Factorability of Lossless Time-Varying Filters and Filter Banks

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***Abstract*— In this paper, we study the factorability of linear time-varying (LTV) *lossless* filters and filter banks. We give a *complete* characterization of all degree-one lossless LTV systems and show that all degree-one lossless systems can be decomposed into a time-dependent unitary matrix followed by a lossless dyadic-based LTV system. The lossless dyadic-based system has several properties that make it useful in the factorization of lossless LTV systems. The traditional lapped orthogonal transform (LOT) is also generalized to the LTV case. We identify two classes of TVLOT's, namely, the invertible inverse lossless (IIL) and noninvertible inverse lossless (NIL) TVLOT's. The minimum number of delays required to implement a TVLOT is shown to be a nondecreasing function of time, and it is a constant if and only if the TVLOT is IIL. We also show that all IIL TVLOT's can be factorized *uniquely* into the proposed degree-one lossless building block. The factorization is *minimal* in terms of delay elements. For NIL TVLOT's, there are factorable and unfactorable examples. Both necessary and sufficient conditions for factorability of lossless LTV systems will be given.**

We also introduce the concept of *strong eternal reachability* (SER) and *strong eternal observability* (SEO) of LTV systems. The SER and SEO of an implementation of LTV systems imply the *minimality* of the structure. Using these concepts, we are able to show that the cascade structure for a factorable IIL LTV system is *minimal*. That implies that if a IIL LTV system is factorable in terms of the lossless dyadic-based building blocks, the factorization is minimal in terms of delays as well as the number of building blocks. We also prove the BIBO stability of the LTV normalized IIR lattice.

I. INTRODUCTION

FIG. 1 SHOWS an M-channel maximally decimated time-varying filter bank (TVFB). In a companion paper [1], we studied some basic properties of TVFB and showed that there are several unusual properties that are not exhibited by the conventional FB's. See [1] for a brief history and references on the topic of TVFB's.

In the linear time-invariant (LTI) case, it is well-known [2], [3] that all LTI paraunitary (PU) FB's can be factorized into degree-one building blocks. The factorization is minimal in terms of delay elements. In this paper, we will study a similar factorizaton for the LTV case. Consider Fig. 1, where $h_k^i(n)$ and $f_k^i(n)$ represent the kth coefficients of the ith analysis and synthesis filters at time n, respectively. The analysis bank is said to be *lossless* if

$$\sum_n |x(n)|^2 = \sum_{i=0}^{M-1} \sum_n |y_i(n)|^2 \tag{1.1}$$

where $y_i(n)$ is the decimated subband signal as shown in Fig. 1. The class of TVFB's with a lossless analysis bank is addressed in detail in [1]. In this paper, we are going to use the results in [1] to study the factorability of this class of TVFB's in terms of lossless LTV building blocks. For convenience, we state some results from [1] in the following:

1) Using the LTV polyphase representation, it was shown in [1] that the M-channel TVFB in Fig. 1 can be redrawn as Fig. 2. We can capture all M-channel TVFB's by characterizing the M-input M-output LTV filters

$$\boldsymbol{y}(n) = \sum_k \boldsymbol{e}_k(n)\boldsymbol{x}(n-k) \tag{1.2a}$$

$$\hat{\boldsymbol{x}}(n) = \sum_k \boldsymbol{r}_k(n-k)\boldsymbol{y}(n-k). \tag{1.2b}$$

In particular, if the $M\times M$ system in (1.2b) is the inverse system of (1.2a), then we have $\hat{\boldsymbol{x}}(n) = \boldsymbol{x}(n)$ for all n. This implies that $x(n) = \hat{x}(n)$ for all n. In this case, we say the TVFB achieves perfect reconstruction (PR). Therefore, in this paper, we will only discuss $M \times M$ LTV systems. The corresponding TVFB can be obtained by using the delay chain and advance chain as shown in Fig. 2.

2) The $M \times M$ system in (1.2a) is lossless [i.e., $\Sigma_n \; \boldsymbol{y}^\dagger(n)\boldsymbol{y}(n) = \Sigma_n \; \boldsymbol{x}^\dagger(n)\boldsymbol{x}(n)$] if and only if the coefficients satisfy

$$\sum_k \boldsymbol{e}_k^\dagger(n+k)\boldsymbol{e}_{k+l}(n+k) = \boldsymbol{I}\delta(l) \tag{1.3}$$

for all n. Furthermore, it is shown in [1] that if the coefficients $\boldsymbol{r}_k(n) = \boldsymbol{e}_{-k}^\dagger(n)$, then the system in (1.2b) is the inverse of the lossless system in (1.2a). Referring to Fig. 1, since $\Sigma_n|x(n)|^2 = \Sigma_n \; \boldsymbol{x}^\dagger(n)\boldsymbol{x}(n)$, the analysis bank is lossless if and only if the LTV system with coefficients $\boldsymbol{e}_k(n)$ is lossless. Similarly, the synthesis bank is lossless only if the LTV system with coefficients $\boldsymbol{r}_k(n)$ is lossless. For a FIR system of order N, we obtain $\boldsymbol{e}_0(n)\boldsymbol{e}_N(n) = \boldsymbol{0}$ from (1.3). We will see that this property helps in the factorization of lossless LTV systems.

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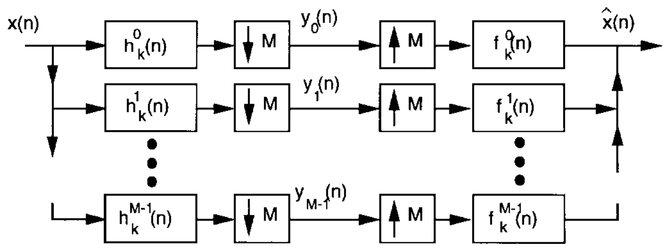

Fig. 1. M-channel maximally decimated time-varying filter bank.

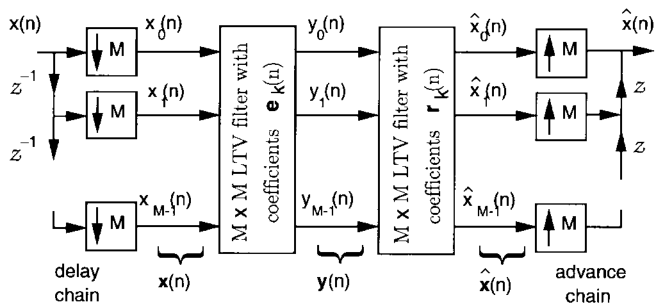

Fig. 2. Polyphase representation of time-varying filter bank.

3) All lossless LTV systems are invertible ([1, Th. 5.2]). This implies that the class of lossless analysis banks can always be inverted. Hence, PR is always possible for this class. However the inverses for lossless LTV systems might *not* be lossless! Therefore, for a PR TVFB, the losslessness of the analysis bank does not always imply the losslessness of the synthesis bank. However, it is shown that the losslessness of the inverse is equivalent to the invertibility of the inverse ([1, Th. 5.3]). According to the invertibility (or equivalently losslessness) of their inverses, we can classify the lossless LTV systems into two groups: i) Invertible inverse lossless (IIL) systems and ii) noninvertible inverse lossless (NIL) systems.

A. Related Works in Literature

Sodagar *et al.* introduced the most general form of time-varying filter banks [4]. In this system, the number of channels, the decimation ratios, and the filter coefficients are all time varying. One of the problems addressed in detail in [4] is the design of a time-varying post filter that reduces the reconstruction error created by the process of switching from one to another analysis/synthesis system. In our paper, however, we take a special case of this general framework, where the number of channels and decimation ratios are fixed. For this case, we address a number of theoretical issues that have not been addressed. One of these is the factorability of the general lossless filter bank (Section IV), and the other is the case of time-varying lapped orthogonal transforms (Section III). A detailed outline will be provided shortly.

Time-invariant lapped orthogonal transforms (LOT's) have been thoroughly studied in [5]. Special instances of the time-varying case have been considered by [6], where the authors propose the design of lossless time-varying filter banks (in particular LOT) by varying the coefficients in the cascaded lossless structure for a PU filter bank. A similar idea was considered earlier by [7], where the coefficients in a two channel lossless lattice were made time varying to obtain a time varying lossless system. In [8], the problem of switching between two LTI PU lattice structures was studied. In this paper, we will see that such techniques do not cover all the possible M-channel FIR lossless time-varying filter banks. This is because of the existence of unfactorable time-varying lossless systems. The thrust of our paper is entirely theoretical, the aim being to focus on the factorability and related theoretical properties. Summarizing, [5]–[8] address practical design issues by using specific instances of the lossless time-varying filter bank, whereas our paper addresses the general factorability issues and the minimality of the cascade of LTV lattice structures.

B. Main Results and Outline of the Paper

With the exception of Sections VI-B and VII, most of the results in this paper are derived for lossless LTV filters and filter banks.

1) In Section II, we will show how to capture all degree-one lossless LTV systems by two time-dependent memoryless unitary matrices. By using the complete parameterization, we will show that all degree-one lossless systems can be realized as a cascade of a TV memoryless unitary matrix followed by a lossless dyadic-based LTV structure. These lossless dyadic-based structures can be used as a building block to form higher degree lossless systems. A number of useful properties (e.g., preservation of losslessness under delay transformation, simple inversion rules, commutivity in cascade, etc.) will be discussed.
2) In Section III, the LOT [5] is extended to the LTV case. We will first show that the minimum number of delay elements required to implement a time-varying LOT (TVLOT) is nondecreasing with respect to time n. Moreover, it is an IIL TVLOT if and only if this minimum number is a constant. Then, we will show that all IIL TVLOT's can be factorized *uniquely* as a cascade of the lossless dyadic-based building blocks introduced in Section II followed by a unitary matrix. The factorization is *minimal* in terms of delay elements. For the NIL TVLOT, we will show factorable as well as unfactorable examples.
3) In Section IV, we will show how to construct higher degree NIL and IIL systems by using the dyadic-based building blocks. We will give several necessary conditions for the factorability of a general lossless LTV system and prove that there are unfactorable IIL systems. A sufficient condition for factorability, which leads to a order reduction procedure, will also be derived.
4) State-space representation of LTV systems will be discussed in Section V. We introduce the concept of *strong eternal reachability* (SER) and *strong eternal observability* (SEO). These concepts can be used to prove that the cascade implementation of factorable IIL systems is *minimal* in terms of delay elements as well as the number of building blocks.

5) In Section VI, we will show that the LTV normalized IIR lattice structure introduced in [9] is bounded input bounded output (BIBO) stable if the TV lattice coefficients $|\alpha_k(n)| \leq \gamma < 1$. For the more efficient two-multiplier IIR LTV lattice, we show that the structure cannot be lossless unless the lattice coefficients have constant magnitude.
6) We will extend the lossless dyadic-based LTV systems to the nonlossless case in Section VII. In the LTI case, these lossless systems reduce to the useful degree-one biorthogonal LTI building blocks introduced in [10] and [11]. Unlike the LTI case, we are unable to prove that the LTV biorthogonal dyadic-based systems are the most general biorthogonal building blocks for the LTV case. However, these LTV nonlossless dyadic-based systems can be used to construct FIR LTV systems with invertible FIR inverse.

All notations and acronyms are the same as described in [1, Sec. 1].

II. The Most General Degree-One Lossless LTV System

Consider the following $M \times M$ first-order system:

$$\boldsymbol{y}(n) = \boldsymbol{e}_0(n)\boldsymbol{x}(n) + \boldsymbol{e}_1(n)\boldsymbol{x}(n-1) \tag{2.1}$$

where $\boldsymbol{e}_k(n)$ are $M \times M$ matrices. Then, we can prove the following theorem:

Theorem 2.1—Complete Characterization of Degree-One Lossless System: The first-order LTV system defined in (2.1) is a degree-one lossless LTV system if and only if for all n there exist unitary matrices $\boldsymbol{U}(n) = [\boldsymbol{u}_0(n)\boldsymbol{u}_1(n)\cdots \boldsymbol{u}_{M-1}(n)]$ and $\boldsymbol{V}(n) = [\boldsymbol{v}_0(n)\boldsymbol{v}_1(n)\cdots\boldsymbol{v}_{M-1}(n)]$ such that

$$\begin{aligned}\boldsymbol{e}_0(n) &= \boldsymbol{U}(n)\begin{bmatrix}0 & \mathbf{0}\\ \mathbf{0} & \boldsymbol{I}_{M-1}\end{bmatrix}\boldsymbol{V}^{\dagger}(n) \quad \text{and}\\ \boldsymbol{e}_1(n) &= \boldsymbol{u}_0(n)\boldsymbol{v}_0^{\dagger}(n-1).\end{aligned} \tag{2.2}$$

