

ciently low compared with the ion plasma frequency, then for reasonable transverse inhomogeneity ($\gamma_n \sim$ order unity)⁵ the propagation will be unaffected by the geometry of the system. However, if the wave frequency is comparable to the ion plasma frequency a detailed examination of $\gamma_n k^2/k_D^2$ is necessary in order to determine the effects of transverse geometry on the ion wave propagation. The above analysis is restricted to a Maxwellian velocity distribution for the electrons. For a different electron velocity distribution the scaling of the parameter $\gamma_n k^2/k_D^2$ will be to another constant determined by the form of the dispersion function for the electrons. Properties of the ion distribution function have not been required to obtain this result, so that it applies for any ion distribution provided the assumption $\omega/k \ll a$ remains valid.

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⁵ Approximate expressions for the lowest eigenvalue γ_0 for various transverse plasma configurations can be found in Ref. 2.

Numerical Study of Collisional Effects on Spatial Ion-Wave Echoes

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Collisional effects on the second-order spatial ion-wave echo are studied numerically. General agreement is obtained with the experimental results of Ikezi, Takahashi, and Nishikawa. A critical argument is raised against the validity of the small collision approximation.

The plasma wave echo has been suggested as a useful tool for investigating collisional processes in a plasma.¹ Detailed experimental investigations have been carried out, particularly for second-order spatial ion-wave echoes,^{2, 3} but some of the observed features still seem to be unexplained by the theory so far reported.^{2, 4-6} One reason for this failure is that the spatial ion waves used in the experiment are highly damped waves and thus involve particles of a wide range of velocities. As a result the approximations used previously in evaluating a certain velocity integral are inappropriate. In order to make a more careful comparison between theory and experiment, a numerical evaluation of

the relevant velocity integral is desirable. In this note, we report results of such a numerical calculation. As we shall show later, agreement between theory and experiment turns out to be reasonably good, despite the fact that the small collision approximation we used is questionable in the relevant range of the parameters.

As was shown by many authors,¹⁻⁶ the first-order perturbed distribution function damps like $\exp[-D\omega^2 z^3/(3v^5)]$ as a result of the velocity-space diffusion due to Coulomb collisions, where D is the velocity-dependent diffusion coefficient, ω is the frequency, z is the distance, and v is the velocity. The formula is valid when D is sufficiently small. For the second-order echo, this collisional damping reduces the amplitude of the second-order perturbed distribution at the echo position by the factor

$$\exp\left(-\frac{D\omega_1^2\omega_2}{3\omega_3 v^5} l^3\right), \quad (1)$$

where ω_1 , ω_2 , and ω_3 are, respectively, the frequencies of the first wave, second wave, and the echo ($\omega_3 = \omega_2 - \omega_1$), and l is the distance between the two exciters. If the echo is due only to those particles whose velocities are close to the phase velocity of the wave, one can replace v in (1) by the phase velocity (hereafter, we call this approximation the *pole approximation*); the echo amplitude will then damp like $\exp(-l^{*3})$ where l^* is a normalized distance between the two exciters. If, on the other hand, the particles of a wide range of velocities can contribute to the echo, the damping factor (1) will act so as to reduce the contribution of slow particles; the damping of the echo amplitude will then be much slower and a slight shift toward greater values of the phase velocity of the echo-wave will occur. A calculation based on the saddle-point approximation³⁻⁶ predicts that the damping of the echo amplitude is of the form $\exp(-l^{*6/(s+7)})$ and that the phase velocity is proportional to $(l^*)^{3/(7+s)}$, where D is assumed to be proportional to v^{-s} . O'Neil pointed out that the saddle-point approximation is appropriate near the peak, whereas in the wing the pole approximation can be used, so that the wing damps faster than the peak and thus the echo becomes sharper because of collisions.

The experimental data for spatial ion waves are clearly summarized in Ref. 2, Secs. IVB and D. The observed attenuation of the peak amplitude appears to be consistent with the theory based on the saddle-point approximation, but the constancy of the phase velocity peak shift toward the exciter with increasing l , and the flattening of the echo line shape by collisions are not explained by the theory described above.

We have numerically evaluated the following integral:

$$I(z, l) = l \int_0^\infty dv \frac{v_i^{\alpha-1}}{v^\alpha} \frac{\exp [i(\omega_3/v)(z - l') - (v^2/v_i^2) - C(\omega_1, l) - C(\omega_3, z - l)]}{\epsilon(-\omega_1/v, -\omega_1)\epsilon(\omega_2/v, \omega_2)\epsilon(\omega_3/v, \omega_3)}, \quad (2)$$

