# PLANETARY MOTION IN A RETARDED NEWTONIAN POTENTIAL FIELD

By Roy J. Kennedy*

Department of Physics, California Institute of Technology

Communicated July 31, 1929

The problem of the motion of a planet within a solar system which is itself in uniform motion with respect to an "absolute" space has been discussed from the time of Laplace up to shortly after the advent of Einstein's gravitational theory. The latest work seems to have been done by Lodge[1] and Eddington,[1] who conclude that the pre-relativity discrepancies between theory and observation are not explicable in terms of variability of mass with velocity of the planet.

Apparently no one has thought of applying to gravitation the idea corresponding to that of effective charge which Lienard[2] and Wiechert[3] developed in electromagnetism. The application is simple; in order to determine the potential of a moving gravitating mass we proceed as follows:

The potential at any point 0 is given by the expression

$$\Phi = k \int\int\int \frac{\rho dV_e}{r_e},$$

integrated over the boundaries of the body, $dV_e$ and $r_e$ being the "effective" volume and radius; the first is the element of volume in a fixed reference system within which each point of the moving element was located at the time when it gave rise to the potential existing at 0 at the slightly later time when the distance from the moving element to 0 is $r$, while $r_e$ is the distance of the center of $dV_e$ from 0. Hence in rectangular fixed axes with $x$-axis in the direction of motion of the mass, if $x$, $y$, $z$, be the instantaneous coördinates of a point of any element of the mass, and $x_e$, $y_e$, $z_e$ the effective coördinates, then

$$\begin{aligned} x_e &= x - vt \\ y_e &= y \\ z_e &= z \end{aligned}$$

where $v$ is the velocity of the body and $t$ is the time of propagation of gravitation from the point to 0. If the coördinates of 0 are $x'$, $y'$, $z'$, $r_e^2 = c^2t^2$ (where $c$ is velocity of gravitation)

$$\begin{aligned} &= (x' - x_e)^2 + (y' - y_e)^2 + (z' - z_e)^2 \\ &= (x' - x)^2 + (y' - y)^2 + (z' - z)^2 + 2vt(x' - x) + v^2t^2. \end{aligned}$$

Or

$$(c^2 - v^2)t^2 - 2v(x' - x)t - r^2 = 0,$$

$$\therefore t = \frac{1}{1-\beta^2}\left\{\frac{v}{c^2}(x'-x) + \frac{r}{c}\left[1 - \frac{\beta^2}{2}\left(1 - \frac{(x'-x)^2}{r^2}\right)\right]\right\}$$

where $$\beta = \frac{v}{c}.$$

$\therefore x_e = x - \beta r - \beta^2(x' - x)$ (if $\beta^3$ and higher powers are neglected) and

$$dx_e = \frac{\partial x_e}{\partial x}dx + \frac{\partial x_e}{\partial r}\frac{\partial r}{\partial x}dx = (1 + \beta\cos\theta + \beta^2)dx$$

where $\theta$ is angle between $x$-axis and $r_e$.

$$\therefore dV_e = dy_e\, dz_e\, dx_e = (1 + \beta\cos\theta + \beta^2)dV.$$

Also $$r_e = ct = r[1 + \beta\cos\theta + \beta^2(1 + \cos^2\theta)]$$

$$\therefore \Phi = k\iiint \frac{\rho(1 + \beta\cos\theta + \beta^2)dV}{r\left[1 + \beta\cos\theta + \frac{\beta^2}{2}(1 + \cos^2\theta)\right]}$$

$$= k\iiint \frac{\rho}{r}\left(1 + \frac{\beta^2}{2}\sin^2\theta\right)dV,$$

if powers of $\beta$ above the second are ignored, as they will be henceforth without mention.

Two distinct cases will be dealt with in determining the orbits; in the first it will be assumed that mass of planet is simply a constant independent of velocity and of direction of acceleration, while in the second the mass will be assumed to correspond to that of the Lorentz electron. In both cases the central mass can be treated as a homogeneous sphere, since only small corrections are being computed; hence, we may write for the present purpose

$$\Phi = \frac{\mu}{r}\left(1 + \frac{\beta^2}{2}\sin^2\theta\right)$$

where $\mu$ is a constant.

*Case I.*—Mass of planet constant.

