

Modeling Position and Momentum in Finite-Dimensional Hilbert Spaces via Generalized Clifford Algebra

Ashmeet Singh* and Sean M. Carroll†

*Walter Burke Institute for Theoretical Physics,
California Institute of Technology, Pasadena, CA 91125*

The finite entropy of black holes suggests that local regions of spacetime are described by finite-dimensional factors of Hilbert space, in contrast with the infinite-dimensional Hilbert spaces of quantum field theory. With this in mind, we explore how to cast finite-dimensional quantum mechanics in a form that matches naturally onto the smooth case, especially the recovery of conjugate position/momentum variables, in the limit of large Hilbert-space dimension. A natural tool for this task is the generalized Clifford algebra (GCA). Based on an exponential form of Heisenberg’s canonical commutation relation, the GCA offers a finite-dimensional generalization of conjugate variables without relying on any *a priori* structure on Hilbert space. We highlight some features of the GCA, its importance in studying concepts such as locality of operators, and point out departures from infinite-dimensional results (possibly with a cutoff) that might play a crucial role in our understanding of quantum gravity. We introduce the concept of “Schwinger locality,” which characterizes how the action of an operator spreads a quantum state along conjugate directions. We illustrate these concepts with a worked example of a finite-dimensional harmonic oscillator, demonstrating how the energy spectrum deviates from the familiar infinite-dimensional case.

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* ashmeet@caltech.edu

† seancarroll@gmail.com

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I. INTRODUCTION

The Hilbert space of a quantum field theory is infinite-dimensional, for three different reasons: wavelengths can be arbitrarily large, and they can be arbitrarily small, and at any one wavelength the occupation number of bosonic modes can be arbitrarily high. Once we include gravity, however, all of these reasons come into question. In the presence of a positive vacuum energy, the de Sitter radius provides a natural infrared cutoff at long wavelengths; the Planck scale provides a natural ultraviolet cutoff at short wavelengths; and the Bekenstein bound [1–3] (or more generally, black hole formation and its consequent finite entropy) provides an energy cutoff. It therefore becomes natural to consider theories where Hilbert space, or at least the factor of Hilbert space describing our observable region of the cosmos, is finite-dimensional [4–12].

Our interest here is in how structures such as fields and spatial locality emerge in a locally finite-dimensional context. Hilbert space is featureless: all Hilbert spaces of a specified finite dimension are isomorphic, and the algebra of observables is simply “all Hermitian operators.” Higher-level structures must therefore emerge from whatever additional data we are given, typically eigenstates and eigenvalues of the Hamiltonian and perhaps the amplitudes of a particular quantum state. To explore how this emergence occurs, in this paper we consider the role of conjugate variables in a finite-dimensional context.

In the familiar infinite-dimensional case, classical conjugate variables such as position (q) and momentum (p) are promoted to linear operators on Hilbert space obeying the Heisenberg canonical commutation relations (CCR),

$$[\hat{q}, \hat{p}] = i, \tag{1}$$

where throughout this paper we take $\hbar = 1$. In field theory, one takes the field and its conjugate momentum as operators labelled by spacetime points and generalizes the CCR to take a continuous form labelled by spacetime locations. The Stone-von Neumann theorem shows that there is a unique

irreducible representation (up to unitary equivalence) of the CCR on infinite-dimensional Hilbert spaces that are separable (possessing a countable dense subset), but also that the operators \hat{q}, \hat{p} must be unbounded. There are therefore no such representations on finite-dimensional spaces.

There is, however, a tool that works in finite-dimensional Hilbert spaces and maps onto conjugate variables in the infinite-dimensional limit: the Generalized Clifford Algebra (GCA). As we shall see, the GCA is generated by a pair of normalized operators \hat{A} and \hat{B} – sometimes written as “clock” and “shift” matrices – that commute up to a dimension-dependent phase,

$$\hat{A}\hat{B} = \omega^{-1}\hat{B}\hat{A}, \quad (2)$$

where $\omega = \exp(2\pi i/N)$ is the N -th primitive root of unity. Any linear operator can be written as a sum of products of these generators. Appropriate logarithms of these operators reduce, in the infinite-dimensional limit, to conjugate operators obeying the CCR. The GCA therefore serves as a starting point for analyzing the quantum mechanics of finite-dimensional Hilbert spaces in a way that matches naturally onto the infinite-dimensional limit.

We will follow a series of papers by Jagannathan, Santhanam, Tekumalla and Vasudevan [13–15] from the 1970-80’s, which developed the subject of finite-dimensional quantum mechanics, motivated by the Weyl’s exponential form of the CCR. These constructions have been discussed in the past by Weyl, Schwinger [16, 17] and many others in various contexts, some representative papers include [18–23] (and references therein). The basic mathematical constructions worked out in this paper are not new; our goal here is to distill the features of the GCA that are useful in the study of locally finite-dimensional Hilbert spaces in quantum gravity, especially the emergence of a classical limit.

The paper is organized as follows. In Section II, we motivate the need for an intrinsic finite-dimensional construction by pointing out the incompatibility of conventional textbook quantum mechanics and QFT with a finite-dimensional Hilbert space; and follow it up by introducing and using the GCA to construct a finite-dimensional generalization of conjugate variables. In Section III we introduce the concept of Schwinger Locality as a means to quantify and study the spread induced by operators along the conjugate variables. Section IV deals with understanding equations of motion for conjugate variables in a finite-dimensional context and how they map to Hamilton’s equations in the large dimension limit, and we explore features of the finite-dimensional quantum mechanical oscillator.

II. FINITE-DIMENSIONAL CONJUGATE VARIABLES FROM THE GENERALIZED CLIFFORD ALGEBRA

A. Prelude

Consider the problem of adapting the Heisenberg CCR (1) to finite-dimensional Hilbert spaces. One way of noticing an immediate obstacle is to take the trace of both sides; the left-hand side vanishes, while the right-hand side does not. To remedy this, Weyl [16] gave an equivalent version of Heisenberg’s CCR in exponential form,

$$e^{i\eta\hat{p}}e^{i\zeta\hat{q}} = e^{i\eta\zeta}e^{i\zeta\hat{q}}e^{i\eta\hat{p}}, \quad (3)$$

for some real parameters η and ζ . This does indeed admit a finite-dimensional representation (which is unique, up to unitary equivalence, as guaranteed by Stone-von Neumann theorem). One can interpret this to be a statement of how the conjugate operators \hat{q} and \hat{p} fail to commute,

although the commutation relation $[\hat{q}, \hat{p}]$ will now no longer have the simple form of a c-number (1).

The GCA offers a natural implementation of Weyl's relation (3) to define a set of intrinsically finite-dimensional conjugate operators. In this section, we will follow the construction laid out in references [13–15] in developing finite-dimensional quantum mechanics based on Weyl's exponential form of the CCR.

B. The Algebra

Consider a finite-dimensional Hilbert space \mathcal{H} of dimension

$$\dim \mathcal{H} = N, \quad (4)$$

with $N < \infty$. Let us associate a Generalized Clifford Algebra (GCA) on the space $\mathcal{L}(\mathcal{H})$ of linear operators acting on \mathcal{H} , by equipping it with two unitary operators as generators of the algebra, call them \hat{A} and \hat{B} , which satisfy the following commutation¹ relation,

$$\hat{A}\hat{B} = \omega^{-1}\hat{B}\hat{A}, \quad (5)$$

where $\omega = \exp(2\pi i/N)$ is the N -th primitive root of unity. This is also known as the Weyl Braiding relation in the physics literature, and is the basic commutation relation offered by the algebra. In addition to being unitary, the generators are normalized as follows,

$$\hat{A}^N = \hat{B}^N = \hat{\mathbb{I}}, \quad (6)$$

where $\hat{\mathbb{I}}$ is the identity operator on \mathcal{H} . The operators are further specified by their spectrum, which will be identical for both GCA generators \hat{A} and \hat{B} ,

$$\text{spec}(\hat{A}) = \text{spec}(\hat{B}) = \{1, \omega^1, \dots, \omega^2, \dots, \omega^{N-1}\}. \quad (7)$$

Thus, in line with our Hilbert-space perspective, specifying just the dimension N of Hilbert space is sufficient to construct the algebra, which determines the spectrum of the generators and the basic commutation relations.

The GCA can be constructed for both even and odd values of N and both cases are important and useful in different contexts. In this section, let us specialize to the case of odd $N \equiv 2l + 1$ for some $l \in \mathbb{Z}^+$, which will be useful in constructing conjugate variables whose eigenvalues can be thought of labelling lattice sites centered around 0. In the case of even dimensions $N = 2m$ for some $m \in \mathbb{Z}^+$, one will be able to define conjugate on a lattice labelled from $\{0, 1, 2, \dots, N - 1\}$ and not on a lattice centered around 0. For the case of $N = 2$, we recover the Pauli matrices, corresponding to $A = \sigma_x$ and $B = \sigma_z$. Operators on qubits can be seen as a special $N = 2$ case of the GCA. While the subsequent construction can be done in a basis-independent way, we choose a hybrid route, switching between an explicit representation of the GCA and abstract vector space relations, to explicitly point out the properties of the algebra.

