A Complete Factorization of Paraunitary Matrices with Pairwise Mirror-Image Symmetry in the Frequency Domain

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***Abstract*—The problem of designing orthonormal (paraunitary) filter banks has been addressed in the past. Several structures have been reported for implementing such systems. One of the structur es reported imposes a pairwise mirror-image symmetry constraint on the frequency responses of the analysis (and synthesis) filters around $\pi/2$. This structure requires fewer multipliers, and the design time is correspondingly less than most other structures. The filters designed also have much better attenuation.**

In this correspodence, we characterize the polyphase matrix of the above filters in terms of a matrix equation. We then prove that the structure reported in a paper by Nguyen and Vaidyanathan, with minor modifications, is complete. This means that every polyphase matrix whose filters satisfy the mirror-image property can be factorized in terms of the proposed structure.

I. Introduction

Digital filter banks have been used in the past to decompose a signal into frequency subbands, and the theory of perfect reconstruction has also been studied extensively. One of the ways of ensuring perfect reconstruction is by choosing the polyphase matrix of the filters to be a paraunitary matrix [1]. Several structures have been developed for implementing paraunitary systems. In [2], the authors have imposed the *additional* condition that the analysis (and synthesis) filters satisfy the pairwise mirror-image symmetry constraint in the frequency domain around $\pi/2$. The advantage of the resulting structure is that it requires fewer parameters. The design time is correspondingly lower than other structures. The filter responses obtained using this structure are also better and have been widely used.

A natural question that arises is whether the structure suggested in [2] is *complete*. Completeness of the structure would imply that every polyphase matrix whose filters satisfy the pairwise mirror-image symmtery property can be factorized in terms of this structure. This question has not been addressed in [2]. The purpose of this work is to prove that the factorization suggested in [2] is indeed complete. We will also show how each of the building blocks can be parametrized by a minimum number of free variables.

Another point worth mentioning is the *minimality* of the structure. A structure is said to be minimal if it uses the minimum number of necessary delay elements [1]. The minimality of our structure can be easily verified using Theorem 14.7.2 in [1].

Most of our notation will be identical to that used in [2]. The number of channels M is even. Other notations will be defined as required.

II. Factorization of Paraunitary Matrices Having Pairwise Mirror-Image Symmetry

In this section, we first obtain a factorization of paraunitary matrices whose filters satisfy the pairwise mirror-image symmetry. The filters have real coefficients, and hence, the polyphase matrices are also real. The approach we take is based on directly manipulating the polyphase matrices and yields a compact derivation. Next, we prove the completeness of this structure, which is the main result of this paper.

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Let the filters at the mth stage be denoted as $H_{m,i}(z)$. If these filters satisfy pairwise mirror-image symmetry, they can be related to each other as (see (33) of [2])

$$H_{m,M-1-k}(z) = z^{-((m+1)M-1)} H_{m,k}(-z^{-1}), \quad k = 0, \cdots, M-1. \tag{2.1}$$

The order of each filter is $(m+1)M-1$. It can be verified that in this case, the polyphase matrix $\boldsymbol{E}_m(z)$ of the filters satisfies the matrix equation

$$z^{-m}\boldsymbol{Q}\boldsymbol{J}_M\boldsymbol{V}_M\boldsymbol{E}_m(z^{-1})\boldsymbol{V}_M\boldsymbol{J}_M = \boldsymbol{E}_m(z). \tag{2.2}$$

In the above equation, the matrix $\boldsymbol{J}_M$ is the antidiagonal matrix of size $M \times M$. For example

$$\boldsymbol{J}_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix $\boldsymbol{V}_M$ is a diagonal matrix of size $M \times M$ with alternating ± 1's on the diagonal starting with $+1$. Hence, if $M/2$ is even, we can write

$$\boldsymbol{V}_M = \begin{pmatrix} \boldsymbol{V}_{M/2} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{V}_{M/2} \end{pmatrix}$$

whereas if $M/2$ is odd

$$\boldsymbol{V}_M = \begin{pmatrix} \boldsymbol{V}_{M/2} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{V}_{M/2} \end{pmatrix}.$$