■

Proof: The first-order system in (2.1) has degree $= 1$ if and only if the rank of $\boldsymbol{e}_1(n)$ is one for all n. Therefore, we can express $\boldsymbol{e}_1(n) = \boldsymbol{w}_0(n)\boldsymbol{v}_0^{\dagger}(n-1)$, where $\boldsymbol{w}_0(n)$ and $\boldsymbol{v}_0(n-1)$ are nonzero vectors. Without lost of generality, we can assume $\boldsymbol{w}_0^{\dagger}(n)\boldsymbol{w}_0(n) = 1$ for all n. Applying the necessary and sufficient condition for losslessness in (1.3) to (2.1), we obtain

$$\boldsymbol{e}_0^{\dagger}(n)\boldsymbol{e}_0(n) + \boldsymbol{e}_1^{\dagger}(n+1)\boldsymbol{e}_1(n+1) = \boldsymbol{I} \tag{2.3a}$$

$$\boldsymbol{e}_0^{\dagger}(n)\boldsymbol{e}_1(n) = \mathbf{0}. \tag{2.3b}$$

Substituting $\boldsymbol{e}_1(n) = \boldsymbol{w}_0(n)\boldsymbol{v}_0^{\dagger}(n-1)$ into the above equation, we get

$$\boldsymbol{e}_0^{\dagger}(n)\boldsymbol{e}_0(n) = \boldsymbol{I} - \boldsymbol{v}_0(n)\boldsymbol{v}_0^{\dagger}(n) \tag{2.4a}$$

$$\boldsymbol{e}_0^{\dagger}(n)\boldsymbol{w}_0(n) = \mathbf{0} \tag{2.4b}$$

where the fact that $\boldsymbol{v}_0(n-1) \neq \mathbf{0}$ has been applied to obtain (2.4b). Let $\boldsymbol{v}_1(n), \cdots, \boldsymbol{v}_{M-1}(n)$ be unit norm vectors perpendicular to $\boldsymbol{v}_0(n)$, i.e., $\boldsymbol{v}_i^{\dagger}(n)\boldsymbol{v}_0(n) = 0$ for $i \neq 0$. We have $[\boldsymbol{I} - \boldsymbol{v}_0(n)\boldsymbol{v}_0^{\dagger}(n)]\boldsymbol{v}_i(n) = \lambda_i\boldsymbol{v}_i(n)$, where the eigenvalues $\lambda_0 = 1 - \boldsymbol{v}_0^{\dagger}(n)\boldsymbol{v}_0(n)$ and $\lambda_i = 1$ for $i \neq 0$. Therefore, the matrix $[\boldsymbol{I} - \boldsymbol{v}_0(n)\boldsymbol{v}_0^{\dagger}(n)]$ is nonsingular unless $\boldsymbol{v}_0^{\dagger}(n)\boldsymbol{v}_0(n) = 1$. However, we know from (2.4b) that $\boldsymbol{e}_0^{\dagger}(n)\boldsymbol{e}_0(n)$ is singular. Thus, it is necessary that $\boldsymbol{v}_0^{\dagger}(n)\boldsymbol{v}_0(n) = 1$ and $\lambda_0 = 0$. Applying the singular value decomposition to the matrix $[\boldsymbol{I} - \boldsymbol{v}_0(n)\boldsymbol{v}_0^{\dagger}(n)]$, we conclude that there is a unitary matrix $\boldsymbol{U}'(n)$ such that

$$\boldsymbol{e}_0(n) = \boldsymbol{U}'(n)\begin{bmatrix}0 & \mathbf{0}\\ \mathbf{0} & \boldsymbol{I}_{M-1}\end{bmatrix}\boldsymbol{V}^{\dagger}(n). \tag{2.5}$$

Substituting (2.5) into (2.4b), we have

$$\boldsymbol{U}'^{\dagger}(n)\boldsymbol{w}_0(n) = [\boldsymbol{u}_0'(n) \quad \boldsymbol{u}_1'(n) \quad \cdots \quad \boldsymbol{u}_{M-1}'(n)]^{\dagger}\boldsymbol{w}_0(n) = \begin{bmatrix}\times\\ 0\\ \vdots\\ 0\\ 0\end{bmatrix} \tag{2.6}$$

where "$\times$" indicates a don't-care term. Equation (2.6) implies that $\boldsymbol{w}_0(n)$ are orthogonal to $\boldsymbol{u}_1'(n), \cdots, \boldsymbol{u}_{M-1}'(n)$. Therefore, $\boldsymbol{w}_0(n) = e^{j\theta(n)}\boldsymbol{u}_0'(n)$ for some real $\theta(n)$. By letting $\boldsymbol{u}_0(n) = e^{j\theta(n)}\boldsymbol{u}_0'(n)$ and $\boldsymbol{u}_i(n) = \boldsymbol{u}_i'(n)$ for $i \neq 0$, we have proved the theorem. ■

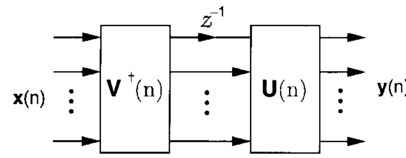

Fig. 3. Most general degree-one lossless LTV system. Here, $\boldsymbol{U}(n)$ and $\boldsymbol{V}(n)$ are unitary matrices.

A. Implementation Using Planar Rotations and Degree of Freedom

Theorem 2.1 tells us how to characterize all the degree-one lossless LTV systems. We can implement (2.2) by using the structure shown in Fig. 3. All lossless degree-one LTV systems can be parameterized by the two time-dependent unitary matrices, $\boldsymbol{U}(n)$ and $\boldsymbol{V}(n)$. For the real coefficient case, the unitary matrices $\boldsymbol{V}(n)$ and $\boldsymbol{U}(n)$ are real and can be implemented by using planar rotations [3], [12]. If the redundant planar rotations of $\boldsymbol{U}(n)$ are moved into $\boldsymbol{V}^{\dagger}(n)$, we can obtain the implementation shown in Fig. 4. Counting the number of rotations, we know that a degree-one lossless LTV system has only $(M-1)(M/2+1)$ degrees of freedom, instead of $2M^2$, which is the number of elements in the coefficients $\boldsymbol{e}_0(n)$ and $\boldsymbol{e}_1(n)$. The implementation based on planar rotations is *minimal* in terms of free parameters, and it remains lossless even when we change the angles in the rotations.

Remark: In the LTI case, it was shown in [12] that the general 2×2 LTI PU matrices can be implemented by using the normalized and denormalized FIR lattice structure shown, respectively, in [12, Figs. 6.4-1 and 6.4-2]. If we make the free parameters (θ_m for the normalized lattice and α_m for

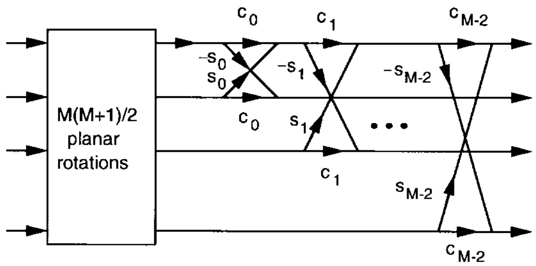

Fig. 4. Implementation of degree-one lossless real coefficient lossless LTV system based on planar rotation. $c_m = \cos(\theta_m(n))$ and $s_m = \sin(\theta_m(n))$.

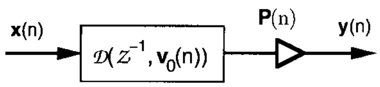

Fig. 5. Implementation of degree-one lossless LTV system using dyadic-based structure $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}_0(n))$, where $\boldsymbol{v}^\dagger(n)\boldsymbol{v}(n) = 1$, and $\boldsymbol{P}(n)$ is unitary. $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n))$ is shown in Fig. 6.

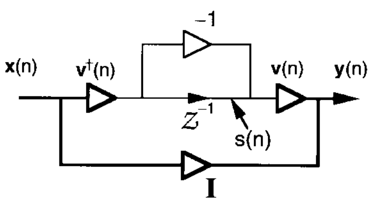

Fig. 6. Dyadic-based structure $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n))$.

the denormalized lattice) time varying, then one can show that the LTV normalized lattice structure in [12, Fig. 6.4-1] will remain lossless while the denormalized lattice structure in [12, Fig. 6.4-2] will no longer be lossless unless $|\alpha_m(n)|$ are independent of n. In both cases, PR can be achieved by inverting the lattice section by section, as shown in [7].

B. Dyadic-Based Building Blocks

The implementation based on planar rotations gives a minimal parameterization of degree-one lossless LTV system. However, the implementation is not efficient in the sense that it requires more multipliers than necessary. In order to obtain a more efficient implementation, we simplify the coefficients $\boldsymbol{e}_0(n)$ and $\boldsymbol{e}_1(n)$ as

$$\begin{aligned}\boldsymbol{e}_0(n) &= \boldsymbol{P}(n)[\boldsymbol{I} - \boldsymbol{v}_0(n)\boldsymbol{v}_0^\dagger(n)]\\ \boldsymbol{e}_1(n) &= \boldsymbol{P}(n)[\boldsymbol{v}_0(n)\boldsymbol{v}_0^\dagger(n-1)]\end{aligned} \tag{2.7}$$

where $\boldsymbol{P}(n) = \boldsymbol{U}(n)\boldsymbol{V}^\dagger(n)$. Since $\boldsymbol{U}(n)$ can be arbitrary unitary matrix, $\boldsymbol{P}(n)$ is an arbitrary unitary matrix unrelated to $\boldsymbol{V}(n)$. Using (2.7), we obtain the implementation as in Fig 5, where the system $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n))$ is shown in Fig. 6. We will call the structure in Fig. 6 a *dyadic-based* structure. Therefore, all degree-one lossless LTV systems can be realized as a cascade of a dyadic-based LTV system $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n))$ with $\boldsymbol{v}^\dagger(n)\boldsymbol{v}(n) = 1$ followed by a time-dependent unitary matrix. The dyadic-based structure has only $2M$ multipliers, which are fewer than $4M - 2$, which is the number of multiplications required for the implementation based on planar rotations.

Remarks:

1) Notice that we can also express the coefficients as

$$\begin{aligned}\boldsymbol{e}_0(n) &= [\boldsymbol{I} - \boldsymbol{u}_0(n)\boldsymbol{u}_0^\dagger(n)]\boldsymbol{U}(n)\boldsymbol{V}^\dagger(n)\\ \boldsymbol{e}_1(n) &= [\boldsymbol{u}_0(n)\boldsymbol{u}_0^\dagger(n-1)]\boldsymbol{U}(n-1)\boldsymbol{V}^\dagger(n-1).\end{aligned} \tag{2.8}$$

By using the above equation, we have another implementation the degree-one lossless system as a cascade of a TV unitary matrix followed by the dyadic-based lossless system $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{u}_0(n))$.

2) In the LTI case, the lossless dyadic-based structure $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n))$ in Fig. 6 reduces to the degree-one building block given in [12, Fig. 14.5-1].

C. Properties of Dyadic-Based Structures $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n))$

The dyadic-based structure $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n))$ in Fig. 6 has several nice properties, and it can be used as a basic building block to factorize some higher degree lossless LTV systems. The system equation for $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n))$ can be expressed as

$$\boldsymbol{y}(n) = [\boldsymbol{I} - \boldsymbol{v}(n)\boldsymbol{v}^\dagger(n)]\boldsymbol{x}(n) + \boldsymbol{v}(n)\boldsymbol{v}^\dagger(n-1)\boldsymbol{x}(n-1). \tag{2.9}$$

Here, we list some of its properties:

1) *Identity System:* If $\boldsymbol{v}(n) = \boldsymbol{0}$, the dyadic-based structure $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{0})$ reduces to the identity system.

2) *Losslessness:* In general, it is not easy to satisfy the condition for losslessness in (1.3). However, for the dyadic-based structure in Fig. 6, if $\boldsymbol{v}^\dagger(n)\boldsymbol{v}(n) = 1$ for all n, then one can show that $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n))$ is lossless. In the presence of quantization, if the vector $\boldsymbol{v}(n)$ is quantized in such a way that the quantized vector $\boldsymbol{v}_q(n)$ satisfies $\boldsymbol{v}_q^\dagger(n)\boldsymbol{v}_q(n) = 1$, then $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}_q(n))$ remains lossless. This implies the implementation in Fig. 6 is structurally lossless.

3) *Scalar Dyadic-Based Systems:* In the single input single output (scalar) case, the degree-one lossless system degenerates to a delay followed by a unit magnitude multiplier, i.e., $y(n) = c(n)x(n-1)$ for some $|c(n)| = 1$.

4) *Simple Inverse System:* It is shown in [1] that the coefficients of the inverse of a lossless system can be obtained as the mirror image and transpose conjugate of the original system. For a lossless dyadic-based system, the inverse is even simpler. It can be verified that if $\boldsymbol{v}^\dagger(n)\boldsymbol{v}(n) = 1$, the inverse of the degree-one lossless building block $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n))$ in Fig. 6 can be obtained by simply replacing the delay with an advance operator as shown in Fig. 7. The inverse system $\mathcal{D}(\mathcal{Z}, \boldsymbol{v}(n))$ can be expressed as

$$\hat{\boldsymbol{x}}(n) = [\boldsymbol{I} - \boldsymbol{v}(n)\boldsymbol{v}^\dagger(n)]\boldsymbol{y}(n) + \boldsymbol{v}(n)\boldsymbol{v}^\dagger(n+1)\boldsymbol{y}(n+1). \tag{2.10}$$

$\mathcal{D}(\mathcal{Z}, \boldsymbol{v}(n))$ is anticausal, FIR, and lossless [can be verified by directly substituting the coefficients into (1.3)]. Therefore, $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n)$ is an IIL system if $\boldsymbol{v}^\dagger(n)\boldsymbol{v}(n) = 1$.

5) *Commutivity:* If we have a cascade of two dyadic-based structures, say $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}_0(n))$ and $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}_1(n))$,

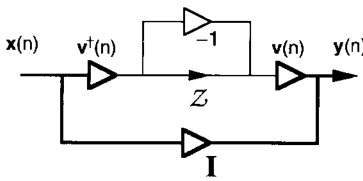

Fig. 7. Inverse system for the lossless dyadic-based system $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n))$ with $\boldsymbol{v}^{\dagger}(n)\boldsymbol{v}(n) = 1$.

where $\boldsymbol{v}_0(n)$ and $\boldsymbol{v}_1(n)$ are unit norm vectors, then we can show that the two building blocks commute with each other if and only if $\boldsymbol{v}_0(n) = \boldsymbol{v}_1(n)$ or $\boldsymbol{v}_0^{\dagger}(n)\boldsymbol{v}_1(n) = 0$ (i.e., perpendicular) for all n. If $\boldsymbol{v}_0^{\dagger}(n)\boldsymbol{v}_1(n) = 0$, the building blocks $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}_0(n))$ and $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}_1(n))$ are said to be *perpendicular*. We will see in the next section that precisely this situation arises in the factorization of the TVLOT. The cascade of k perpendicular building blocks can be expressed as $\Pi_{i=0}^{k-1}\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}_i(n))$. The ordering of these sections does not matter. The cascade system $\prod_{i=0}^{k-1}\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}_i(n))$ has order one and can be expressed as

$$\boldsymbol{y}(n) = [\boldsymbol{I} - \boldsymbol{V}_k(n)\boldsymbol{V}_k^{\dagger}(n)]\boldsymbol{x}(n) + \boldsymbol{V}_k(n)\boldsymbol{V}_k^{\dagger}(n-1)\boldsymbol{x}(n-1) \tag{2.11}$$

where the $M \times k$ matrix $\boldsymbol{V}_k(n) = [\boldsymbol{v}_0(n) \cdots \boldsymbol{v}_{k-1}(n)]$. For $M \times M$ systems, if we cascade M such perpendicular lossless building blocks $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}_i(n))$, the resulting system reduces to $\boldsymbol{y}(n) = \boldsymbol{P}(n)\boldsymbol{x}(n-1)$ for some unitary $\boldsymbol{P}(n)$.

