Kinematical Constraints on Helicity Amplitudes*

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The kinematical constraints satisfied by helicity amplitudes are derived by two methods. One uses extra kinematical symmetries, which are present whenever there are constraints. The other uses the connection between M functions and helicity amplitudes. It is shown that the threshold and pseudothreshold constraints do not imply conspiracies between different Regge trajectories. The transitions between the different sets of $s=0$ constraints (which change when masses become equal) are described.

I. INTRODUCTION

KINEMATICAL constraints[1–4] have been found among helicity amplitudes at thresholds, pseudothresholds [$s=$ (difference of masses)^2], and at $s=0$. In this paper we will show how these constraints may be derived from Lorentz invariance and the special kinematical symmetries which appear at these points. The method of derivation and the form in which the relations are obtained will allow a discussion of some of their general features. We will also show that some of the constraints are very closely related to the kinematical singularities[5] of the helicity amplitudes. We will present an independent derivation of these relations, based on the analyticity of M functions,[6] which yields both the constraints and singularities simultaneously, and which may be extended to two-body thresholds and pseudothresholds in production reactions. Finally, we describe the behavior of the helicity amplitudes near $s=0$ in the transition from unequal to equal mass kinematics.

Except for an extension of the threshold and pseudothreshold relations to two-body thresholds and pseudothresholds in multiparticle reactions, none of the constraints which we discuss here are new. Many of the constraints were first found for restricted values of the masses and spins of the reacting particles, and Cohen-Tannoudji, Morel, and Navelet,[3] and Fox[4] have given unified treatments of these relations. The usual method involves an analysis of the singularities of helicity amplitudes and crossing matrices.[7,8]

The symmetry approach, which we will use to derive all the relations, was first used in this connection by Bardakci and Segrè,[9] who used it to derive the $s=0$ constraints in (equal mass) $\rightarrow$ (equal mass) scattering. The idea it exploits is this. In two-body scattering reactions there are ordinarily three linearly independent momenta. It may happen though, that only two of the momenta are independent, and when this happens there exist Lorentz transformations which do not change any of the momenta, and act only on the spins of the reacting particles. These Lorentz transformations do not connect processes going on in different rest frames, but instead relate a process with one set of spins to the same process but with different spins. Lorentz invariance then yields a set of linear relations between the helicity amplitudes for the process. In the center-of-mass frame this phenomenon occurs in forward and backward scattering, at $s=0$, and at thresholds and pseudothresholds. The kinematical constraints of forward and backward scattering, the vanishing of each amplitude like $\sin^n\theta$, where n is the spin flip along the direction of motion, have been known since antiquity and will receive no discussion here.

Threshold constraints in two-body scattering were noted by Jones,[10] and Jackson and Hite,[11] and Franklin[12] showed that these constraints are a consequence of the threshold behavior of partial wave amplitudes. In Sec. II we apply the analysis to thresholds and pseudothresholds and discover the relations derived by Cohen-Tannoudji, Morel, and Navelet. The analysis begins by constructing the momenta near these values of s. It is the form which the momenta take at these special points that determines which Lorentz transformations leave them unchanged, and through that the form of the kinematical constraints. These are found by expanding the Lorentz transformation rule for the amplitudes to first order in the Lorentz transformation, and then evaluating it for the Lorentz transformations which leave the momenta unchanged. The constraints on the derivatives of the amplitudes are then found by a repetition of this process.

The result is simplest when expressed in terms of amplitudes with spins quantized along the axis perpendicular to the reaction plane. That amplitude in

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[2] M. Gell-Mann and E. Leader, in *Proceedings of the Thirteenth International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, Cal., 1967).

[3] G. Cohen-Tannoudji, A. Morel, and H. Navelet, Ann. Phys. (N. Y.) **46**, 239 (1968).

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[9] K. Bardakci and G. Segrè, Nuovo Cimento **53A**, 56 (1968).

[10] H. F. Jones, Nuovo Cimento **50A**, 814 (1967).

[11] J. D. Jackson and G. E. Hite, Phys. Rev. **169**, 1248 (1968).

[12] J. Franklin, Phys. Rev. **170**, 1606 (1968).

which the combined spin of the two particles at threshold or pseudothreshold is greatest is the largest, and for each unit of spin less than the maximum, the amplitude vanishes like $(s-s_0)^{1/2}$ with respect to the most singular amplitude, where s_0 is the threshold or pseudothreshold value of s. Only the spins of the particles at the threshold or pseudothreshold are relevant to the constraints. The beahvior of the leading amplitude is determined by the kinematical singularities which occur at the same points as these constraints. For a maximum total spin of S, the most singular amplitude goes like $(s-s_0)^{-S/2}$. Thus, if s_y labels the total perpendicular-axis spin of the two particles at the threshold or pseudothreshold, the linear combination of helicity amplitudes corresponding to this spin has the threshold kinematical behavior

$$f_{s_y}\sim(s-s_0)^{-s_y/2}.$$

We argue that the relations at thresholds and at pseudothresholds for unequal-mass particles are not conspiracy relations; that is, they cannot be satisfied at large momentum transfer by the cancellation of contributions from Regge trajectories with different quantum numbers. Although we defer the details until Sec. II, the line of argument is as follows. The threshold and unequal-mass pseudothreshold constraints are the statement that the raising operator for the total perpendicular-axis spin of the outgoing particles gives zero when applied to the helicity amplitudes. Because the outgoing particles can be taken to have zero three-momenta, this is equivalent to the statement that the raising operator for the total angular momentum of these particles annihilates the helicity amplitudes

$$\langle f|J_{(y)}{}^{+}(S-1)|i\rangle=0.$$

Since the quantum numbers which label trajectories commute with angular momentum, the relations can always be written so as to involve amplitudes which receive contributions from Regge trajectories with only one set of values of these quantum numbers in each relation. Thus they are not conspiracy relations.