The matrix $\boldsymbol{Q}$ is by definition

$$\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{V}_{M/2} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{V}_{M/2} \end{pmatrix}$$

for even $M/2$, whereas

$$\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{V}_{M/2} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{V}_{M/2} \end{pmatrix}$$

for odd $M/2$. In either case, (2.2) may be simplified to

$$z^{-m}\boldsymbol{W}\boldsymbol{E}_m(z^{-1})\boldsymbol{V}_M\boldsymbol{J}_M = \boldsymbol{E}_m(z) \tag{2.3}$$

where

$$\boldsymbol{W} = \boldsymbol{Q}\boldsymbol{J}_M\boldsymbol{V}_M = \begin{pmatrix} \boldsymbol{0} & -\boldsymbol{J}_{M/2} \\ \boldsymbol{J}_{M/2} & \boldsymbol{0} \end{pmatrix}. \tag{2.4}$$

Suppose we add another stage to the cascade. Then, the polyphase matrix corresponding to the filters on the next stage is given by

$$\boldsymbol{E}_{m+1}(z) = \boldsymbol{K}_{m+1}\boldsymbol{A}(z)\boldsymbol{E}_m(z) \tag{2.5}$$

where

$$\boldsymbol{A}(z) = \begin{pmatrix} \boldsymbol{I}_{M/2} & \boldsymbol{0} \\ \boldsymbol{0} & z^{-1}\boldsymbol{I}_{M/2} \end{pmatrix}. \tag{2.6}$$

$\boldsymbol{I}_N$ is the identity matrix of size $N \times N$ in the above. If $\boldsymbol{K}_{m+1}$ is an orthogonal matrix, we have

$$\boldsymbol{E}_m(z) = \boldsymbol{A}(z^{-1})\boldsymbol{K}_{m+1}^T\boldsymbol{E}_{m+1}(z). \tag{2.7}$$

For the filters at the next stage to retain the pairwise mirror-image property, we need

$$z^{-(m+1)}\boldsymbol{W}\boldsymbol{E}_{m+1}(z^{-1})\boldsymbol{V}_M\boldsymbol{J}_M = \boldsymbol{E}_{m+1}(z). \tag{2.8}$$

Using (2.7) in (2.3), we obtain the following equation:

$$\begin{aligned} z^{-m}\boldsymbol{W}\boldsymbol{\Lambda}(z)\boldsymbol{K}^T_{m+1}\boldsymbol{E}_{m+1}(z^{-1})\boldsymbol{V}_M\boldsymbol{J}_M \\ = \boldsymbol{\Lambda}(z^{-1})\boldsymbol{K}^T_{m+1}\boldsymbol{E}_{m+1}(z). \end{aligned}$$

Using the identity $\boldsymbol{\Lambda}(z)\boldsymbol{W}\boldsymbol{\Lambda}(z) = z^{-1}\boldsymbol{W}$, we see that the necessary and sufficient condition on $\boldsymbol{K}_{m+1}$ for (2.8) to hold is $\boldsymbol{K}_{m+1}\boldsymbol{W}\boldsymbol{K}^T_{m+1} = \boldsymbol{W}$. By partitioning $\boldsymbol{K}_{m+1}$ as $\begin{pmatrix} \boldsymbol{A}' & \boldsymbol{C}' \\ \boldsymbol{B}' & \boldsymbol{D}' \end{pmatrix}$, we can verify that the necessary and sufficient condition for (2.8) to hold is that the matrix $\boldsymbol{K}_{m+1}$ be of the form

$$\boldsymbol{K}_{m+1} = \begin{pmatrix} \boldsymbol{A}' & \boldsymbol{C}' \\ -\boldsymbol{J}_{M/2}\boldsymbol{C}'\boldsymbol{J}_{M/2} & \boldsymbol{J}_{M/2}\boldsymbol{A}'\boldsymbol{J}_{M/2} \end{pmatrix}. \tag{2.9}$$