6) *Delay Transformation:* It is shown in [1] that if the delay $\mathcal{Z}^{-1}$ in an implementation of a lossless system is replaced by $\mathcal{Z}^{-L}$, the losslessness will usually be destroyed. However, the lossless dyadic-based structure $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n))$ preserves the lossless property under such delay transformation. That is, if the delay in Fig. 6 is replaced with $\mathcal{Z}^{-L}$ for arbitrary integer L (possibly negative), the new system $\mathcal{D}(\mathcal{Z}^{-L}, \boldsymbol{v}(n))$ remains lossless. In this case, the system equation is

$$\boldsymbol{y}(n) = [\boldsymbol{I} - \boldsymbol{v}(n)\boldsymbol{v}^{\dagger}(n)]\boldsymbol{x}(n) + \boldsymbol{v}(n)\boldsymbol{v}^{\dagger}(n-L)\boldsymbol{x}(n-L). \tag{2.12}$$

Moreover, we can show that $\mathcal{D}(\mathcal{Z}^{-L}, \boldsymbol{v}(n)) = \Pi_{Ltimes}\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n))$.

Example 2.1—Lossless and Nonlossless System Obtained from $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n))$:

1) If the vector $\boldsymbol{v}(n)$ in $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n))$ is switched from a zero vector to a unit norm vector at $n = 0$, we know from [1, Example 6.3] that the system is lossless. In this case, the system equation is given in [1, (6.6)], which we reproduce here for convenience:

$$\boldsymbol{y}(n) = \begin{cases} \boldsymbol{x}(n), & \text{for } n < 0 \\ [\boldsymbol{I} - \boldsymbol{v}(0)\boldsymbol{v}^{\dagger}(0)]\boldsymbol{x}(0), & \text{for } n = 0 \\ [\boldsymbol{I} - \boldsymbol{v}(n)\boldsymbol{v}^{\dagger}(n)]\boldsymbol{x}(n) + \boldsymbol{v}(n)\boldsymbol{v}^{\dagger}(n-1)\boldsymbol{x}(n-1) & \\ & \text{for } n > 0. \end{cases} \tag{2.13}$$

The above system is lossless because the coefficients satisfy (1.3), and its inverse is not lossless (see [1, Example 6.3]). Hence, it is a NIL system. The fact that the inverse is not invertible also follows from Theorem 3.1, which will be proved in the next section.

2) Consider another example: If we switch the vector $\boldsymbol{v}(n)$ in $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n))$ from a unit norm vector to a zero vector at $n = 0$, the resulting system is

$$\boldsymbol{y}(n) = \begin{cases} [\boldsymbol{I} - \boldsymbol{v}(n)\boldsymbol{v}^{\dagger}(n)]\boldsymbol{x}(n) & \\ \quad + \boldsymbol{v}(n)\boldsymbol{v}^{\dagger}(n-1)\boldsymbol{x}(n-1), & \text{for } n < 0 \\ \boldsymbol{x}(n), & \text{for } n \geq 0. \end{cases} \tag{2.14}$$

Notice that $\boldsymbol{x}(-1)$ appears only in the expression of $\boldsymbol{y}(-1)$, and it is premultiplied by the singular matrix $[\boldsymbol{I} - \boldsymbol{v}(-1)\boldsymbol{v}^{\dagger}(-1)]$. Therefore, the system in (2.14) is not invertible because $\boldsymbol{x}(-1)$ can never be recovered from $\boldsymbol{y}(n)$. Hence, it cannot be lossless (from [1, Th. 5.2], which says that all lossless systems are invertible).

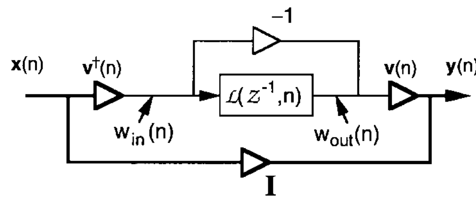

Fig. 8. Lossless system obtained from the dyadic-based structure. Here, $\mathcal{L}(\mathcal{Z}^{-1}, n)$ is an arbitrary lossless scalar system.

D. IIR Lossless LTV Systems Obtained from Dyadic-Based Structures

More generally, if the delay $\mathcal{Z}^{-1}$ in Fig. 6 is replaced with a BIBO stable scalar lossless LTV system $\mathcal{L}(\mathcal{Z}^{-1}, n)$ (possibly IIR) as shown in Fig. 8, does the system $\mathcal{D}(\mathcal{L}(\mathcal{Z}^{-1}, n), \boldsymbol{v}(n))$ remain lossless? The answer is yes. The BIBO stability of $\mathcal{D}(\mathcal{L}(\mathcal{Z}^{-1}, n), \boldsymbol{v}(n))$ can be shown as follows: Since $\boldsymbol{v}(n)$ has unit norm, the scalar quantity $w_{\text{in}}(n)$ is bounded for bounded input $\boldsymbol{x}(n)$. Therefore, the scalar $w_{\text{out}}(n)$ is also bounded as $\mathcal{L}(\mathcal{Z}^{-1}, n)$ is a BIBO stable scalar system. Therefore, the output vector $\boldsymbol{y}(n)$ is bounded. To show the losslessness, we write the output as

$$\boldsymbol{y}(n) = [\boldsymbol{I} - \boldsymbol{v}(n)\boldsymbol{v}^{\dagger}(n)]\boldsymbol{x}(n) + w_{\text{out}}(n)\boldsymbol{v}(n) \tag{2.15}$$

where $\Sigma_n|w_{\text{out}}(n)|^2 = \Sigma_n|w_{\text{in}}(n)|^2 = \Sigma_n|\boldsymbol{v}^{\dagger}(n)\boldsymbol{x}(n)|^2$ because the scalar system $\mathcal{L}(\mathcal{Z}^{-1}, n)$ is lossless. Computing the energy of $\boldsymbol{y}(n)$ from (2.15), we get

$$\sum_n \boldsymbol{y}^{\dagger}(n)\boldsymbol{y}(n) = \sum_n (\boldsymbol{x}^{\dagger}(n)\boldsymbol{x}(n) - |\boldsymbol{v}^{\dagger}(n)\boldsymbol{x}(n)|^2 + |w_{\text{out}}(n)|^2). \tag{2.16}$$

Using the fact that $\Sigma_n|w_{\text{out}}(n)|^2 = \Sigma_n|\boldsymbol{v}^{\dagger}(n)\boldsymbol{x}(n)|^2$, one can immediately show that $\mathcal{D}(\mathcal{L}(\mathcal{Z}^{-1}, n), \boldsymbol{v}(n))$ is lossless. Furthermore, it can be verified by direct substitution that the inverse of $\mathcal{D}(\mathcal{L}(\mathcal{Z}^{-1}, n), \boldsymbol{v}(n))$ is given by $\mathcal{D}(\mathcal{L}^{-1}(\mathcal{Z}^{-1}, n), \boldsymbol{v}(n))$, where $\mathcal{L}^{-1}(\mathcal{Z}^{-1}, n)$ is the inverse of $\mathcal{L}(\mathcal{Z}^{-1}, n)$. The existence of inverse $\mathcal{L}^{-1}(\mathcal{Z}^{-1}, n)$ is guaranteed by the losslessness of $\mathcal{L}(\mathcal{Z}^{-1}, n)$.

In particular, the scalar allpass function $A(z)$ is a scalar lossless system. If $\mathcal{L}(\mathcal{Z}^{-1}, n)$ is taken as the stable Nth-order allpass function $A_N(z)$, then we can get a subclass of MIMO IIR lossless LTV systems $\mathcal{D}(A_N(z), \boldsymbol{v}(n))$ with degree N. However, in this case, the inverse is anticausal IIR. For the details of implementing IIR anticausal systems when the input is infinite, see [10].

Remark: More generally, we can obtain a class of invertible nonlossless LTV system by replacing $\mathcal{Z}^{-1}$ in Fig. 6 with an invertible scalar system $\mathcal{T}(\mathcal{Z}^{-1}, n)$ (not necessary lossless). In this case, the BIBO stability of $\mathcal{D}(\mathcal{T}(\mathcal{Z}^{-1}, n), \boldsymbol{v}(n))$ is guaranteed by that of $\mathcal{T}(\mathcal{Z}^{-1}, n)$. Moreover, the inverse of $\mathcal{D}(\mathcal{T}(\mathcal{Z}^{-1}, n), \boldsymbol{v}(n))$ can still be obtained simply as $\mathcal{D}(\mathcal{T}^{-1}(\mathcal{Z}^{-1}, n), \boldsymbol{v}(n))$.

III. Time-Varying Lapped Orthogonal Transform (TVLOT)

LOT's have been shown to be very useful in subband coding of image and video signals [5]. They provide satisfactory coding gain and good perceptual quality in these applications with low complexity. In this section, we will generalize the theory of the conventional LOT system to the time-varying case. Consider the $M \times M$ first-order system

$$\boldsymbol{y}(n) = \boldsymbol{e}_0(n)\boldsymbol{x}(n) + \boldsymbol{e}_1(n)\boldsymbol{x}(n-1). \tag{3.1}$$

If the above system is lossless, then it is called a TVLOT. From [1, Th. 5.3], we know that a lossless system is always invertible. Hence, a TVLOT is always invertible, and its inverse has also order one. However, we also know from [1, Th. 5.3] that the inverse system may not be lossless. That means the inverse of a TVLOT may not be a TVLOT system! This is a very different situation from the LTI LOT case. If the inverse of a TVLOT is also lossless, then it is called an *invertible inverse lossless* (IIL) TVLOT. In this case, its inverse is also a TVLOT. If the inverse is not invertible, then it is called a *noninvertible inverse lossless* (NIL) TVLOT. Note that a dyadic-based structure with unit norm vector in Fig. 6 is an IIL TVLOT. The existence of NIL TVLOT is shown by Example 2.1.

In (3.1), since $\boldsymbol{e}_1(n)$ is time-varying, its rank could also be time-varying. Therefore, the degree of a TVLOT is not a constant. We will call the rank of $\boldsymbol{e}_1(n)$ the *instantaneous degree* since this is the minimum number of delays required at time n. In the following, we will first show that for a TVLOT (either IIL or NIL), the rank of $\boldsymbol{e}_1(n)$ cannot decrease with n. Moreover, the rank of $\boldsymbol{e}_1(n)$ is time-invariant if and only if it is an IIL TVLOT. In the second part of this section, we will show that an IIL TVLOT system can *always* be realized as a *unique* cascade of *perpendicular* degree-one building blocks $\mathcal{D}(z^{-1}, \boldsymbol{v}(n))$ introduced in the Section II. We will also provide an example to show there exist unfactorable NIL TVLOT's.

A. The Instantaneous Degree of TVLOT

Theorem 3.1: Let $\rho(n)$ be the instantaneous degree of a $M \times M$ TVLOT. Then, $\rho(n)$ cannot be decreasing. ■

Proof: If the system is lossless in (3.1), then from the lossless condition in (1.3), we have

$$\boldsymbol{e}_0^\dagger(n)\boldsymbol{e}_1(n) = \boldsymbol{0} \tag{3.2a}$$

$$\boldsymbol{e}_0^\dagger(n)\boldsymbol{e}_0(n) + \boldsymbol{e}_1^\dagger(n+1)\boldsymbol{e}_1(n+1) = \boldsymbol{I} \tag{3.2b}$$

for all n. By definition, $\rho(n)$ = rank of $\boldsymbol{e}_1(n)$. From (3.2a), we see that rank of $\boldsymbol{e}_0(n)$ is at most $M - \rho(n)$. Using the facts that the rank of $\boldsymbol{e}_0^\dagger(n)\boldsymbol{e}_0(n) =$ the rank of $\boldsymbol{e}_0(n)$ and the rank of $(\boldsymbol{I} - \boldsymbol{e}_1^\dagger(n+1)\boldsymbol{e}_1(n+1))$ is at least $M - \rho(n+1)$, we conclude that

$$\rho(n) \leq M - \text{rank of} \boldsymbol{e}_0(n) \leq \rho(n+1). \tag{3.3}$$

■

Applying the above theorem to Example 2.1, we conclude that the lossless system in (2.13) is a NIL system since its instantaneous degree increases from zero to one at time $n = 0$. This result is consistent with that obtained from [1, Th. 5.3].

Theorem 3.2: A TVLOT is invertible inverse lossless (IIL) TVLOT if and only if its instantaneous degree $\rho(n)$ is time-invariant. ■

Proof:

1) "If" part: See Section III-B for a constructive proof.
2) "Only if" part: Assume that the system given in (3.1) is an IIL TVLOT. From [1, Sec. V], we know that the unique inverse is given by $\mathcal{G}$: $\hat{\boldsymbol{x}}(n) = \boldsymbol{e}_0^\dagger(n)\boldsymbol{y}(n) + \boldsymbol{e}_1^\dagger(n+1)\boldsymbol{y}(n+1)$. Consider the system $\mathcal{W}$: $\boldsymbol{w}(n) = \boldsymbol{e}_1^\dagger(n)\boldsymbol{y}(n) + \boldsymbol{e}_0^\dagger(n)\boldsymbol{y}(n-1)$. Clearly, the system $\mathcal{W}$ is lossless because the system $\mathcal{G}$ is lossless. Therefore, we can apply Theorem 3.1 to the system $\mathcal{W}$ to obtain the following result:

$$\text{rank of } \boldsymbol{e}_0(n+1) \leq M - \text{rank of } \boldsymbol{e}_1(n) \leq \text{rank of} \boldsymbol{e}_0(n) \tag{3.4}$$

where $\rho(n) =$ rank of $\boldsymbol{e}_1(n)$. Combining (3.3) and (3.4), we have proved that $\rho(n)$ is a constant. ■

Theorem 3.1 gives a simple test of the nonlosslessness of first-order LTV systems. If the instantaneous degree of a first-order LTV system decreases for some n, then it is guaranteed to be nonlossless. Theorem 3.2 can be used to verify the losslessness of the inverse of a TVLOT system.