Let us follow the convention that all indices used in this section (for the case of odd $N = 2l + 1$) for labelling states or matrix elements of an operator in some basis will run over

$$i, j, k \in -l, (-l + 1), \dots, -1, 0, 1, \dots, l - 1, l. \quad (8)$$

¹ This is a GCA with two generators that follow an ordered commutation relation. In general, a GCA can be defined with more generators and their braiding relations. For instance, the Clifford algebra of the ‘‘gamma’’ matrices used in spinor QFT and the Dirac equation is a particular GCA with 4 generators [24].

The eigenspectrum of both GCA generators \hat{A} and \hat{B} can be relabelled as,

$$\text{spec}(\hat{A}) = \text{spec}(\hat{B}) = \{\omega^{-l}, \omega^{-l+1}, \dots, \omega^{-1}, 1, \omega^1, \dots, \omega^{l-1}, \omega^l\}. \quad (9)$$

There exists a unique irreducible representation (up to unitary equivalence) [24] of the generators of the GCA defined via Eqs. (2) and (6) in terms of $N \times N$ matrices

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}_{N \times N}, \quad B = \begin{bmatrix} \omega^{-l} & 0 & 0 & \cdots & 0 \\ 0 & \omega^{-l+1} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \omega^l \end{bmatrix}_{N \times N}. \quad (10)$$

The $\hat{\cdot}$ has been removed to stress that these matrices are representations of the operators \hat{A} and \hat{B} in a particular basis, in this case, the eigenbasis of \hat{B} (so that B is diagonal). More compactly, the matrix elements of operators \hat{A} and \hat{B} in this basis are,

$$[A]_{jk} \equiv \langle b_j | \hat{A} | b_k \rangle = \delta_{j,k+1}, \quad [B]_{jk} \equiv \langle b_j | \hat{B} | b_k \rangle = \omega^j \delta_{j,k}, \quad (11)$$

with the indices j and k running from $-l, \dots, 0, \dots, l$ and δ_{jk} is the Kronecker delta function. The generators obey the following trace condition,

$$\text{Tr}(\hat{A}^j) = \text{Tr}(\hat{B}^j) = N \delta_{j,0}. \quad (12)$$

Let us now further understand the properties of the eigenvectors of \hat{A} and \hat{B} and the action of the algebra on them. Consider the set $\{|b_j\rangle\}$ of eigenstates of \hat{B} ,

$$\hat{B} |b_j\rangle = \omega^j |b_j\rangle. \quad (13)$$

As can be seen in the matrix representation of \hat{A} in Eq. (10), the operator \hat{A} acts to generate cyclic shifts for the eigenstates of \hat{B} , mapping an eigenstate to the next,

$$\hat{A} |b_j\rangle = |b_{j+1}\rangle. \quad (14)$$

The unitary nature of these generators implies a cyclic structure in which one identifies $|b_{l+1}\rangle \equiv |b_{-l}\rangle$, so that $\hat{A} |b_l\rangle = |b_{-l}\rangle$.

The operators \hat{A} and \hat{B} have the same relative action on the eigenstates of one another, as there is nothing in the algebra which distinguishes between the two. The operator \hat{B} generates unit shifts in eigenstates of \hat{A} ,

$$\hat{B} |a_k\rangle = |a_{k+1}\rangle, \quad (15)$$

with cyclic identification $|a_{l+1}\rangle \equiv |a_{-l}\rangle$. Hence we have a set of operators that generate shifts in the eigenstates of the other, which is precisely the way in which conjugate variables act and which is why the GCA provides a natural structure to define conjugate variables on Hilbert space. While should think of these eigenstates of \hat{A} and \hat{B} to be marked by their eigenvalues on a lattice: there is no notion of a scale or physical distance at this point, just a lattice of states labelled by their eigenvalues in a finite-dimensional construction along with a pair of operators which translate each other's states by unit shifts, respectively.

To further reinforce this conjugacy relation between \hat{A} and \hat{B} , we see that they are connected to each other under a discrete Fourier transformation implemented by Sylvester's circulant matrix S , which is an $N \times N$ unitary matrix connecting A and B via $SAS^{-1} = B$. Sylvester's matrix in the $\{|b_j\rangle\}$ basis has the form $[S]_{jk} = \omega^{jk}/\sqrt{N}$. The GCA generators \hat{A} and \hat{B} have been studied in various contexts in quantum mechanics and are often referred to as ‘‘clock and shift’’ matrices. They offer a higher dimensional, non-hermitian generalization of the Pauli matrices.

The set of N^2 linearly independent unitary matrices $\{B^b A^a | b, a = -l, (-l+1), \dots, 0, \dots, (l-1), l\}$, which includes the identity for $a = b = 0$, form a unitary basis for $\mathcal{L}(\mathcal{H})$. Schwinger [17] studied the role of such unitary basis, hence this operator basis is often called Schwinger's unitary basis. Any operator $\hat{M} \in \mathcal{L}(\mathcal{H})$ can be expanded in this basis,

$$\hat{M} = \sum_{b,a=-l}^l m_{ba} \hat{B}^b \hat{A}^a. \quad (16)$$

Since from the structure of the GCA we have $\text{Tr} \left[\left(\hat{B}^{b'} \hat{A}^{a'} \right)^\dagger \left(\hat{B}^b \hat{A}^a \right) \right] = N \delta_{b,b'} \delta_{a,a'}$, we can invert Eq. (16) to get the coefficients m_{ba} as,

$$m_{ba} = \frac{1}{N} \text{Tr} \left[\hat{A}^{-a} \hat{B}^{-b} \hat{M} \right]. \quad (17)$$

Thus, in addition to playing the role of conjugate variables in a finite-dimensional construction, the GCA fits in naturally with the program of minimal quantum mechanics in Hilbert space [12, 25] by being able to define a notion of conjugate variables, one is able to classify and use any other operator on this space, including the Hamiltonian that governs the dynamics. This notion will be important to us when we define the idea of conjugate spread of operators, the so-called ‘‘Schwinger Locality,’’ in Section III.

C. Finite-Dimensional Conjugate Variables

We are now prepared to define a notion of conjugate variables on a finite-dimensional Hilbert space. The defining notion for a pair of conjugate variables is identifying two self-adjoint operators that each generate translations in the eigenstates of the other. For instance, in textbook quantum mechanics, the momentum operator \hat{p} generates translations in the eigenstates of its conjugate variable, the position operator \hat{q} , and vice-versa. Taking this as our defining criterion, we would like to define a pair of conjugate operators acting on a finite-dimensional Hilbert space, each of which is the generator of translations in the eigenstates of its conjugate.

We define a pair $\hat{\phi}$ and $\hat{\pi}$ to be conjugate operators by making the following identification,

$$\hat{A} \equiv \exp(-i\alpha\hat{\pi}), \quad \hat{B} \equiv \exp(i\beta\hat{\phi}), \quad (18)$$

where α and β are non-zero real parameters which set the scale of the eigenspectrum of the operators $\hat{\phi}$ and $\hat{\pi}$. These are bounded operators on \mathcal{H} , and due to the virtue of the GCA generators \hat{A} and \hat{B} being unitary, the conjugate operators $\hat{\phi}$ and $\hat{\pi}$ are self-adjoint, satisfying $\hat{\phi}^\dagger = \hat{\phi}$ and $\hat{\pi}^\dagger = \hat{\pi}$. The operator $\hat{\pi}$ is the generator of translations of $\hat{\phi}$ and vice-versa. The apparent asymmetry in the sign in the exponential in Eq. (18) when identifying $\hat{\phi}$ and $\hat{\pi}$ is to ensure that the j -th column (with $j = -l, -l+1, \dots, 0, \dots, l-1, l$) of Sylvester's matrix S that diagonalizes A is an eigenstate of $\hat{\pi}$ with eigenvalue proportional to j , and hence on an ordered lattice. Of course, $\hat{\phi}$ has common eigenstates with those of \hat{B} and $\hat{\pi}$ shares eigenstates with \hat{A} . Let us label the eigenstates of $\hat{\phi}$

as $|\phi_j\rangle$ and those of $\hat{\pi}$ as $|\pi_j\rangle$ with the index j running from $-l, \dots, 0, \dots, l$. The corresponding eigenvalue equations for $\hat{\phi}$ and $\hat{\pi}$ can be easily deduced using Eqs. (18) and (9),

$$\hat{\phi}|\phi_j\rangle = j \left(\frac{2\pi}{(2l+1)\beta} \right) |\phi_j\rangle, \quad \hat{\pi}|\pi_j\rangle = j \left(\frac{2\pi}{(2l+1)\alpha} \right) |\pi_j\rangle. \quad (19)$$