Thus, $\boldsymbol{K}_{m+1}$ can be rewritten as

$$\boldsymbol{K}_{m+1} = \underbrace{\begin{pmatrix} \boldsymbol{I}_{M/2} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J}_{M/2} \end{pmatrix}}_{\boldsymbol{P}} \underbrace{\begin{pmatrix} \boldsymbol{A}_{m+1} & \boldsymbol{C}_{m+1} \\ -\boldsymbol{C}_{m+1} & \boldsymbol{A}_{m+1} \end{pmatrix}}_{\boldsymbol{T}_{m+1}} \underbrace{\begin{pmatrix} \boldsymbol{I}_{M/2} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J}_{M/2} \end{pmatrix}}_{\boldsymbol{P}} \tag{2.10}$$

where $\boldsymbol{A}_{m+1} = \boldsymbol{A}'$, and $\boldsymbol{C}_{m+1} = \boldsymbol{C}'\boldsymbol{J}_{M/2}$. The result of [2] is equivalent to this result. The conditions on the matrix $\boldsymbol{K}_0$ that initializes the process can be worked out similarly, as has been done in [2].

We now address the converse.

Theorem: Let $\boldsymbol{E}_{m+1}(z)$ be a FIR paraunitary matrix whose filters have pairwise mirror-image symmetry in the frequency domain (i.e., $\boldsymbol{E}_{m+1}(z)$ satisfies (2.8)). Then, it can always be factored as

$$\boldsymbol{E}_{m+1}(z) = \boldsymbol{K}_{m+1}\boldsymbol{\Lambda}(z)\boldsymbol{K}_m\boldsymbol{\Lambda}(z)\cdots\boldsymbol{\Lambda}(z)\boldsymbol{K}_0 \tag{2.11}$$

where $\boldsymbol{\Lambda}(z)$ is as in (2.6), and $\boldsymbol{K}_i$ are as in (2.10).

Proof: We perform the "order-reduction" process as outlined below. Let

$$\begin{aligned} \boldsymbol{E}_{m+1}(z) = &\, \boldsymbol{e}_{m+1}(0) + \boldsymbol{e}_{m+1}(1)z^{-1} + \boldsymbol{e}_{m+1}(2)z^{-2} \\ &+ \cdots + \boldsymbol{e}_{m+1}(m+1)z^{-(m+1)}, \\ &\boldsymbol{e}_{m+1}(m+1) \neq \boldsymbol{0} \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} \boldsymbol{E}_m(z) = &\, \boldsymbol{e}_m(0) + \boldsymbol{e}_m(1)z^{-1} + \boldsymbol{e}_m(2)z^{-2} \\ &+ \cdots + \boldsymbol{e}_m(m)z^{-m}, \boldsymbol{e}_m(m) \neq \boldsymbol{0}. \end{aligned} \tag{2.13}$$

Let $\boldsymbol{E}_{m+1}(z)$ satisfy (2.8). Specifically, we will now show that it can always be written as

$$\boldsymbol{E}_{m+1}(z) = \underbrace{\boldsymbol{P}\boldsymbol{T}_{m+1}\boldsymbol{P}}_{\boldsymbol{K}_{m+1}}\boldsymbol{\Lambda}(z)\boldsymbol{E}_m(z) \tag{2.14}$$

where $\boldsymbol{E}_m(z)$ satisfies (2.3), and the matrices $\boldsymbol{P}, \boldsymbol{T}_{m+1}$, and $\boldsymbol{\Lambda}(z)$ have the form described in (2.10) and (2.6).

Paraunitariness of $\boldsymbol{E}_m(z)$ follows by noting that

$$\boldsymbol{E}_m(z) = \boldsymbol{\Lambda}(z^{-1})\boldsymbol{P}\boldsymbol{T}^T_{m+1}\boldsymbol{P}\boldsymbol{E}_{m+1}(z) \tag{2.15}$$

where all matrices on the right-hand side of this equation are paraunitary.

