

## POINT SPECTRUM AND MIXED SPECTRAL TYPES FOR RANK ONE PERTURBATIONS

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ABSTRACT. We consider examples  $A_\lambda = A + \lambda(\varphi, \cdot)\varphi$  of rank one perturbations with  $\varphi$  a cyclic vector for  $A$ . We prove that for any bounded measurable set  $B \subset I$ , an interval, there exist  $A, \varphi$  so that  $\{E \in I \mid \text{some } A_\lambda \text{ has } E \text{ as an eigenvalue}\}$  agrees with  $B$  up to sets of Lebesgue measure zero. We also show that there exist examples where  $A_\lambda$  has a.c. spectrum  $[0, 1]$  for all  $\lambda$ , and for sets of  $\lambda$ 's of positive Lebesgue measure,  $A_\lambda$  also has point spectrum in  $[0, 1]$ , and for a set of  $\lambda$ 's of positive Lebesgue measure,  $A_\lambda$  also has singular continuous spectrum in  $[0, 1]$ .

### §1. INTRODUCTION

In this note we will consider families of operators

$$A_\lambda = A + \lambda(\varphi, \cdot)\varphi$$

where  $A$  is a self-adjoint operator on a separable Hilbert space  $\mathcal{H}$  and  $\varphi \in \mathcal{H}$  is a cyclic vector for  $A$ . It will be convenient to consider also the value  $\lambda = \infty$ , which is the operator  $QAQ$  on  $Q\mathcal{H}$  where  $Q$  is the projection onto the operators orthogonal to  $\varphi$ . Let  $d\mu_\lambda$  be the spectral measure for  $A_\lambda$  with vector  $\varphi$  and  $d\rho_\lambda = (1 + \lambda^2)d\mu_\lambda$ . It is known [3] that  $d\rho_\lambda$  has a weak limit as  $\lambda \rightarrow \infty$ ,  $d\rho_\infty$ , which is a spectral measure for  $A_\infty$ .

Define for  $x \in \mathbb{R}$ ,

$$G_\lambda(x) = \int \frac{d\rho_\lambda(y)}{(x-y)^2}$$

where  $G$  may be infinite.

Also define for  $z \in \mathbb{C}$  with  $\text{Im } z > 0$ ,

$$F_\lambda(z) = \int \frac{d\rho_\lambda(E)}{E-z} = (1 + \lambda)^2(\varphi, (A_\lambda - z)^{-1}\varphi).$$

(This differs from the standard  $F$  [6] by a factor of  $(1 + \lambda^2)$ .) It is known [2], [6] that

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**Theorem 0.** *The sets*

$$\begin{aligned}
 P &= \{E \mid G_\lambda(E) < \infty\} \cup \{E \mid E \text{ is an eigenvalue of } A_\lambda\}, \\
 L &= \{E \mid \lim_{\epsilon \downarrow 0} F_\lambda(E + i\epsilon) \equiv F_\lambda(E + i0) \text{ exists and } \operatorname{Im} F_\lambda(E + i0) > 0\}, \\
 S &= \mathbb{R} \setminus P \cup L
 \end{aligned}$$

are  $\lambda$  independent for  $\lambda \in \mathbb{R}$ , and for every  $\lambda \in \mathbb{R} \cup \{\infty\}$ :

$$\begin{aligned}
 (1a) \quad & \rho_\lambda^{\text{pp}}(\cdot) = \rho_\lambda(\cdot \cap P), \\
 (1b) \quad & \rho_\lambda^{\text{ac}}(\cdot) = \rho_\lambda(\cdot \cap L), \\
 (1c) \quad & \rho_\lambda^{\text{sc}}(\cdot) = \rho_\lambda(\cdot \cap S),
 \end{aligned}$$

where  $\rho_\lambda^{\text{pp}}$ ,  $\rho_\lambda^{\text{ac}}$ ,  $\rho_\lambda^{\text{sc}}$  are the pure point, absolutely continuous, and singular continuous parts of the measure  $\rho_\lambda$ . Moreover,

$$P = \bigcup_{\lambda \in \mathbb{R} \cup \{\infty\}} \{E \mid E \text{ is an eigenvalue of } A_\lambda\}$$

and for any set  $C$ ,

$$(2) \quad \int \frac{\rho_\lambda(C)}{(1 + \lambda^2)} d\lambda = |C|,$$

the Lebesgue measure of  $C$ . In particular, by (1a)

$$(3) \quad \int \frac{\rho_\lambda^{\text{pp}}(C)}{(1 + \lambda^2)} d\lambda = |C \cap P|$$

and similarly for  $L$  and  $S$ .