Let us now solve for the conjugate operators $\hat{\phi}$ and $\hat{\pi}$ explicitly by finding their matrix representations in the $|\phi_j\rangle$ basis. By virtue of being diagonal, the principle logarithm of B is

$$\log B = (\log \omega) \text{diag}(-l, -l+1, \dots, 0, \dots, l-1, l). \quad (20)$$

Hence we have the matrix representation of $\hat{\phi}$,

$$\langle \phi_j | \hat{\phi} | \phi_{j'} \rangle = j \left(\frac{2\pi}{(2l+1)\beta} \right) \delta_{jj'}, \quad (21)$$

which is diagonal in the $|\phi_j\rangle$ basis as expected. To find a representation of $\hat{\pi}$ in this basis, we notice that \hat{A} is diagonalized by Sylvester's matrix, hence we can get its principle logarithm as $\log A = S^{-1}(\log B)S$. In the case of odd dimension $N = 2l + 1$, the principle logarithms of A and B are well-defined, and we are able to find explicit matrix representations for operators $\hat{\phi}$ and $\hat{\pi}$ as above. The conjugate operators $\hat{\phi}$ and $\hat{\pi}$ are connected through Sylvester's operator,

$$\hat{\pi} = \left(\frac{-\beta}{\alpha} \right) \hat{S}^{-1} \hat{\phi} \hat{S}, \quad \hat{\phi} = \left(\frac{-\alpha}{\beta} \right) \hat{S} \hat{\pi} \hat{S}^{-1}. \quad (22)$$

The following parity relations are obeyed, since S^2 is the parity operator, $[S^2]_{jk} = \delta_{j,-k}$,

$$\hat{S}^4 = \hat{\mathbb{I}}, \quad \hat{S}^2 \hat{\phi} \hat{S}^{-2} = -\hat{\phi}, \quad \hat{S}^2 \hat{\pi} \hat{S}^{-2} = -\hat{\pi}. \quad (23)$$

These relations have the same form as in infinite-dimensional quantum mechanics.

Using the expression $\log A = S^{-1}(\log B)S$, the matrix representation for $\hat{\pi}$ in the $|\phi_j\rangle$ basis is,

$$\langle \phi_j | \hat{\pi} | \phi_{j'} \rangle = \begin{cases} 0, & \text{if } j = j' \\ \left(\frac{i\pi}{(2l+1)\alpha} \right) \text{cosec} \left(\frac{2\pi l(j-j')}{2l+1} \right), & \text{if } j \neq j' \end{cases}$$

The eigenstates of both $\hat{\phi}$ and $\hat{\pi}$ each individually are orthonormal bases for the Hilbert space \mathcal{H} ,

$$\langle \phi_j | \phi_{j'} \rangle = \delta_{j,j'}, \quad \sum_{j=-l}^l |\phi_j\rangle \langle \phi_j| = \hat{\mathbb{I}}, \quad \langle \pi_j | \pi_{j'} \rangle = \delta_{j,j'}, \quad \sum_{j=-l}^l |\pi_j\rangle \langle \pi_j| = \hat{\mathbb{I}}. \quad (24)$$

Thus, using the generators of the GCA, we are able to naturally identify a notion of conjugate operators, each of which is the generator of translations for the eigenstates of the other as seen by Eqs. (18), (14) and (15). While the GCA provides us with a notion of dimensionless conjugate variables that have familiar "position/momentum" properties, there is no notion of a physical length scale as yet. The operators we will ultimately identify as classical position and momentum operators depend on a non-generic decomposition of Hilbert space into subsystems that makes emergent classicality manifest. This is the so called quantum factorization problem [26–28], sometimes referred to as the set selection problem [29].

D. The Commutator

In this section, we will work out the commutation relation between conjugate operators $\hat{\phi}$ and $\hat{\pi}$ as defined from the GCA in a finite-dimensional Hilbert space and understand how they deviate from the usual Heisenberg CCR and converge to it in the large dimension limit. In the infinite limit, the conjugate operators $\hat{\phi}$ and $\hat{\pi}$ obey Heisenberg's form of the CCR $[\hat{\phi}, \hat{\pi}] = i$, while our conjugate variables based on Eq. (18) satisfy the the GCA commutation relation,

$$\exp(-i\alpha\hat{\pi})\exp(i\beta\hat{\phi}) = \exp\left(-\frac{2\pi i}{2l+1}\right)\exp(i\beta\hat{\phi})\exp(-i\alpha\hat{\pi}). \quad (25)$$

On expanding the left-hand side of the GCA braiding relation Eq. (25) and using the Baker-Campbell-Hausdorff Lemma we obtain,

$$\exp\left(i\beta\hat{\phi} + [-i\alpha\hat{\pi}, i\beta\hat{\phi}] + \frac{1}{2!}[-i\alpha\hat{\pi}, [-i\alpha\hat{\pi}, i\beta\hat{\phi}]] + \dots\right)\exp(-i\alpha\hat{\pi}) = \exp\left(\frac{2\pi i}{2l+1}\right)\exp(i\beta\hat{\phi})\exp(-i\alpha\hat{\pi}). \quad (26)$$

While this holds for arbitrary real, non-zero α and β for any dimension $N = 2l + 1$, let us focus on the infinite limit when $\hat{\phi}$ and $\hat{\pi}$ should satisfy Heisenberg's CCR of Eq. (1). Substituting this in Eq. (26) we obtain,

$$\exp(i\beta\hat{\phi} - i\alpha\beta)\exp(-i\alpha\hat{\pi}) = \exp\left(-\frac{2\pi i}{2l+1}\right)\exp(i\beta\hat{\phi})\exp(-i\alpha\hat{\pi}), \quad (27)$$

which immediately gives us a constraint on the parameters α and β ,

$$\alpha\beta = \frac{2\pi}{2l+1}, \quad (28)$$

such that the commutation relation in the infinite-dimensional limit maps onto the Weyl form of the CCR, Eq. (25). Thus, when Eq. (28) is satisfied, the commutator of $\hat{\phi}$ and $\hat{\pi}$ will obey Heisenberg's CCR in the infinite-dimensional limit.

We will show this explicitly later in this section, but before that, let us first compute the commutator of $\hat{\phi}$ and $\hat{\pi}$ in finite dimensions. The matrix representation of $[\hat{\phi}, \hat{\pi}]$ in the $\{|\phi_j\rangle\}$ basis is,

$$\langle\phi_j|[\hat{\phi}, \hat{\pi}]|\phi_{j'}\rangle = \frac{4\pi^2(j-j')}{(2l+1)^3\alpha\beta} \sum_{n=-l}^l n \exp\left(\frac{2\pi i(j-j')n}{2l+1}\right) = \frac{2\pi(j-j')}{(2l+1)^2} \sum_{n=-l}^l n \exp\left(\frac{2\pi i(j-j')n}{2l+1}\right). \quad (29)$$

Imposing $\alpha\beta(2l+1) = 2\pi$ and performing the sum, the commutator becomes

$$\langle\phi_j|[\hat{\phi}, \hat{\pi}]|\phi_{j'}\rangle = \begin{cases} 0, & \text{if } j = j' \\ \frac{i\pi(j-j')}{(2l+1)} \operatorname{cosec}\left(\frac{2\pi l(j-j')}{2l+1}\right), & \text{if } j \neq j'. \end{cases} \quad (30)$$

Under the constraint of Eq. (28), the matrix elements of $\hat{\phi}$ and $\hat{\pi}$ become,

$$\langle\phi_j|\hat{\phi}|\phi_{j'}\rangle = j\alpha\delta_{j,j'}, \quad \langle\phi_j|\hat{\pi}|\phi_{j'}\rangle = \left(\frac{\beta}{2l+1}\right) \sum_{n=-l}^l n \exp\left(\frac{2\pi i(j-j')n}{2l+1}\right). \quad (31)$$

While we need α and β to satisfy Eq. (28) to obtain the correct limit of Heisenberg's CCR in infinite dimensions, there is still freedom to choose one of the two parameters independently. One possibility is that their values are determined by the eigenvalues and functional dependence of the Hamiltonian on these conjugate operators. (Since powers of $\hat{\phi}$ and $\hat{\pi}$ generate Schwinger's unitary basis of Eq. (16), any operator can be expressed as a function of these conjugate operators.) Alternatively, since there is no sense of scale at this level of construction and the conjugate operators are dimensionless and symmetric, one could by fiat impose $\alpha = \beta = \sqrt{2\pi/(2l+1)}$ and accordingly change the explicit functional form of the Hamiltonian, which should have no bearing on the physics.