Pairwise Mirror-Image Property: We want to show that $\boldsymbol{E}_m(z)$ satisfies (2.3). Substituting (2.14) into (2.8), we get

$$\begin{aligned} z^{-(m+1)}\boldsymbol{W}\boldsymbol{P}\boldsymbol{T}_{m+1}\boldsymbol{P}\boldsymbol{\Lambda}(z^{-1})\boldsymbol{E}_m(z^{-1})\boldsymbol{V}_M\boldsymbol{J}_M \\ = \boldsymbol{P}\boldsymbol{T}_{m+1}\boldsymbol{P}\boldsymbol{\Lambda}(z)\boldsymbol{E}_m(z). \end{aligned} \tag{2.16}$$

Since $\boldsymbol{P}^{-1} = \boldsymbol{P}$ and $\boldsymbol{E}_m(z)$ is paraunitary, and noting that $\boldsymbol{W}^{-1} = -\boldsymbol{W}$, we get

$$\begin{aligned} z^{-(m+1)}\boldsymbol{\Lambda}(z^{-1})\boldsymbol{E}_m(z^{-1})\boldsymbol{V}_M\boldsymbol{J}_M\tilde{\boldsymbol{E}}_m(z) \\ = -\boldsymbol{P}\boldsymbol{T}^T_{m+1}\boldsymbol{P}\boldsymbol{W}\boldsymbol{P}\boldsymbol{T}_{m+1}\boldsymbol{P}\boldsymbol{\Lambda}(z). \end{aligned} \tag{2.17}$$

If $\boldsymbol{T}_{m+1}$ is an orthogonal matrix of the form described in (2.10), and $\boldsymbol{P}$ has the form described in (2.10), then it can be verified that $\boldsymbol{P}\boldsymbol{T}^T_{m+1}\boldsymbol{P}\boldsymbol{W}\boldsymbol{P}\boldsymbol{T}_{m+1}\boldsymbol{P} = \boldsymbol{W}$. Hence, we get

$$z^{-(m+1)}\boldsymbol{\Lambda}(z^{-1})\boldsymbol{E}_m(z^{-1})\boldsymbol{V}_M\boldsymbol{J}_M\tilde{\boldsymbol{E}}_m(z) = -\boldsymbol{W}\boldsymbol{\Lambda}(z). \tag{2.18}$$

It follows that

$$z^{-m}[z^{-1}\boldsymbol{\Lambda}(z^{-1})\boldsymbol{W}\boldsymbol{\Lambda}(z^{-1})]\boldsymbol{E}_m(z^{-1})\boldsymbol{V}_M\boldsymbol{J}_M\tilde{\boldsymbol{E}}_m(z) = \boldsymbol{I}_M. \tag{2.19}$$

It can be verified that $[z^{-1}\boldsymbol{\Lambda}(z^{-1})\boldsymbol{W}\boldsymbol{\Lambda}(z^{-1})] = \boldsymbol{W}$. Sustituting this into (2.19), and rearranging the terms, we get (2.3).

Causality: It only remains to show that there exists a matrix $\boldsymbol{T}_{m+1}$ such that $\boldsymbol{E}_m(z)$ obtained from (2.15) is causal. Both the pairwise mirror image property and the paraunitary property continue to hold for the reduced system as long as the matrix $\boldsymbol{T}_{m+1}$ is *any* orthogonal matrix of the required form (2.10). Indeed, it is the causality condition on the reduced system that determines the particular choice of the matrix $\boldsymbol{T}_{m+1}$.

From (2.15) we get

$$\begin{aligned} \boldsymbol{E}_m(z) = &\begin{pmatrix} \boldsymbol{I}_{M/2} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}\boldsymbol{P}\boldsymbol{T}^T_{m+1}\boldsymbol{P}\boldsymbol{E}_{m+1}(z) \\ &+ \begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & z\boldsymbol{I}_{M/2} \end{pmatrix}\boldsymbol{P}\boldsymbol{T}^T_{m+1}\boldsymbol{P}\boldsymbol{E}_{m+1}(z). \end{aligned} \tag{2.20}$$