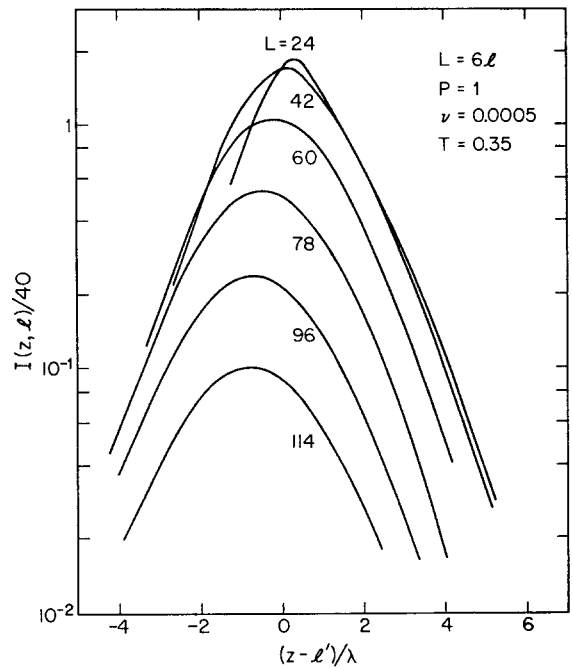
where v_i is the ion thermal velocity, $l' (= \omega_2 l / \omega_3)$ is the expected position of the echo, $C(\omega, z)$ is the collisional damping factor given by

$$C(\omega, z) = \frac{D(v)}{3v^5} \omega^2 z^3, \quad (3)$$

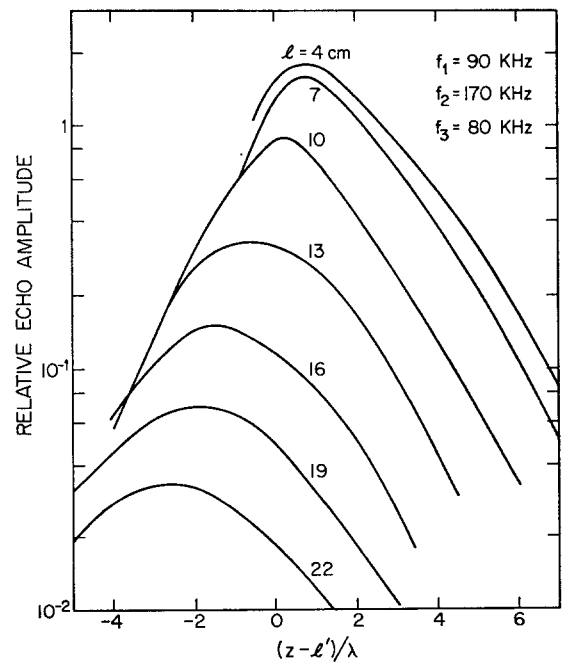
and $\epsilon(k, \omega)$ is the dielectric function divided by the static dielectric function

$$\epsilon(k, \omega) = 1 - \frac{T_e}{2T_i} Z' \left(\frac{\omega}{kv_i} \right); \quad (4)$$

T_e , T_i , and $Z'(\zeta)$ being the electron temperature, ion temperature, and the derivative of the plasma dispersion function, respectively. In Eq. (2) α is an integer which depends on the physical quantity of interest (e.g., density, current, electric field) and on the excitation mechanism of the wave. If we set $\alpha = 8$ and $z = l'$ in $C(\omega_3, z - l)$,⁷ then $I(z, l)$ becomes proportional to Eq. (19) of Ref. 5. The choice of the value of α is relatively unimportant as long as the contribution from small velocities is negligible, that is, as long as the collisional effect is not very small. However, when the collisional effect is small, $I(z, \omega)$ shows a pronounced peak near $z = l'$ if α is large. Such a pronounced peak is not observed in the experiment quoted above. We note that since the observed quantity is an ion current and the actual excitation mechanism of ion wave by a grid is different from that by an ideal dipole grid, then α can be as small as two. We, therefore, set $\alpha = 2$ in our calculation to give the closest agreement with experiment in the region of small collisions. An appropriate velocity dependence of $D(v)$ is assumed, since D varies appreciably in the important range of velocities. We adopt a simple model $D(v) = v_i^2(1 + v^2/v_i^2)^{-3/2}$ in order to represent the basic features, $D(v) \sim \text{const}$ for $v \ll v_i$ and $D(v) \sim v^{-3}$ for $v \gg v_i$. The dimensionless parameters in (2) are $\omega_3/\omega_1 = L$, $\beta/3\omega_1 = \nu$, $T_i/T_e = T$, etc. We have chosen these parameters so as to simulate the experimental situation of Ref. 2, Fig. 9: $p = 1$, $T = 0.35$, $L = 6l = 6(4 + 3n)$ ($n = 0, 1, 2, \dots$). The value of T was chosen so that the ratio of the damping length to the wavelength agrees with the measured value.⁸ The value of ν was chosen so that the calculated line shape of the echo and its attenuation show the best fit to the experimental data. We found $\nu = 0.0005$, or a collision frequency $\beta = 9 \times 10^2/\text{sec}$. Thompson's formula gives $\beta = 2 \times 10^2/\text{sec}$ for $n = 10^9/\text{cm}^3$ and $T_i = 0.25$ eV. For $\nu = 0.0005$ we calculated the echo shape $|I(z, l)|$ and the phase velocity of the echo wave for various values of L . The results for the line shape are plotted in Fig. 1(a). For comparison we cite Fig. 9 of Ref. 2