In Sec. III we turn our attention to the relations which obtain at $s=0$. As has been known since their discovery, these relations do connect different Regge trajectories, and in the literature they have become known as conspiracy conditions. The first of these was found in $N\bar{N}\to N\bar{N}$ scattering by Gribov and Volkov. (Equal mass) $\to$ (unequal mass) constraints have been investigated by Diu and Le Bellac,[13] Hogaasen and Salin,[14] and by Stack,[15] while Frautschi and Jones[16] have examined (unequal mass) $\to$ (unequal mass) scattering. As we have noted, the case of (equal mass) $\to$ (equal mass) scattering has been examined by this method by Bardakci and Segrè. The analysis proceeds just as at thresholds. Depending on whether two particles have equal or unequal masses, the form that their momenta take at $s=0$ is very different. Since the kinematical constraints follow from the momentum configuration, we must treat the cases where neither one or both at the incoming and outgoing channels contain equal-mass particles separately; we expect, and obtain, different constraints in the three cases.

The results are as follows: Take the scattering plane to be the xz plane and treat helicity as spin along the z axis. For (equal mass) $\to$ (equal mass) scattering, the linear combinations of helicity amplitudes corresponding to total x-axis spin m (total of all four spins) go like

$$f_m\sim s^{|m|/2}$$

near $s=0$. If the masses of one pair of particles are unequal, only the spins of the other pair are relevant, and the relations turn out to be just the same as the pseudothreshold constraints at pseudothreshold in the equal-mass particles. The linear combination of helicity amplitudes corresponding to y-axis spins τ_1 and τ_2 of the equal-mass particles behaves like

$$f_{\tau_1\tau_2}\sim s^{-(\tau_1-\tau_2)/2}.$$

When both sets of particles have unequal masses, we find no constraints.

The algebraic form of the constraints is not by itself sufficient to determine the presence or absence of conspiracies, and the argument excluding them at thresholds and unequal-mass pseudothresholds also uses the momentum configurations at these points. The conventional analysis for the presence of conspiracies involves first evaluating the contribution of each Regge trajectory to the helicity amplitudes, and then seeing if different kinds of trajectories are related by the constraints. The contribution of any given trajectory to a helicity amplitude depends, among other things, on the kinematical factors associated with the branch points at the boundary of the physical region (the "half-angle factors")[17] and these factors are very different at $s=0$ than they are at unequal-mass pseudothresholds. The difference reflects the fact that the momentum configurations are different in the two situations. Thus a kinematical constraint can imply conspiracies at $s=0$,[13–15] even though it cannot at unequal-mass pseudothresholds.

A useful distinction can be made between those constraints which are accompanied by singularities in the helicity amplitudes and those which are not. The latter category contains only the constraints at forward and backward scattering, and at $s=0$ in (equal mass) $\to$ (equal mass) scattering. For these constraints the previous discussion is complete. But when singularities are present, they must first be removed in order to

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find the constraints. Also, when singularities are present, the kinematical symmetry which we exploit appears only in Lorentz frames in which some of the momentum components are infinite, and so the existence of the symmetry argument seems to rest on the presence of the singularities.

In Sec. IV we demonstrate the closeness of this relationship by deriving the singularities and constraints simultaneously from a single argument which makes no use of the symmetry considerations which dominate Secs. II and III. This derivation is based on M functions, whose relevant properties we review, and their connection to helicity amplitudes. Stapp[18] has used this method to derive the kinematical singularities in helicity amplitudes, and Stack[15] has used it to get the $s=0$ constraints in (equal mass) $\rightarrow$ (unequal mass) scattering. The idea here is just to write down the connection between M functions and helicity amplitudes, and to explicitly display the singularities and zeros implicit in the relation. M functions are supposed to be free of kinematical singularities and zeros, so the singularities and zeros in the expression which relates them to helicity amplitudes must appear in the helicity amplitudes. The singularities are just those of Wang, and the zeros are our previously discussed constraints.

M functions are finite only when the momenta at which they are evaluated are finite, and so we must relate center-of-mass frame helicity amplitudes to M functions evaluated in a frame where the momenta are finite. Thus not only do we not use the kinematical symmetry in this derivation, but we are concerned with M functions only in frames where there is no kinematical symmetry. Parenthetically, we may note that the assumption that the M functions are free of kinematical singularities is used in the derivation of the helicity-amplitude crossing relations, and so this assumption is present in the derivations of kinematical constraints which proceed from an analysis of the singularities of crossing matrices.

The constraints at $s=0$ in (equal mass) $\rightarrow$ (equal mass) scattering have no singularities associated with them, they are not the result of a singular connection between nonsingular M functions and helicity amplitudes, and so they are not derivable by this method. All the other constraints that we discuss, together with the singularities, are equivalent to the statement that M functions are free of kinematical singularities and zeros, but at $s=0$ in (equal mass) $\rightarrow$ (equal mass) scattering the M functions do have kinematical zeros. It is from these, rather than from the connection between M functions and helicity amplitudes, that the (equal mass) $\rightarrow$ (equal mass) conspiracy relations arise.

By contrast, the symmetry method does give all the constraints, but it requires as additional input a knowledge of the kinematical singularities. It is also very appropriate for the examination of the question of which constraints imply conspiracies. However, it does not generalize in a simple way to multiparticle reactions, which the M-function method does.

The singularities in the connection between M functions and helicity amplitudes at thresholds and pseudothresholds result only from the configuration of the momenta of the two particles whose total energy is the sum or difference of their masses, and the relations depend only on the spins of these particles. This momentum configuration is the same no matter how many other particles are involved in the reaction, and so the threshold and pseudothreshold constraints hold at a two-body threshold or pseudothreshold in a reaction involving any number of particles.

