *Physika*, **5**, 817–832 (1938).

[9] I. Fankuchen, "A Condensing Monochromator for X-Rays," *Nature*, **139**, 193 (1937).

[10] Ralph H. V. M. Dawton, "The Integration of Large Numbers of X-Ray Crystal Reflections," *Proc. Phys. Soc.*, **50**, 919–925 (1938).

THE AERODYNAMICS OF REACTING SUBSTANCES

By H. Bateman

Guggenheim Graduate School of Aeronautics, Pasadena, California

Communicated May 22, 1939

Speaking broadly, the subject of aerodynamics is concerned not only with the forces exerted by a single fluid on solid bodies but also with the behavior of mixtures such as those which occur in the gasoline engine and in the atmosphere. In a complete study of the motion of a fluid attention must be paid to reactions between its constituents, evaporation, conduction and radiation of heat, diffusion, viscosity and other phenomena.

It is uncertain to what extent equations governing the various processes can be derived from a single variational principle as some of the phenomena mentioned are known to present difficulties, but the situation may be clarified a little by an examination of the equations derived from a variational principle of a very general type.

We consider n superposed substances $S_1, S_2, \ldots S_n$. The density of the substance S_r at the point (x, y, z) at time t will be denoted by ρ_r and a set of component velocities in an irrotational motion of this substance will be supposed to be derivable from a potential, ϕ_r, which is a function of x, y, z and t. The changes taking place in the complex will be regarded as governed more or less by a governing function, f, which is a function of x, y, z, t and of $\rho_1, \rho_2, \ldots \rho_n, \phi_1, \phi_2, \ldots \phi_n$. Writing $u_r = \frac{\partial\phi_r}{\partial x}$, $v_r = \frac{\partial\phi_r}{\partial y}$, $w_r = \frac{\partial\phi_r}{\partial z}$ we consider the variational principle

$$O = \delta \int d(x, y, z, t)\left[f - \sum_{s=1}^{n} \rho_s\left(\frac{\partial \phi_s}{\partial t} + {}^1/_2 u_s^2 + {}^1/_2 v_s^2 + {}^1/_2 w_s^2\right)\right] = \delta \int d(x, y, z, t)L. \quad (1)$$

in which the quantities $\rho_1, \rho_2, \ldots, \rho_n, \phi_1, \phi_2, \ldots, \phi_n$ are to be varied independently.

The equations of Euler and Lagrange for this variational problem are

$$o = f^s - \left(\frac{\partial \phi_s}{\partial t} + {}^1/_2 u_s^2 + {}^1/_2 v_s^2 + {}^1/_2 w_s^2\right), \; s = 1, 2, \ldots, n \quad (2)$$

$$o = f_s + \frac{\partial \rho_s}{\partial t} + \frac{\partial}{\partial x}(\rho_s u_s) + \frac{\partial}{\partial y}(\rho_s v_s) + \frac{\partial}{\partial z}(\rho_s w_s), \; s = 1, 2, \ldots, n \quad (3)$$

where the symbol f^s is used to denote $\partial f/\partial \rho_s$ and the symbol f_s is used to denote $\partial f/\partial \phi_s$.

Differentiating the first equation with respect to x we get

$$o = df^s/dx - (\partial u_s/\partial t + u_s \partial u_s/\partial x + v_s \partial u_s/\partial y + w_s \partial u_s/\partial z) \quad (4)$$

where $df^s/dx = \partial f^s/\partial x + \sum_{r=1}^{n} (f^{rs}\partial \rho_r/\partial x + f_r^s \partial \phi_r/\partial x)$.

Let us now write

$$p = f - \sum_{r=1}^{n} \rho_r \partial f/\partial \rho_r, \quad (5)$$

then

$$\begin{aligned} dp/dx &= \partial f/\partial x - \sum_{r=1}^{n} \rho_r \partial f^r/\partial x + \sum f_r \partial \phi_r/\partial x - \sum \rho_r (f^{rs}\partial \rho_s/\partial x + f_s^r \partial \phi_s/\partial x) \\ &= \partial f/\partial x - \sum_{r=1}^{n}\left[\frac{\partial}{\partial t}(\rho_r u_r) + (\partial/\partial x)(\rho_r u_r^2) + (\partial/\partial y)(\rho_r u_r v_r) + (\partial/\partial z)(\rho_r u_r w_r)\right]. \end{aligned} \quad (6)$$

This is one of the dynamical equations for the system as a whole. The quantity p can be regarded as the total pressure. The mean motion (u, v, w) representing the motion of the center of mass of each fluid element is generally rotational until the differences in velocity of the constituents S_r are annulled.

Adding the equations (3) we see that Lavoisier's principle of the indestructibility of mass is fulfilled in the complete system if

$$\sum_{s=1}^{n} f_s = 0. \quad (7)$$

This equation is satisfied if f is a function of the differences $\phi_1 - \phi_s$. In the simple case in which $f = \sum_{s=1}^{n} F_s(\rho_s, \phi_1, \phi_2, \ldots, \phi_n)$ we may write

$$p_s = F_s - \rho_s \partial F_s / \partial \rho_s$$
$$p = \sum_{s=1}^{n} p_s. \quad (8)$$

The quantity p_r can then be regarded as the partial pressure of the constituent S_r. We have also

$$(\partial/\partial t)(\rho_r u_r) + (\partial/\partial x)(\rho_r u_r^2) + (\partial/\partial y)(\rho_r u_r v_r) + (\partial/\partial z)(\rho_r u_r w_r) = \partial F_r/\partial x - \partial p_r/\partial x. \quad (9)$$

These equations are like the ordinary dynamical equations for a single gas.