One can ask what kind of sets can occur as a  $P$ . We have a partial answer given in Section 2:

**Theorem 1.** *For any bounded measurable set  $B$  and any interval  $I \supset B$ , there exists a measure  $d\mu$  on  $I$  so that (where a.e. means with respect to Lebesgue measure)*

$$G_0(x) = \begin{cases} < \infty & \text{a.e. } x \in B, \\ = \infty & \text{a.e. } x \in I \setminus B. \end{cases}$$

The measure  $d\mu$  may be chosen purely a.c., or purely s.c., or purely p.p.

*Remarks.* 1. By Theorem 0, this says something about allowed sets of eigenvalues.

2. We will also show that if  $B$  is open, we can drop the a.e. We believe that this can be done for an arbitrary  $F_\delta$ , but have not proven it.

Using a technical result in Section 3, we will prove our second main result in Section 4:

**Theorem 2.** *There exists an example  $A$  so that*

- (i)  $\sigma_{\text{ac}}(A_\lambda) = [0, 1]$  for all  $\lambda$ .
- (ii)  $\{\lambda \mid \sigma_{\text{pp}}(A_\lambda) \cap [0, 1] \neq \emptyset\}$  has positive Lebesgue measure; indeed, for any interval  $I \subset [0, 1]$ ,  $\{\lambda \mid \sigma_{\text{pp}}(A_\lambda) \cap I \neq \emptyset\}$  has positive measure.
- (iii)  $\{\lambda \mid \sigma_{\text{sc}}(A_\lambda) \neq \emptyset\}$  has positive Lebesgue measure; indeed, for any interval  $I \subset [0, 1]$ ,  $\{\lambda \mid \sigma_{\text{sc}}(A_\lambda) \cap I \neq \emptyset\}$  has positive measure.

There also exist examples where (i) is replaced by  $\sigma_{\text{ac}}(A_\lambda) = \emptyset$ .

One can translate these results into ones for variations on boundary conditions for Schrödinger operators  $-u'' + Vu$  on  $[0, \infty)$  in two steps:

- (a) Extend the theory to  $\varphi \in \mathcal{H}_{-1}(A)$  and rewrite the Sturm-Liouville/Schrödinger operator in this language [6].
- (b) Appeal to the Gel'fand-Levitan construction [5], which implies that for any measure  $\mu$  on a bounded interval  $I$ , we can find a continuous  $V$  on  $[0, \infty)$  with  $-u'' + Vu$  limit point at infinity and boundary condition  $\theta$  at  $x = 0$  so that the spectral measure  $d\rho_\theta$  restricted to  $I$  is  $d\mu$ . Typical of the result is:

**Theorem 1'.** *For any bounded measurable set  $B$  and interval  $I \supset B$ , there is a continuous function  $V$  on  $[0, \infty)$  so that up to sets of Lebesgue measure zero,*

$$\{E \mid -u'' + Vu = Eu \text{ has a solution } L^2 \text{ at infinity}\}$$

*is precisely  $B$ .*

Because the Gel'fand-Levitan construction gives no information on  $V$  at infinity (for example, it could be unbounded below), we regard these translations as being of limited interest.

§2. THE SET WHERE  $G$  IS FINITE

Recall that a perfect set is a closed set with no isolated points. We will also need the following notion.

**Definition.** A closed subset  $C \subset \mathbb{R}$  will be called *minimal* if and only if for all  $x \in C$  and  $\epsilon > 0$ ,  $|(x - \epsilon, x + \epsilon) \cap C| > 0$ .

The name comes from the fact that among all closed sets  $D$  with  $|D \Delta C| = 0$ ,  $C$  is the minimal such set. We will see below that any closed set  $D$  has a minimal closed set  $C$  contained in it so that  $|D \setminus C| = 0$ .