The most important feature of the finite-dimensional commutator is its non-centrality, departing from being a commuting c-number (as it is in infinite dimensions). Many characteristic features of quantum mechanics and quantum field theory hinge on this property of a central commutator of conjugate operators. It is expected that the presence of a non-central commutator will induce characteristic changes in familiar results, such as computing the zero-point energy. Non-centrality allows for a richer structure in quantum mechanical models, as we will discuss in Section IV B. Let us write down the finite-dimensional commutator as $[\hat{\phi}, \hat{\pi}] = i\hat{Z}$, where \hat{Z} is a hermitian operator whose matrix elements in the $\hat{\phi}$ -basis can be read off from Eq. (30). The matrix Z (i.e. the matrix elements of \hat{Z} in the $\hat{\phi}$ basis) is a real, traceless (null entries on the diagonal), symmetric Toeplitz matrix. Such structure can be exploited to better understand deviations of the commutator in finite dimensions as compared to the usual infinite-dimensional results.

We now turn to recovering conventional notions associated with conjugate variables in quantum mechanics based on an infinite-dimensional Hilbert space. In the infinite-dimensional case of continuum quantum mechanics, we take $l \rightarrow \infty$ and at the same time make the spectral differences of $\hat{\phi}$ and $\hat{\pi}$ infinitesimally small so that they are now labelled by continuous indices on the real line \mathbb{R} , while at the same time respecting the constraint $\alpha\beta(2l+1) = 2\pi$. While finite-dimensional Hilbert spaces in the $N \rightarrow \infty$ limit are not isomorphic to infinite-dimensional ones (even with countably finite dimensions), there is a way in which we can recover Heisenberg's CCR as $N \rightarrow \infty$.

In the expression for the commutator in Eq. (29), replace $n/(2l+1)$ with a continuous variable $x \in \mathbb{R}$ and replace the sum with an integral with $dx \equiv 1/(2l+1)$ playing the role of the integration measure,

$$\langle \phi_j | [\hat{\phi}, \hat{\pi}] | \phi_{j'} \rangle = 2\pi(j - j') \int_{-\infty}^{\infty} dx x \exp(2\pi i(j - j')x). \quad (32)$$

Since the labels j and j' are continuous, we can re-write the integral above as,

$$\langle \phi_j | [\hat{\phi}, \hat{\pi}] | \phi_{j'} \rangle = 2\pi(j - j') \frac{1}{2\pi i} \frac{d}{d(j - j')} \int_{-\infty}^{\infty} dx \exp(2\pi i(j - j')x), \quad (33)$$

$$= -i(j - j') \frac{d}{d(j - j')} \delta(j - j'), \quad (34)$$

$$= i\delta(j - j'), \quad (35)$$

where we have used $y\delta'(y) = -\delta(y)$. Thus, we are able to recover Heisenberg's CCR as the infinite-dimensional limit of the Weyl braiding relation. It can be shown on similar lines that in the infinite-dimensional limit, $\hat{\pi}$ has the familiar representation of $-id/d\phi$ in the $\hat{\phi}$ basis. Hence, finite-dimensional quantum mechanics based on the GCA reduces to known results in the infinite-dimensional limit, while at the same time offering more flexibility to tackle finite-dimensional problems, as might be the case for local spatial regions in quantum gravity. As we will discuss

in Sections IV B and V, infinite-dimensional quantum mechanics with cutoffs is very different from an intrinsic finite-dimensional theory; these difference could affect our understanding of fine-tuning problems due to radiative corrections, such as the hierarchy and cosmological-constant problems. Also, finite-dimensional constructions can offer new features in the spectrum of possible Hamiltonians, as we discuss in Section (IV B).

III. SCHWINGER LOCALITY: THE CONJUGATE SPREAD OF OPERATORS

The concept of locality manifests itself in different ways in conventional physics. In field theory, commutators of spacelike-separated fields vanish, the Hamiltonian can be written as a spatial integral of a Hamiltonian density $\hat{H} = \int d^3x \hat{\mathbf{H}}(\vec{x})$, and Lagrangians typically contain local interaction terms and kinetic terms constructed from low powers of the conjugate momenta. Higher powers of the conjugate momenta are interpreted as non-local effects and are expected to be suppressed. Haag’s formulation of algebraic QFT [30, 31] is also based on an understanding of locality. When we think about sub-systems in quantum mechanics as a tensor product structure $\mathcal{H} = \bigotimes_j \mathcal{H}_j$, the interaction Hamiltonian is taking to be k -local on the graph [32], thereby connecting only k -tensor factors for some small integer k , thus reinforcing the local character of physical interactions.

In a theory with gravity, the role of locality is more subtle. On general grounds, considering the metric as a quantum operator (or as a field to be summed over in a path integral) makes it impossible to define local observables, since there is no unique way to associate given coordinate values with “the same” points of spacetime. More specifically, the black-hole information puzzle and the principles of holography and complementarity [33–35] strongly suggest that the fundamental degrees of freedom in quantum gravity are not locally distributed in any simple way. En route to understanding how spacetime emerges from quantum mechanics, we would like to understand these features better in a finite-dimensional construction without imposing any addition structure or implicit assumptions of a preferred decomposition of Hilbert space, preferred observables, locality etc.

But we can also consider an even more primitive notion of locality. Given a pair of conjugate variables, a dynamics worthy of the label “local” should have the feature that a state localized around a given position should not instantly evolve into a delocalized state. (This requirement can be thought of as a precursor to relativistic causality, although a version of the notion is still relevant in non-relativistic quantum mechanics.) The Hamiltonian for a single non-relativistic particle, for example, typically takes the special form $\hat{H} \sim \hat{p}^2/2 + \hat{V}(\hat{x})$ for classical conjugate variables of position \vec{x} and momentum \vec{p} . Both the quadratic nature of the kinetic term and fact that the Hamiltonian is additively separable in the conjugate variables serve to enforce this kind of locality.

Within our framework, this primitive kind of dynamical locality can be understood by studying how operators in general (and the Hamiltonian in particular) act to spread eigenstates of conjugate variables in Hilbert space. In this section, we develop a notion of the conjugate spread of an operator. This quantity helps characterize the support of an operator along the two conjugate directions.

As discussed in Section II, the Schwinger unitary basis $\{B^b A^a |b, a = -l, (-l+1), \dots, 0, \dots, (l-1), l\}$ offers a complete basis for linear operators in $\mathcal{L}(\mathcal{H})$. The GCA generator \hat{A} corresponds to a unit shift in the eigenstates of $\hat{\phi}$, and \hat{B} generates unit shifts in the eigenstates of $\hat{\pi}$; hence, a basis element $B^b A^a$ generates a units of shift in eigenstates of $\hat{\phi}$ and b units in eigenstates of $\hat{\pi}$, respectively (up to overall phase factors).

For more general operators, the shifts implemented by the GCA generators turn into spreading

of the state. Consider a self-adjoint operator $\hat{M} \in \mathcal{L}(\mathcal{H})$ expanded in terms of GCA generators,

$$\hat{M} = \sum_{b,a=-l}^l m_{b,a} \hat{B}^b \hat{A}^a. \quad (36)$$

Since \hat{M} is self-adjoint $\hat{M}^\dagger = \hat{M}$, we get a constraint on the expansion coefficients, $\omega^{-ba} m_{-b,-a}^* = m_{b,a}$, which implies $|m_{b,a}| = |m_{-b,-a}|$ since $\omega = \exp(2\pi i/(2l+1))$ is a primitive root of unity. The coefficients $m_{b,a}$ are a set of basis-independent numbers that quantify the spread induced by the operator \hat{M} along each of the conjugate variables $\hat{\phi}$ and $\hat{\pi}$. To be precise, $|m_{b,a}|$ represents the amplitude of b shifts along $\hat{\pi}$ for an eigenstate of $\hat{\pi}$ and a shifts along $\hat{\phi}$ for an eigenstate of $\hat{\phi}$. The indices of $m_{b,a}$ run from $-l, \dots, 0, \dots, l$ along both conjugate variables and thus, characterize shifts in both increasing (a or $b > 0$) and decreasing (a or $b < 0$) eigenvalues on the cyclic lattice. The action of \hat{M} on a state depends on details of the state, and in general will lead to a superposition in the eigenstates of the chosen conjugate variable as our basis states, but the set of numbers $m_{b,a}$ quantify the spread along conjugate directions by the operator \hat{M} independent of the choice of state. The coefficient m_{00} accompanies the identity $\hat{\mathbb{I}}$, and hence corresponds to no shift in either of the conjugate variables.