In view of the fact that the $s=0$ constraints are so different depending on whether a pair of masses is equal or not, it is pertinent to ask how the amplitudes behave when two unequal masses approach equality. In Sec. V we give a qualitative description of these transitions. We show how helicity amplitudes for equal-mass particles are obtained from those for unequal masses as a continuous limit, despite the fact that the $s=0$ kinematical relations change abruptly as the masses become equal. The transition from all masses unequal to one pair equal is very simple to describe. There are no $s=0$ relations for unequal masses, but there are relations at the pseudothreshold. The pseudothreshold relations simply continue to hold when, because the mass difference becomes zero, the pseudothreshold moves to $s=0$.

The transition from one pair of masses equal to both equal is more complicated. It is best described if we remember that kinematical constraints and singularities are a direct result of momentum configurations. For $|s|$ much greater than the (mass)^2 difference the momenta are substantially what they would be if the mass difference were zero, while for $|s|$ less than the (mass)^2 difference they are very rapidly varying functions of s. These rapid variations, with singularities at $s=0$ and pseudothreshold, are responsible for the kinematical constraints at these points. As the mass difference approaches zero, the rapid variations of the amplitudes are confined to a progressively smaller range of s, while equal-mass kinematics governs the momentum configuration progressively nearer to $s=0$. When the mass difference becomes zero, the range over which the rapid variations take place collapses to a point, and equal-mass kinematics becomes correct over the entire range of s, establishing the equal-mass constraints. The use of equal-mass kinematics when mass differences are very small but finite is justified because the kinematical effects of the unequal masses are confined to very small range at s.

II. THRESHOLD AND PSEUDOTHRESHOLD CONDITIONS

At a threshold or unequal-mass pseudothreshold for a two-body scattering process, the center-of-mass

[18] H. P. Stapp, Phys. Rev. **160**, 1251 (1967).

momentum of one pair of particles vanishes, and so all the momenta lie in a plane in four-space. The freedom to make rotations which leave all the momenta unchanged leads to a set of linear relations among the helicity amplitudes. In contrast to the $s=0$ relations, the form of these does not depend on the relative sizes of masses involved. We will treat the case of threshold or pseudothreshold for the outgoing particles.

In order to find the rotation which leaves all the momenta unchanged, we will construct the momenta near these special points. We take the momenta of the outgoing particles along the z axis, and the momenta of the incoming particles in the xz plane. At threshold, they are

$$\begin{aligned} p_1&=(E_1,p_x,0,p_z)\,,\\ p_2&=(E_2,\,-p_x,\,0,\,-p_z)\,,\\ p_3&=((m_3^2+\epsilon^2)^{1/2},0,0,\epsilon)\,,\\ p_4&=((m_4^2+\epsilon^2)^{1/2},\,0,\,0,\,-\epsilon)\,. \end{aligned} \tag{1}$$

The values of E_1, E_2, and $p^2=p_x^2+p_z^2$ are fixed by s or, equivalently ϵ, and have some finite limit as $\epsilon\to 0$. This is not true of the components p_x and p_z, however, since in order for the momentum transfer t to differ from the value it obtains at threshold in the physical region $(m_1^2+m_3^2-2m_3E_1)$, the limit of ϵp_z must be finite as $\epsilon\to 0$. Since $p^2=p_x^2+p_z^2$ is finite as $\epsilon\to 0$, both p_x and p_z must diverge simultaneously. For real t, as $\epsilon\to 0$, they become

$$\begin{aligned} p_z&\to\infty\sim 1/\epsilon\,,\\ p_x&\to ip_z\,. \end{aligned} \tag{2}$$

At a pseudothreshold for unequal-mass particles the same phenomenon occurs. The only differences are the numerical values of E_1, E_2, and p^2, and p_4 which becomes

$$p_4=(-(m_4^2+\epsilon^2)^{1/2},\,0,\,0,\,-\epsilon)\,. \tag{3}$$

The values of p_x and p_z still diverge as shown by Eq. (2).

Let us define $\mathcal{J}$ to be the generator of rotations which leave $(p_x,0,p_z)$ unchanged in the limit $\epsilon\to 0$. Then $\mathcal{J}$ is the limit of

$$\mathcal{J}^{(\epsilon)}=(p_x/p_z)J_x+J_z\,. \tag{4}$$

At $\epsilon=0$, the differential operator representing $\mathcal{J}$ annihilates all the momenta; from Eq. (2), $\mathcal{J}^{(\epsilon)}$ has the limit

$$\mathcal{J}=iJ_x+J_z=J_{(y)}{}^+\,, \tag{5}$$

where by the last symbol we indicate that $\mathcal{J}$ is a raising operator for angular momentum quantized along the y axis. Because $\mathbf{J}_{(y)}{}^+$ annihilates all the momenta, applying it to a helicity amplitude at $\epsilon=0$ will lead to a set of linear relations among the amplitudes.