We also define  $G_\mu$  by

$$G_\mu(x) = \int \frac{d\mu(y)}{(x - y)^2}.$$

With these notions out of the way, we can state the two main theorems of this section:

- Theorem 2.1.**
- (a) *Let  $C$  be any closed set in  $\mathbb{R}$ . Then there exists a pure point measure  $\mu$  supported on  $C$  so that  $\{x \mid G_\mu(x) = \infty\} = C$ .*
  - (b) *Let  $C$  be any perfect set. Then there exists a singular continuous measure  $\mu$  supported on  $C$  so that  $\{x \mid G_\mu(x) = \infty\} = C$ .*
  - (c) *Let  $C$  be any minimal closed set. Then there exists an absolutely continuous measure  $\mu$  supported on  $C$  so that  $\{x \mid G_\mu(x) = \infty\} = C$ .*

*Remarks.* 1. The assumptions on the closed sets are optimal in that if  $x$  is an isolated point of  $C$ , then  $G_\mu(x) < \infty$  for any singular continuous measure  $\mu$  supported on  $C$ ; and if  $x \in C$  is a point with  $|(x - \epsilon, x + \epsilon) \cap C| = 0$  for some  $\epsilon > 0$ , then  $G_\mu(x) < \infty$  for any a.c. measure supported on  $C$ .

2. In general,  $\{x \mid G_\mu(x) = \infty\}$  is only a  $G_\delta$ , not a closed set. It is open if “closed” in this theorem can be replaced by  $G_\delta$ .

3. If  $B$  is any measurable set, we can apply the methods of proof below and get a  $\mu$  supported on  $B$  with  $\{x \mid G_\mu(x) = \infty\} \supset B$ . If  $B$  is arbitrary, we can take  $\mu$  pure point. If  $B$  has no isolated points, we can take  $\mu$  singular continuous, and if

$B$  has no essentially isolated points (i.e., no points  $x$  with  $|(x - \epsilon, x + \epsilon) \cap B| = 0$  for some  $\epsilon > 0$ ), we can take  $\mu$  absolutely continuous.

If we are willing to throw out sets of measure zero, we can go beyond Theorem 2.1. We write  $A \equiv B$  to mean  $|A \Delta B| = 0$ . Then we will prove that:

**Theorem 2.2** ( $\equiv$  **Theorem 1**). *For  $B$  an arbitrary measurable subset of an interval  $I$ , we can find  $\mu$  supported on  $I$  so that*

$$\{x \in I \mid G_\mu(x) < \infty\} \equiv B.$$

$\mu$  can be chosen to be purely absolutely continuous or purely singular continuous or pure point. In the a.c. case,  $\mu$  can be chosen so that the essential support of  $\mu$  is  $I \setminus B$ .

In understanding perfect and minimal closed sets, it is useful to have the following pair of results, which we will also need in proving Theorem 2.2.

**Proposition 2.3.** *Any closed set  $S$  in  $\mathbb{R}$  can be written as  $S = C \cup D$  where  $C$  is perfect and  $D$  is countable.*

*Proof.* Let  $C = \{x \in S \mid \forall \epsilon > 0, (x - \epsilon, x + \epsilon) \cap S \text{ is uncountable}\}$  and  $D = S \setminus C$ . It is easy to see that  $C$  is closed. If we show  $D$  is countable, then each  $(x - \epsilon, x + \epsilon) \cap C$  is uncountable, so not empty and  $C$  is perfect.

If  $x \notin C$ , we can find  $a$  and  $b$  rational so  $x \in (a, b)$  and  $(a, b) \cap S$  is countable. Since there are only countably many  $(a, b)$  with  $a, b$  rational, we can find a countable family of  $\{O_n\}_{n=1}^\infty$  with each  $O_n \cap S$  countable, so  $D \subset \bigcup_n (O_n \cap S)$  is countable.  $\square$

**Proposition 2.4.** *Any closed set  $S$  in  $\mathbb{R}$  can be written as  $S = C \cup D$  where  $C$  is minimal closed and  $|D| = 0$ .*