Here we can apply Lagrange's equations in usual form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = Q_\alpha. \qquad (1.1)$$

Choosing polar coördinates $r$, $\theta$ in plane of orbit, and using $\beta$ for ratio of component of "absolute" velocity of the sun in that plane to velocity of gravitation, we have, since $T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$

$$\frac{\partial T}{\partial \dot{\theta}} = m\dot{\theta}r^2 \quad , \quad \frac{\partial T}{\partial \theta} = 0$$

$$\frac{\partial T}{\partial \dot{r}} = m\dot{r} \quad , \quad \frac{\partial T}{\partial r} = mr\dot{\theta}^2$$

$$Q_\theta = \frac{\partial \Phi}{\partial \theta} = \frac{\mu\beta^2}{2r}\sin 2\theta \tag{1.2}$$

$$Q_r = \frac{\partial \Phi}{\partial r} = -\frac{\mu}{r^2}\left(1 + \frac{\beta^2}{2}\sin^2\theta\right). \tag{1.3}$$

Substituting these values in (1.1) we get the simultaneous equations

$$\frac{d}{dt}(r^2\dot{\theta}) = \frac{\mu\beta^2}{2}\frac{\sin 2\theta}{r} \text{ and} \tag{1.4}$$

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{\mu}{r^2}\left(1 + \frac{\beta^2}{2}\sin^2\theta\right). \tag{1.5}$$

Multiplying (1.4) by $2r^2\dot{\theta}$ we get

$$\frac{d}{dt}(r^2\dot{\theta})^2 = \mu\beta^2 r \sin 2\theta \,.\, \dot{\theta}\,.$$

Now, approximately,

$$r = \frac{h^2}{\mu}\frac{1}{1 + e\cos(\theta - \omega)} \tag{1.6}$$

where $h$ and $\omega$ are constants, and $e$ is eccentricity of orbit.

$$\therefore \frac{d}{dt}(r^2\dot{\theta})^2 = \frac{h^2\,\beta^2\sin 2\theta}{1 + e\cos(\theta - \omega)}\frac{d\theta}{dt}$$

whence, on integration,

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{h}{r^2}\left(\int\frac{\beta^2\sin 2\theta\, d\theta}{1 + e\cos(\theta - \omega)}\right)^{\frac{1}{2}} = \frac{h}{r^2}I^{\frac{1}{2}}, \text{ say.} \tag{1.7}$$

Hence the operator

$$\frac{d}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta} = \frac{h}{r^2}I^{\frac{1}{2}}\frac{d}{d\theta}.$$

$$\therefore \frac{d^2r}{dt^2} = \frac{h^2I^{\frac{1}{2}}}{r^2}\frac{d}{d\theta}\left(\frac{I^{\frac{1}{2}}}{r^2}\frac{dr}{d\theta}\right) = \frac{h^2I^{\frac{1}{2}}}{r^2}\left[\frac{I^{\frac{1}{2}}}{r^2}\frac{d^2r}{d\theta^2} + \frac{1}{2rI^{\frac{1}{2}}}\frac{dI}{d\theta}\frac{dr}{d\theta} - \frac{2I^{\frac{1}{2}}}{r^3}\left(\frac{dr}{d\theta}\right)^2\right].$$

Substituting right side of this equation in (1.5) we get, after rearranging,

$$\frac{d^2r}{d\theta^2} + \frac{1}{2I}\frac{dI}{d\theta}\frac{dr}{d\theta} - \frac{2}{r}\left(\frac{dr}{d\theta}\right)^2 - r = -\frac{\mu r^2}{h^2 I}\left(1 + \frac{\beta^2}{2}\sin^2\theta\right).$$

Changing the dependent variable in this equation by setting $u = \frac{1}{r}$, so that

$$\frac{dr}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta} \quad \text{and} \quad \frac{d^2r}{d\theta^2} = -\frac{1}{u^2}\frac{d^2u}{d\theta^2} + \frac{2}{u^3}\left(\frac{du}{d\theta}\right)^2,$$

we obtain, on reduction,

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2 I}\left(1 + \frac{\beta^2}{2}\sin^2\theta - \frac{h^2}{2\mu}\frac{dI}{d\theta}\frac{du}{d\theta}\right).$$

By (1.7)

$$\frac{dI}{d\theta} = \frac{\beta^2 \sin 2\theta}{1 + e\cos(\theta - \omega)} \quad \text{and by (1.6)} \quad \frac{du}{d\theta} = -\frac{\mu}{h^2}e\sin(\theta - \omega),$$

to a sufficient approximation. Hence the differential equation becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2 I}\left[1 + \frac{\beta^2}{2}\sin^2\theta + \frac{\beta^2}{2}e\frac{\sin 2\theta \sin(\theta - \omega)}{1 + e\cos(\theta - \omega)}\right]. \quad (1.8)$$