The second term on the right-hand side of this equation is responsible for the noncausality. In particular, the noncausal part of the second term is given by

$$\begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & z\boldsymbol{I}_{M/2} \end{pmatrix}\boldsymbol{P}\boldsymbol{T}^T_{m+1}\boldsymbol{P}\boldsymbol{e}_{m+1}(0). \tag{2.21}$$

We have to show that there exists a matrix $\boldsymbol{T}_{m+1}$ of the form

$$\boldsymbol{T}_{m+1} = \begin{pmatrix} \boldsymbol{A}_{m+1} & \boldsymbol{C}_{m+1} \\ -\boldsymbol{C}_{m+1} & \boldsymbol{A}_{m+1} \end{pmatrix} \tag{2.22}$$

which makes the above noncausal term equal to zero. Simplifying (2.21), we find that $\boldsymbol{T}_{m+1}$ should be such that

$$\begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J}_{M/2} \end{pmatrix}\begin{pmatrix} \boldsymbol{A}^T_{m+1} & -\boldsymbol{C}^T_{m+1}\boldsymbol{J}_{M/2} \\ \boldsymbol{C}^T_{m+1} & \boldsymbol{A}^T_{m+1}\boldsymbol{J}_{M/2} \end{pmatrix}\boldsymbol{e}_{m+1}(0) = \boldsymbol{0}. \tag{2.23}$$

Hence, it is sufficient to find $\boldsymbol{A}_{m+1}$ and $\boldsymbol{C}_{m+1}$ such that

$$(\boldsymbol{C}^T_{m+1} \quad \boldsymbol{A}^T_{m+1}\boldsymbol{J}_{M/2})\boldsymbol{e}_{m+1}(0) = \boldsymbol{0}. \tag{2.24}$$

Consider the matrix $\boldsymbol{W}$ (in (2.4)). The eigenvalues of this matrix are $\pm j$, where $j = \sqrt{-1}$. The eigenvectors corresponding to the eigenvalue j are

$$\boldsymbol{s}_0 = \begin{pmatrix} j \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \cdots, \boldsymbol{s}_{M/2-1} = \begin{pmatrix} \vdots \\ 0 \\ j \\ 1 \\ 0 \\ \vdots \end{pmatrix}. \tag{2.25}$$

We will denote the space spanned by these vectors as $\mathcal{E}_1$. Similarly, the eigenvectors corresponding to the eigenvalue $-j$ are

$$\boldsymbol{a}_0 = \begin{pmatrix} -j \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \cdots, \boldsymbol{a}_{M/2-1} = \begin{pmatrix} \vdots \\ 0 \\ -j \\ 1 \\ 0 \\ \vdots \end{pmatrix}. \tag{2.26}$$

We will denote the space spanned by these vectors as $\mathcal{E}_2$.

Since the matrix $\boldsymbol{W}$ is skew-symmetric, these eigenvectors together span the entire space. Now, consider any vector $\boldsymbol{y}$. It can always be written as $\boldsymbol{y} = \boldsymbol{u} + \boldsymbol{v}$, where $\boldsymbol{u} \in \mathcal{E}_1$ and $\boldsymbol{v} \in \mathcal{E}_2$. Moreover, if $\boldsymbol{y}$ is real, we have $\boldsymbol{u} = \boldsymbol{v}^*$, where the asterisk superscript denotes conjugation (see the Appendix). This therefore implies that

$$\boldsymbol{u}^\dagger \boldsymbol{u} = \boldsymbol{v}^\dagger \boldsymbol{v} \tag{2.27}$$

where the superscript $\dagger$ denotes conjugate transpose.