*Proof.* Let  $C = \{x \in S \mid \forall \epsilon > 0, |(x - \epsilon, x + \epsilon) \cap S| > 0\}$  and  $D = S \setminus C$ . Now just mimic the proof of Proposition 2.3.  $\square$

We need one more preliminary:

**Proposition 2.5.** (a) *For any non-empty closed set  $C$ , there exists a point measure supported by  $C$ .*

(b) *For any non-empty perfect set  $C$ , there exists a singular continuous measure supported by  $C$ .*

(c) *For any non-empty minimal closed set  $C$ , there is an absolutely continuous measure supported by  $C$ .*

*Proof.* (a) is trivial and stated for parallelism. (c) is also trivial (take  $d\mu = \chi_C dx$ ). That leaves (b); so let  $C$  be perfect. If  $C$  contains an entire interval  $[a, b]$ , place a scaled Cantor measure on  $(a, b)$  and use that for  $d\mu$ . So we need only consider a nowhere dense perfect set. By intersecting it with a suitable bounded interval and scaling, we will suppose it is a subset of  $[0, 1]$ .

We claim such a  $C$  is homeomorphic to  $\{0, 1\}^\mathbb{N}$ , the infinite sequences of 0's and 1's. Use that homeomorphism to transfer the two mutually singular measures

$$d\alpha_1 = \bigotimes_{n=1}^{\infty} \left[ \frac{1}{2} (\delta_0 + \delta_1) \right] \quad \text{and} \quad d\alpha_2 = \bigotimes_{n=1}^{\infty} \left( \frac{1}{3} \delta_0 + \frac{2}{3} \delta_1 \right).$$

$d\alpha_1$  may be purely absolutely continuous (as it is if  $C$  is a symmetric positive measure Cantor set), but then  $d\alpha_2$  is purely singular continuous. Either way, either  $d\alpha_1$  or  $d\alpha_2$  has a non-zero singular continuous component.

To prove the claim (known, but the proof is so short that we give it) that a nowhere closed perfect subset  $C$  of  $[0, 1]$  is homeomorphic to  $\{0, 1\}^{\mathbb{N}}$ , let  $a_- = \min(C)$ ,  $a_+ = \max(C)$ , and  $\ell_1 = a_+ - a_-$ , the length of  $C$ . Since  $C$  is perfect,  $\ell_1 > 0$ . Let  $J = (\frac{a_-+a_+}{2} - \frac{\ell_1}{6}, \frac{a_-+a_+}{2} + \frac{\ell_1}{6})$ , the middle third of  $(a_-, a_+)$ . Since  $C$  is nowhere dense, we can find  $x_1 \in J \setminus C$ . Let  $C_0 = C \cap (-\infty, x_1)$ ,  $C_1 = C \cap (x_1, \infty)$ . Then  $C_0, C_1$  are perfect and  $\text{diam}(C_1) \leq \frac{2}{3}$ . Now repeat this process, and so find  $C_{m_1 \dots m_\ell}$  ( $m_i \in \{0, 1\}$ ) inductively so that  $\text{diam}(C_{m_1 \dots m_\ell}) \leq (\frac{2}{3})^\ell$ ,  $C_{m_1 \dots m_\ell} = C_{m_1 \dots m_\ell 0} \cup C_{m_1 \dots m_\ell 1}$ , each  $C_{m_1 \dots m_\ell}$  is perfect. Define  $a_\ell : C \rightarrow \{0, 1\}$  by  $a_\ell = 0$  on each  $C_{m_1 \dots m_\ell 0}$  and  $a_\ell = 1$  on each  $C_{m_1 \dots m_\ell 1}$ . Each  $a_\ell$  is continuous since each  $C_{m_1 \dots m_\ell}$  is closed. Map  $C \rightarrow \{0, 1\}^\ell$  by  $x \rightarrow (a_1(x), a_2(x), \dots)$ . This map is onto since for any fixed  $m_1, \dots, \bigcap_{\ell=1}^\infty C_{m_1 \dots m_\ell} \neq \emptyset$  by compactness. This map is one-one since  $\text{diam}(C_{m_1 \dots m_\ell}) \rightarrow 0$  to  $\ell \rightarrow \infty$  uniformly in the choice of  $m_\ell$ . A continuous bijection is a homeomorphism.  $\square$