From $m_{b,a}$, which encodes amplitudes of shifts in both $\hat{\phi}$ and $\hat{\pi}$ eigenstates, we would like to extract profiles which illustrate the spreading features of \hat{M} in each conjugate variable separately. Since the coefficients $m_{b,a}$ depend on details of \hat{M} , in particular its norm, we define normalized amplitudes $\tilde{m}_{b,a}$ for these shifts,

$$\tilde{m}_{b,a} = \frac{m_{b,a}}{\sum_{b',a'=-l}^l |m_{b',a'}|}. \quad (37)$$

Then we define the $\hat{\phi}$ -shift profile of \hat{M} by marginalizing over all possible shifts in $\hat{\pi}$,

$$m_a^{(\phi)} = \sum_{b=-l}^l |\tilde{m}_{b,a}| = \frac{\sum_{b=-l}^l |m_{b,a}|}{\sum_{b',a'=-l}^l |m_{b',a'}|}, \quad (38)$$

which is a set of $(2l+1)$ positive numbers characterizing the relative importance of \hat{M} spreading the $\hat{\phi}$ variable by a units, $a = -l, \dots, 0, \dots, l$. Thus, \hat{M} acting on an eigenstate of $\hat{\phi}$, say $|\phi = j\rangle$, will in general, result in a superposition over the support of the basis of the $\hat{\phi}$ eigenstates $\{|\phi = j + a \pmod{l}\rangle\} \forall a$, such that the relative importance (absolute value of the coefficients in the superposition) of each such term is upper bounded by $m_a^{(\phi)}$.

Let us now quantify this spread by defining *Schwinger localities* for each conjugate variable. Consider the ϕ -shift profile first. Operators with a large $m_a^{(\phi)}$ for small $|a|$ will have small spread in the $\hat{\phi}$ -direction, while those with larger $m_a^{(\phi)}$ for larger $|a|$ can be thought of connecting states further out on the lattice for each eigenstate. Following this motivation, we define the Schwinger ϕ -locality S_ϕ of the operator \hat{M} as,

$$S_\phi(\hat{M}) = \sum_{a=-l}^l m_a^{(\phi)} \exp\left(-\frac{|a|}{2l+1}\right). \quad (39)$$

The exponential function suppresses the contribution of large shifts in our definition of locality. There is some freedom in our choice of the decay function in our definition of Schwinger locality, and using an exponential function as in Eq. (39) is one such choice. Thus, an operator with a

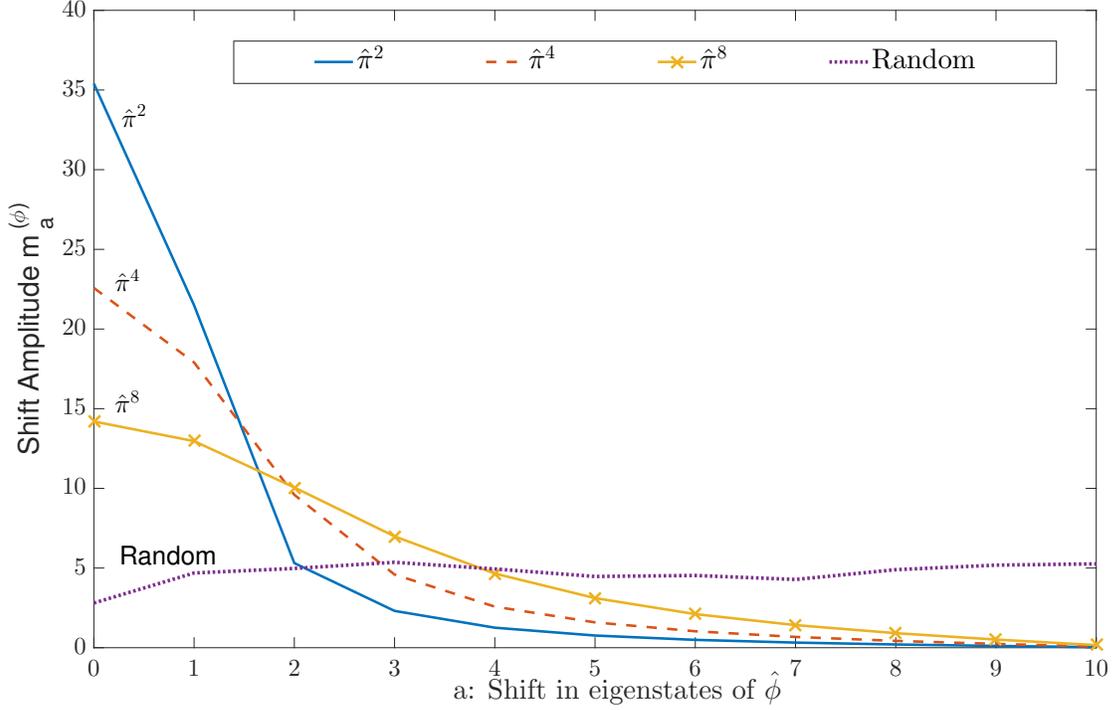


FIG. 1: Plot showing $\hat{\phi}$ -shift profiles of various powers of $\hat{\pi}$. The quadratic operator $\hat{\pi}^2$ is seen to have the most localized profile in the Schwinger locality sense, implying that this operator does the least to spread the state in the conjugate direction. Also plotted is the profile for a random hermitian operator, for which the spread is approximately uniform.

larger S_ϕ is highly Schwinger-local in the $\hat{\phi}$ -direction and does not spread out eigenstates with support on a large number of basis states on the lattice.

On similar lines, one can define the π -shift profile for \hat{M} as,

$$m_b^{(\pi)} = \sum_{a=-l}^l |\tilde{m}_{b,a}| = \frac{\sum_{a=-l}^l |m_{b,a}|}{\sum_{b',a'=-l}^l |m_{b',a'}|}, \quad (40)$$

and a corresponding Schwinger π -locality S_π with a similar interpretation as the $\hat{\phi}$ -case,

$$S_\pi(\hat{M}) = \sum_{b=-l}^l m_b^{(\pi)} \exp\left(-\frac{|b|}{2l+1}\right). \quad (41)$$

Operators such as $\hat{M}(\hat{\pi})$ that depend on only one of the conjugate variables will only induce spread in the $\hat{\phi}$ direction since they have $m_{b,a} = m_{0,a}\delta_{b,0}$, hence they possess maximum Schwinger π -locality, $S_\pi(\hat{M}) = 1$, as they do not spread eigenstates of $\hat{\pi}$ at all. Having a large contribution from terms such as $m_{0,0}, m_{b,0}, m_{0,a}$ will ensure larger Schwinger locality, since there are conjugate direction(s) where the operator has trivial action and does not spread the relevant eigenstates.

In general, we expect that operators which are additively separable in their arguments, $\hat{M}(\hat{\phi}, \hat{\pi}) = \hat{M}_\phi(\hat{\pi}) + \hat{M}_\pi(\hat{\phi})$, will have higher Schwinger locality as compared to a generic non-separable \hat{M} . Let us focus on operators depending only on one conjugate variable, say $\hat{M} \equiv \hat{M}(\hat{\pi})$. While the maximum value of $S_\pi(\hat{M}(\hat{\pi}))$ can be at most unity, one can easily see that the hermitian

operator,

$$\hat{M}(\hat{\pi}) = \frac{A + A^\dagger}{2} = \frac{\exp(-i\alpha\hat{\pi}) + \exp(i\alpha\hat{\pi})}{2} = \cos(\alpha\hat{\pi}) = \hat{\mathbb{1}} - \frac{\alpha^2\hat{\pi}^2}{2} + \frac{\alpha^4\hat{\pi}^4}{4} - \dots, \quad (42)$$

has the least non-zero spread along the $\hat{\phi}$ direction: it connects only ± 1 shifts along eigenstates of $\hat{\phi}$ and hence has highest (non-unity) Schwinger ϕ -locality $S_\phi(\hat{M})$. Thus, one can expect operators which are quadratic in conjugate variables are highly Schwinger local. We see that the fact that real-world Hamiltonians include terms that are quadratic in the momentum variables (but typically not higher powers) helps explain the emergence of classicality: it is Hamiltonians of that form that have higher Schwinger locality, and therefore induce minimal spread in the position variable.