B. Factorization of TVLOT

In this section, we will show that all IIL TVLOT's (i.e., TVLOT with constant degree $\rho(n) = \rho$) can be factorized *uniquely* as a cascade of ρ perpendicular, degree-one building blocks $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n))$ (see Section II-A). Since there is only one delay in each building block, the factorization is minimal in terms of delay. Moreover, the building blocks $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n))$ are invertible, and their inverses have the form $\mathcal{D}(\mathcal{Z}, \boldsymbol{v}(n))$. Therefore, the unique inverse of IIL TVLOT is also factorable. Similar to the case of degree-one lossless system, the coefficients of an TVLOT system satisfy the following theorem.

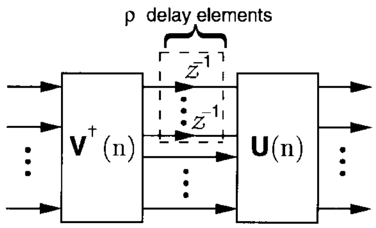

Fig. 9. Complete characterization of IIL TVLOT of degree ρ. Here, $U(n)$ and $V(n)$ are unitary matrices.

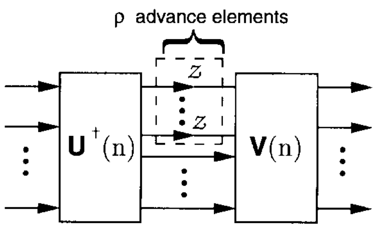

Fig. 10. Inverse system of the IIL TVLOT in Fig. 9.

Theorem 3.3—Complete Characterization of IIL TVLOT's: The system in (3.1) is a TVLOT with a constant degree ρ if and only if the coefficients can be expressed as

$$\begin{aligned} e_0(n) &= U(n)\begin{bmatrix} 0 & 0 \\ 0 & I_{M-\rho} \end{bmatrix} V^{\dagger}(n), \quad \text{and} \\ e_1(n) &= U_\rho(n)V_\rho^{\dagger}(n-1) \end{aligned} \tag{3.5}$$

where $U(n) = [u_0(n)u_1(n)\cdots u_{M-1}(n)]$ and $V(n) = [v_0(n)v_1(n)\cdots v_{M-1}(n)]$ are arbitrary unitary matrices, and $U_\rho(n)$ and $V_\rho(n)$ are submatrices defined, respectively, as

$$\begin{aligned} U_\rho(n) &= [u_0(n)u_1(n)\cdots u_{\rho-1}(n)] \\ V_\rho(n) &= [v_0(n)v_1(n)\cdots v_{\rho-1}(n)]. \end{aligned} \tag{3.6}$$

■

The above theorem can be proved by using a procedure similar to the proof of Theorem 2.1. Theorem 3.3 tells us how all IIL TVLOT's can be captured by two unitary matrices. For all IIL TVLOT's, the linear span of columns of $e_0(n)$ is in the orthogonal complement of the columns of $e_1(n)$; the linear span of rows of $e_0(n)$ is in the orthogonal complement of the rows of $e_1(n+1)$.

C. Implementations and Degree of Freedom

From (3.5) and (3.6), we have the implementation shown in Fig. 9. The inverse for Fig. 9 is given by Fig. 10. Since the system in Fig. 10 is a cascade of lossless systems (two unitary matrices and a diagonal system with only advanced elements), the inverse system is also lossless. This is consistent with the fact that the inverse of an IIL system is also lossless [1, Th. 5.3]. In the real coefficient case, the unitary matrices $U(n)$ and $V^{\dagger}(n)$ are real and can be implemented by using TV planar rotations, and we can obtain an implementation similar to Fig. 4. Counting the free parameters, we conclude that for a degree ρ IIL TVLOT, the degree of freedom is $0.5M(M-1)+M\rho-0.5\rho(\rho+1)$. The implementation based on planar rotations gives a *minimal* characterization of IIL TVLOT.

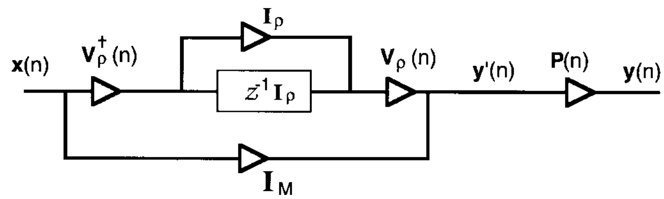

Fig. 11. Another characterization of IIL TVLOT of degree ρ. The matrix $V_\rho(n)$ is defined in (3.6), and $P(n)$ is unitary.

Remarks:

1) We see that for a TVLOT with a constant degree, the inverse shown in Fig. 3.2 is lossless. Therefore a TVLOT with a constant degree is an IIL system. The proof for part 1 of Theorem 3.2 is complete.
2) When $\rho = 0$, the IIL TVLOT reduces to the special case of LTV transform coding (i.e., a time-dependent unitary matrix); when $\rho = M$, it reduces to a LTV transform coding followed by a pure delay.

1) Complete Factorization of IIL TVLOT's: Similar to the degree-one lossless case, we can simplify the coefficients as

$$\begin{aligned} e_0(n) &= P(n)[I - V_\rho(n)V_\rho^{\dagger}(n)] \\ e_1(n) &= P(n)[V_\rho(n)V_\rho^{\dagger}(n-1)] \end{aligned} \tag{3.7}$$

where $P(n) = U(n)V^{\dagger}(n)$. Since $U(n)$ can be arbitrary unitary matrix, the unitary matrix $P(n)$ is arbitrary. Using the above equation, we arrive at the implementation shown in Fig. 11. Since $V_\rho(n)$ is a submatrix of a unitary matrix, we have $V_\rho^{\dagger}(n)V_\rho(n) = I_\rho$. This implies that the vectors $v_k(n)$ for $0 \le k \le \rho - 1$ are perpendicular to each other. Recall from Section II-A and (2.11) that the LTV system from $x(n)$ to $y'(n)$ in Fig. 11 is a cascade of ρ perpendicular lossless dyadic-based building blocks. Using this fact, we arrive at the factorization in Fig. 12. The ordering of the lossless dyadic-based systems $\mathcal{D}(\mathcal{Z}^{-1}, v_i(n))$ in Fig. 11 does not matter because the building blocks are perpendicular. From Fig. 12, it is clear that the inverse system can be obtained by inverting the building blocks and $P(n)$. Therefore, the inverse of an IIL TVLOT can be realized as a cascade of $\mathcal{D}(\mathcal{Z}, v_i(n))$ followed by $P^{\dagger}(n)$, as shown in Fig. 13. Summarizing all the results, we have proved the following theorem.

Theorem 3.4—Complete Factorization of IIL TVLOT: The first-order system in (3.1) is an IIL TVLOT with degree ρ if and only if it can be factorized in the perpendicular lossless dyadic-based building blocks as shown in Fig. 12. Moreover, the inverse is given by Fig. 13. ■.

Remark: We can also simplify the coefficients as in the form similar to (2.8). In this case, we can obtain another implementation of the IIL TVLOT as a cascade of $P(n)$ *followed by* the lossless dyadic-based building blocks $\mathcal{D}(\mathcal{Z}^{-1}, v_i(n))$.

Example 3.1—Unfactorable NIL TVLOT: Consider the first-order system

$$e_0(n) = \begin{cases} I, & \text{for } n < 0 \\ I - (1+\sqrt{1-\gamma^2})u(0)u^{\dagger}(0), & \text{for } n = 0 \\ I - u(n)u^{\dagger}(n), & \text{for } n > 0 \end{cases}$$

$$e_1(n) = \begin{cases} 0, & \text{for } n < 1 \\ \gamma u(1)u^{\dagger}(0), & \text{for } n = 1 \\ u(n)u^{\dagger}(n-1), & \text{for } n > 1 \end{cases} \tag{3.8}$$

Fig. 12. Complete factorization of the IIL TVLOT of degree ρ. Here, $\boldsymbol{v}_i^\dagger(n)\boldsymbol{v}_j(n) = \delta(i-j)$, and $\boldsymbol{P}(n)$ is unitary.

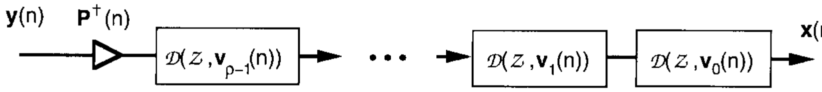

Fig. 13. Implementation of the inverse of IIL TVLOT in factorized form.

where $|\gamma| \leq 1$ and $\boldsymbol{u}(n)$ are unit norm vectors. It can be verified by direct substitution into (1.3) that the above first-order system is lossless; therefore, it is a TVLOT. It is clear that it is a NIL TVLOT since its instantaneous degree increases (Theorem 3.2). However, unless $|\gamma| = 1$, the NIL TVLOT in (3.8) cannot be factorized into the dyadic-based building block. If it could, $\boldsymbol{e}_0(n)$ should always be of the form $[\boldsymbol{I} - \boldsymbol{v}(n)\boldsymbol{v}^\dagger(n)]\boldsymbol{P}(n)$, where $\boldsymbol{v}(n)$ are either zero or unit norm vectors, and $\boldsymbol{P}(n)$ is a unitary matrix. This means that $\boldsymbol{e}_0(n)$ should be either a singular matrix or a unitary matrix. However, from (3.8), we see that $\boldsymbol{e}_0(0)$ is neither singular nor unitary for $|\gamma| \neq 1$.

Recall from Example 2.1 that the system in (2.13) is a NIL TVLOT. This NIL TVLOT is factorable because it is already in factorized form. Combining this result and the result in Example 3.1, we conclude that there are factorable and unfactorable NIL TVLOT's.

IV. Factorability of Higher Order Lossless Systems

In the previous section, we proved that all IIL TVLOT can be factorized into the degree-one building blocks. We also know that there are factorable and unfactorable NIL systems. However, we still don't know if all IIL systems are factorable. More generally, how to determine if a lossless system is factorable? In this section, we will give several necessary conditions for a factorable lossless system. These necessary conditions give simple tests for unfactorable systems. Using these tests, we are able to show some unfactorable IIL examples. Therefore, unlike TVLOT, an IIL system of order >1 could be unfactorable. Moreover, we will also give a sufficient condition for factorability of lossless LTV systems.

A. Building Higher Order Lossless Systems and Some Necessary Conditions for Factorability

The TVLOT's are first-order lossless LTV systems. One way to generate higher order lossless systems is to cascade N sections of the dyadic-based building blocks $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}_i(n))$ with $\boldsymbol{v}_i^\dagger(n)\boldsymbol{v}_i(n) = 1$. If none of the adjacent building blocks are perpendicular to each other (in the sense defined in Property 5 in Section II-A), then the result of the cascade is an Nth-order lossless system. If some of the adjacent building blocks are perpendicular, then the order can be smaller than N. In the extreme case of TVLOT, all the building blocks are perpendicular. The lossless systems constructed by this method have the same number of building blocks for all time n. Since the inverses of the building blocks are lossless, so is their cascade. Therefore, we conclude that the above construction will always gives IIL systems.

To construct examples of higher order NIL systems, recall from (2.13) of Example 2.1 that if the vector $\boldsymbol{v}(n)$ in a dyadic building block $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n))$ changes from a zero vector to a unit norm vector, then $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n))$ is a lossless system with nonlossless inverse, i.e., it is a NIL system. By cascading N sections of such $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}_i(n))$, where $\boldsymbol{v}_i(n)$ are allowed to switch from zero to unit norm vectors, we can get a NIL system of order N. In addition, recall from Example 2.1 that if the vector $\boldsymbol{v}(n)$ changes from a zero vector to a unit norm vector, the dyadic building block $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n))$ is no longer lossless. Therefore, we conclude that the number of building blocks in a factorable lossless system cannot be decreasing. Summarizing the results, we have the following theorem.

Theorem 4.1: If a lossless LTV system is factorable in terms of degree-one lossless building blocks, the number of building blocks cannot decrease with time. Moreover, the factorable lossless system is IIL if and only if the number of building blocks is a constant with respect to time. ■

The above theorem can be used to determine if a cascade of building blocks is NIL or IIL. However, it is not very useful for testing the factorability of a lossless system because it assumes that the system is given in factorized form. In the following, we will give some other necessary conditions that lead to simple tests.

B. Unfactorability of Nontrivial Scalar Lossless Systems

In the scalar case, we know from Section II that the degree-one building block $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n))$ reduces to a delay followed by a unit magnitude multiplier. If a scalar lossless system is factorable using these building blocks, then it should be the cascade of these trivial building blocks. Therefore, the output of the factorable scalar lossless system can always be written as

$$y(n) = a(n)x(n - k(n)) \tag{4.1}$$

where $k(n)$ is a nondecreasing (because of Theorem 4.1) positive integer, and $|a(n)| = 0$ or 1 (because it is a product of either zero or unit norm multipliers). Thus, we conclude that *all nontrivial* (with at least two non zero coefficients at the same time) *lossless scalar systems are unfactorable* in terms of degree-one building blocks. Therefore, the lossless scalar LTV system given in [1, Example 6.1] is an unfactorable IIL system.