Equation (2.8) implies in particular that $\boldsymbol{W}\boldsymbol{e}_{m+1}(0)\boldsymbol{V}_M\boldsymbol{J}_M = \boldsymbol{e}_{m+1}(m+1)$. Paraunitariness of $\boldsymbol{E}(z)$, on the other hand, implies that $\boldsymbol{e}_{m+1}^T(m+1)\boldsymbol{e}_{m+1}(0) = \boldsymbol{0}$. Hence

$$\boldsymbol{J}_M^T \boldsymbol{V}_M^T \boldsymbol{e}_{m+1}^T(0) \boldsymbol{W}^T \boldsymbol{e}_{m+1}(0) = \boldsymbol{0}. \tag{2.28}$$

Using the facts that $\boldsymbol{W}^T = -\boldsymbol{W}$ and $\boldsymbol{e}_{m+1}(0)$ is real, we get $\boldsymbol{e}_{m+1}^\dagger(0)\boldsymbol{W}\boldsymbol{e}_{m+1}(0) = \boldsymbol{0}$. We therefore have

$$\boldsymbol{U}^\dagger \boldsymbol{e}_{m+1}^\dagger(0) \boldsymbol{W} \boldsymbol{e}_{m+1}(0) \boldsymbol{U} = \boldsymbol{0} \tag{2.29}$$

for any matrix $\boldsymbol{U}$. Let the matrix $\boldsymbol{U}$ be so chosen that the first r columns of the matrix $\boldsymbol{e}_{m+1}(0)\boldsymbol{U}$ form an orthonormal basis of real vectors $\boldsymbol{x}_i$ for the columns of the matrix $\boldsymbol{e}_{m+1}(0)$. (This is possible, since the matrix $\boldsymbol{e}_{m+1}(0)$ is itself real. Hence, r is the rank of the matrix $\boldsymbol{e}_{m+1}(0)$). These vectors $\boldsymbol{x}_i$ being orthonormal satisfy $\boldsymbol{x}_i^\dagger \boldsymbol{x}_j = 0, (i \neq j)$. Let $\boldsymbol{x}_i = \boldsymbol{u}_i' + \boldsymbol{v}_i'$, where $\boldsymbol{u}_i' \in \mathcal{E}_1$ and $\boldsymbol{v}_i' \in \mathcal{E}_2$. Therefore, $(\boldsymbol{u}_i' + \boldsymbol{v}_i')^\dagger(\boldsymbol{u}_j' + \boldsymbol{v}_j') = 0$, which simplifies to

$$\boldsymbol{u}_i'^\dagger \boldsymbol{u}_j' + \boldsymbol{v}_i'^\dagger \boldsymbol{v}_j' = 0. \tag{2.30}$$

The real vectors $\boldsymbol{x}_i$ satisfy $\boldsymbol{x}_i^\dagger \boldsymbol{W} \boldsymbol{x}_i = \boldsymbol{0}$ since $\boldsymbol{W}$ is antisymmetric. Hence

$$(\boldsymbol{u}_i' + \boldsymbol{v}_i')^\dagger \boldsymbol{W} (\boldsymbol{u}_j' + \boldsymbol{v}_j') = 0 \tag{2.31}$$

i.e.

$$\boldsymbol{u}_i'^\dagger \boldsymbol{W} \boldsymbol{u}_j' + \boldsymbol{v}_i'^\dagger \boldsymbol{W} \boldsymbol{u}_j' + \boldsymbol{u}_i'^\dagger \boldsymbol{W} \boldsymbol{v}_j' + \boldsymbol{v}_i'^\dagger \boldsymbol{W} \boldsymbol{v}_j' = 0. \tag{2.32}$$

Noting that the vectors $\boldsymbol{u}_i'$ and $\boldsymbol{v}_j'$ are orthogonal, for all i, j, and the fact that $\boldsymbol{W}\boldsymbol{v}_i' = -j\boldsymbol{v}_i', \boldsymbol{W}\boldsymbol{u}_i' = j\boldsymbol{u}_i'$, we get

$$\boldsymbol{u}_i'^\dagger \boldsymbol{u}_j' = \boldsymbol{v}_i'^\dagger \boldsymbol{v}_j'. \tag{2.33}$$