When it is recalled that for an undisturbed Newtonian orbit, $r^2\dot{\theta} = h$, it is evident from (1.7) that

$$I = 1 + \int_0^\theta \frac{\beta^2 \sin 2\theta\, d\theta}{1 + e\cos(\theta - \omega)}.$$

Hence (1.8) may be written

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h}\left[1 + \frac{\beta^2}{2}\sin^2\theta + \frac{\beta^2 e}{2}\frac{\sin 2\theta\sin(\theta - \omega)}{1 + e\cos(\theta - \omega)} - \beta^2\int_0^\theta \frac{\sin 2\theta\, d\theta}{1 + e\cos(\theta - \omega)}\right]. \quad (1.9)$$

Solution of an equation of this form, i.e.,

$$\frac{d^2\mu}{d\theta^2} + u = f(\theta)$$

is

$$u = A\cos\theta + B\sin\theta + \int^\theta f(\xi)\sin(\theta - \xi)d\xi. \quad (1.10)$$

In expanded form, $f(\theta)$ evidently consists solely of terms in $\sin^n\theta\cos^m\theta$ so that solution will be derived from terms

$$\int^\theta \sin^n\xi\cos^m\xi\sin(\theta - \xi)d\xi = \sin\theta\int^\theta \sin^n\xi\cos^{m+1}\xi d\xi - \cos\theta\int^\theta \sin^{n+1}\xi\cos^m\xi d\xi.$$

Such integrals yield purely periodic terms (which contribute no observable effects) unless one of the exponents $n$ or $m$ is even while the other is odd. Furthermore, terms containing powers of the eccentricity $e$ above the second are negligibly small. Thus the significant terms on the right side of (1.9) are

$$\frac{\beta^2 e}{2} \sin 2\theta \sin (\theta - \omega) + \beta^2 e \int_0^\theta \sin 2\theta \cos (\theta - \omega) d\theta = f(\theta).$$

Using this result in (1.10) and discarding the small periodic terms as they appear, we get as equation of orbit,

$$u = \frac{\mu}{h^2}\left[1 + e \cos (\theta - \omega) - \frac{\beta^2 e \theta}{8} \sin (\theta - \omega)\right]$$

$$= \frac{\mu}{h^2}\left[1 + e \cos (\theta - \omega) - \frac{\beta^2 e}{8} \cos 2\omega . \theta \sin (\theta - \omega) - \frac{\beta^2 e}{8} \sin 2\omega . \theta \cos (\theta - \omega)\right]$$

which may be written with sufficient approximation

$$u = \frac{\mu}{h^2}\left[1 + \left(e - \frac{\beta^2 e}{8} \sin 2\omega . \theta\right) \cos \left(\theta - \omega + \frac{\beta^2}{8} \cos 2\omega . \theta\right)\right].$$

Hence the perihelion apse advances at rate

$$\delta\omega_0 = -\frac{2\pi}{8}\beta^2 \cos 2\omega$$

per revolution, while eccentricity increases by $\delta e_0 = -\frac{2\pi}{8}\beta^2 e \sin 2\omega$ per revolution. Evidently $\delta e_0 = \tan 2\omega . e\delta\omega_0$. Hence, using values in table[4] on right below for Mercury, $\tan 2\omega = -\frac{8.8}{8.48} = -0.104$.

$\therefore 2\omega = -6°$ or $+174°$. Since $\delta\omega$ is positive, $\cos 2\omega$ must be negative; $\therefore 2\omega = 174°$. The advance per century $\delta\omega = -\frac{2\pi}{8}\beta^2 \cos 2\omega \times \frac{100}{T}$ (where $T$ is period of revolution expressed in years), or $e\delta\omega = -\frac{25\pi e\beta^2 \cos 2\omega}{T}$. Hence for Mercury, for which $e = 0.206$ and $T = 0.24$,

$$\beta^2 = -\frac{8.48 \times 0.24 \times 5 \times 10^{-6}}{25\pi \times 0.206 \times \cos 174°} = 63.5 + 10^{-8}.$$

$\therefore\ \beta = 7.97 \times 10^{-4}$ and if we assume the velocity of gravitation to equal that of light, the velocity of system is $v = \beta c = 7.97 \times 10^{-4} \times 3 \times 10^{+5} = 239$ km. per sec. This is by no means unreasonably large.