To determine the exact relation between $a(n)$ and $k(n)$, assume that we start the system at $n = n_0$ with the initial conditions

$$a(n_0 - 1) = 1, \quad k(n_0 - 1) = 0, \quad x(n_0 - 1) = 0. \quad (4.2)$$

Then, it can be shown that the coefficient $|a(n)| = 1$ whenever n satisfies the condition $n - k(n) = \hat{n} - k(\hat{n}) + 1$, where $\hat{n}$ is the largest integer $<n$ such that $a(\hat{n}) = 1$.

C. Necessary Condition for Factorability

Consider the $M \times M$ causal lossless system given in (1.2a) with the coefficients $\boldsymbol{e}_k(n)$. Suppose that the system is FIR and that is there is an N such that $\boldsymbol{e}_k(n) = \boldsymbol{0}$ for $k > N$ for all n. Let $i(n)$ be the largest integers such that $\boldsymbol{e}_k(n) = \boldsymbol{0}$ for $k < i(n)$. Therefore, we have $\boldsymbol{e}_{i(n)}(n) \neq \boldsymbol{0}$. If the system is factorable, then it is a cascade of the dyadic-based building blocks $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}_k(n))$. Since the first coefficient of $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}_k(n))$ is of the form $[\boldsymbol{I} - \boldsymbol{v}_k(n)\boldsymbol{v}_k^\dagger(n)]$, the quantity $\boldsymbol{e}_{i(n)}(n)$ is a product of matrices $[\boldsymbol{I} - \boldsymbol{v}_k(n)\boldsymbol{v}_k^\dagger(n)]$ followed by a unitary matrix $\boldsymbol{P}(n)$, where $\boldsymbol{v}_k(n)$ is either a zero or unit norm vector. This means that $\boldsymbol{e}_{i(n)}(n)$ is singular unless $\boldsymbol{v}_k(n) = \boldsymbol{0}$ for all k, which implies that the system is a trivial system that contains only one nonzero coefficient. Therefore, we conclude that for a nontrivial factorable lossless system, the first nonzero coefficient is singular for each n. This gives a quick test for unfactorable lossless systems. Applying this result to [1, Example 3.1] since the first non zero coefficient is nonsingular for all n, the system is an unfactorable IIL system.

Summarizing the results on factorability of lossless systems we have so far, we can make the following conclusions:

i) All IIL TVLOT's are factorable (Section III).
ii) All nontrivial SISO lossless systems are unfactorable (Section IV).
iii) There are factorable and unfactorable NIL systems (Examples 2.1 i) and 3.1, respectively).
iv) There are factorable and unfactorable IIL systems (IIL TVLOT and [1, Example 3.1], respectively).

D. Sufficient Condition for Factorability

Consider the Nth-order FIR LTV system $\mathcal{H}$

$$\boldsymbol{y}(n) = \sum_{k=0}^{N} \boldsymbol{e}_k(n)\boldsymbol{x}(n-k). \quad (4.3)$$

Supposing that the system is lossless, i.e., the coefficients satisfy (1.3), then we can prove the following theorem.

Theorem 4.2—Order Reductibility: Consider the lossless system $\mathcal{H}$ given in (4.3). If the highest order coefficient $\boldsymbol{e}_N(n)$ has a constant rank ρ for all n, then the system $\mathcal{H}$ can be factorized as a cascade of a causal lossless system $\mathcal{H}'$ whose order is at most $N-1$ followed by an IIL TVLOT block of degree ρ. ■

Before proving the theorem, we notice a few things. If $\mathcal{H}'$ also satisfies the condition in the above theorem, we can apply the above order reduction procedure to $\mathcal{H}'$ to further reduce the order. If the order reduction procedure is applicable at every step, we will finally reduce the lossless system $\mathcal{H}$ to a zeroth-order lossless system, which is a unitary matrix $\boldsymbol{P}(n)$. In this case, the lossless system can be realized as a cascade of IIL TVLOT's, which implies the lossless system itself is IIL. In the special case of LTI systems, this order reduction procedure is always possible because the coefficient always has a fixed rank. Therefore, a LTI PU system is always factorable. The order reduction for the LTI case is also given in [8]. Before we prove Theorem 4.2, it should be mentioned that the constant rank condition on $\boldsymbol{e}_N(n)$ is not a necessary condition as shown next.

Example 4.1—Factorable System that Violates the Constant Rank Condition: Consider a cascade of two degree one building blocks $\mathcal{D}(z^{-1}, \boldsymbol{v}_1(n))\mathcal{D}(z^{-1}, \boldsymbol{v}_0(n))$. Let $\boldsymbol{v}_0(n)$ and $\boldsymbol{v}_1(n)$ be unit norm vectors such that $\boldsymbol{v}_0^\dagger(2n)\boldsymbol{v}_1(2n) = 0$ and $\boldsymbol{v}_0^\dagger(2n-1)\boldsymbol{v}_1(2n-1) \neq 0$. Then, one can verify that the highest order coefficient of $\mathcal{D}(z^{-1}, \boldsymbol{v}_1(n))\mathcal{D}(z^{-1}, \boldsymbol{v}_0(n))$, which is denoted as $\boldsymbol{e}_2(n)$, has the form $\boldsymbol{e}_2(2n-1) = \boldsymbol{0}$ and $\boldsymbol{e}_2(2n) = (\boldsymbol{v}_1^\dagger(2n-1)\boldsymbol{v}_0(2n-1))\boldsymbol{v}_1(2n)\boldsymbol{v}_0^\dagger(2n-2)$. It is clear that this IIL system does not satisfy the constant rank condition of Theorem 4.2 although it is a cascade of two degree-one factors (hence "factorable").

Proof of Theorem 4.2: If $\rho = M$, then one can verify that the condition for losslessness in (1.3) implies that $\boldsymbol{e}_k(n) = \boldsymbol{0}$ for $0 \leq k \leq N-1$ and $\boldsymbol{e}_N^\dagger(n)\boldsymbol{e}_N(n) = \boldsymbol{I}$. The system $\mathcal{H}$ reduces to the trivial system $\boldsymbol{y}(n) = \boldsymbol{e}_N(n)\boldsymbol{x}(n-N)$ for unitary $\boldsymbol{e}_N(n)$. Therefore, we can assume that $1 \leq \rho \leq M-1$. Since $\boldsymbol{e}_N(n)$ has rank ρ, it can be written as $\boldsymbol{e}_N(n) = \boldsymbol{U}_\rho(n)\boldsymbol{V}_\rho(n)$, where the $M \times \rho$ matrices $\boldsymbol{U}_\rho(n)$ and $\boldsymbol{V}_\rho(n)$ are given in (3.6). Since $\boldsymbol{u}_i(n)$ are independent, we can apply the invertible Gram–Schmidt orthonormalization procedure [13] so that $\boldsymbol{u}_i(n)$ are orthonormal. Therefore, without lost of generality, we can assume that $\boldsymbol{U}_\rho^\dagger(n)\boldsymbol{U}_\rho(n) = \boldsymbol{I}_\rho$. Consider the system $\mathcal{H}'$, which is a cascade of $\mathcal{H}$ followed by the anticausal system

$$\mathcal{F}(\mathcal{Z}) = \prod_{k=0}^{\rho-1} \mathcal{D}(\mathcal{Z}, \boldsymbol{u}_k(n)). \quad (4.4)$$

The above LTV system $\mathcal{F}(\mathcal{Z})$ is lossless since it is the inverse of an IIL TVLOT. Note that in this case, the ordering of $\mathcal{D}(\mathcal{Z}, \boldsymbol{u}_k(n))$ does not matter because these building blocks are perpendicular (see Section II-A, Property 5). The system $\mathcal{H}'$ has order at most equal to N (because the IIL TVLOT is anticausal), and the Nth-order coefficient $\boldsymbol{e}'_N(n)$ can be written as

$$\boldsymbol{e}'_N(n) = [\boldsymbol{I} - \boldsymbol{U}_\rho(n)\boldsymbol{U}_\rho^\dagger(n)]\boldsymbol{U}_\rho(n)\boldsymbol{V}_\rho^\dagger(n) = \boldsymbol{0}. \quad (4.5)$$

Therefore $\mathcal{H}'$ has order $\leq N-1$. It remains to show that the system $\mathcal{H}'$ is causal and lossless. The losslessness of $\mathcal{H}'$ follows directly from that of $\mathcal{H}$ and $\mathcal{F}(\mathcal{Z})$. To prove the causality, recall from (1.3) that we have the condition $\boldsymbol{e}_N^\dagger(n)\boldsymbol{e}_0(n) = \boldsymbol{0}$ for all n. Since the vectors $\boldsymbol{v}_k(n)$ are independent, the above condition implies that $\boldsymbol{U}_\rho^\dagger(n)\boldsymbol{e}_0(n) = \boldsymbol{0}$. Therefore, we have

$$\boldsymbol{e}'_{-1}(n) = \boldsymbol{U}_\rho(n)\boldsymbol{U}_\rho^\dagger(n)\boldsymbol{e}_0(n) = \boldsymbol{0}. \quad (4.6)$$

Thus, the causality of $\mathcal{H}'$ follows. Inverting the anticausal system $\mathcal{F}(\mathcal{Z})$, we conclude that $\mathcal{H}$ is a cascade of a causal lossless

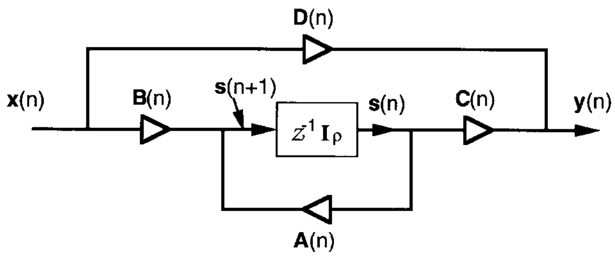

Fig. 14. State-space implementation of a LTV system.

system $\mathcal{H}'$ whose order is at most $N-1$ followed by the causal IIL TVLOT block $\mathcal{F}(\mathcal{Z}^{-1}) = \Pi_{k=0}^{\rho-1} \mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{u}_k(n))$. ■

V. State-Space Manifestation of Factorable IIL Systems

In this section, we will consider the state-space representation of LTV systems. The theory is well known in the LTI case [12], [14], [15]. We will generalize the concept of reachability and observability to the LTV case in a way most suited for our purpose. We will prove that for the cascade system of an arbitrary number of dyadic building blocks $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}_k(n))$, the realization matrix is *unitary*. Furthermore, the cascade system is *strongly eternally reachable* and *observable*. We will also prove that the strong eternal reachability and observability imply that the *minimality* of the structure. Thus, the implementation based on factorization is *minimal* in terms of delays as well as the number of building blocks $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}_k(n))$. A brief introduction to the continuous-time reachability and observability of LTV systems is given in [14]. In the following, we will develop the theory for the discrete-time LTV case.

A. State-Space Representation of LTV Systems

Consider the state-space realization of an $M \times M$ LTV system

$$\boldsymbol{s}(n+1) = \boldsymbol{A}(n)\boldsymbol{s}(n) + \boldsymbol{B}(n)\boldsymbol{x}(n), \text{(state equation)} \quad (5.1a)$$
$$\boldsymbol{y}(n) = \boldsymbol{C}(n)\boldsymbol{s}(n) + \boldsymbol{D}(n)\boldsymbol{x}(n), \text{(output equation)} \quad (5.1b)$$

where $\boldsymbol{s}(n) = [s_1(n)s_2(n)\cdots s_\rho(n)]^T$ is the state vector, and $\boldsymbol{x}(n)$ and $\boldsymbol{y}(n)$ are, respectively, the input and output vectors. The integer ρ is called the dimension of the state space. In (5.1), we have assumed that ρ is time invariant. According to Theorem 4.1, the instantaneous degree of a factorable lossless system could be increasing with time n. Thus, the constant degree assumption is a loss of generality. However, since a factorable IIL system has a constant number of building blocks (Theorem 4.1), we will see that all factorable IIL lossless systems have constant ρ. From (5.1), we have the implementation in Fig. 14. Note that the system in Fig. 14 is always causal. The realization matrix $\boldsymbol{R}(n)$ is given as

$$\boldsymbol{R}(n) = \begin{bmatrix} \boldsymbol{A}(n) & \boldsymbol{B}(n) \\ \boldsymbol{C}(n) & \boldsymbol{D}(n) \end{bmatrix}. \quad (5.2)$$

1) Time-Varying Impulse Response: Assuming that we start the system at $n = n_0$ with zero initial condition, using (5.1), the output of the system can be expressed as

$$\boldsymbol{y}(n) = \sum_{k=0}^{n-1-n_0} \boldsymbol{C}(n)\Phi(n, n-k)\boldsymbol{B}(n-1-k) \cdot \boldsymbol{x}(n-1-k) + \boldsymbol{D}(n)\boldsymbol{x}(n) \quad (5.3)$$

where the $\rho \times \rho$ state transition matrix $\Phi(n, m)$ is defined as

$$\Phi(n+1, m) = \boldsymbol{A}(n)\Phi(n, m), \quad \text{and} \quad \Phi(m, m) = \boldsymbol{I}. \quad (5.4)$$

Comparing (5.3) and the direct-form implementation $\boldsymbol{y}(n) = \sum_k \boldsymbol{e}_k(n)\boldsymbol{x}(n-k)$, we conclude that the impulse response coefficients are

$$\boldsymbol{e}_{k+1}(n) = \boldsymbol{C}(n)\Phi(n, n-k)\boldsymbol{B}(n-1-k) \quad k \geq 0 \quad \text{and} \quad \boldsymbol{e}_0(n) = \boldsymbol{D}(n). \quad (5.5)$$

B. Reachability of LTV Systems

For the LTI case, there are several equivalent definitions of reachability [12], [14], [15]. In the following, we generalize the one given in [12, ch. 13] to the LTV case. Since the implementation is time varying, we have to differentiate between the instantaneous and eternal reachabilities, which are defined as follows.