The principle is readily extended to the case of constituents in rotational motion.[1] The quantity under square brackets in (1) is replaced by

$$f - \sum_{s=1}^{n} \rho_s(D_s\phi_s + \alpha_s D_s \beta_s - {}^1/_2 u_s^2 - {}^1/_2 v_s^2 - {}^1/_2 w_s^2) \quad (10)$$

where $D_s = \partial/\partial t + u_s\partial/\partial x + v_s\partial/\partial y + w_s\partial/\partial z$,

and f is now a function of $x, y, z, t, \rho_1, \rho_2, \ldots, \rho_n, \phi_1, \phi_2, \ldots, \phi_n, \alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n, u_s, v_s, w_s \ldots$. In forming the equations of Euler and Lagrange the quantities $\rho_s, \phi_s, \alpha_s, \beta_s, u_s, v_s, w_s$ are to be varied independently. The principle may also be extended to the case in which the axes are those at a place on the rotating earth. The form of the Lagrangian function is readily inferred from the remark in the author's review of Ertel's book.

It is thought that these variational principles may be of some use in obtaining the characteristics of the set of partial differential equations characterizing the motion and, in particular, for the determination of the velocity and properties of a surface of discontinuity in the atmosphere or in an explosion engine. It will be seen from equations (3) that when the quantities f_s are not zero the equation of continuity for a constituent substance is not satisfied. This allows, then, for the production or destruction of the substance by a chemical transformation and the generalized equation of continuity should also be a generalized form of the law of mass action of chemical dynamics. Such an equation should be applicable in particular to the reaction zone which forms a region of continuity between burnt and unburnt gases in an explosion. It is just in such a region that a more complete treatment than that usually given is desirable.

In their recent work Lewis and von Elbe have devoted some attention to the part played by diffusion. To include diffusion in the present analysis new terms must be added to the Lagrangian function L. If we add terms of type

$$\sum_{s=1}^{n} G_s\left(\frac{\partial\rho_s}{\partial x}\frac{\partial\phi_s}{\partial x} + \frac{\partial\rho_s}{\partial y}\frac{\partial\phi_s}{\partial y} + \frac{\partial\rho_s}{\partial z}\frac{\partial\phi_s}{\partial z}\right)$$

the left-hand side of equation (3) must be replaced by

$$\frac{\partial}{\partial x}\left(G_s\frac{\partial\rho_s}{\partial x}\right) + \frac{\partial}{\partial y}\left(G_s\frac{\partial\rho_s}{\partial y}\right) + \frac{\partial}{\partial z}\left(G_s\frac{\partial\rho_s}{\partial z}\right)$$

and a term

$$-\left[\frac{\partial}{\partial x}(G_s u_s) + \frac{\partial}{\partial y}(G_s v_s) + \frac{\partial}{\partial z}(G_s w_s)\right]$$

must be added to the right-hand side of equation (2). Thus in a non-uniform flow there may be forces on a constituent which arise from diffusion.

[1] An appropriate form of the principle was suggested by the author in a physical seminar at the California Institute of Technology in 1938 and also in a review of H. Ertel's "Methoden und Probleme der Dynamischen Meteorologie," *Zentralblatt für Mathematik und ihre Grenzgebiete*, **18**, 311 (1938). The analysis for the case of a single gas has been given in detail by H. Ertel, *Meteorologische Zeit.*, 105–108 (1939).

[2] Bernard Lewis and Guenther von Elbe, "Combustion, Flames and Explosion of Gases," Cambridge Univ. Press (1938). See also J. B. Zeldovich and D. A. Frank-Kameneckij, "On the Theory of Uniform Flame Propagation," *Compt. rend.* (*Doklady*) *de l'acad. des sciences de l'U. R. S. S.*, **19**, 693–697 (1938).

A QUESTION IN GENERAL RELATIVITY

BY L. I. SCHIFF

DEPARTMENT OF PHYSICS, UNIVERSITY OF CALIFORNIA

Communicated June 7, 1939

I.—The following question, which provides an illuminating application of general relativity to electrodynamics, has been put to me by Professor Oppenheimer. Consider two concentric spheres with equal and opposite total charges uniformly distributed over their surfaces. When the spheres are at rest, the electric and magnetic fields outside the spheres vanish. When the spheres are in uniform rotation about an axis through their center, the electric field outside vanishes, while the magnetic field does not, since the magnetic moment of each of the spheres is proportional to the square of its radius. Suppose that the spheres are stationary; then an observer traveling in a circular orbit around the spheres should find no field, for since all of the components of the electromagnetic field tensor vanish in one coördinate system, they must vanish in all coördinate systems.