A. Helicity Amplitudes

Helicity eigenstates are obtained from states at rest and with spin quantized along the z axis by the application of prescribed Lorentz transformations, which we will call boosts. A boost is a pure Lorentz transformation along the $+z$ axis followed by a rotation through an angle less than or equal to π about an axis in the xy plane. The helicity is the same as the z component of the spin of the state at rest. The boosts are determined by the momenta of the state, and we will denote them by $B(p)$. The Lorentz transformation properties of these states are

$$\begin{aligned} \Lambda|p\lambda\rangle&=D_{\lambda\lambda'}{}^{(s)T}(W(\Lambda,p))|\Lambda p\lambda'\rangle\\ &=D_{\lambda\lambda'}{}^{(s)*}(W^{-1}(\Lambda,p))|\Lambda p\lambda'\rangle\,, \end{aligned} \tag{6}$$

where the Wigner rotation is given by

$$W(\Lambda,p)=B^{-1}(\Lambda p)\Lambda B(p)\,. \tag{7}$$

Helicity amplitudes are matrix elements of $S-1$ between two particle helicity states. Their Lorentz transformation law is, in a matrix notation (p represents all the momenta),

$$f(p)=D(W^{-1}(\Lambda,p))f(\Lambda p)\,. \tag{8}$$

The representation D is the direct product of two representations of Wigner rotations for the outgoing particles with two conjugate representations for the incoming particles. The infinitesimal form of this relation will be useful in our applications. If $\Lambda=1-i\alpha M$ is an infinitesimal Lorentz transformation, $\mathbf{M}$ a representation of M as a differential operator acting on functions of momenta, and $W(\Lambda,p)=1-i\alpha W(M,p)$, then Eq. (8) becomes

$$[D(W(M,p))-\mathbf{M}]f(p)=0\,. \tag{9}$$

Here the representation D is the direct sum of representations of the generators of the Wigner rotations.

B. Thresholds

We wish to express the threshold constraints in terms of helicity amplitudes, and so we use helicity states for the incoming particles. However, at a threshold the helicity state basis is singular. The singularity is a result of the fact that the limit

$$\lim_{|\mathbf{p}|\to 0}|p\lambda\rangle$$

is ill defined, since it depends on the direction through which $\mathbf{p}$ approaches zero. To avoid this coordinate singularity, we will use Wigner basis states for the outgoing particles. Like helicity states, these are obtained from states at rest by a Lorentz transformation which we will also call a boost, but with B a pure Lorentz transformation along the $\hat{p}$ axis. The connection between Wigner and helicity states is indicated by the comparison of their boosts. For the momenta we have chosen for particles 3 and 4, Eq. (1) ($\epsilon>0$), the con-

nection is

$$|p_3\lambda\rangle^{\text{Wigner}}=|p_3\lambda\rangle^{\text{helicity}},$$
$$|p_4\lambda\rangle^{\text{Wigner}}=(-1)^{s_4+\lambda}|p_4-\lambda\rangle^{\text{helicity}}. \quad (10)$$

It is thus a triviality to transform the Wigner-amplitude relations we will obtain into helicity-amplitude relations.

We now assume that the threshold kinematical singularities have been removed from the mixed Wigner helicity amplitudes $f(p)$ by multiplication with a common scalar factor $[s-(m_3+m_4)]^{\alpha/2}$ or ϵ^α, and that there are no other singularities there. This does not change their Lorentz transformation properties. Then, because $\mathbf{J}_{(y)}{}^+p\rightarrow 0$ as $\epsilon\rightarrow 0$, we have

$$\mathbf{J}_{(y)}{}^+f(p)|_{\epsilon=0}=0. \quad (11)$$

Equation (9), the general statement of Lorentz invariance, then implies that

$$D(W(J_{(y)}{}^+,p))f(p)|_{\epsilon=0}=0. \quad (12)$$

We must simply calculate the infinitesimal Wigner rotations corresponding to $J_{(y)}{}^+$ in order to find the kinematical constraints. For the outgoing states this is a triviality, since because $J_{(y)}{}^+$ generates rotations, $W(J_{(y)}{}^+,p_{3,4})$ is just $J_{(y)}{}^+$ itself. For the incoming states, Eq. (7) gives

$$\lim_{\epsilon\rightarrow 0} W(J_{(y)}{}^+,p_{1,2})=\lim_{\epsilon\rightarrow 0}(|p|/p_z)J_{(y)}{}^+=0. \quad (13)$$

The reason for this is that we are subjecting a generator pointing in the $(i,0,1)$ direction to the inverse of the rotation which takes a finite vector lying along the z axis into an infinite vector parallel to $(i,0,1)$. Thus the threshold conditions only involve the spins of the outgoing particles and are

$$D^{(\text{out})}(J_{(y)}{}^+)f(p,s=(m_3+m_4)^2)=0. \quad (14)$$

Because the linear combinations of helicity states which correspond to a definite value of s_y at threshold are, except for a phase, the same as the corresponding linear combinations of Wigner states, this is directly a helicity-amplitude constraint.

The content of these relations is that if S is the total spin of the outgoing state ($|s_3-s_4|\leq S\leq|s_3+s_4|$), then only the amplitude whose spin along the y axis is as large as possible, $s_y=+S$, can be nonvanishing; the others vanish. The fact that only the spins of the outgoing particles enter into this kinematical constraint will also be true for pseudothresholds.

We can deduce how quickly the states of different s_y vanish by a simple repetition of this argument. The amplitudes with $s_y\leq S-1$ all vanish with ϵ, and so are finite when divided by ϵ. Applying Lorentz invariance, we get Eq. (14) applied to $\epsilon^{-1}D^{(\text{out})}(J_y{}^+)f(p)$, rather than to $f(p)$, so that we find that $[D^{(\text{out})}(J_y{}^+)]^2f(p)$ vanishes like ϵ^2 as $\epsilon\rightarrow 0$. This procedure may be repeated to yield the result that $[D^{(\text{out})}(J_y{}^+)]^nf(p)$ vanishes like ϵ^n. We may state this result as a set of derivative relations on the amplitude:

$$[D^{(\text{out})}(J_y{}^+)]^{n+1}(d^n/d\epsilon^n)f(p)|_{\epsilon=0}=0. \quad (15)$$