Definition 5.1—Reachability: An implementation is said to be *reachable* at time n_f if we can reach any specified final state $\boldsymbol{s}^f$ at time n_f (i.e., $\boldsymbol{s}(n_f) = \boldsymbol{s}^f$) starting from any initial state by application of an appropriate *finite length* input. If the implementation is reachable for all n, then we say that it is *eternally reachable* (ER).

Let $(\boldsymbol{A}(n), \boldsymbol{B}(n), \boldsymbol{C}(n), \boldsymbol{D}(n))$ be the state space representation of an implementation of a LTV system as in (5.1). Next, we are going to show how the reachability of an implementation depends only on $\boldsymbol{A}(n)$ and $\boldsymbol{B}(n)$. Assuming that we start the system at $n = n_0$ with initial condition $\boldsymbol{s}(n_0)$, from (5.1a), we have

$$\boldsymbol{s}(n) - \Phi(n, n_0)\boldsymbol{s}(n_0) = \sum_{k=0}^{n-1-n_0} \Phi(n, n-k)\boldsymbol{B}(n-1-k)\boldsymbol{x}(n-1-k) \quad (5.6)$$

where $\Phi(n, n-k)$ is defined in (5.3). From (5.6), we see that an implementation is reachable at n if there is a finite integer L such that the following $\rho \times ML$ matrix $\boldsymbol{R}_{\boldsymbol{A},\boldsymbol{B}}(n-1, n-L)$ has full column rank (i.e. column rank $= \rho$, which is the dimension of the state space):

$$\boldsymbol{R}_{\boldsymbol{A},\boldsymbol{B}}(n, n-L) = [\boldsymbol{B}(n-1) \quad \Phi(n, n-1)\boldsymbol{B}(n-2) \quad \cdots \quad \Phi(n, n-L+1)\boldsymbol{B}(n-L)]. \quad (5.7)$$

In the LTI case, we know [12], [14], [15] that if we cannot reach a particular final state by applying an input of length ρ, then the final state cannot be reached by applying more inputs (because of the Cayley–Hamilton Theorem). In the LTV case, a similar statement does not hold. The fact that the matrix $\boldsymbol{R}_{\boldsymbol{A},\boldsymbol{B}}(n-1, n-\rho)$ does not have full column rank does not imply that $\boldsymbol{R}_{\boldsymbol{A},\boldsymbol{B}}(n, n-L)$ will not have full column rank for all $L > \rho$. Therefore, we cannot determine in finite time if a state is unreachable. Therefore, in the LTV case, we have the following definition of strong reachability:

Definition 5.2—Strong Reachability: An implementation is said to be *strongly reachable* at time n if the $\rho \times M\rho$ matrix $\boldsymbol{R}_{\boldsymbol{A},\boldsymbol{B}}(n, n-\rho)$ defined in (5.7) has full column rank, i.e., $\boldsymbol{R}_{\boldsymbol{A},\boldsymbol{B}}(n, n-\rho)\boldsymbol{R}^{\dagger}_{\boldsymbol{A},\boldsymbol{B}}(n, n-\rho)$ is nonsingular. If an implementation is strongly reachable for all n, then we say that it is *strongly eternally reachable* (SER).

C. Observability of LTV Systems

Similar to the case of reachability, we will generalize the definition of LTI observability given in [12, ch. 13] to the LTV case.

Definition 5.3—Observability: An implementation is said to be *observable* at time n if the state $\boldsymbol{s}(n)$ can be determined *uniquely* by observing a *finite-length* segment of the output. If the implementation is observable for all n, then we say that it is *eternally observable* (EO).

One can show that a state at time n is observable if and only if there is a finite L such that the following $ML \times \rho$ matrix $\boldsymbol{O}_{\boldsymbol{A},\boldsymbol{C}}(n+L-1, n)$ has full row rank:

$$\boldsymbol{O}_{\boldsymbol{A},\boldsymbol{C}}(n+L-1, n) = \begin{bmatrix} \boldsymbol{C}(n) \\ \boldsymbol{C}(n+1)\Phi(n+1,n) \\ \vdots \\ \boldsymbol{C}(n+L-1)\Phi(n+L-1,n) \end{bmatrix}. \tag{5.8}$$

We cannot determine in finite time if a state is not observable. Therefore, similar to the case of reachability, we have the following definition.

Definition 5.4—Strong Observability: An implementation is said to be *strongly observable* at time n if the $M\rho \times \rho$ matrix $\boldsymbol{O}_{\boldsymbol{A},\boldsymbol{C}}(n+\rho-1, n)$ defined in (5.8) has full row rank, i.e., $\boldsymbol{O}^{\dagger}_{\boldsymbol{A},\boldsymbol{C}}(n+\rho-1, n)\boldsymbol{O}_{\boldsymbol{A},\boldsymbol{C}}(n+\rho-1, n)$ is nonsingular. If the implementation is strongly observable for all n, then we say that it is *strongly eternally observable* (SEO).

D. Minimality of LTV Systems

The reason we introduce the concepts of SER and SEO as in Definitions 5.2 and 5.4 is that it leads to the minimality of LTV systems. Let $(\boldsymbol{A}(n), \boldsymbol{B}(n), \boldsymbol{C}(n), \boldsymbol{D}(n))$ be the state-space representation of the system $\boldsymbol{y}(n) = \Sigma_k\, \boldsymbol{e}_k(n)\boldsymbol{x}(n-k)$. By using (5.5), (5.7), and (5.8), one can verify that the $ML \times ML$ product matrix of $\boldsymbol{O}_{\boldsymbol{A},\boldsymbol{C}}(n+L-1, n)\boldsymbol{R}_{\boldsymbol{A},\boldsymbol{B}}(n, n-L)$ is related to the impulse coefficients as in (5.9), shown at the bottom of the page. By using the above equation, we can show (see Appendix A) that if an implementation is SER and SEO, then we cannot reduce the number of state variables. That is, the implementation is *minimal*. Thus, we have the following theorem.

Theorem 5.1—Minimality: If an implementation of an LTV is SER and SEO, then it is minimal. ■

E. State-Space Representation of Factorable IIL Systems

First, let us consider the IIL TVLOT studied in Section III-B. For an IIL TVLOT of degree ρ, we know that the coefficients can be characterized as in (3.7). If we take the output of the delay elements in the dyadic-based structure as the state variable, then state vector is $\boldsymbol{s}_{\rho}(n) = \boldsymbol{V}^{\dagger}_{\rho}(n-1)\boldsymbol{x}(n-1)$. The state-space representation of the system becomes

$$\begin{bmatrix} \boldsymbol{s}(n+1) \\ \boldsymbol{y}(n) \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{0} & \boldsymbol{V}^{\dagger}_{\rho}(n) \\ \boldsymbol{P}(n)\boldsymbol{V}_{\rho}(n) & \boldsymbol{P}(n)[\boldsymbol{I} - \boldsymbol{V}_{\rho}(n)\boldsymbol{V}^{\dagger}_{\rho}(n)] \end{bmatrix}}_{\boldsymbol{R}(n)} \begin{bmatrix} \boldsymbol{s}(n) \\ \boldsymbol{x}(n) \end{bmatrix} \tag{5.10}$$

where $\boldsymbol{P}(n)$ is an arbitrary unitary matrix, and $\boldsymbol{V}^{\dagger}_{\rho}(n)\boldsymbol{V}_{\rho}(n) = \boldsymbol{I}_{\rho}$. One can verify by direct substitution that the realization matrix $\boldsymbol{R}(n)$ is unitary, i.e., $\boldsymbol{R}^{\dagger}(n)\boldsymbol{R}(n) = \boldsymbol{I}_{M+\rho}$. For the special case of degree-one IIL TVLOT, the system reduces to the dyadic-based structure $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n))$ in Fig. 6 with $\boldsymbol{v}^{\dagger}(n)\boldsymbol{v}(n) = 1$. In this case, the state vector is the scalar quantity $s(n)$ as indicated in Fig. 6. For the more general case of the cascade of arbitrary number of building blocks, we have the following theorem:

Theorem 5.2—Unitariness of Realization Matrix: Consider the cascade implementation of factorable IIL system $\mathcal{H} = \mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}_k(n)) \cdots \mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}_1(n))$ with $\boldsymbol{v}^{\dagger}_i(n)\boldsymbol{v}_i(n) = 1$ for $1 \le i \le k$. Then, for any integer k, the realization matrix $\boldsymbol{R}(n)$ of the cascade implementation $\mathcal{H}$ satisfies

a) $\boldsymbol{A}(n)$ in (5.2) is a lower (or upper) triangular matrix with zero diagonal elements.

b) $\boldsymbol{R}(n)$ is unitary. ■

Proof: Part a) is clear since the state variable of the ith section does not depend the state variables of the jth section for all $j \ge i$. To prove part b), we denote the output of the ith section as $\boldsymbol{y}_i(n)$. Since the realization matrix of the dyadic-based structure is unitary, we have

$$\boldsymbol{y}^{\dagger}_{i-1}(n)\boldsymbol{y}_{i-1}(n) + |s_i(n)|^2 = \boldsymbol{y}^{\dagger}_i(n)\boldsymbol{y}_i(n) + |s_i(n+1)|^2, \quad \text{for } 1 \le i \le k \tag{5.11}$$

$$\boldsymbol{O}_{\boldsymbol{A},\boldsymbol{C}}(n+L-1, n)\boldsymbol{R}_{\boldsymbol{A},\boldsymbol{B}}(n, n-L) = \begin{bmatrix} \boldsymbol{e}_1(n) & \boldsymbol{e}_2(n) & \cdots & \boldsymbol{e}_L(n) \\ \boldsymbol{e}_2(n+1) & \boldsymbol{e}_3(n+1) & \cdots & \boldsymbol{e}_{L+1}(n+1) \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{e}_L(n+L-1) & \boldsymbol{e}_{L+1}(n+L-1) & \cdots & \boldsymbol{e}_{2L-1}(n+L-1) \end{bmatrix}. \tag{5.9}$$

where $\boldsymbol{y}_0(n)$ is the input $\boldsymbol{x}(n)$, and $\boldsymbol{y}_k(n)$ is the output $\boldsymbol{y}(n)$. Summing up all the k terms in (5.11), we have

$$\sum_{i=1}^{k} |s_i(n+1)|^2 + \boldsymbol{y}^\dagger(n)\boldsymbol{y}(n) = \sum_{i=1}^{k} |s_i(n)|^2 + \boldsymbol{x}^\dagger(n)\boldsymbol{x}(n), \quad \forall n. \tag{5.12}$$

Since the right-hand sight of (5.12) is arbitrary, we conclude that the realization matrix of the cascade system $\mathcal{H}$ is unitary. ■

From Theorem 4.1, we know that if an IIL system is factorable, then the number of building blocks $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}_i(n))$ is time invariant. Therefore, combining the results in Theorems 4.1 and 5.2, we conclude that *there is always a unitary realization matrix for any factorable IIL system*.

F. Minimal Factorization of IIL Systems

Consider the dyadic-based structure $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}_i(n))$ with unit norm vector $\boldsymbol{v}_i(n)$. Taking the state variable to be $s_i(n) = \boldsymbol{v}_i^\dagger(n)\boldsymbol{x}(n)$, then it is clear that the dyadic-based structure is SER and SEO. More generally, we can prove the following theorem.

Theorem 5.3—Strong Reachability and Observability: Consider the factorized system $\mathcal{H} = \mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}_k(n)) \cdots \mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}_1(n))$ with $\boldsymbol{v}_i^\dagger(n)\boldsymbol{v}_i(n) = 1$ for $1 \le i \le k$. This cascade implementation of $\mathcal{H}$ is SER and SEO and, hence, minimal. ■

Proof: Because of Theorem 5.1, we need only to prove the SER and SEO of the structure. Let $s_i(n)$ and $\boldsymbol{y}_i(n)$ be, respectively, the state variable and the output of the ith section $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}_i(n))$.

1) *Strong Reachability:* The proof is based on induction on k. If $k = 1$, the cascade system $\mathcal{H}$ reduces to the dyadic-based structure, which is SCR. Assuming that the theorem is true for $k = L$, we will show that it is true for $k = L + 1$. Let $\boldsymbol{s}^f = [s_1^f \cdots s_{L+1}^f]^T$ be the final state vector we want to reach. We will construct $\{\boldsymbol{x}(n-1), \cdots, \boldsymbol{x}(n-L-2)\}$ such that $\boldsymbol{s}(n) = \boldsymbol{s}^f$. Let $\boldsymbol{y}_L^f(n-1)$ be such that $s_{L+1}(n) = \boldsymbol{v}_{L+1}^\dagger(n-1)\boldsymbol{y}_L^f(n-1) = s_{L+1}^f$ (this is always possible because $\boldsymbol{v}_{L+1}(n) \neq \boldsymbol{0}$). Therefore, the problem reduces to choosing the input $\boldsymbol{x}(n)$ such that $\boldsymbol{y}_L(n-1) = \boldsymbol{y}_L^f(n-1)$ and the state vector $\hat{\boldsymbol{s}}(n-1) = [s_1(n-1) \cdots \boldsymbol{s}_L(n-1)]^T$ satisfy

$$\boldsymbol{R}_L(n-1)\begin{bmatrix} \hat{\boldsymbol{s}}(n-1) \\ \boldsymbol{x}(n-1) \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{s}}^f(n) \\ \boldsymbol{y}_L^f(n-1) \end{bmatrix} \tag{5.13}$$

where $\boldsymbol{R}_L(n)$ is the unitary realization matrix of the cascade of first L sections (see Theorem 5.2). By the hypothesis of the induction that $\mathcal{H}$ is strongly reachable for $k = L$, we can choose $\{\boldsymbol{x}(n-2), \cdots, \boldsymbol{x}(n-L-2)\}$ so that $\hat{\boldsymbol{s}}(n-1)$ satisfies (5.12). Therefore, the cascade of $L+1$ sections is SER.