*Proof of Theorem 2.1.* This is motivated by a construction in [7]. For  $n = 1, 2, \dots$  and  $j = 0, \dots, 2^n - 1$ , let  $C_j^{(n)} = (\frac{j}{2^n}, \frac{j+1}{2^n}) \cap C$  which is  $C \cap [\frac{j}{2^n}, \frac{j+1}{2^n}]$  with the endpoints dropped if they would be isolated. Then if  $C$  is perfect (minimal), so is each non-empty  $C_j^{(n)}$ . For such non-empty  $C_j^{(n)}$ , let  $\mu_j^{(n)}$  be a measure of the requisite type (i.e., pure point, singular continuous, or absolutely continuous) of unit measure and supported on  $C_j^{(n)}$ . Such measures exist by Proposition 2.5. Let

$$\mu = \sum_{n=1}^\infty n^{-2} 2^{-n} \sum_{\substack{j=1 \\ j \text{ so that} \\ C_j^{(n)} \neq \emptyset}}^{2^n} \mu_j^{(n)}.$$

Then  $\mu$  is a finite measure of the requisite type supported on  $C$ . If  $y \notin C$ , then  $G_\mu(y) \leq \text{dist}(y, C)^{-2} \int d\mu < \infty$  since  $C$  is closed. On the other hand, if  $y \in C$  and  $y \in (\frac{j}{2^n}, \frac{j+1}{2^n})$ , then  $C_j^{(n)} \neq \emptyset$  and  $\int \frac{d\mu_j^{(n)}(x)}{(x-y)^2} \geq (2^{-n})^2$ , and if  $y \in \{\frac{j}{2^n}\}_{j=0}^{2^n} \cap C$ , either  $C_j^{(n)}$  or  $C_{j-1}^{(n)}$  is non-empty. It follows that

$$\int \frac{d\mu(x)}{(x-y)^2} \geq \sum_{n=1}^\infty 2^{2n} n^{-2} 2^{-n} = \infty,$$

so  $\{y \mid G_\mu(y) = \infty\} = C$ .  $\square$

*Proof of Theorem 2.2.* This uses an explicit version of an argument of Howland [4] as in [1]. Since Lebesgue measure is inner regular, we can find  $C_1, \dots, C_n, \dots$  and  $K_1, \dots, K_n, \dots$  closed with  $C_1 \subset C_2 \subset \dots \subset I \setminus B$  and  $K_1 \subset K_2 \subset \dots \subset B$  and with  $|B \cup_n K_n| = 0$ ,  $|(I \setminus B) \setminus \cup_n C_n| = 0$ .

By Proposition 2.3, we can suppose that  $C_n$ 's are minimal closed (and so, perfect) without loss of generality. We can also suppose each  $C_n$  is non-empty (if  $|I \setminus B| = 0$ , we just take  $\mu = 0$ ).

Let  $\mu_n$  be a unit measure of the requisite type supported on  $C_n$  with

$$C_n = \{x \mid G_{\mu_n}(x) = \infty\}.$$

Let

$$\mu = \sum_{n=1}^{\infty} 2^{-n} \text{dist}(K_n, C_n)^2 \mu_n.$$

Since  $K_n$  and  $C_n$  are compact and disjoint,  $\text{dist}(K_n, C_n) > 0$  and thus,  $G_{\mu}(x) \geq 2^{-n} \text{dist}(K_n, C_n)^2 G_{\mu_n}(x) = \infty$  on  $C_n$  and so on  $\cup C_n$  and so a.e. on  $I \setminus B$ .

On the other hand, since  $K_n \subset K_{n+1}, \dots$ ,  $\text{dist}(K_n, C_m) \geq \text{dist}(K_m, C_m)$  if  $m \geq n$  and so if  $x \in K_n$ ,

$$G_{\mu}(x) = \sum_{\ell=1}^{n-1} 2^{-\ell} \text{dist}(K_{\ell}, C_{\ell})^2 G_{\mu_{\ell}}(x) + \sum_{\ell=n}^{\infty} 2^{-n} < \infty,$$

and so  $G_{\mu} < \infty$  on  $\cup K_n$  and thus a.e. on  $B$ .