These relations describe the kinematical behavior of amplitudes whose final states have definite values of s_y.[19] In terms of helicity states these are

$$\langle s_y=\tau|=C_{\tau\lambda}\langle h=\lambda|,$$
$$C_{\tau\lambda}=\langle s_z=\tau|R_x(\pi/2)|s_z=\lambda\rangle. \quad (16)$$

The actual kinematical behavior of the helicity amplitudes must also include the kinematical singularities at thresholds. Borrowing from Sec. IV, we use the fact that the amplitude with $s_y=S$ has a kinematical singularity ϵ^{-S}. Thus the threshold kinematical behavior of the amplitudes $f_{\tau_3\tau_4;\lambda_1\lambda_2}$, connected to helicity amplitudes by Eq. (16), is

$$f_{\tau_3\tau_4;\lambda_1\lambda_2}\sim\epsilon^{-(\tau_3+\tau_4)}. \quad (17)$$

C. Pseudothresholds

At an unequal-mass pseudothreshold, $s=(m_3-m_4)^2$, substantially the preceding analysis yields constraints. The only difficulty is that near pseudothreshold p_4 approaches $(-m_4,\mathbf{0})$ and both the Wigner and helicity bases are singular. We therefore use modified Wigner states for particle 4, obtained from states at rest by choosing the boost to be a pure Lorentz transformation along the z axis of $i\pi$, followed by a pure Lorentz transformation along the direction of $\mathbf{p}_4$. For the indicated value of p_4, the connection between the states $|p_4\lambda\rangle^{W'}$ and the helicity states is just the same as in Eq. (10), except for an extra phase of (-1) for fermions, which will never enter into any relations.

The same rotation leaves all the momenta unchanged, since $p_{1,2}$ are as before. All the Wigner rotations are as before, except for $W(J_{(y)}{}^+,p_4)$, which because of the extra $L_z(i\pi)$ is

$$W(J_{(y)}{}^+,p_4)=iJ_x-J_z=-J_{(y)}{}^-. \quad (18)$$

This yields the pseudothreshold conditions on the helicity amplitudes

$$[D^{(3)}(J_{(y)}{}^+)\oplus D^{(4)}(-J_{(y)}{}^-)]\times f(p,s=(m_3-m_4)^2)=0. \quad (19)$$

The derivative constraints follow by exactly the same reasoning as at threshold and are

$$[D^{(3)}(J_{(y)}{}^+)\oplus D^{(4)}(-J_{(y)}{}^-)]^{n+1}\times(d^n/d\epsilon^n)f(p)|_{\epsilon=0}=0. \quad (20)$$

In terms of the linear combinations of helicity amplitudes appearing in Eq. (16), and including the threshold singularities, the pseudothreshold behaviors of the

[19] These are the same as the transversity amplitudes of Cohen-Tannoudji *et al.* If they are used to describe all the spins, then parity conservation implies $(-1)^{\Sigma\tau}=-\Pi$ (intrinsic parities).

amplitudes are

$$f_{\tau_3\tau_4;\lambda_1\lambda_2}\sim\epsilon^{-(\tau_3-\tau_4)}. \tag{21}$$

Since the state $\langle\tau_4, E\sim -m_4|$ has y-axis spin $-\tau_4$, the interpretation of these relations is just the same as that of the threshold constraints.

D. No Conspiracies

The constraints at threshold or unequal-mass pseudothresholds for the outgoing particles are the statement that the raising operator for total y-axis spin of the outgoing particles, $S_{(y)}{}^{+(\text{out})}$, annihilates the helicity amplitudes at the thresholds and unequal-mass pseudothreshold, Eqs. (14) and (19). We may choose the momenta of the outgoing particles, Eqs. (1) and (3), so that they vanish there,[20] and so $\mathbf{J}_{(y)}{}^{+(\text{out})}$, a differential operator which generates rotations of $\mathbf{p}_3$ and $\mathbf{p}_4$, trivially annihilates the helicity amplitudes at the threshold and pseudothreshold. We may thus write these constraints as

$$[S_{(y)}{}^{+(\text{out})}+\mathbf{J}_{(y)}{}^{+(\text{out})}]f(p)=0. \tag{22}$$

The object in brackets is just the $z+ix$ component of the total angular momentum, acting on the final state. Thus these constraints are as well the statement that the raising operator for total angular momentum along the y axis, acting on the outgoing particles, annihilates the helicity amplitudes.

Phrased in this way, it is clear that these relations never imply correlations between Regge trajectories, which are supposed to describe amplitudes at large momentum transfer, of different quantum numbers. The reason is that whatever quantum numbers label a trajectory—internal quantum numbers, parity, charge conjugation, signature, etc.—all commute with each component of the total angular momentum. Thus the relations in Eq. (22) can be written so that each relation involves amplitudes with only one set of values of those quantum numbers. At large t then, each becomes a constraint on one kind of Regge trajectory only. Since they cannot be satisfied by the cancellation of the contributions of two different kinds of Regge trajectories, they are not conspiracy relations.

E. Phase of p_x/p_z

Throughout the preceding derivations we have taken the phase of p_x so that (Eq. (2)) $p_x/p_z\to+i$. There is clearly no reason for preferring this choice to $p_x/p_z\to -i$. This freedom reflects the choice of path in continuing the scattering amplitudes around the kinematical singularities at $|\cos\theta|=1$ (the "half-angle factors"), i.e., whether t is given a positive or negative imaginary part in circumventing the branch points at the boundary of the physical region. Equivalently, the choice of phase of p_x/p_z reflects the choice of phase at the center-of-mass scattering angle for a given value of $\cos\theta$. The choice we have made is $\theta\to+i\infty$; the other choice is $\theta\to-i\infty$.