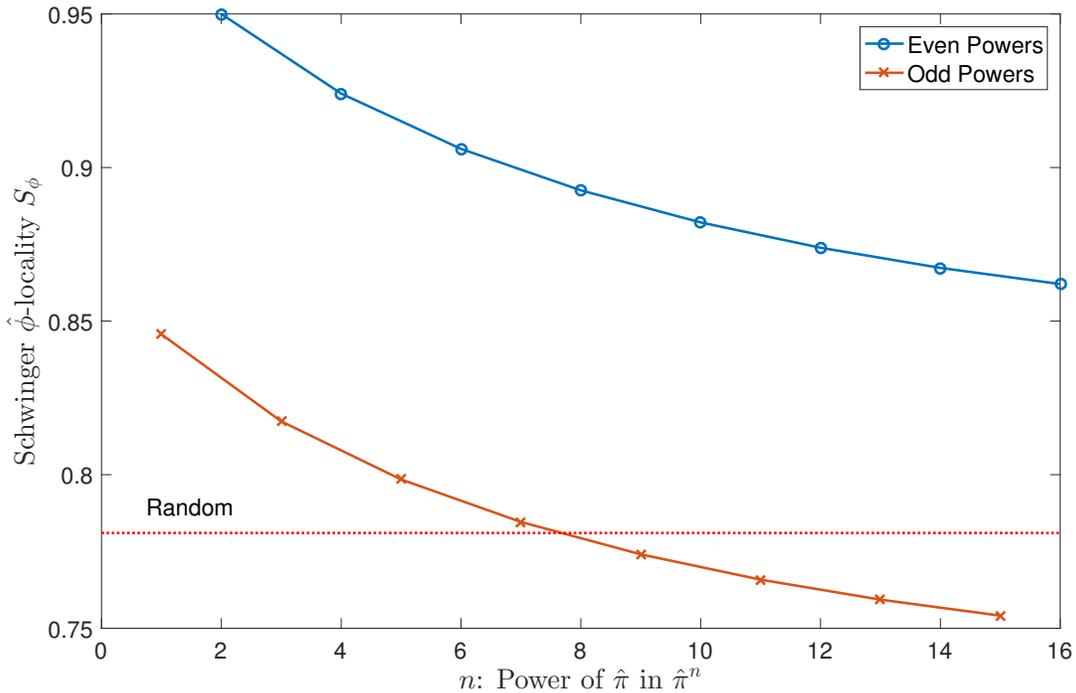


FIG. 2: Schwinger ϕ -locality of various powers of $\hat{\pi}$. Even powers are seen to have systematically larger values of Schwinger locality. Also plotted for comparison is a line marking the Schwinger ϕ -locality of a random hermitian operator.

Let us follow this idea further. The quadratic operator $\hat{\pi}^2$ has higher Schwinger ϕ -locality than any other integer power $\hat{\pi}^n$, $n \geq 1$, $n \neq 2$. There is a difference between odd and even powers of $\hat{\pi}$, with even powers systematically having larger Schwinger localities than the odd powers. This is because odd powers of $\hat{\pi}$ do not have support of the identity $\hat{\mathbb{1}}$ term in the Schwinger unitary basis expansion (and hence have $m_{00} = 0$), and having an identity contribution boosts locality since it contributes to the highest weight in S_ϕ by virtue of causing no shifts. In Figure (1), we plot the ϕ -shift profiles for a few powers of $\hat{\pi}$ and it is explicitly seen that quadratic $\hat{\pi}^2$ has the least spreading and hence is most Schwinger $\hat{\phi}$ -local, values for which are plotted in Figure (2). Note that due to the symmetry $|m_{b,a}| = |m_{-b,-a}|$, we only needed to plot the positive half for $a > 0$, which captures all the information about the spread. Also, for comparison, we also plot the ϕ -spread and its Schwinger $\hat{\phi}$ -locality of a random Hermitian operator (with random matrix elements in the $\hat{\phi}$ basis); such operators spread states almost evenly and thus have low values of Schwinger locality.

IV. FINITE-DIMENSIONAL QUANTUM MECHANICS

A. Equations of Motion for Conjugate Variables

We would next like to understand equations of motion of conjugate variables defined by the GCA in a finite-dimensional Hilbert space evolving under a given Hamiltonian (and a continuous time parameter). In the large-dimension limit, and when appropriate classical structure has been identified on Hilbert space, conjugate variables $\hat{\phi}$ and $\hat{\pi}$ can be identified as position and momenta which satisfy Hamilton's equations of motion. As we will see, the structure of Hamilton's equations of motion is seen to emerge from basic algebraic constructions of the GCA when accompanied by an evolution by the Hamiltonian. Note that using the GCA, one can work with finite-dimensional phase space constructions, such as the Gibbon-Wooters construction ([36, 37] and references therein). We will not discuss such finite-dimensional phase space ideas here but rather focus on understanding the equations of motion for conjugate variables and how they connect to Hamilton's equations.

Consider a Hamiltonian operator $\hat{H}^\dagger = \hat{H}$ on \mathcal{H} which acts as the generator of time translations. We wish to construct operators corresponding to “ $\partial H/\partial\phi$ ” and “ $\partial H/\partial\pi$,” and be able to connect them with time derivatives of $\hat{\phi}$ and $\hat{\pi}$. We saw that the operator \hat{A} from the GCA generates translations in the eigenstates of $\hat{\phi}$, and \hat{B} generates translations in eigenstates of $\hat{\pi}$. Notice that one can define a change in the ϕ variable as a finite central difference (we have used constraint $\alpha\beta = 2\pi(2l+1)$ from Eq. (28) which gives the eigenvalues of $\hat{\phi}$ from Eq. (31)),

$$\delta_\phi \hat{\phi} \equiv \left(\hat{A}^\dagger \hat{\phi} \hat{A} - \hat{A} \hat{\phi} \hat{A}^\dagger \right) \implies \langle \phi_j | \delta_\phi \hat{\phi} | \phi_{j'} \rangle = 2j\alpha \delta_{j,j'} , \quad (43)$$

up to “edge” terms in the matrix where the finite-difference scheme will not act as in the usual way it does on a lattice due to the cyclic structure of the GCA eigenstates. Following this, we can write the change in \hat{H} due to a change in the ϕ variable (translation in $\hat{\phi}$) as a central difference given by,

$$\delta_\phi \hat{H} \equiv \left(\hat{A}^\dagger \hat{H} \hat{A} - \hat{A} \hat{H} \hat{A}^\dagger \right) . \quad (44)$$

This allows us to define an operator corresponding to $\partial H/\partial\phi$ based on these finite central difference constructions,

$$\left(\frac{\partial \hat{H}}{\partial \phi} \right) = \frac{1}{2\alpha} \left(\hat{A}^\dagger \hat{H} \hat{A} - \hat{A} \hat{H} \hat{A}^\dagger \right) , \quad (45)$$

and similarly, for the change with respect to the other conjugate variable $\hat{\pi}$,

$$\left(\frac{\partial \hat{H}}{\partial \pi} \right) = \frac{1}{2\beta} \left(\hat{B}^\dagger \hat{H} \hat{B} - \hat{B} \hat{H} \hat{B}^\dagger \right) . \quad (46)$$

The central difference is one possible construction of the finite derivative on a discrete lattice. One could use other finite difference schemes, but in the large-dimension limit, as we approach a continuous spectrum, any well-defined choice will converge to its continuum counterpart.

With this basic construction, let us now make contact with equations of motion (EOM) for the set of conjugate variables $\hat{\phi}$ and $\hat{\pi}$. We will work in the Heisenberg picture, where operators rather than states are time-dependent, even though we do not explicitly label our operators with a time

argument. The Heisenberg equation of motion for an operator $\hat{\mathcal{O}}$ that is explicitly time-independent ($\partial_t \hat{\mathcal{O}} = 0$) is,

$$\frac{d}{dt} \hat{\mathcal{O}} = i [\hat{H}, \hat{\mathcal{O}}] . \quad (47)$$

In particular, for the time evolution of $\hat{\pi}$, we expand the right hand side of Eq. (45) using the Baker-Campbell-Hausdorff formula, and isolate the commutator $i [\hat{H}, \hat{\pi}]$ that will be the time rate of change of $\hat{\pi}$. One can easily show that,

$$\frac{d}{dt} \hat{\pi} = i [\hat{H}, \hat{\pi}] = - \left(\frac{\partial H}{\partial \phi} \right)_{op} + \sum_{n=3}^{\text{odd}} \frac{i^n}{n!} \alpha^{n-1} [\hat{\pi}, \hat{H}]_n , \quad (48)$$

where we have defined $[\hat{\pi}, \hat{H}]_n$ as the n -point nested commutator in $\hat{\pi}$,

$$[\hat{\pi}, \hat{H}]_n = \left[\hat{\pi}, \left[\hat{\pi}, \left[\hat{\pi} \cdots (n \text{ times}), \hat{H} \right] \cdots \right] \right] . \quad (49)$$

The corresponding equation for $\hat{\phi}$ is likewise,

$$\frac{d}{dt} \hat{\phi} = i [\hat{H}, \hat{\phi}] = \left(\frac{\partial H}{\partial \pi} \right)_{op} + \sum_{n=3}^{\text{odd}} \frac{i^n}{n!} \beta^{n-1} [\hat{\phi}, \hat{H}]_n . \quad (50)$$

In the infinite-dimensional limit we take $l \rightarrow \infty$, and α and β are taken to be infinitesimal but obeying $\alpha\beta(2l+1) = 2\pi$ to recover back the Heisenberg CCR. As expected, the equations of motion simplify to resemble Hamilton's equations of motion from classical mechanics,

$$\frac{d}{dt} \hat{\pi} = i [\hat{H}, \hat{\pi}] = - \left(\frac{\partial H}{\partial \phi} \right)_{op} , \quad (51)$$

and

$$\frac{d}{dt} \hat{\phi} = i [\hat{H}, \hat{\phi}] = \left(\frac{\partial H}{\partial \pi} \right)_{op} . \quad (52)$$

These are intrinsically quantum equations for a set of conjugate variables from the GCA. They resemble the form of the classical equations of motion, but they do not necessarily describe quasiclassical dynamics. The emergence of quasiclassicality and identification of $\hat{\phi}$ and $\hat{\pi}$ with the classical conjugate variables of position and momentum is possible only in special cases when the substructure in Hilbert space allows for decoherence and robustness in the conjugate variables chosen. This is the concern of the quantum factorization problem of our upcoming work [26].