In the a.c. case, we can take  $\mu_n = \frac{1}{|C_n|} \chi_{C_n} dx$ , in which case it is evident that the essential support of  $\mu$  is  $\cup C_n = I \setminus B$  as claimed.  $\square$

### §3. ESSENTIALLY DENSE SETS

**Definition.** A measurable set  $S \subset I$ , an interval, is called essentially dense if for every subinterval  $J \subset I$ , we have  $|J \cap S| > 0$ .

**Theorem 3.1.** *There exist disjoint measurable subsets  $A, B, C \subset [0, 1]$  whose union is  $[0, 1]$  so that each is essentially dense.*

*Remarks.* 1. Our proof shows that one can assert the same for sets  $A_1, \dots, A_n$  rather than three sets or even construct a countable disjoint decomposition, each of which is essentially dense.

2. Our construction is related to a construction in [2].

*Proof.* Let  $n_j = (2j + 1)^2$ , the square of the  $j^{\text{th}}$  odd number. Given  $x \in [0, 1]$ , we define  $a_j(x)$  by requiring

$$x = \sum_{j=1}^{\infty} \frac{a_j(x)}{n_1 \dots n_j}$$

with  $a_j(x) \in \{0, 1, \dots, n_j - 1\}$  and the requirement that if  $x$ 's expansion can end in all 0's, we do that (to settle the ambiguity between  $\dots a(n_j - 1)(n_{j+1} - 1) \dots$  and  $\dots (a + 1)00 \dots$ ). This is a standard positive measure Cantor set construction. Define  $m_j = \frac{1}{2}(n_j - 1)$ . Let

$$A = \{x \mid \text{the number of } j\text{'s with } a_j(x) = m_j \text{ is } 1, 4, \dots \text{ or infinite}\},$$

$$B = \{x \mid \text{the number of } j\text{'s with } a_j(x) = m_j \text{ is } 2, 5, 8, \dots\},$$

$$C = \{x \mid \text{the number of } j\text{'s with } a_j(x) = m_j \text{ is } 3, 6, 9, \dots\}.$$

This is obviously a decomposition. We need only to show that each set is essentially dense. It suffices to show that  $|B \cap J| > 0$  for any interval of the form  $J = \{x \mid a_1(x) = \alpha_1, \dots, a_k(x) = \alpha_k\}$  since every interval contains such a  $J$ . By increasing  $k$  by 1 or 2 and shrinking  $J$  by taking  $\alpha_{k+1} = m_{k+1}$  (and perhaps  $\alpha_{k+2} = m_{k+2}$ ), we can suppose that  $\#\{j \in \{1, \dots, k\} \mid \alpha_j = m_j\} \equiv 2 \pmod{3}$ . In that case, by

looking at  $x$ 's with no further  $a_\ell(x) = m_\ell$ , we have

$$|B \cap J| \geq \prod_{\ell=k+1}^{\infty} \left(1 - \frac{1}{n_\ell}\right) > 0$$

since  $\sum \frac{1}{n_\ell} < \infty$ . □

#### §4. MIXED SPECTRA

*Proof of Theorem 2.* Decompose  $[0, 1] = A \cup B \cup C$  into three disjoint essentially dense sets. Pick a measure  $d\mu_1$  which is absolutely continuous with essential support  $A$  so that  $G_{\mu_1}(x) < \infty$  a.e. on  $B \cup C$  and a s.c. measure  $\mu_2$  supported on  $B$  so that  $G_{\mu_2}(x) < \infty$  on  $A \cup C$  and  $G_{\mu_2}(x) = \infty$  a.e. on  $B$ . Let  $d\mu = d\mu_1 + d\mu_2$ .

By Theorem 0 (recall  $X \equiv Y$  means  $|X \Delta Y| = 0$ ),

$$P \equiv C \cup (\mathbb{R} \setminus [0, 1]),$$

$$L \equiv A,$$

$$S \equiv B.$$

By equation (3) and its analogs for a.c. and s.c., we have the claimed assertions (i)–(iii). For the examples with  $\sigma_{ac}(A_\lambda) = \emptyset$ , just use  $d\mu = d\mu_2$ . □

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