2) *Strong Observability:* We want to determine $\boldsymbol{s}(n)$ uniquely by observing $\{\boldsymbol{y}(n), \cdots, \boldsymbol{y}(n+k-1)\}$. First, note that $\boldsymbol{y}(n) = \boldsymbol{v}_k(n)s_k(n) + [\boldsymbol{I} - \boldsymbol{v}_k(n)\boldsymbol{v}_k^\dagger(n)]\boldsymbol{y}_{k-1}(n)$. Using the fact that $\boldsymbol{v}_k^\dagger(n)\boldsymbol{v}_k(n) = 1$, we find that $s_k(n) = \boldsymbol{v}_k^\dagger(n)\boldsymbol{y}(n)$. Since $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}_k(n))$ is invertible, knowing $\boldsymbol{y}(n+1), \cdots, \boldsymbol{y}(n+k-1)$ and $s_k(n)$, we can uniquely determine $\boldsymbol{y}_{k-1}(n), \cdots, \boldsymbol{y}_{k-1}(n+k-2)$, from which we can find $s_{k-1}(n)$. Repeating the above process, we can determine all the state variables $s_i(n)$ for $1 \le i \le k$. The cascade system $\mathcal{H}$ is therefore SEO. ■

Combining the results in Theorems 4.1, 5.1, and 5.3, we have proved that the factorization of IIL system is *minimal*. We cannot find a structure that has a smaller number of delays.

VI. IIR Lattice Structres for Lossless LTV Systems

In all the previous discussions, we have considered only the FIR case (except Section II-B). In the IIR LTV case, it is not easy to ensure the stability. In the LTV case, there are several types of stability [9], [14]. In this section, we will study only two of them, namely, the BIBO stability and l_2 stability, which are defined as follows:

Definitions 6.1—BIBO and l_2 Stability: A system is said to be BIBO stable if bounded input produces bounded output. A system is said to be l_2 stable if a finite energy input generates a finitew energy output.

In general, BIBO stability and l_2 stability are different. To see this, consider the idea LTI lowpass filter and the LTV system $y(n) = x(0)$. The former is l_2 stable but not BIBO stable, whereas the latter is BIBO stable but not l_2 stable.

A. Stability of LTV Normalized IIR Lattice

Consider the LTV normalized IIR lattice structure given in Fig. 15, where the number of delays ρ is time invariant. For an introduction to the theory of LTI IIR lattice, see [16, ch. 7]. In the LTV case, it was shown in [9] that the system in Fig. 15 preserves the energy from input to output. Using this energy balance property, the authors in [9] showed that the normalized IIR lattice structure in Fig. 15 is l_2 stable if the time-varying lattice coefficients $|\alpha_k(n)| \le \gamma < 1$. In this section, we will show that the structure in Fig. 15 is BIBO stable in addition to being l_2 stable. To prove the BIBO stability of the normalized lattice, we need the following lemma and the definition of matrix norm.

Definition 6.2—Matrix Norm [13]: The norm of a matrix $\boldsymbol{A}$ (which is denoted as $||\boldsymbol{A}||$) is defined as $\max_{||\boldsymbol{v}||=1} \boldsymbol{v}^\dagger \boldsymbol{A}^\dagger \boldsymbol{A}\boldsymbol{v}$.

It can be shown [13] that $||\boldsymbol{A}\boldsymbol{v}|| \le ||\boldsymbol{A}||\,||\boldsymbol{v}||$ and $||\boldsymbol{A}_1\boldsymbol{A}_2|| \le ||\boldsymbol{A}_1||\,||\boldsymbol{A}_2||$. By using these norm properties, we can prove the following lemma.

Lemma 6.1: Let $(\boldsymbol{A}(n), \boldsymbol{B}(n), \boldsymbol{C}(n), D(n))$ be a state-space description of a LTV system such that the realization matrix $\boldsymbol{R}(n)$ is unitary. Let ρ be the dimension of the state space. Then, for all n, $||\Phi(n+\rho, n)|| \le 1$ with equality if and only if the LTV system is not SEO. ■

Proof: From (5.4), we have $||\Phi(n+\rho, n)|| = ||\boldsymbol{A}(n+\rho-1)\Phi(n+\rho-1, n)|| \le \Pi_{i=0}^{\rho-1} ||\boldsymbol{A}(n+i)||$. Since $\boldsymbol{R}(n)$ is unitary, we have

$$\boldsymbol{A}^\dagger(n)\boldsymbol{A}(n) + \boldsymbol{C}^\dagger(n)\boldsymbol{C}(n) = \boldsymbol{I} \tag{6.1}$$

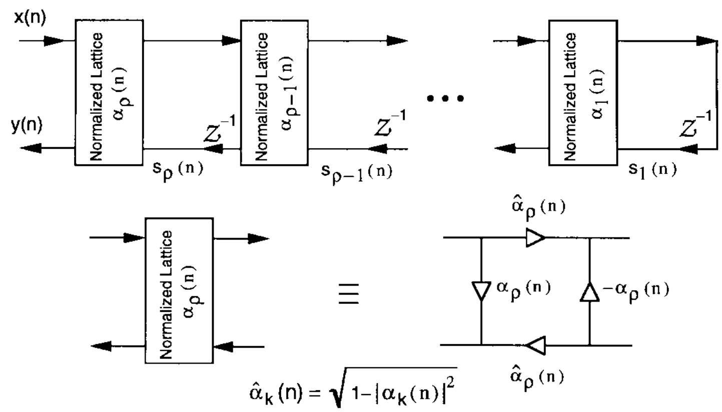

Fig. 15. LTV normalized IIR lattice structure.

for all n. It immediately follows from (6.1) that $\|\Phi(n+\rho,n)\| \le 1$. Using (6.1) and the recursive formula in (5.4), we can expand $\Phi^\dagger(n+\rho,n)\Phi(n+\rho,n)$ as

$$\begin{aligned}\Phi^\dagger(n+\rho,n)\Phi(n+\rho,n) &= \boldsymbol{I} - \boldsymbol{C}^\dagger(n)\boldsymbol{C}(n) - \Phi^\dagger(n+1,n)\boldsymbol{C}^\dagger(n+1)\boldsymbol{C}(n+1)\\ &\quad \cdot \Phi(n+1,n)\\ &\quad - \cdots - \Phi^\dagger(n+\rho-1,n)\boldsymbol{C}^\dagger(n+\rho-1)\boldsymbol{C}(n+\rho-1)\\ &\quad \cdot \Phi(n+\rho-1,n). \end{aligned} \tag{6.2}$$

By using the definition of $\boldsymbol{O}_{\boldsymbol{A},\boldsymbol{C}}(n+\rho-1,n)$ in (5.8), we see that

$$\Phi^\dagger(n+\rho,n)\Phi(n+\rho,n) = \boldsymbol{I} - \boldsymbol{O}^\dagger_{\boldsymbol{A},\boldsymbol{C}}(n+\rho-1,n)\boldsymbol{O}_{\boldsymbol{A},\boldsymbol{C}}(n+\rho-1,n). \tag{6.3}$$

Therefore, we conclude $\|\Phi(n+\rho,n)\| = 1$ if and only if $\boldsymbol{O}_{\boldsymbol{A},\boldsymbol{C}}(n+\rho-1,n)$ is singular [which implies that the LTV system is not SEO (Definition 5.4)]. ■

For the LTI case, it was shown in [16, ch. 7] that the realization matrix $\boldsymbol{R}$ of a normalized IIR lattice structure is unitary. This property continues to hold for the LTV case. Therefore, the LTV normalized IIR lattice satisfies the condition given in Lemma 6.1. Furthermore, it is shown in Appendix B that the system in Fig. 15 is SEO if $|\alpha_k(n)| \le \gamma < 1$. Therefore, there is some fixed $\epsilon > 0$ such that $\boldsymbol{O}^\dagger_{\boldsymbol{A},\boldsymbol{C}}(n+\rho-1,n)\boldsymbol{O}_{\boldsymbol{A},\boldsymbol{C}}(n+\rho-1,n) \ge \epsilon\boldsymbol{I}$ (Appendix B). From (6.3), we have $\Phi^\dagger(n+\rho,n)\Phi(n+\rho,n) \le (1-\epsilon)\boldsymbol{I}$, which implies that $\|\Phi(n+\rho,n)\| \le (1-\epsilon)$. Using (5.3), the output $y(n)$ of the IIR LTV system in Fig. 15 satisfies

$$\begin{aligned}|y(n)| &\le |Dx(n)| + \sum_{k=0}^{n-1-n_0} \|\boldsymbol{C}(n)\|\|\Phi(n,n-k)\|\\ &\quad \cdot \|\boldsymbol{B}(n-1-k)\||x(n-1-k)|\\ &\le |x(n)| + \sum_{k=0}^{n-1-n_0} \|\Phi(n,n-k)\||x(n-1-k)| \end{aligned} \tag{6.4}$$

where we have used the fact that $\|\boldsymbol{C}(n)\| \le 1$ and $\|\boldsymbol{B}(n-1-k)\| \le 1$ [which follow from the unitariness of $\boldsymbol{R}(n)$] in the second inequality. If there is a $B < \infty$ such that the input $|x(n)| \le B$, then using the fact that $\|\Phi(n+\rho,n)\| \le (1-\epsilon)$, we get

$$|y(n)| \le B + B\sum_{k=0}^{n-1-n_0} (1-\epsilon)^{\lfloor k/\rho \rfloor} \tag{6.5}$$

where $\lfloor k/\rho \rfloor$ denotes the largest integer $\le k/\rho$. From (6.5), we conclude that $|y(n)| \le (\rho\epsilon^{-1}+1)B$ for all n. Therefore, the output is bounded. Summarizing the result, we have the following theorem.

Theorem 6.1—Stability of LTV Normalized IIR Lattice: The LTV normalized IIR lattice structure in Fig. 6.1 is both BIBO stable and l_2 stable if the lattice coefficients $|\alpha_k(n)| \le \gamma < 1$. ■

Remarks:

1) In the LTI case, Lemma 6.1 reduces to the following [12]: If $\boldsymbol{A}$ is a $\rho \times \rho$ unit norm stable matrix, then $\|\boldsymbol{A}^\rho\| < \boldsymbol{I}$. It is shown in [16], [17] that the condition $\|\boldsymbol{A}^\rho\| < \boldsymbol{I}$ is sufficient for preventing zero-input limit cycles.
2) Since $\|\Phi(n+\rho,n)\| < 1$ for a normalized IIR lattice structure with $|\alpha_k(n)| \le \gamma < 1$, the energy of the state vector has to decrease after ρ time interval if there is no input. Therefore, we conclude that the structure is free from zero-input limit cycles.

B. Stability of the Two-Multiplier IIR Lattice Structures

In the LTI case, we know that the normalized IIR lattice is not efficient in terms of computation, although it has a better noise performance. There is a more efficient two-multiplier IIR lattice [9], [12], [16]. In this subsection, we will generalize the LTI two-multiplier lattice structure to the LTV denormalized IIR lattice as shown in Fig. 16, where the number of sections ρ is a constant independent of n. After some simplifications, it can be shown that the LTV system in Fig. 16 is equivalent to that in Fig. 17. The structure in Fig. 17 is very similar to that of the normalized IIR lattice structure in Fig. 15, except the time-dependent multipliers $\hat{\alpha}_k(n) = \sqrt{1-|\alpha_k(n)|^2}$ between the sections. Because of these multipliers, it can be shown that the LTV system in Fig. 17 can *never* be lossless unless $\hat{\alpha}_k(n)$ are time independent, which is equivalent to saying that the magnitude of the lattice coefficients $|\alpha_k(n)|$

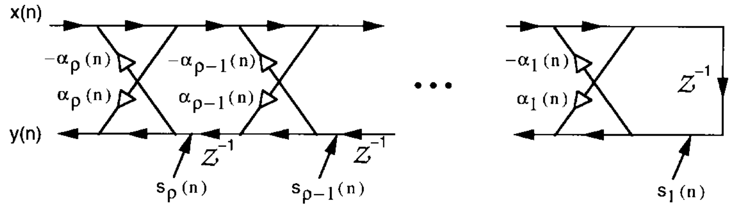

Fig. 16. LTV denormalized IIR lattice structure.

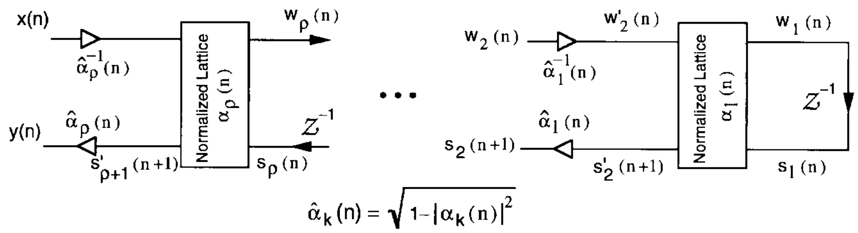

Fig. 17. Redrawing of Fig. 16 in terms of normalized building blocks.

is a constant independent of n. To see this, we consider Fig. 17. Since the system from $w'_2(n)$ to $s'_2(n+1)$ is lossless, we have $\Sigma_n|w'_2(n)|^2 = \Sigma_n|s'_2(n+1)|^2$. This implies that $\Sigma_n|w_2(n)/\hat{\alpha}_1(n)|^2 = \sigma_n|s_2(n)/\hat{\alpha}_1(n)|^2$. Therefore, $s_2(n)$, in general, does not have the same energy as $w_2(n)$ unless $\hat{\alpha}_1(n)$ is a constant. This proves that the first-order denormalized IIR lattice is, in general, not lossless. Continuing the process, we can show that the output $y(n)$ in general does not have the same energy as the input $x(n)$ unless $|\alpha_k(n)|$ are time independent.

In general, we cannot prove either the BIBO or l_2 stability of the two-multiplier IIR lattice in Fig. 16. However, in the special case when $|\alpha_k(n)|$ are constant independent of n, both the BIBO and l_2 stability of the structure is guaranteed by the condition $|\alpha_k(n)| \leq \gamma < 1$. The reason is because in this case, the time-independent multipliers $\hat{\alpha}_k(n)$ can be moved to the left, and the resulting structure is very similar to the normalized IIR lattice in Fig. 15.