B. The Finite-Dimensional Quantum Harmonic Oscillator

With this technology of conjugate variables from the GCA, we can revisit some important models in quantum mechanics from a finite-dimensional perspective to compare the results with the usual infinite-dimensional results on $\mathbb{L}_2(\mathbb{R})$. All such results from finite-dimensional models will converge to the conventional infinite-dimensional ones when we take the limit $\dim \mathcal{H} \rightarrow \infty$.

We will focus on a finite-dimensional version of the harmonic oscillator. Consider the following Hamiltonian \hat{H} operator for an oscillator with “frequency” Ω on a finite-dimensional Hilbert space \mathcal{H} with $\dim \mathcal{H} = 2l + 1$, and let $\hat{\phi}$ and $\hat{\pi}$ be conjugate operators from the GCA,

$$\hat{H} = \frac{1}{2}\hat{\pi}^2 + \frac{1}{2}\Omega^2\hat{\phi}^2 = \Omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} [\hat{a}, \hat{a}^\dagger] \right). \quad (53)$$

At this stage $\hat{\phi}$ and $\hat{\pi}$ are dimensionless operators, and Ω is a dimensionless parameter, so the Hamiltonian is also dimensionless. One can define a change of variables,

$$\hat{a} = \sqrt{\frac{\Omega}{2}}\hat{\phi} + \frac{i}{\sqrt{2\Omega}}\hat{\pi}, \quad \hat{a}^\dagger = \sqrt{\frac{\Omega}{2}}\hat{\phi} - \frac{i}{\sqrt{2\Omega}}\hat{\pi}, \quad (54)$$

but as we will see, these will *not* serve as ladder or annihilation/creation operators in the finite-dimensional case, since the non-central nature of the commutator carries through, $[\hat{\phi}, \hat{\pi}] = i[\hat{a}, \hat{a}^\dagger] = i\hat{Z}$.

Due to finite-dimensionality of Hilbert space, and finite separation between eigenvalues of the conjugate variables, standard textbook results such as a uniformly spaced eigenspectrum will no longer hold. Depending on the interplay of eigenvalues of $\hat{\pi}$ and $\Omega\hat{\phi}$, there is an effective separation of scales, and correspondingly, the eigenvalue spectrum will have different features to reflect this. In the infinite-dimensional case, for any finite Ω the spectra of $\hat{\pi}$ and $\Omega\hat{\phi}$ match, since the conjugate operators have continuous, unbounded eigenvalues (the reals \mathbb{R}). In this sense, there is more room for non-trivial features in the finite-dimensional oscillator as compared to the infinite case.

In the eigenbasis of $\hat{\phi}$, the matrix elements of the Hamiltonian are,

$$[\hat{H}]_{jj'} = \begin{cases} \sum_{n \neq j} \frac{\pi}{4(2l+1)} \csc^2 \left(\frac{2\pi l}{2l+1} (j-n) \right) + \frac{\Omega^2 \pi}{2l+1} j^2, & \text{if } j = j' \\ \sum_{n \neq j, n \neq j'} \frac{\pi}{4(2l+1)} \csc \left(\frac{2\pi l}{2l+1} (j-n) \right) \csc \left(\frac{2\pi l}{2l+1} (n-j') \right), & \text{if } j \neq j' \end{cases}$$

where we have used the constraint $\alpha = \beta = \sqrt{2\pi/2l+1}$ as described in Section (II), and all sums and indices run from $-l, \dots, 0, \dots, l$. In the infinite-dimensional case, one can solve for the spectrum of the harmonic oscillator and obtain equispaced eigenvalues, which we refer to as the “vanilla” spectrum,

$$\lambda_n^{(\text{vanilla})} = \left(n + \frac{1}{2} \right) \Omega, \quad n = 0, 1, 2, \dots. \quad (55)$$

The finite-dimensional case is more involved and we were unable to find an analytic, closed form for the spectrum $\{\lambda_k\}$ in terms of l and Ω . We can solve for the spectrum numerically for different values of l and Ω , and here we point out a few important features.

First consider the spectra of various oscillators with different Ω and how they compare with the vanilla, infinite-dimensional spectrum. In Figure (3), we plot the spectrum for a $\dim \mathcal{H} = 401$ ($l = 200$) finite-dimensional oscillator for different values of Ω . Depending on how much Ω breaks the symmetry between eigenstates of $\hat{\pi}$ and $\Omega\hat{\phi}$ (corresponding to $\max(\Omega, 1/\Omega)$), the spectrum of the finite oscillator deviates from the vanilla, infinite-dimensional case and is no longer uniformly spaced. For the lower eigenvalues (what constitutes “lower” depends on Ω), both spectra match, and for larger eigenvalues, the finite-dimensional oscillator is seen to have larger values as compared to the vanilla case. On the same figure, we have also plotted part of the equispaced vanilla spectrum (which holds in infinite dimensions) for comparison. Another important feature to consider is the

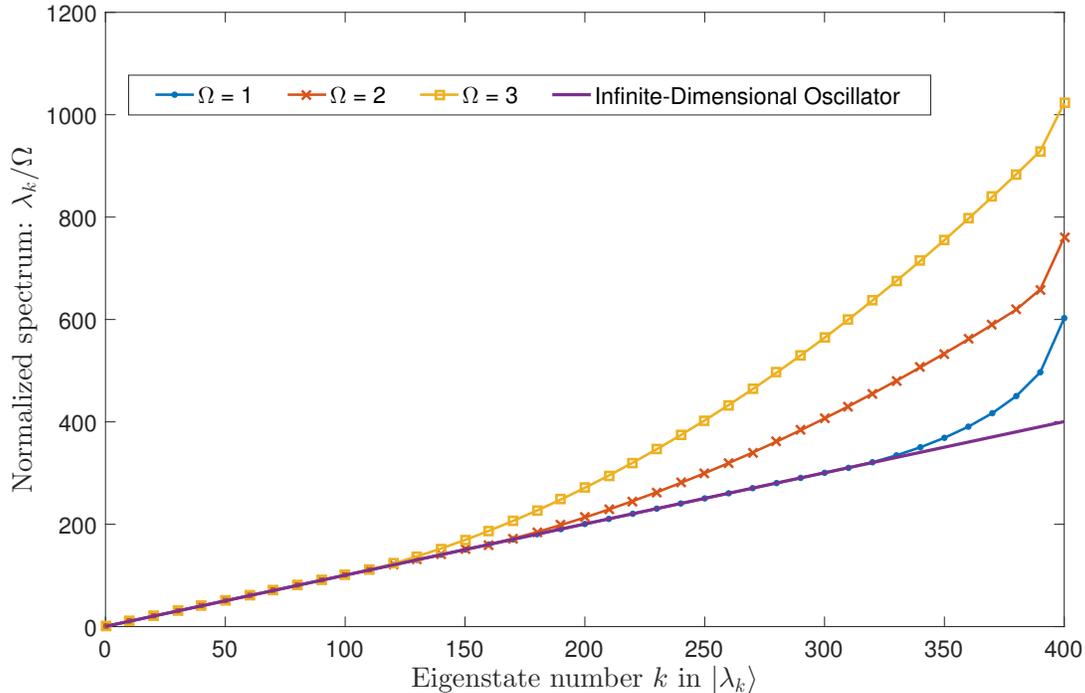


FIG. 3: Eigenspectrum (normalized by Ω) for a $\dim \mathcal{H} = 401 (l = 200)$ finite-dimensional oscillator for different values of Ω . Depending on the value of Ω , the spectrum deviates from the vanilla spectrum of the infinite-dimensional oscillator, which has also been plotted for comparison.

maximum eigenvalue of \hat{H} , λ_{\max} . While there is no maximum eigenvalue in the infinite-dimensional case, we find that λ_{\max} has almost linear behavior in the dimension $\dim \mathcal{H}$ of Hilbert space, as plotted in Figure (4).