Equations (2.30) and (2.33) together imply that $\boldsymbol{u}_i'^\dagger \boldsymbol{u}_j' = 0$, and $\boldsymbol{v}_i'^\dagger \boldsymbol{v}_j' = 0$. The vectors $\boldsymbol{u}_i', i = 1, \cdots, r$ and $\boldsymbol{v}_i', i = 1, \cdots, r$ therefore form orthonormal bases for r-dimensional subspaces of $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. Moreover, since the $\boldsymbol{x}_i, i = 1, \cdots, r$ are real, we have (again referring to Appendix A)

$$\boldsymbol{u}_i' = \boldsymbol{v}_i'^*. \tag{2.34}$$

In $\mathcal{E}_1$, there exist $p = M/2 - r$ orthogonal vectors $\boldsymbol{u}_i', i = r+1, \cdots, M/2$ which are also orthogonal to the previously mentioned set of r vectors $\boldsymbol{u}_i', i = 1, \cdots, r$. Similarly, in $\mathcal{E}_2$, there exist $p = M/2 - r$ orthogonal vectors $\boldsymbol{v}_i', i = r+1, \cdots, M/2$, which are also orthogonal to the previously mentioned set of r vectors $\boldsymbol{v}_i', i = 1, \cdots, r$. Clearly, a particular choice of these p vectors in $\mathcal{E}_2$ could be the conjugates of the p vectors chosen in $\mathcal{E}_1$.

Now, using these additional p orthonormal vectors from $\mathcal{E}_1$ and adding them to their conjugates from $\mathcal{E}_2$, we can form p orthonormal real vectors $\boldsymbol{x}_i$.

Let $\boldsymbol{X}^T$ be the matrix of size $(M/2) \times M$ whose rows are the vectors $\boldsymbol{x}_i, i = 1, \cdots, M/2 - 1$. This matrix satisfies the following properties:

1) $\boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{I}_{M/2}$ (from the fact that $\boldsymbol{x}_i$ are orthonormal).
2) $\boldsymbol{X}^T\boldsymbol{W}\boldsymbol{e}_{m+1}(0) = \boldsymbol{0}$ (by the construction outlined above).

Partition $\boldsymbol{X}^T = [\boldsymbol{Y}^T \quad \boldsymbol{Z}^T]$. Then, with $\boldsymbol{C}_{m+1}^T = \boldsymbol{Z}^T\boldsymbol{J}_{M/2}$ and $\boldsymbol{A}_{m+1}^T = -\boldsymbol{Y}^T$, the matrix $\boldsymbol{T}_{m+1}$ defined in (2.22) is orthogonal and satisfies (2.23). This proves that $\boldsymbol{E}_m(z)$ is causal.

Order Reduction: Given the fact that $\boldsymbol{E}_m(z)$ is causal and that it satisfies (2.3), we can see that the order of $\boldsymbol{E}_m(z)$ is m. Thus, there is a reduction in order by one. Hence, for a system of order N, the factorization process is guaranteed to terminate in N steps.

This concludes the proof of the theorem.

Appendix

As in Section II, we denote the two eigenspaces of the matrix $\boldsymbol{W}$ by $\mathcal{E}_1$ and $\mathcal{E}_2$, corresponding to the eigenvalues j and $-j$, respectively. Let $\boldsymbol{y} = \boldsymbol{u} + \boldsymbol{v}$, where $\boldsymbol{u} \in \mathcal{E}_1$ and $\boldsymbol{v} \in \mathcal{E}_2$. Let

$$\boldsymbol{u} = \alpha_1\boldsymbol{s}_1 + \cdots + \alpha_{M/2-1}\boldsymbol{s}_{M/2-1}$$

and

$$\boldsymbol{v} = \beta_1\boldsymbol{a}_1 + \cdots + \beta_{M/2-1}\boldsymbol{a}_{M/2-1}$$

where the $\boldsymbol{s}_i$ and the $\boldsymbol{a}_i$ are the basis vectors for the two subspaces as in Section II. From their definition, it is clear that if $\boldsymbol{y}$ is real, $\alpha_i\boldsymbol{s}_i + \beta_i\boldsymbol{a}_i$ is real for $i = 1, \cdots, M/2-1$. Since $\boldsymbol{s}_i = \boldsymbol{a}_i^*$, it follows that $\alpha_i = \beta_i^*$. Hence, $\boldsymbol{u} = \boldsymbol{v}^*$.

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