If we were to carry through the preceding with $p_x/p_z\to -i$, we would find similar relations, but with $J_{(y)}{}^-$ and $J_{(y)}{}^+$ substituted for each other everywhere. Their significance would be just the opposite of those we found, namely, the amplitude with minimum y spin is nonvanishing, and amplitudes with successively larger y spins vanish more and more strongly as $s\to(m_3+m_4)^2$. The two kinematical configurations, with their attendant kinematical constraints, are transformed into one another by a rotation of π about the z axis, and so the two sets of relations imply each other.

An equivalent ambiguity is present in the constraints at $s=0$ in (equal mass) $\to$ (unequal mass) scattering.

III. $s=0$ CONDITIONS

At $s=0$, in the center-of-mass frame, the total four-momentum vector vanishes, and so here too there are only two linearly independent momenta, the differences of the initial and of the final momenta. The previous analysis will again yield linear relations among the helicity amplitudes. These are the constraints which have become known as conspiracy conditions. Their form depends upon whether the incoming and outgoing particles have equal or unequal masses. To see why this is so, we need only construct the relative momenta,

$$q_{12}=p_1-p_2=\left(\frac{m_1{}^2-m_2{}^2}{s^{1/2}},\hat{e}_{12}\left[\frac{(m_1{}^2-m_2{}^2)^2}{s}+s-2(m_1{}^2+m_2{}^2)\right]^{1/2}\right) \tag{23}$$

and similarly for q_{34}. As $s\to 0$, when the masses are equal,

$$q\to(0,2i\hat{e}m), \tag{24}$$

while when they are unequal [δ is the (mass)^2 difference],

$$q\to(\delta/s^{1/2})(1,\hat{e}). \tag{25}$$

The momentum configurations depend on the masses, and thus so will the symmetry transformations and their associated Wigner generators which determine the relations. We will discuss how the helicity amplitudes behave as masses approach equality in Sec. V.

A. (Equal Mass) $\to$ (Equal Mass)

In this case, $m_1=m_2$ and $m_3=m_4$, both relative momenta have the form indicated by Eq. (24). If $\hat{e}_{12}$ and $\hat{e}_{34}$ are chosen to lie in the xz plane (their relative orientation determines t), then the Lorentz transformation which leaves all the momenta unaffected at $s=0$ is a pure Lorentz transformation along the y axis.

[20] The three-momenta of a pair of equal-mass particles cannot be chosen to vanish at $s=0$, so the following argument does not apply to that case.

These amplitudes have no singularities at $s=0$, so the differential operator $\mathbf{N}_y$ generating y-axis Lorentz transformations annihilates the helicity amplitudes $f(p)$, and thus so must the infinitesimal Wigner rotation corresponding to this Lorentz generator.

The boosts appropriate to these momenta are a pure z-axis Lorentz transformation through $i\frac{1}{2}\pi$ followed by a y-axis rotation, and the infinitesimal Wigner rotations can be easily evaluated. The result for each of the spins is the same, $-iJ_x$. Thus the (equal mass) $\rightarrow$ (equal mass) conspiracy conditions are

$$D(J_x)f(p)=0. \tag{26}$$

The spins of all the particles are involved in this relation.

To find how quickly the vanishing amplitudes vanish as $s\rightarrow 0$, we consider the linear combinations of helicity amplitudes which are eigenvectors of $D(J_x)$,

$$D(J_x)f_{(m)}(p)=mf_{(m)}(p). \tag{27}$$

The differential operator which generates a linear change in center-of-mass energy, or $s^{1/2}$, is a sum of generators of Lorentz transformations, along different directions for the different particles. It may be written as the sum of two terms, $\mathbf{d}^+ + \mathbf{d}^-$, which act as raising and lowering operators for $i\mathbf{N}_y$:

$$[i\mathbf{N}_y,\mathbf{d}^\pm]=\pm\mathbf{d}^\pm. \tag{28}$$

Thus if we apply $d/d(s^{1/2})$ to $f_{(m)}(p)$, we find it can be nonvanishing for $|m|=1$. Similarly, the first nonvanishing derivative of $f_{(m)}(p)$ is $d^{(m)}/d(s^{1/2})^{(m)}$. This is equivalent to the kinematical constraints

$$f_{(m)}(p)\sim(s^{1/2})^{|m|},\quad s\rightarrow 0. \tag{29}$$

There are no kinematical singularities in this case, and so this is the final form of the constraint.

B. (Unequal Mass) $\rightarrow$ (Equal Mass)

In this case, $m_1\neq m_2$ and $m_3=m_4$, both the limits Eqs. (24) and (25) are involved. For definiteness, we choose $\hat{e}_{12}=(0,0,1)$, and $\hat{e}_{34}$ in the xz plane. $\hat{e}_{34}$ must approach the x axis in the limit $s\rightarrow 0$ in order for t to remain finite. Thus the two independent momenta lie along (1,0,0,1) and (0,1,0,0), and the Lorentz transformations which affect neither are generated by N_y+J_x. For this mass configuration, at $s=0$, the helicity amplitudes are annihilated by $\mathbf{N}_y+\mathbf{J}_x$, and so, from Eq. (9), by its associated infinitesimal Wigner rotation.

As always, we need the boosts corresponding to the momenta at $s=0$. For the outgoing particles, they are as described in the previous section, with rotations of $\mp\frac{1}{2}\pi$. The infinitesimal Wigner rotations are, then,

$$W(N_y+J_x,\,p_{3,4})=-iJ_x\mp J_z=\mp J_{(y)}{}^{\pm}. \tag{30}$$

The boosts corresponding to p_1 and p_2 are divergent Lorentz transformations at $s=0$, because the momenta themselves are, their difference being given by Eq. (25). Thus the infinitesimal Wigner rotations must be calculated for $s\rightarrow 0$ as a limit, and when this is done, they are seen to vanish. The reason is that the generator of the Lorentz transformation is $M_{02}+M_{32}$, which acts as a null vector under z-axis Lorentz transformations, and the calculation of the infinitesimal Wigner rotation involves an infinite Lorentz transformation along the $(-z)$ axis, whence it vanishes.