A bound for λ_{\max} can easily be given,

$$\lambda_{\max} \leq \frac{1}{2} (1 + \Omega^2) (l\alpha)^2 = \frac{\pi l^2}{2l + 1} (1 + \Omega^2), \quad (56)$$

where we have used the fact that for hermitian matrices P, Q and R such that $P = Q + R$, the maximum eigenvalue of P is at most the sum of maximum eigenvalues of Q and R .

At the other end, while the minimum eigenvalue, normalized by Ω , has a constant $1/2$ value for the vanilla, infinite-dimensional oscillator, we find a richer structure for the minimum eigenvalue of the finite oscillator, plotted in Figure (5). This is itself a reflection of the non-centrality of the commutator $[\hat{a}, \hat{a}^\dagger] \neq 1$, and we see how the lowest eigenvalue normalized by Ω is suppressed for larger values of Ω for a given Hilbert space. These features of the finite-dimensional oscillator spectrum could play a crucial role in the physics of locally finite-dimensional models of quantum gravity.

V. DISCUSSION

Quantum-mechanical models have been extensively studied in both finite- and infinite-dimensional Hilbert spaces; the connection between the two contexts is less well-understood, and has been our

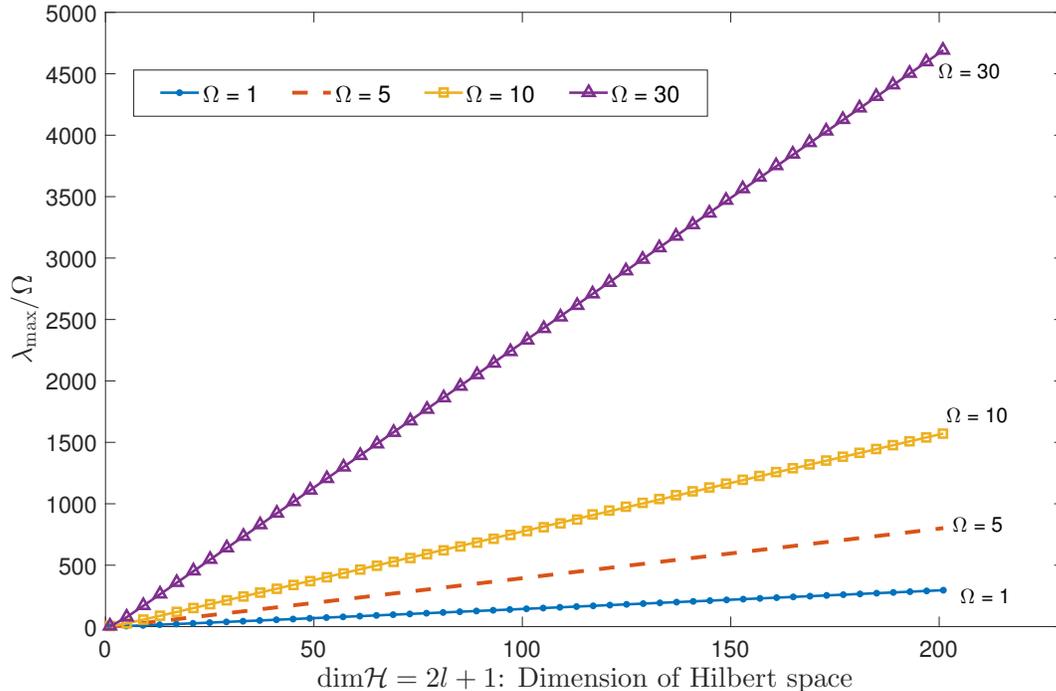


FIG. 4: Plot of the maximum eigenvalue (normalized by Ω) of the finite-oscillator as a function of dimension $\dim \mathcal{H} = 2l + 1$ for different values of Ω . A linear trend is observed.

focus in this paper. Infinite-dimensional models are often constructed by quantizing classical systems that have a description in terms of phase space and conjugate variables. We have therefore studied the notion of a Generalized Clifford Algebra as a tool for adapting a form of conjugate variables to the finite-dimensional case, including the appropriate generalization of the Heisenberg canonical commutation relations.

An advantage of the GCA is that it is completely general, not relying on any pre-existing structure or preferred algebra of observables. This makes it a useful tool for investigating situations where we might not know ahead of time what such observables should be, such as in quantum gravity. We have investigated the development of position/momentum variables, and an associated notion of locality, within this framework. This analysis revealed hints concerning the special nature of the true Hamiltonian of the world, especially the distinction between position and momentum and the emergence of local interactions (and therefore of space itself).

As we have seen, features of a theory based on an intrinsic finite-dimensional Hilbert space can be very different than one based on naive truncation of an infinite-dimensional one. This is particularly seen in the example of the finite-dimensional quantum harmonic oscillator discussed in Section IV B, where the spectrum of the oscillator differs from a simple truncation of the vanilla spectrum based on the infinite-dimensional oscillator. A consistent finite-dimensional construction applied to field theory could have important consequences for issues such as the hierarchy problem, the cosmological constant problem, and Lorentz violation, and may lead to corrections in Feynman diagrams for given scattering problems. In addition to its possible role in field theory, modifications to the commutation relation of conjugate variables (departure from it being a commuting number) can further lead to modifications to uncertainty relations. It has been shown [38–41] (and references therein) that taking into account gravitational effects will lead to modified commutation relations,

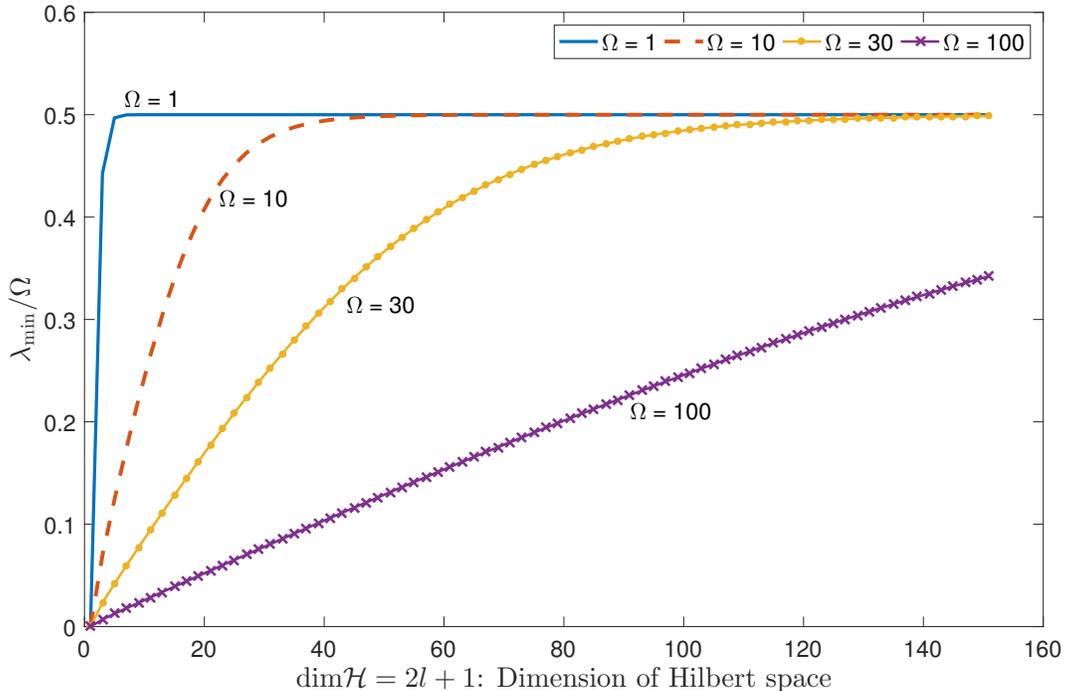


FIG. 5: Plot of the minimum eigenvalue(normalized by Ω) of the finite-oscillator as a function of dimension $\dim \mathcal{H} = 2l + 1$ for different values of Ω . A gradual build-up is noticed, with more suppression for larger Ω for a given \mathcal{H} , which saturates to the vanilla, infinite-dimensional result of $\lambda_{\min}/\Omega = 0.5$.

and the GCAs can provide a natural way to understand these in terms of the local dimension of Hilbert space in a theory with gravity. The GCA can also play an important role in our understanding of emergent classicality in a finite-dimensional setting, where in some preferred factorization of Hilbert space into sub-systems, the conjugate variables can be identified as classical conjugates such as positions and momenta.

Constructions based on the GCA have also been shown to be important in quantum error correction and fault tolerance [42], where one can further try and quantify robustness of different operators based on a notion of Schwinger locality. Once dynamics is added to the problem, one can study the Schwinger locality of operators as a function of time, understanding how their support on Hilbert space evolves, and this can be connected with ideas in quantum chaos and out-of-time-ordered-correlators (OTOCs) [43].

In future work we plan to further explore the emergence of spacetime and quantum field theory in a locally finite-dimensional context.

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