Computing $e\delta\omega$ and $\delta e$ for the other inner planets on basis of above values for Mercury and table of longitudes of perihelion, and subtracting results from outstanding pre-relativity values of discrepancies we find as remaining discrepancies from observations per century:

| | PRESENT THEORY | | NEWTONIAN THEORY | |
|---|---|---|---|---|
| | $e\delta\omega$ | $\delta e$ | $e\delta\omega$ | $\delta e$ |
| Mercury | $0.00 \pm 0.43$ | $0.00 \pm 0.50$ | $8.48 \pm 0.43$ | $-0.88 \pm 0.50$ |
| Venus | $-0.05 \pm 0.25$ | $0.09 \pm 0.31$ | $-0.05 \pm 0.25$ | $+0.21 \pm 0.31$ |
| Earth | $-0.03 \pm 0.13$ | $-0.10 \pm 0.10$ | $0.10 \pm 0.13$ | $0.02 \pm 0.10$ |
| Mars | $-1.18 \pm 0.34$ | $0.05 \pm 0.27$ | $0.75 \pm 0.34$ | $0.29 \pm 0.27$ |

Thus the present theory is in good agreement with observation, the discrepancy in perihelial motion of Mars being the only one greater than the probable error. Possibly the application of this theory to the perturbation of Mars by massive Jupiter would remove the latter discrepancy.

*Case II.*—Mass of planet variable.

Here it is assumed that the inertial mass of the planet is dependent on direction of acceleration in same way as in the case of the Lorentz electron, i.e., the mass for acceleration in direction of motion is $\frac{m_0}{(1 - \beta^2)^{3/2}}$ while for acceleration in perpendicular direction it is $\frac{m_0}{(1 - \beta^2)^{1/2}}$, where $\beta$ is ratio of absolute velocity of planet in orbital plane to velocity of gravitation; the gravitational force depends only on the constant $m_0$. The treatment is a little less simple in this case, since the usual form of Lagrange's equations is not applicable.

If rectangular axes are chosen with sun at origin, abscissa in direction of projection of absolute constant velocity $v_0$ of sun, and $xy$-plane in plane of orbit, and if $\theta$, $\varphi$, $\omega$ and $\xi$ are angles from $x$-axis to radius vector to planet, direction of force, direction of perihelial apse and direction of instantaneous absolute motion of planet, respectively, then

$$m_0\alpha_\xi = f_\xi(1 - \beta^2)^{3/2} \quad \text{and} \quad m_0\alpha_n = f_n\,(1 - \beta^2)^{1/2},$$

where $\alpha$ and $f$ refer to acceleration and gravitational force, and subscripts to directions $\xi$ and perpendicular to $\xi$.

$$\therefore\quad m_0\alpha_x = m_0\left[\alpha_\xi \cos\xi + \alpha_n \cos\left(\xi + \frac{\pi}{2}\right)\right].$$

Also

$$f_\xi = f\cos(\xi - \varphi) \quad \text{and}$$

$$f_n = -f\sin(\xi - \varphi)\cdot$$

Hence

$$m_0\alpha_x = f[(1-\beta^2)^{3/2}\cos\xi\cos(\xi-\varphi) + (1-\beta^2)^{1/2}\sin\xi\sin(\xi-\varphi)]$$

$$= f\left\{\left(1-\frac{\beta^2}{2}\right)[\cos\xi\cos(\xi-\varphi) + \sin\xi\sin(\xi-\varphi)] - \beta^2\cos\xi\cos(\xi-\varphi)\right\}$$

$$= f\left[\left(1-\frac{\beta^2}{2}\right)\cos\varphi - \beta^2\cos\xi\cos(\xi-\varphi)\right].$$

Similarly,

$$m_0\alpha_y = f\left[\left(1-\frac{\beta^2}{2}\right)\sin\varphi - \beta^2\sin\xi\cos(\xi-\varphi)\right],$$

$$\therefore\ m_0\alpha = m_0(\alpha_x^2 + \alpha_y^2)^{1/2} = f\left[1 - \frac{\beta^2}{2}(1 + 2\cos^2(\xi-\varphi))\right].$$

Also

$$\cot \widehat{x\alpha} = \frac{\alpha_x}{\alpha_y} = \frac{\cos\varphi(1-\frac{1}{2}\beta^2) - \beta^2\cos\xi\cos(\xi-\varphi)}{\sin\varphi(1-\frac{1}{2}\beta^2) - \beta^2\sin\xi\cos(\xi-\varphi)}$$

$$= \cot\varphi + \beta^2\frac{\cos(\xi-\varphi)}{\sin\varphi}(\sin\xi\cot\varphi - \cos\xi)\cdot$$