The conspiracy relation involves only the spins of the equal mass particles and is

$$[D^{(3)}(J_y{}^{(+)})\oplus D^{(4)}(-J_y{}^{(-)})]f(p)=0. \tag{31}$$

If the final particles had unequal masses and the initial particles equal masses, the conspiracy relation would be just the same, but with initial spins replacing final ones. If the procedure followed at thresholds is adapted to this case, we get the derivative constraints

$$[D^{(3)}(J_y{}^{(+)})\oplus D^{(4)}(-J_y{}^{(-)})]^{n+1}\times(d^n/d(s^{1/2})^n)f(p)=0. \tag{32}$$

Combining this with the kinematical singularity of Sec. IV, and using the linear combinations of amplitudes described by Eq. (16), we may write the $s\rightarrow 0$ kinematic behavior as

$$f_{\tau_3\tau_4;\lambda_1\lambda_2}\sim(s^{1/2})^{-(\tau_3-\tau_4)}. \tag{32'}$$

These are the same relations as the pseudothreshold results. However, the presence or absence of conspiracies depends not just on the kinematical constraints, but also on kinematical factors which affect the contributions of trajectories to the helicity amplitudes. Because these factors are very different at $s=0$ than they are at unequal-mass pseudothresholds, we cannot deduce anything about the necessity for conspiracies at $s=0$ from the fact the kinematical constraints are the same there as they are at pseudothresholds.

C. (Unequal Mass) $\rightarrow$ (Unequal Mass)

In this case all the momenta diverge as $s\rightarrow 0$, and they must become parallel in order to keep t finite. However, the magnitudes of the momenta diverge inversely as their relative angle approaches zero, so that there are two independent momenta in the limit. If we choose $\hat{e}_{12}$ along the z axis, and $\hat{e}_{34}$ in the xz plane, then the independent momenta are exactly as in the previous case. Furthermore, the momenta are just the same as those of the unequal-mass particles in the previous case, so the infinitesimal Wigner rotations are just those of the unequal-mass particles in the unequal $\rightarrow$ equal conspiracy relations. We have seen that these vanish, so that there are no conspiracy conditions at all for (unequal mass) $\rightarrow$ (unequal mass) scattering at $s=0$. Here we have a symmetry but no associated relations.

IV. KINEMATICAL SINGULARITIES

The kinematical singularities of helicity amplitudes can be found by examining the crossing relations, as Wang[5] did, or by examining the connection between helicity amplitudes and M functions, as Stapp[6] did. The methods are equivalent because the connection between helicity amplitudes and M functions and the assumption that M functions are free of kinematical singularities are the ingredients that go into the derivation of the crossing relations. For the specific linear combinations of helicity amplitudes with which we are concerned, Eq. (16), the latter method is especially simple, and so we will use it.

M functions may be thought of as the pole residues of Green's functions, with fields chosen to transform according to the $(j,0)$ representation of the Lorentz group. [We denote these representations by $D_L(\Lambda)$.] Thus, under Lorentz transformations, they behave as

$$M(p)=D_L(\Lambda^{-1})M(\Lambda p)\,, \tag{33}$$

where, as usual, D_L is a direct product of representations for the outgoing particles with conjugate representations for the incoming particles. The connection between M functions and helicity amplitudes, which is compatible with the transformation rule for helicity amplitudes, Eq. (8), is

$$f(p)=D_L(B^{-1}(p))M(p)\,. \tag{34}$$

$B(p)$ is, for each particle, its helicity-state boost. We will find the kinematical singularities, and also the associated constraints, by noting the singularities of $B(p)$. If the momenta are infinite, so that M need not be (kinematically) finite, we will use (33) to relate it to an M function at finite momenta.

For thresholds and pseudothresholds, we will use, instead of the frame specified by Eqs. (1) and (2), one related to it by a y-axis rotation. The helicity amplitudes are the same because, by construction, they are invariant under rotations in the scattering plane. The incoming momenta are to be finite in the limit $\epsilon\sim(s-s_{\text{threshold}})^{1/2}\to 0$, while the outgoing momenta take the form

$$p_{3,4}\to(m_{3,4},\mp ip,0,\pm p)\,. \tag{35}$$

The singularity comes from the rotation $R_y(\theta)$ which takes the momenta $(m_{34},0,0,\pm\epsilon)$ into the form of Eq. (35). For this rotation, $\cos\theta\cong i\sin\theta\sim 1/\epsilon$, and absorbing all the nonsingular factors into M, we get

$$f(p)=\epsilon^{-D^{(\text{out})}(J_y)}M(p) \tag{36a}$$

or

$$D^{(\text{out})}(R_x(\tfrac{1}{2}\pi))f(p)=\epsilon^{-D^{(\text{out})}(J_z)}M(p)\,, \tag{36b}$$

where an extra factor of $D^{(\text{out})}(R_x(\frac{1}{2}\pi))$ has been absorbed into M. This not only displays the threshold singularities of the amplitudes Eq. (16), but is exactly the matrix form of Eq. (17). Thus the singular connection between M functions and helicity amplitudes provides not only the singularities at threshold, but the constraints as well.