(a)



(b)

FIG. 1. (a) Calculated line shape of the second-order ion-wave echo. The abscissa is the distance from the expected position of the echo divided by the wavelength λ of the echo wave. (b) Observed line shape of the second-order ion-wave echo plotted in the same scale as in (a). The data are taken from Ref. 2, Fig. 9.

plotted in the same scale in Fig. 1(b). A comparison of these two figures shows a general agreement between theory and experiment. In addition, the following properties of the integral $I(z, l)$ are disclosed by this calculation:

- (1) The echo shape becomes flatter as L is increased, in agreement with the experimental result.
- (2) The peak amplitude damps like $\exp[-l^{*1.4}]$.
- (3) The phase velocity for $L = 24$ is exactly equal to the phase velocity calculated from the dispersion relation ($v_{ph} = 1.86 v_i$) but it increases to 1.16 times that value as L is increased to 114.
- (4) A slight shift in the echo peak toward the exciter is obtained as L is increased.

Point 1 may be interpreted as follows. As noted above, the main effect of collisions is to reduce the contribution of slow particles. These particles have a significant contribution only near the echo peak; in the wing their contributions phase mix to zero {note the phase factor $\exp[i(\omega/v)(z - l)]$ }. Therefore, the collisional effect is more important near the peak than in the wing and the echo shape becomes flatter by collisions. Points 2 and 3 indicate a closer agreement between theory and experiment than any of the previous approximate calculations. Point 4 is a result of the factor $\exp[-C(\omega_3, z - l)]$ which causes a stronger damping at greater values of z . However, this peak shift is smaller than the observed peak shift. One reason for this is that we have considered only that part of the collisional effects which affects the damping of the distribution. In fact, collisions also produce a modification of the phase of the distribution. Although this effect is of higher order in the limit of small collisions, it is important near the peak where the phase vanishes.

Finally, we examine the validity of the small collision approximation. For this purpose we substitute $f \sim \exp[i(\omega/v)z - (D/3)(\omega^2/v^5)z^3]$ into the collision term $D(\partial^2/\partial v^2)f$, and we obtain $D(\partial/\partial v)\{-i(\omega z/v^2) - (\omega^2 z^3/3)\partial/\partial v(D/v^5)\}f$. The small collision approximation is violated when the second term in the bracket becomes comparable to the first term. Assuming $D(v) = \beta v_i^5/v^3$ the ratio of the second term to the first term is given by

$$\frac{8(\omega z)}{3(v_i)} \left(\frac{\beta}{\omega}\right) \left(\frac{v_i}{v}\right)^7. \quad (5)$$

Setting $z = l$, $\beta/\omega = 0.0005$, and replacing v by the phase velocity, the above ratio becomes $0.01 \sim 0.25$ for $L = 24 \sim 114$. It should be noted that for electrons whose velocity is less than the phase velocity the above ratio can easily become greater than unity. Thus, the use of the small collision approximation in the entire range of velocities is questionable in the present calculations.⁹

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⁷ If we use O'Neil's method, this factor should be replaced by $\{C(\omega_3, z - l') - C[\omega_3, (l - l')]\}$. The difference becomes important when $(z - l')$ is large, but there both O'Neil's formula and our formula become inappropriate.

⁸ In actual plasmas T is about unity, but the distribution function deviates considerably from Maxwellian and as a result the damping is smaller than that evaluated by assuming a Maxwellian distribution. Here we simulate this situation by assuming a Maxwellian distribution and by choosing T to be small.

⁹ In the case of the experiment by Wong and Baker,³ the ratio (5) with $v = v_{ph}$ ranges between 0.5 and several hundreds, so the small collision approximation seems to be inappropriate for the purpose of comparing with their data.

Instabilities in Finite Beam-Plasma Systems

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The relationship between the temporal growth of linear fluctuations in a finite beam-plasma system, and the spatial growth of the resulting stationary oscillations is discussed.

During the past several years there has been some theoretical¹⁻³ and considerable experimental⁴⁻⁹ research on the inhomogeneous turbulent state produced by an electron beam injected into a plasma. The stationary oscillations are found to grow in space as one proceeds from the injection point into the plasma. In contrast to the temporal instabilities, which cannot be resolved experimentally, this spatial instability affords an opportunity for a detailed check of theoretical ideas. The purpose of this note is to point out several erroneous suggestions about the linear development of instabilities in such a system and to obtain a consistent picture