Since directions of force and acceleration are but little different, we have approximately on expansion of $\cot \widehat{x\alpha}$ in Taylor's series

$$\cot\widehat{x\alpha} = \cot\varphi + (\widehat{x\alpha} - \varphi)\frac{d}{d\varphi}\cot\varphi + \ldots$$

$$= \cot\varphi - (\widehat{x\alpha} - \varphi)\csc^2\varphi + \ldots$$

Comparing the last two equations we find for angle from force to acceleration vector

$$\widehat{x\alpha} - \varphi = -\frac{\beta^2\cos(\xi-\varphi)}{\sin\varphi\csc^2\varphi}(\sin\xi\cot\varphi - \cos\xi)$$

$$= -\frac{\beta^2}{2}\sin 2(\xi-\varphi) = -\frac{\beta^2}{2}\sin 2(\xi-\theta), \text{ approximately.}$$

Also it follows from (1.2) and (1.3) that angle from force vector to radius vector is

$$\theta - \varphi = \frac{\beta_0^2}{2} \sin 2\theta$$

where $\beta_0 = \frac{v_0}{c}$. Hence angle from radius vector to acceleration vector is

$$\nu = \overset{\wedge}{x\alpha} - \theta = (x\alpha - \varphi) - (\theta - \varphi) = \frac{\beta^2}{2} \sin 2(\theta - \xi) - \frac{\beta_0^2}{2} \sin 2\theta. \quad (2.1)$$

Now since corrections to orbit are very small we can of course compute $\beta$ from velocity in a truly elliptic orbit. By an elementary theorem the orbital velocity may then be resolved into a constant component $ev_1$ inclined at constant angle $\omega + \frac{\pi}{2}$ with $x$-axis and another constant component $v_1$ perpendicular to radius vector, i.e., inclined at angle $\theta + \frac{\pi}{2}$ with $x$-axis. Hence the orbital component of absolute velocity $v_1'$ of planet has components

$$v_x' = v_0 + ev_1 \cos\left(\omega + \frac{\pi}{2}\right) + v_1 \cos\left(\theta + \frac{\pi}{2}\right)$$

$$= v_0 - ev_1 \sin \omega - v_1 \sin \theta \text{ and}$$

$$v_y' = ev_1 \cos \omega + v_1 \cos \theta\cdot$$

$$\therefore \beta^2 = \frac{v'^2}{c^2} = \frac{v_x'^2 + v_y'^2}{c^2}$$

$$= \beta_0^2 + \frac{1}{c^2} \{ [v_1^2(1 + e^2) - 2ev_1v_0 \sin \omega] + 2ev_1^2 \cos (\theta - \omega) - 2v_1v_0 \sin \theta \}. \quad (2.2)$$

(2.1) may be written

$$\nu = \frac{\beta^2}{2} [\sin 2\theta(1 - 2 \sin^2 \xi) - 2 \cos 2\theta \sin \xi \cos \xi] - \frac{\beta_0^2}{2} \sin 2\theta. \quad (2.3)$$

Now

$$\sin \xi = \frac{v_y'}{v'} = \frac{v_1'}{c\beta} (\cos \theta + e \cos \omega)$$

and

$$\cos \theta = \frac{v_x'}{v'} = \frac{v_0}{c\beta}\left[1 - \frac{v_1}{v_0} (\sin \theta + e \sin \omega)\right].$$

Substituting these values in (2.3), and then for $\beta^2$ from (2.2) we obtain after a few transformations

$$\nu = -\frac{v_1 v_0}{c^2}[\cos\theta + e\cos(2\theta - \omega)] - \frac{v_1^2}{c^2}\left[e\sin(\theta - \omega) + \frac{e^2}{2}\sin 2(\theta - \omega)\right] \quad (2.4)$$

If component of acceleration perpendicular to radius in direction of motion is $\alpha_\theta$, we have

$$\alpha_\theta = \alpha\cos\left(\frac{\pi}{2} + \nu\right) = -\alpha\sin\nu = -\alpha\nu = -\frac{f}{m_0}\nu = -\frac{\mu}{r^2}\nu \quad (2.5)$$

to usual approximation.