At pseudothresholds, the previous analysis must be modified by the inclusion of an extra factor $L_z(i\pi)$ in the boost associated with particle 4. It may be anticommuted through the J_y and absorbed into M, yielding

$$f(p)=\epsilon^{-[D^{(3)}(J_y)\oplus -D^{(4)}(J_y)]}M(p) \tag{37a}$$

or

$$D^{(\text{out})}(R_x(\tfrac{1}{2}\pi))f(p)=\epsilon^{-[D^{(3)}(J_z)\oplus -D^{(4)}(J_z)]}M(p)\,, \tag{37b}$$

which are matrix forms of Eq. (21).

At $s=0$, in (equal mass) $\to$ (equal mass) scattering, all the boosts are finite Lorentz transformations, and so there are evidently no kinematical singularities. Also, the constraints cannot be found by comparing helicity amplitudes to M functions; they are the result of a symmetry which is not produced by a kinematical singularity. For this case the discussion of Sec. III is complete.

For (unequal mass) $\to$ (equal mass) scattering, however, there are singularities associated with the divergence of the components of the momenta of the incoming particles. To use M functions at finite momenta, we must combine Eqs. (33) and (34), as

$$f(p)=D_L(B^{-1}(p)\Lambda^{-1})M(\Lambda p)\,, \tag{38}$$

where we choose Λ to be the inverse of the boost associated with the heavier of the incoming particles. Because the momenta of the outgoing particles become orthogonal to the direction of motion of the incoming particles at $s=0$, all the momenta Λp are finite as $s\to 0$. Also, the Lorentz transformations $B^{-1}(p_{1,2})\Lambda^{-1}$ are finite, and only $B^{-1}(p_{3,4})\Lambda^{-1}$ are singular. Multiplying by $D_L^{(\text{out})}(L_z(i\frac{1}{2}\pi)B(p))$, absorbing some nonsingular factors into M, and displaying the matrices appropriate to the configuration of Sec. III, we have ($\delta=m_1^2-m_2^2$)

$$[D^{(3)}(R_x(\tfrac{1}{2}\pi))\oplus D^{(4)}(R_x(-\tfrac{1}{2}\pi))]f(p) = (\delta^2/s)^{D^{(\text{out})}(J_z)/2}M(\Lambda p)\,. \tag{39}$$

This, of course, is the matrix form of Eq. (32). We could convert this into Eq. (37b) by simply multiplying by $D^{(4)}(R_x(\pi))$.

If we try to apply this analysis to (unequal mass) $\to$ (unequal mass) scattering, we find that the same choice of Λ results in finite Λp as $s\to 0$. However, all the $B^{-1}(p)\Lambda^{-1}$ Lorentz transformations are also finite in that limit, showing that there are no kinematic singularities, and yielding no constraints. Here both the symmetry and singularity which appear as $s=0$ are without effects.

V. TRANSITIONS BETWEEN MASS CONFIGURATIONS

In view of the fact that the $s=0$ constraints are different depending on whether the masses of the in-

coming or the outgoing particles are different or not, some remarks are appropriate as to how amplitudes behave when two unequal masses are allowed to approach equality. We will describe these transitions and see that although the constraints do change discontinuously as two masses become equal, the amplitudes, considered as analytic functions of s, survive the transitions quite smoothly. The description will illustrate why it is appropriate to use equal-mass kinematical behavior when a pair of particles have very slightly different masses, for example, in the scattering of two different members of an isotopic multiplet.

The transformation from the case when both the initial and final masses are unequal to when one pair are equal presents no difficulties in continuity. When both pairs of masses are unequal, there are no constraints at $s=0$, but there are at pseudothreshold. When two masses approach equality, the pseudothreshold moves to $s=0$, and the constraints that obtain there are just the pseudothreshold constraints. The smoothness of this transition is reflected in the fact that the two momentum configurations which are obtained by taking the limits (1) s goes to pseudothreshold, followed by letting the masses approach equality, and (2) masses set equal, followed by s going to zero, are related to each other by a *finite* Lorentz transformation, despite the fact that infinite momentum components arise in both configurations.

When one pair of masses are qual, and the other pair are allowed to approach equality, the transition is not so smooth. So long as the masses are unequal, there are different singularities and constraints at $s=0$ and at the pseudothreshold, but when the masses become equal, all the singularities disappear, and there are new constraints which look quite different from either of the unequal-mass constraints. A picture of how this transition occurs follows from the fact that the singularities and constraints are determined by the configuration of the momenta. For values of $|s|$ many times the (mass)^2 difference the momenta are substantially the same as if the mass difference were zero, while for $|s|$ comparable to the (mass)^2 difference the momenta are rapidly varying functions of s. They become singular at $s=0$ and at pseudothreshold. Thus we can divide the range of s into two regions: an outer region, $|s|$ many times the (mass)^2 difference, and an inner region. Equal-mass kinematics governs the outer region, while amplitudes fluctuate rapidly in the inner region. (Of course, these fluctuations are such as to satisfy the unequal-mass constraints.) As the mass difference approaches zero, the inner region becomes narrower, and the fluctuations are confined to a smaller region. When the mass difference becomes zero, the inner region elapses to a point, and the fluctuations become removable discontinuities in the amplitudes. With these point discontinuities neglected, the entire range of s becomes the outer region, and the equal-mass kinematical constraints hold at $s=0$. This is the behavior of the amplitudes which allows a very different set of kinematical constraints and singularities depending upon whether a pair of masses are equal or not.

This picture of how helicity amplitudes behave when a pair of masses become equal is equivalent to the observation that the unequal-mass constraints vanish when masses become equal.[14] They are equivalent because an analytic function with a zero is small over a region comparable to the inverse coefficient of the zero, and so the vanishing of the inverse coefficient (i.e., the disappearance of the zero) is the same as the confining of the region in which unequal-mass kinematics is relevant to a null range of s.

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