VII. Nonlossless FIR LTV Systems with FIR Inverses

In this section, we will show how to construct nonlossless FIR LTV systems with FIR inverses. The following two classes will be considered: i) Causal FIR LTV systems with causal FIR inverses, which are also called the LTV unimodular systems (just by analogy to the LTI case), and ii) causal FIR LTV systems with anticausal FIR inverses (abbreviated as LTV CAFACAFI). For a detailed discussion on LTI CAFACAFI systems, see [10] and [11]. First, we will construct a degree-one system that can be used to form higher degree systems with FIR inverses.

Theorem 7.1: Consider the degree-one LTV system $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n), \boldsymbol{u}(n))$

$$\boldsymbol{y}(n) = [\boldsymbol{I} - \boldsymbol{u}(n)\boldsymbol{v}^\dagger(n)]\boldsymbol{x}(n) + \boldsymbol{u}(n)\boldsymbol{v}^\dagger(n-1)\boldsymbol{x}(n-1) \quad (7.1)$$

where $\boldsymbol{u}(n)$ and $\boldsymbol{v}(n-1)$ are non zero vectors. We then have the following:

a) If $\boldsymbol{u}^\dagger(n)\boldsymbol{v}(n) = 0$ for all n, then $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n), \boldsymbol{u}(n))$ is a LTV unimodular system. Its unique causal FIR inverse is invertible and can be described as

$$\hat{\boldsymbol{x}}(n) = [\boldsymbol{I} + \boldsymbol{u}(n)\boldsymbol{v}^\dagger(n)]\boldsymbol{y}(n) - \boldsymbol{u}(n)\boldsymbol{v}^\dagger(n-1)\boldsymbol{y}(n-1). \quad (7.2)$$

b) If $\boldsymbol{u}^\dagger(n)\boldsymbol{v}(n) = 1$ for all n, then $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n), \boldsymbol{u}(n))$ is a LTV CAFACAFI system. Its unique anticausal FIR inverse is invertible and can be described as

$$\hat{\boldsymbol{x}}(n) = [\boldsymbol{I} - \boldsymbol{u}(n)\boldsymbol{v}^\dagger(n)]\boldsymbol{y}(n) + \boldsymbol{u}(n)\boldsymbol{v}^\dagger(n+1)\boldsymbol{y}(n+1). \quad (7.3)$$

Moreover, the LTV CAFACAFI system is lossless if and only if $\boldsymbol{u}(n) = \boldsymbol{v}(n)$.

The above theorem can be proved by direct substitution. Since the cascade of LTV unimodular systems (or LTV CAFACAFI) is also a LTV unimodular (LTV CAFACAFI) system, we can generate higher degree systems by using the corresponding degree-one system $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n), \boldsymbol{u}(n))$ given in Theorem 7.1. However, it should be mentioned that we do not know if the degree-one system in (7.1) is a most general LTV unimodular system (or LTV CAFACAFI system). Thus, the above construction of higher degree systems might not be complete.

1) LTV Unimodular Lapped Transform (ULT) and Biorthogonal Lapped Transform (BOLT) [11]: Consider the cascade of $M \times M$ dyadic-based systems: $\mathcal{H} = \mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}_\rho(n), \boldsymbol{u}_\rho(n)) \cdots \mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}_1(n), \boldsymbol{u}_1(n))$. Assuming that $\rho < M$, it can be shown that if the vectors $\boldsymbol{v}_j^\dagger(n)\boldsymbol{u}_i(n) = 0$ for all $i < j$, then the system $\mathcal{H}$ has order one. In this case, we can get either a LTV ULT if $\boldsymbol{v}_i(n)\boldsymbol{u}_i(n) = 0$ or a LTV BOLT if $\boldsymbol{v}_i(n)\boldsymbol{u}_i(n) = 1$.

VIII. Concluding Remarks

In this paper, we showed how to capture all degree-one lossless LTV systems by two unitary matrices (Theorem 2.1) and proved that they can be realized as a cascade of a lossless dyadic-based building block and a unitary matrix (Fig. 5). The dyadic-based building block in Fig. 6 has many useful properties (Section II-A). The theory of LOT [5] is extended to the LTV case (Section III). We showed that the instantaneous degree of a TVLOT is nondecreasing with time n, and it is a constant if and only if it is an IIL TVLOT (Theorems 3.1 and 3.2). All IIL TVLOT can be factorized uniquely into perpendicular dyadic-based building blocks (Fig. 12), and the inverse is also factorable (Fig. 13). For NIL TVLOT systems, there are factorable examples [(2.13) of Example 2.1] and unfactorable examples (Example 3.1). Factorability of higher order lossless LTV systems is also studied (Section IV). By using the test for factorability (Section IV-A), we demonstrated that there are unfactorable IIL systems ([1, Example 3.1]). A sufficient condition for factorability that leads to an order-reduction procedure is also given (Theorem 4.2). We also introduced the concept of SER and SEO (Section V-A). If an implementation of a LTV system is SER and SEO, then it is minimal (Theorem 5.1). In particular, we show that the implementation in terms of building blocks is minimal in terms of delay elements (Theorem 5.3). The LTV normalized IIR lattice is proved to be BIBO stable as well as l_2 stable if the lattice coefficients $|\alpha_k(n)| \le \gamma < 1$ (Theorem 6.1).

However, there are still many unsolved problems related to the topic of lossless LTV systems. Some of these are stated as follows. From Section III, we know that there exist unfactorable lossless systems. However, in all of our unfactorable lossless examples, their instantaneous degree is time dependent. This leads us to ask if all lossless systems with a time-independent degree are factorable in terms of the degree-one lossless building block $\mathcal{D}(\mathcal{Z}^{-1}, \boldsymbol{v}(n))$ introduced in Section II. In the more general TV biorthogonal case, it is still unknown as to whether the system given in (7.1) is the most general degree-one TV unimodular (or TV CAFA-CAFI) system. A complete characterization of TV ULT or TV BOLT systems is still unknown. In the LTI case, a complete parameterization of the BOLT systems is given in [11], and it is shown that BOLT systems can always be factorized into degree-one building blocks. In the LTV case, the factorability of TV BOLT is currently under study.

Appendix A
Proof of Theorem 5.1

Let $(\boldsymbol{A}(n), \boldsymbol{B}(n), \boldsymbol{C}(n), \boldsymbol{D}(n))$ be a SER and SEO realization of the LTV system $\mathcal{H}$ with $\boldsymbol{A}(n)$ being $\rho \times \rho$ matrix. Suppose that there is another SER and SEO realization $(\boldsymbol{A}'(n), \boldsymbol{B}'(n), \boldsymbol{C}'(n), \boldsymbol{D}'(n))$ with smaller state space dimension. That is, $\boldsymbol{A}'(n)$ is $\rho' \times \rho'$ with $\rho' < \rho$. By using (5.9), we know that

$$\begin{aligned}\boldsymbol{O}_{\boldsymbol{A},\boldsymbol{C}}&(n+\rho-1,n)\boldsymbol{R}_{\boldsymbol{A},\boldsymbol{B}}(n,n-\rho)\\ &= \boldsymbol{O}_{\boldsymbol{A}',\boldsymbol{C}'}(n+\rho-1,n)\boldsymbol{R}_{\boldsymbol{A}',\boldsymbol{B}'}(n,n-\rho). \quad \text{(A.1)}\end{aligned}$$

Premultiplying and postmultiplying both sides of (A.1), respectively, by $\boldsymbol{O}^{\dagger}_{\boldsymbol{A},\boldsymbol{C}}(n+\rho-1,n)$ and $\boldsymbol{R}^{\dagger}_{\boldsymbol{A},\boldsymbol{B}}(n,n-\rho)$, we get

$$\begin{aligned}&\underbrace{\boldsymbol{O}^{\dagger}_{\boldsymbol{A},\boldsymbol{C}}\boldsymbol{O}_{\boldsymbol{A},\boldsymbol{C}}}_{\rho\times\rho}\underbrace{\boldsymbol{R}_{\boldsymbol{A},\boldsymbol{B}}\boldsymbol{R}^{\dagger}_{\boldsymbol{A},\boldsymbol{B}}}_{\rho\times\rho}\\ &\quad = \underbrace{\boldsymbol{O}^{\dagger}_{\boldsymbol{A},\boldsymbol{C}}\boldsymbol{O}_{\boldsymbol{A}',\boldsymbol{C}'}}_{\rho\times\rho'}\underbrace{\boldsymbol{R}_{\boldsymbol{A}',\boldsymbol{B}'}\boldsymbol{R}^{\dagger}_{\boldsymbol{A},\boldsymbol{B}}}_{\rho'\times\rho} \qquad \text{(A.2)}\end{aligned}$$

where we have dropped the indices for notational simplicity. Because $(\boldsymbol{A}(n), \boldsymbol{B}(n), \boldsymbol{C}(n), \boldsymbol{D}(n))$ is SER and SEO, the left-hand side of (A.2) is a $\rho\times\rho$ nonsingular matrix. The rank of the matrix on the right-hand side of (A.2) is at most $\rho' < \rho$, which is a contradiction! Therefore, we cannot find a realization with fewer than ρ delays.

Appendix B
Proof of SEO of Normalized IIR Lattice

Consider Fig. 15. Since $\hat{\alpha}_k(n) = \sqrt{1-|\alpha_k(n)|^2} \ge \beta > 0$ for all n, we have

$$\begin{bmatrix} w_i(n) \\ s_i(n) \end{bmatrix} = \underbrace{\frac{1}{\hat{\alpha}_k(n)}\begin{bmatrix} 1 & -\alpha_k(n) \\ -\alpha_k(n) & 1 \end{bmatrix}}_{\boldsymbol{T}_k(n)}\begin{bmatrix} w_{i+1}(n) \\ s_{i+1}(n+1) \end{bmatrix} \qquad \text{(B.1)}$$

where $w_{\rho+1}(n)$ is the input $x(n)$, and $s_{\rho+1}(n+1)$ is the output $y(n)$. Knowing the input $x(i)$ and the output $y(i)$ for $n \le i \le n+\rho-1$, we can determine $s_\rho(i)$ and $w_\rho(i)$ for $n \le i \le n+\rho-1$ by using (B.1). The information of $s_\rho(i)$ and $w_\rho(i)$ for $n \le i \le n+\rho-1$ can be used to determine $s_{\rho-1}(i)$ and $w_{\rho-1}(i)$ for $n \le i \le n+\rho-2$. Continuing this procedure, we can determine $s_k(n)$ for $1 \le k \le \rho$. The structure in Fig. 15 is therefore SEO. Furthermore, since $\|\boldsymbol{T}_k(n)\| \le \text{constant} < \infty$, we have $\|\boldsymbol{O}_{\boldsymbol{A},\boldsymbol{C}}(n+\rho,n)\| \ge \epsilon > 0$.

References

[1] S.-M. Phoong, and P. P. Vaidyanathan, "Time-varying filters and filter banks: Some basic principles," *IEEE Trans. Signal Processing*, vol. 44, pp. 2971–2987, Dec. 1996.

[2] P. P. Vaidyanathan and P.-Q. Hoang, "Lattice structures for optimal design and robust implementation of two-channel perfect reconstruction QMF banks," *IEEE Trans. Acoust. Speech, Signal Processing*, vol. 36, pp. 81–94, Jan. 1988.

[3] Z. Doganata, P. P. Vaidyanathan, and T. Q. Nguyen, "General synthesis procedures for FIR lossless transfer matrices, for perfect-reconstruction multirate filter bank applications," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 36, pp. 1561–1574, Oct. 1988.

[4] I. Sodagar, K. Nayebi, T. P. Barnwell, and M. J. T. Smith, "Time-varying analysis-synthesis systems based on filter banks and post-filtering," *IEEE Trans. Signal Processing*, vol. 43, pp. 2512–2521, Nov. 1995.

[5] H. S. Malvar, *Signal Processing with Lapped Transform.* Norwood, MA: Artech House, 1992.

[6] R. L. de Queiroz and K. R. Rao, "Time-varying lapped transform and wavelet packets," *IEEE Trans. Signal Processing*,, vol. 41, pp. 3293–3305, Dec. 1993.

[7] J. L. Arrowood, Jr. and M. J. T. Smith, "Exact reconstruction analysis/synthesis filter banks with time-varying filters," in *Proc. Int. Conf. Acoust. Speech, Signal Processing*, Minneapolis, MN, Apr. 1993, pp. 233–236.

[8] R. A. Gopinath, and C. S. Burrus, "Factorization approach to unitary time-varying filter bank trees and wavelets," *IEEE Trans. Signal Processing*, vol. 43, pp. 666–680, Mar. 1995.

[9] A. H. Gray, and J. D. Markel, "A normalized digital filter structure," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-23, pp. 268–277, June 1975.

[10] P. P. Vaidyanathan and T. Chen, "Role of anticausal inverses in multirate filter banks, part I: System-theoretic fundamentals," *IEEE Trans. Signal Processing*, vol. 43, pp. 1090–1102, May 1995.

[11] ——, "Role of anticausal inverses in multirate filter banks, Part II: The FIR case, factorizations, and biorthonormal lapped transforms (BOLT)," *IEEE Trans. Signal Processing*, vol. 43, pp. 1103–1115, May 1995.

[12] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*. Englewood Cliffs, NJ: Prentice-Hall, 1993.

[13] R. A. Horn, and C. R. Johnson, *Matrix Analysis.* Cambridge, U.K.: Cambridge Univ. Press, 1990.

[14] R. A. Decarlo, *Linear Systems: A State Variable Approach with Numerical Implementation*. Englewood Cliffs, NJ: Prentice-Hall, 1989.

[15] T. Kailath, *Linear Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1980.

[16] S. K. Mitra and J. F. Kaiser, *Handbook for Digital Signal Processing*. New York: Wiley, 1993.

[17] P. P. Vaidyanathan and V. Liu, "An improved sufficient condition for absence of limit cycles in digital filters," *IEEE Trans Cirtcuits Syst.*, vol. CAS-34, pp. 319–332, 1987.

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