Similarly the acceleration along radius is

$$\alpha_r = \alpha\cos\nu = -\frac{\mu}{r^2}\left\{1 - \frac{\beta^2}{2}[1 + 2\cos^2(\theta - \xi)]\right\}\left(1 - \frac{\nu^2}{2}\right).$$

$\nu^2$ is negligible. Expanding $\cos^2(\theta - \xi)$ and substituting above values of $\sin\xi$, $\cos\xi$ and $\beta^2$ and dropping small constant terms and terms in $\sin^m\theta\cos^n\theta$ for which both $m$ and $n$ are even or odd, we find

$$\alpha_r = -\frac{\mu}{r^2}\left[1 + \frac{v_1 v_0}{c^2}\sin\theta - \frac{v_1^2}{c^2}e\cos(\theta - \omega)\right]. \quad (2.6)$$

In general, the components of acceleration along and perpendicular to radius vector are, respectively, $\ddot{r} - r\dot{\theta}^2$ and $r\ddot{\theta} + 2\dot{r}\dot{\theta}$. Equating these values to (2.6) and (2.5),

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}\left[1 + \frac{v_1 v_0}{c^2}\sin\theta - \frac{v_1^2}{c^2}e\cos(\theta - \omega)\right]$$

and

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -\frac{\mu}{r^2}\nu.$$

Treating these equations just as the corresponding ones in Case I were treated, we obtain the differential equation

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2 I}\left[1 + \frac{v_1 v_0}{c^2}\sin\theta - \frac{v_1^2}{c^2}e\cos(\theta - \omega) - \frac{h^2}{2\mu}\frac{dI}{d\theta}\frac{du}{d\theta}\right],$$

where

$$I = \int^{\theta} \frac{-\nu d\theta}{1 + e\cos(\theta - \omega)} = 1 - \int_0^{\theta} \frac{\nu d\theta}{1 + e\cos(\theta - \omega)}$$

Hence finally, if terms containing third and higher powers of $e$ are dropped,

$$\frac{d^2\mu}{d\theta^2} + u = \frac{\mu}{h^2}\Bigg[1 + \frac{v_1v_0}{c^2}\sin\theta - \frac{v_1^2}{c^2}e\cos(\theta - \omega) - \frac{e}{2}\frac{\nu\sin(\theta - \omega)}{1 + e\cos(\theta - \omega)} + \int_0^{\theta}\frac{\nu d\theta}{1 + e\cos(\theta - \omega)}$$

Solving this equation by method used with Case I, we obtain

$$u = \frac{\mu}{h^2}\left\{1 + e\cos(\theta - \omega) + \frac{5}{16}e^2\frac{v_1v_0}{c^2}\cos\omega.\theta\cos(\theta - \omega) + \frac{1}{16}e^2\frac{v_1v_0}{c^2}\sin\omega.\theta\sin(\theta - \omega) + \frac{e}{2}\frac{v_1v_0}{c^2}\cos\omega.\theta\right\}$$

$$= \frac{\mu}{h^2}\left\{1 + \left(e + \frac{5}{16}e^2\frac{v_1v_0}{c^2}\cos\omega.\theta\right)\cos\left(\theta - \omega - \frac{1}{16}e\frac{v_1v_0}{c^2}\sin\omega.\theta\right) + \frac{e}{2}\frac{v_1v_0}{c^2}\cos\omega.\theta\right\}$$

to approximation previously used.

Thus the orbits will undergo changes of eccentricity and perihelial phase per revolution of $2\pi.\frac{5}{16}e^2\frac{v_1v_0}{c^2}\cos\omega$ and $+\ 2\pi.\frac{1}{16}e\frac{v_1v_0}{c^2}\sin\omega$, respectively, together with a slow variation in radius. If $\omega$ is computed from the observed motion of Mercury, it is found that the elements for other planets are nearly zero; the agreement with observation is therefore good. The rate of variation of radius even for Mercury is small, somewhat below the limit of observation. However, the value of velocity of solar system in its own plane turns out so large (about 18,000 km. per sec.) that it would surely have shown itself in discrepancies in the times of transit of Jupiter's moons; the value of $v_0$ computed in Case I is just too small to be observable in this way.

In view of the requirement of covariance set forth by Bateman and Einstein, a theory of the kind developed in this paper can scarcely be given great weight; nevertheless it does seem to show that the classical potential theory is capable of accounting for planetary motion about as accurately as does the relativity theory, if less probably.

* Guggenheim Fellow in Physics.

[1] *Phil. Mag.*, **34** and **35**.

[2] *L'Éclairage électrique,* **16,** (1898).

[3] *Arch. Néerl.*, (2) **5,** (1900).

[4] Jeffreys, *Monthly Notices of Royal Ast. Soc.*, Dec., 1916.